

Algebraic and topological interplay of algebraic  
varieties.  
11~17 June 2023. Jaca (Artal 60 & Melle 55)

# Topological and Enumerative Problems of Hyperplane Arrangements.

(Minicourse 14.15.16.June)

Masahiko Yoshinaga (Osaka U.)

# Plan of this course:

(1) Introduction to hyperplane arrangements.

- the characteristic polynomial and DAY 1  
Orlik-Solomon alg.

(2) Topology of Hyperplane Arrangements

- the minimality and  $\pi_1$ .
- Milnor fibers. DAY 2
- $g$ -analogue of Artin complexes. DAY 2

(3) Geometry behind enumerative (quasi) polynomials. DAY 3

- characteristic quasi-polynomials.
- Ehrhart quasi-poly. and reciprocity.
- Log concavity (J. Huh's work)

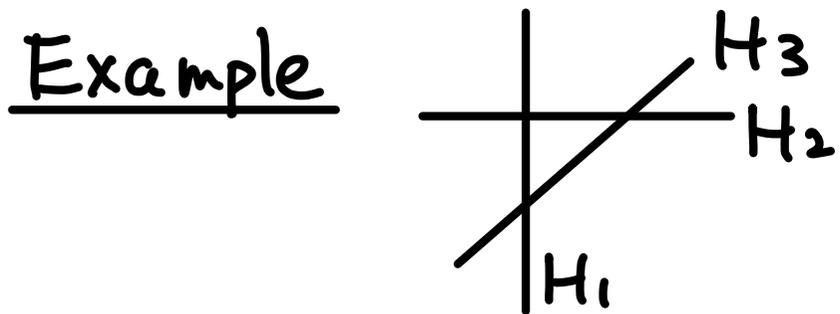
# 1. Hyperplane Arrangements

$V = \mathbb{K}^l$  : a  $l$ -dim affine space /  $\mathbb{K}$ .

A **hyperplane arrangement** is a finite collection

$$A = \{H_1, H_2, \dots, H_n\}.$$

of affine hyperplanes  $H_i \subset V$ .



Notation  $M(A) := \mathbb{K}^l \setminus \bigcup_{i=1}^l H_i$ , (mainly  $\mathbb{K} = \mathbb{C}$ ).

The characteristic polynomial  $\chi(A, t) \in \mathbb{Z}[t]$  measures a kind of "size" of the complement  $M(A)$ .

# 1. Hyperplane Arrangements

For finite sets  $A_1, A_2 \subseteq X$ , everyone knows

$$|X - A_1 \cup A_2| = |X| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

More generally, for  $A_1, \dots, A_n \subseteq X$ .

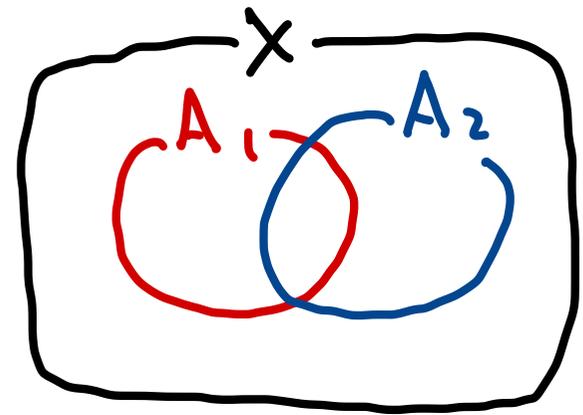
$$|X - \bigcup_{i=1}^n A_i| = \sum_{I \subseteq [n]} (-1)^{|I|} \cdot |A_I|,$$

where,  $[n] = \{1, 2, \dots, n\}$ ,  $A_I = \bigcap_{i \in I} A_i$ , with  $A_\emptyset = X$ .

Def. For an arr. of hyperplanes  $A = \{H_1, \dots, H_n\}$ , the **characteristic polynomial**  $\chi(A, t)$  is

$$\chi(A, t) := \sum_{\substack{I \subseteq [n] \\ H_I \neq \emptyset}} (-1)^{|I|} \cdot t^{\dim H_I},$$

where,  $\dim H_I$  is the dim. of the affine space  $H_I = \bigcap_{i \in I} H_i$ ,  $H_\emptyset = V$ .



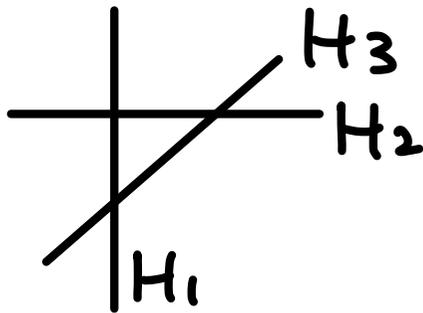
# 1. Hyperplane Arrangements

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Example

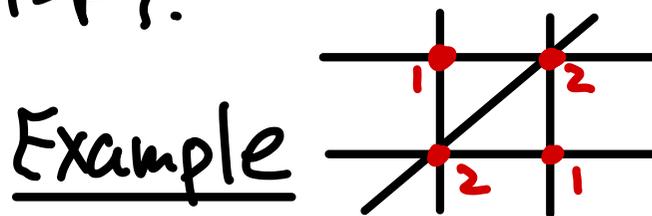


$$\begin{aligned} \chi(A, t) &= t^2 - \underset{H_1}{t} - \underset{H_2}{t} - \underset{H_3}{t} + \underset{H_{12}}{t^2} + \underset{H_{13}}{t^2} + \underset{H_{23}}{t^2} \\ &= t^2 - 3t + 3 \end{aligned}$$

More generally, for a line arr.  $A = \{H_1, \dots, H_n\}$ ,

$$\chi(A, t) = t^2 - |A| \cdot t + \sum_{P: \text{intersection}} (|A_P| - 1).$$

where  $A_P = \{H \in A \mid H \supseteq P\}$ .



$$\chi(A, t) = t^2 - 5t + 6$$

# 1. Hyperplane Arrangements

Def. For an arr. of hyperplanes  $A = \{H_1, \dots, H_n\}$ , the characteristic poly.  $\chi(A, t)$  is

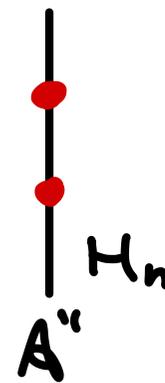
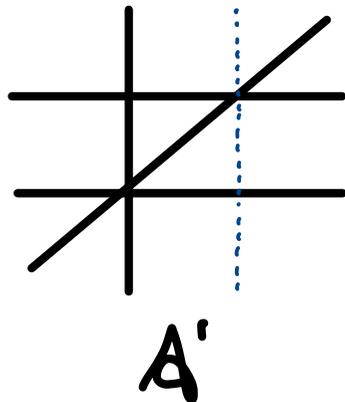
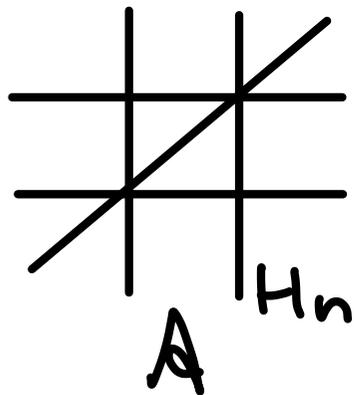
$$\chi(A, t) := \sum_{\substack{I \subseteq [n] \\ H_I \neq \emptyset}} (-1)^{|I|} \cdot t^{\dim H_I},$$

where,  $\dim H_I$  is the dim. of the affine space  $H_I = \bigcap_{i \in I} H_i$ ,  $H_\emptyset = V$ .

Basic property:

$$A = \{H_1, \dots, H_n\}, \quad A' := \{H_1, \dots, H_{n-1}\} = A \setminus \{H_n\}, \quad A'' := H_n \cap A'$$

Arr. defined on  $H_n$ .



Prop.  $M(A') = M(A) \cup M(A'')$

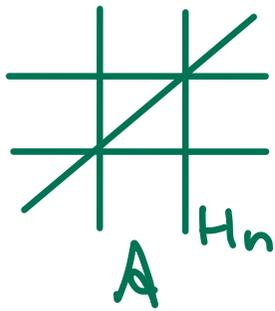
Hence,  $M(A) = M(A') \setminus M(A'')$ . This leads us ...

# 1. Hyperplane Arrangements

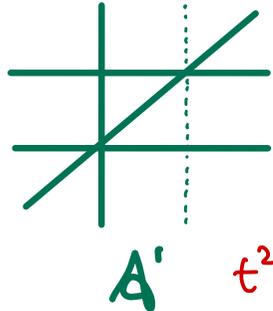
Def.  $\chi(A, t) := \sum_{\substack{I \subseteq [n] \\ H_I \neq \emptyset}} (-1)^{|I|} \cdot t^{\dim H_I}$ ,

Basic property:

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$t^2 - 5t + 6$



$t^2 - 4t + 4$



$t - 2$

Prop.  $M(A') = M(A) \cup M(A'')$ . Hence,  $M(A) = M(A') \setminus M(A'')$ .

Thm. (Deletion-restriction formula.)

$$\chi(A, t) = \begin{cases} t^{\dim V} & \text{if } A = \emptyset \\ \chi(A', t) - \chi(A'', t) & \text{if } A \neq \emptyset. \end{cases}$$

Rem. In  $K_0(\text{Var}_{\mathbb{K}})$ ,  $[M(A)] = \chi(A, [A'_{\mathbb{K}}])$ .

# 1. Hyperplane Arrangements

Thm. (Deletion-restriction formula.)

$$\chi(A, t) = \begin{cases} t^{\dim V} & \text{if } A = \emptyset \\ \chi(A', t) - \chi(A'', t) & \text{if } A \neq \emptyset \end{cases}$$

Several well-known results

① (Crapo-Rota) If  $K = \mathbb{F}_2$ ,  $|M(A)| = \chi(A, 8)$  ↕ det

② (Zaslavsky) If  $K = \mathbb{R}$ , and  $A$  is **essential**, then

the set of chambers  $|ch(A)| = (-1)^l \cdot \chi(A, -1)$

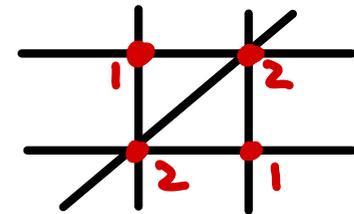
$$|bch(A)| = (-1)^l \cdot \chi(A, 1)$$

**bda chambers**

③ (Orlik-Solomon)  $K = \mathbb{C}$ , then

$$\text{Poin}(M(A), t) = (-t)^l \cdot \chi(A, -\frac{1}{t}).$$

∃ 0-dim intersection



$$\chi(A, t) = t^2 - 5t + 6$$

$$\chi(A, -1) = 12, \quad \chi(A, 1) = 2.$$

$$\text{Poin}(M, t) = 1 + 5t + 6t^2.$$

# 1. Hyperplane Arrangements

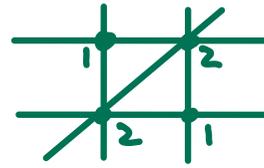
## Several well-known results

① (Crapo-Rota) If  $\mathbb{K} = \mathbb{F}_2$ ,  $|M(A)| = \chi(A, 0)$

② (Zaslavsky) If  $\mathbb{K} = \mathbb{R}$ , and  $A$  is essential, then

$$|ch(A)| = (-1)^l \cdot \chi(A, -1)$$

$$|bch(A)| = (-1)^l \cdot \chi(A, 1)$$



$$\chi(A, t) = t^2 - 5t + 6$$

$$\chi(A, -1) = 12, \quad \chi(A, 1) = 2.$$

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③ (Orlik-Solomon)  $\mathbb{K} = \mathbb{C}$ , then

$$\text{Poin}(M(A), t) = (-t)^l \cdot \chi(A, -\frac{1}{t}).$$

## A More recent well-known result:

Thm (J. Huh 2012, Conjectured by Read 1968)

$$\text{Let } \chi(A, t) = t^l - a_1 t^{l-1} + a_2 t^{l-2} - \dots + (-1)^l a_l. \quad (a_i \geq 0)$$

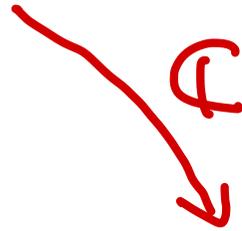
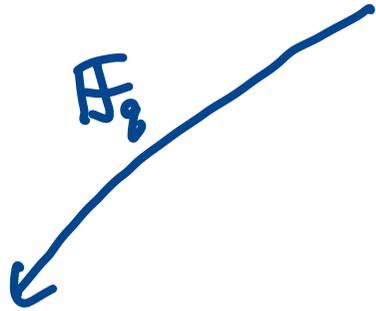
Then

$$a_i^2 \geq a_{i-1} \cdot a_{i+1}.$$

# 1. Hyperplane Arrangements

Example (Braid (trenza!) arr.)  $H_{ij} = \{(x_1, \dots, x_\ell) \in \mathbb{K}^\ell \mid x_i = x_j\}$

$$Br(\ell) = \{H_{ij} \mid 1 \leq i < j \leq \ell\}.$$



$$M(\Lambda) = \{(x_1, \dots, x_\ell) \in \mathbb{F}_q^\ell : x_i \neq x_j\}$$

{chambers}

$\exists$  fibration

$$M(Br(\ell)) \rightarrow M(Br(\ell-1))$$

$$(x_1, \dots, x_{\ell-1}, x_\ell) \mapsto (x_1, \dots, x_{\ell-1})$$

with fiber

$$\mathbb{C} \setminus \{x_1, \dots, x_{\ell-1}\}$$

$$\text{Poin}(M(Br(\ell)), t)$$

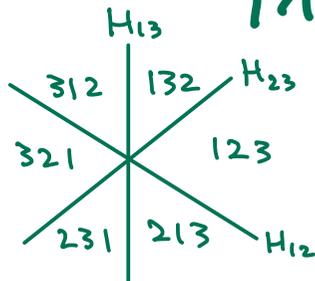
$$= (1+t)(1+2t)\dots(1+(\ell-1)t)$$



$$|M(\Lambda)| = q \cdot (q-1) \cdot \dots \cdot (q-\ell+1) \quad |ch(Br(\ell))| = \ell!$$

$$= \chi(Br(\ell), q)$$

$$= |\chi(Br(\ell), -1)|$$



# 1. Hyperplane Arrangements

$\chi(A, t)$  is related to many enumerative problems.

$G = (V, E)$  : a finite simple graph.  $V = \{1, 2, \dots, \ell\}$ .



Def. A map  $c: V \rightarrow [k]$  is a  $k$ -coloring if it satisfies  $(i, j) \in E \implies c(i) \neq c(j)$ .

(Adjacent vertices should have different colors.)

Def.  $\chi(G, k) = \#\{c: V \rightarrow [k] : k\text{-coloring of } G\}$

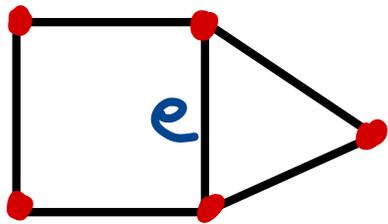
Example  $\chi(\text{---}, k) = k(k-1)$ ,  $\chi(\triangle, k) = k(k-1)(k-2)$

# 1. Hyperplane Arrangements

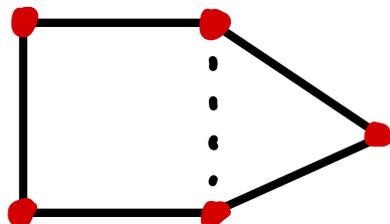
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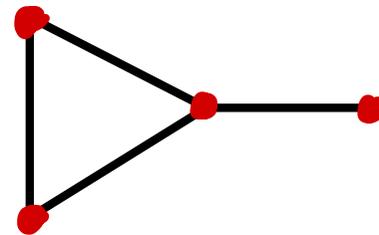
Def Let  $e \in E$ .  $G \setminus e$  is the deletion,  $G/e$  is the contraction.



$G$



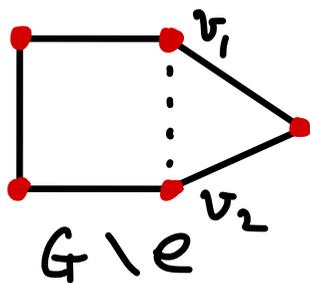
$G \setminus e$



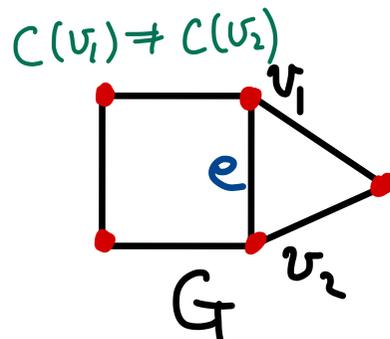
$G/e$

Suppose  $e$  connects  $v_1$  and  $v_2 \in V$ .

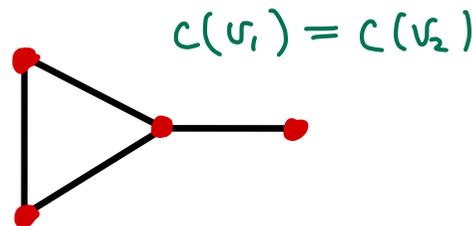
$\{k\text{-coloring of } G \setminus e\} = \{ \text{of } G \} \sqcup \{ \text{of } G/e \}$



$G \setminus e$



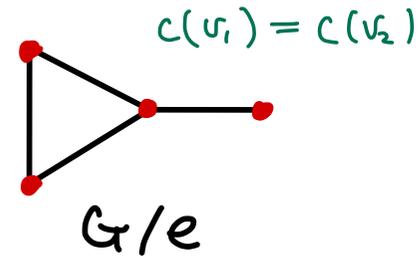
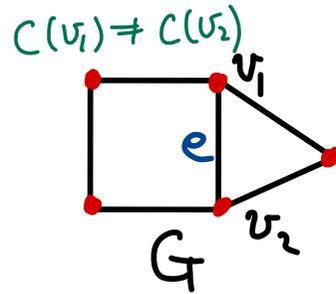
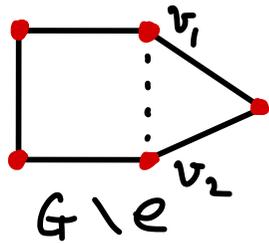
$G$



$G/e$

# 1. Hyperplane Arrangements

$$\{\mathbb{k}\text{-coloring of } G \setminus e\} = \{\text{of } G\} \sqcup \{\text{of } G/e\}$$



Prop Let  $G = (V, E)$ ,  $V = [q]$ . Then

$$\chi(G, \mathbb{k}) = \begin{cases} \mathbb{k}^q & \text{if } E = \emptyset \\ \chi(G \setminus e, \mathbb{k}) - \chi(G/e, \mathbb{k}) & \text{if } E \neq \emptyset. \end{cases}$$

Cor.  $\chi(G, \mathbb{k})$  is a polynomial in  $\mathbb{k}$ . (chromatic poly.)

# 1. Hyperplane Arrangements

Prop Let  $G=(V,E)$ ,  $V=[\ell]$ . Then

$$\chi(G, \mathbb{k}) = \begin{cases} \mathbb{k}^\ell & \text{if } E = \emptyset \\ \chi(G \setminus e, \mathbb{k}) - \chi(G/e, \mathbb{k}) & \text{if } E \neq \emptyset. \end{cases}$$

The above formula very much looks like

Thm. (Deletion-restriction formula.)

$$\chi(A, t) = \begin{cases} t^{\dim V} & \text{if } A = \emptyset \\ \chi(A', t) - \chi(A'', t) & \text{if } A \neq \emptyset. \end{cases}$$

Actually it is.

Def For a graph  $G=(V,E)$ ,  $V=[\ell]$ , let

$A_G := \{H_{ij} \mid (ij) \in E\}$ . Then  $\chi(A_G, \mathbb{k}) = \chi(G, \mathbb{k})$ .

$\uparrow$   $\{x_i - x_j = 0\}$

# 1. Hyperplane Arrangements

## Cohomology ring

Recall  $H^*(\mathbb{C}^{\times n}, \mathbb{Z}) \cong \Lambda E$ , where  $E = \bigoplus_{i=1}^n \mathbb{Z} \cdot e_i$ .  
*← exterior algebra.*

Let  $A = \{H_1, \dots, H_n\}$  be an arr. in  $\mathbb{C}^{\ell}$ .

$d_i$ : the defining equation of  $H_i = \{d_i = 0\}$ .

$$\begin{array}{ccc} \mathbb{C}^{\ell} & \xrightarrow{(d_1, \dots, d_n)} & \mathbb{C}^n \\ \cup & & \cup \\ M(A) & \longrightarrow & (\mathbb{C}^{\times})^n \end{array}$$

induces an algebra homomorphism

$$H^*(M(A)) \longleftarrow H^*(\mathbb{C}^{\times n}) \cong \Lambda E.$$

This is surjective, and Orlik-Solomon gives ...

# 1. Hyperplane Arrangements

Cohomology ring Let  $A = \{H_1, \dots, H_n\}$ ,  $H_i = \{d_i = 0\}$ ,  $E = \bigoplus_{i=1}^n \mathbb{Z} \cdot e_i$ .

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{(d_1, \dots, d_n)} & \mathbb{C}^n \\ \cup & & \cup \\ M(A) & \longrightarrow & (\mathbb{C}^*)^n \end{array} \text{ induces } H^*(M(A), \mathbb{Z}) \longleftarrow H^*((\mathbb{C}^*)^n) \cong \wedge E.$$

Def  $I_A \subseteq \wedge E$  is an ideal generated by

- $e_S = e_{s_1} \wedge e_{s_2} \wedge \dots \wedge e_{s_k}$ , for  $S = \{s_1, \dots, s_k\} \subset [n]$  with  $H_S = \emptyset$ .

- $\sum_{i=1}^{k-1} (-1)^{i-1} e_{s_1} \wedge \dots \wedge \widehat{e_{s_i}} \wedge \dots \wedge e_{s_k}$ , for  $S \subset [n]$  with  $\text{codim } H_S < |S|$

Thm (Orlik-Solomon)

$$H^*(M(A), \mathbb{Z}) \cong \wedge E / I_A. \quad (\text{Note } e_i \leftrightarrow \frac{1}{2\pi\sqrt{-1}} \frac{dd_i}{d_i} \in H^1(M).)$$

(dependent subset)

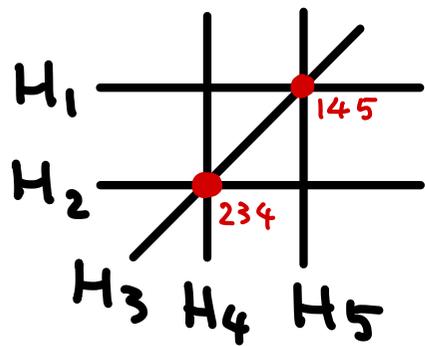
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- $e_S = e_{s_1} \wedge e_{s_2} \wedge \dots \wedge e_{s_k}$ , for  $S = \{s_1, \dots, s_k\} \subset [n]$  with  $H_S = \emptyset$ .
- $\sum_{i=1}^{k-1} (-1)^{i-1} e_{s_1} \wedge \dots \wedge \widehat{e_{s_i}} \wedge \dots \wedge e_{s_k}$ , for  $S \subset [n]$  with  $\text{codim } H_S < k$ .

Thm (Orlik-Solomon)  $H^*(M(A), \mathbb{Z}) \cong \Lambda E / I_A$ . (Note  $e_i \leftrightarrow \frac{1}{2\pi\sqrt{-1}} \frac{dd_i}{\alpha_i} \in H^1(M)$ .)

Example



$$E = \bigoplus_{i=1}^5 \mathbb{Z} e_i$$

$$H^*(M(A)) \cong \Lambda E / \left( \begin{array}{l} e_{12}, e_{45}, e_{23} - e_{24} + e_{34} \\ e_{13} - e_{15} + e_{35} \end{array} \right)$$

$e_1 \wedge e_2 (H_1 \cap H_2 = \emptyset)$

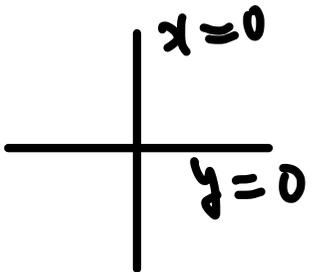
Def  $R$ : comm. ring,  $A_R(A) = H^*(M(A), R)$ ,  $\omega \in A_R(A)$

$(A_R(A), \omega^\wedge)$  is called the **Aomoto complex**.

## 2. Topology of arrangements

Mainly, we consider **complexified real arr**, meaning  $A = \{H_1, \dots, H_n\}$  is defined /  $\mathbb{R}$ , consider  $M = M(A) = \mathbb{C}^\ell \setminus \bigcup_{H \in A} H \otimes \mathbb{C}$ .

Ex.

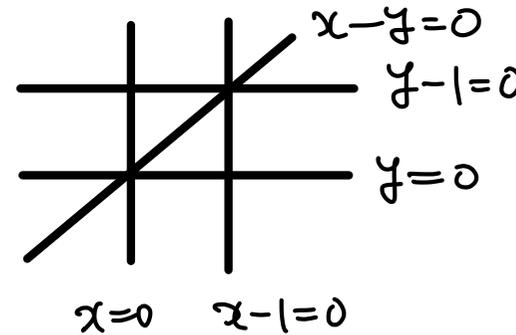


$$M(A) = \{(x, y) \in \mathbb{C}^2 \mid x \neq 0, y \neq 0\}$$

$$= \mathbb{C}^* \times \mathbb{C}^*$$

$$\simeq (S^1)^2 \quad (\text{Cor. } \pi_1(M) = H_1(M) = \mathbb{Z}^2)$$

Ex.



$$M(A) = \mathbb{C}^2 \setminus \left\{ \begin{array}{l} x(x-1)y(y-1) \\ \times (x-y)=0 \end{array} \right\}$$

By combining Zaslavsky's and Orlik-Solomon's results,

Prop Let  $A$ : arr. in  $\mathbb{R}^\ell$ .

$$|\text{ch}(A)| = \sum_i b_i(M), \quad |\text{bch}(A)| = (-1)^\ell \cdot \chi(M)$$

## 2. Topology of arrangements

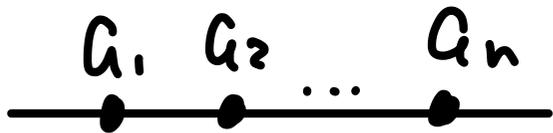
Prop Let  $A$ : arr. in  $\mathbb{R}^l$ .

$$|ch(A)| = \sum_i b_i(M), \quad |bch(A)| = (-1)^l \cdot \chi(M)$$

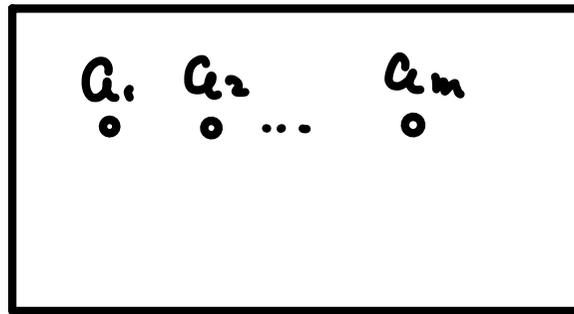
Ex ( $l=1$ )

$$|ch(A)| = n+1$$

$$M = M(A) = \mathbb{C} \setminus \{a_1, \dots, a_n\}$$



$$|bch(A)| = n-1$$



$$b_0 = 1, \quad b_1 = n, \quad \chi = 1 - n.$$

Prop Let  $X$  be a finite CW complex. Let  $S_k$  be the set of all  $k$ -cells. Then  $|S_k| \geq b_k(X)$ .

(proof)  $X \rightsquigarrow$  chain cpx  $C_k = \bigoplus_{\sigma \in S_k} \mathbb{Z} \cdot \sigma \rightsquigarrow b_k = \text{rk } H_k(C, \mathbb{Z})$

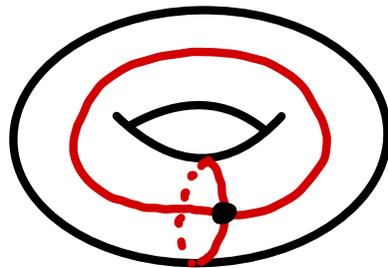
$\nearrow$   
 $\text{rk} = |S_k|$

## 2. Topology of arrangements

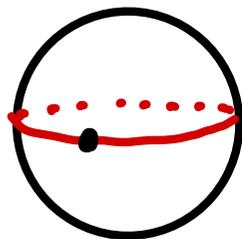
Prop Let  $X$  be a finite CW complex. Let  $S_k$  be the set of all  $k$ -cells. Then  $|S_k| \geq b_k(X)$ .

Def.  $X$  is a minimal CW cpx if  $|S_k| = b_k(X)$ .

Example



are minimal complexes.



are not.

Prop Let  $X$  be a minimal CW cpx. Then  $H_k(X, \mathbb{Z}) \cong \mathbb{Z}^{|S_k|}$ .

In particular,  $H_k(X, \mathbb{Z})$  is torsion free.

(proof) If the boundary map  $C_{k+1} = \bigoplus_{\sigma \in S_{k+1}} \mathbb{Z} \cdot \sigma \xrightarrow{\partial} C_k$

does not vanish,  $b_k$  drops. Hence it must be  $\partial = 0$ . //

## 2. Topology of arrangements

Prop Let  $X$  be a minimal CW cpx. Then  $H_*(X, \mathbb{Z}) \cong \mathbb{Z}^{|S|}$ .

In particular,  $H_*(X, \mathbb{Z})$  is torsion free.

Rem •  $\nexists$  minimal CW cpx s.t.  $X \cong \mathbb{R}P^2$ .

Because  $H_1(\mathbb{R}P^2, \mathbb{Z}) = \mathbb{Z}/2$ .

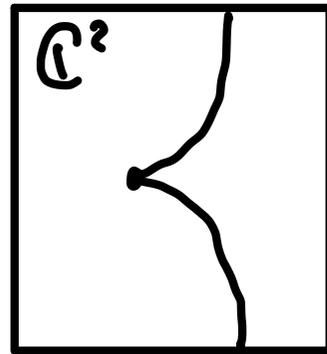
•  $\nexists$  minimal CW-cpx  $X$  s.t.  $X \cong M := \mathbb{C}^2 \setminus \{x^2 = y^3\}$

because

$$\pi_1(M) \cong \langle \gamma_1, \gamma_2 \mid \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \rangle$$

( $\ncong \mathbb{Z}$ )

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z}.$$



Thm (Dimca-Papadima, Randell 2002)

For any complex arr.  $A$  in  $\mathbb{C}^l$ ,  $\exists$  minimal

$l$ -dim CW-cpx  $X$  s.t.  $X \cong M(A)$ .

## 2. Topology of arrangements

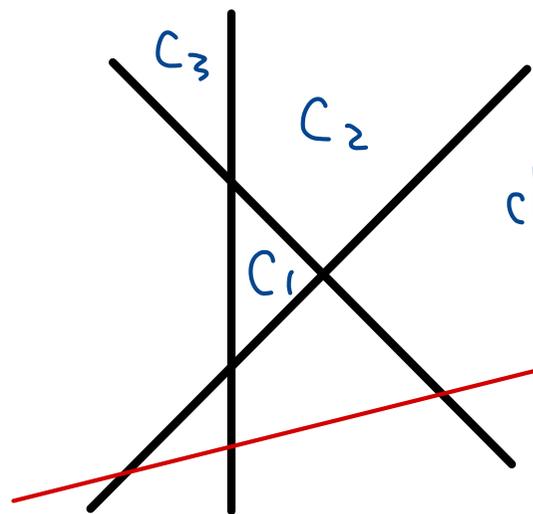
Thm (Dimca-Papadima, Randell 2002)

For any complex arr.  $A$  in  $\mathbb{C}^l$ ,  $\exists$  minimal  $l$ -dim CW-cpx  $X$  s.t.  $X \simeq M(A)$ .

Sketch of proof (Y. 2007) for  $A / \mathbb{R}$

$$A = \{H_1, \dots, H_n\}, \quad H_i = \{d_i = 0\}.$$

Let  $F = \{f=0\}$  be a generic hyperplane.



$$ch_F(A) = \{C_1, C_2, C_3\} \quad \text{Consider}$$

$$\text{Def. } ch_F(A) = \{C \in ch(A) \mid C \cap F = \emptyset\}$$

(Note:  $|ch_F(A)| = b_\emptyset$ )

$$\varphi : M(A) \rightarrow \mathbb{R}_{\geq 0}$$

$$x \mapsto \left| \frac{f(z)^{n+1}}{\prod d_i} \right|.$$

Observation:  $\varphi|_C$  diverges near  $\partial C$ .

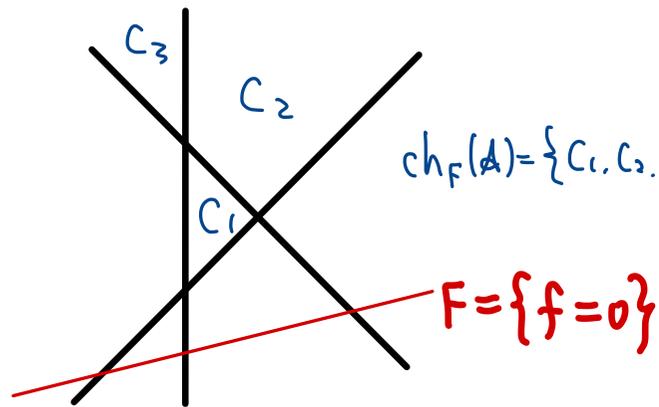
For each  $C \in ch_F(A)$ ,  $\exists p_C \in \text{Crit}(\varphi|_C)$ .

## 2. Topology of arrangements

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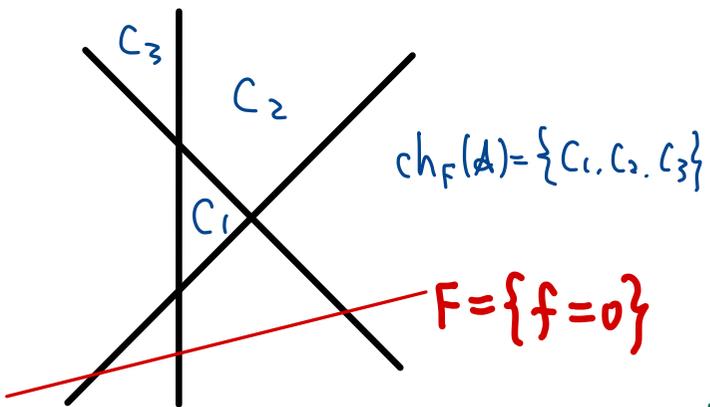
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Prop (after little perturbation) This gives

- (i) 1-1 corresp.  $ch_F(A) \leftrightarrow \text{Crit}(\varphi)$
- (ii)  $p_C$  has Morse index  $l$ .
- (iii)  $C$  is the stable manifolds of  $p_C$ .

# 2. Topology of arrangements



$$\varphi: M(A) \rightarrow \mathbb{R}_{\geq 0}$$

$$x \mapsto \left| \frac{f(z)^{n+1}}{\prod d_i} \right|$$

$$\varphi^{-1}(0) = F \cap M(A)$$

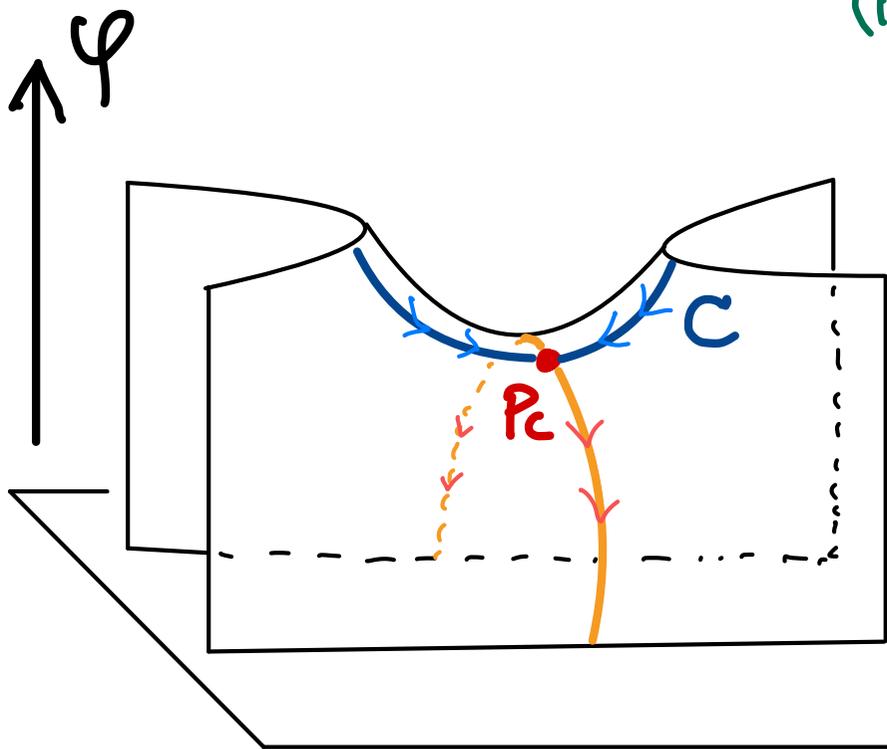
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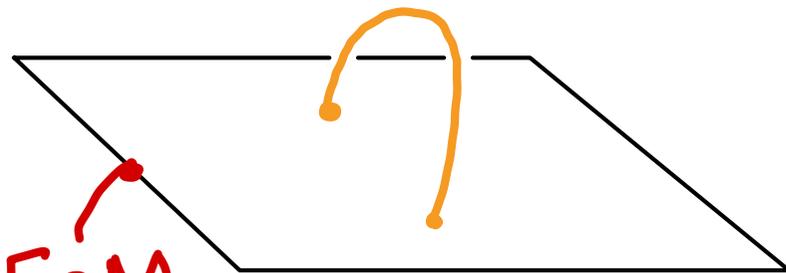
(iii)  $C$  is the stable manifolds of  $p_c$ .

$$C \in ch_F(A), \exists p_c \in \text{Crit}(\varphi|_C)$$

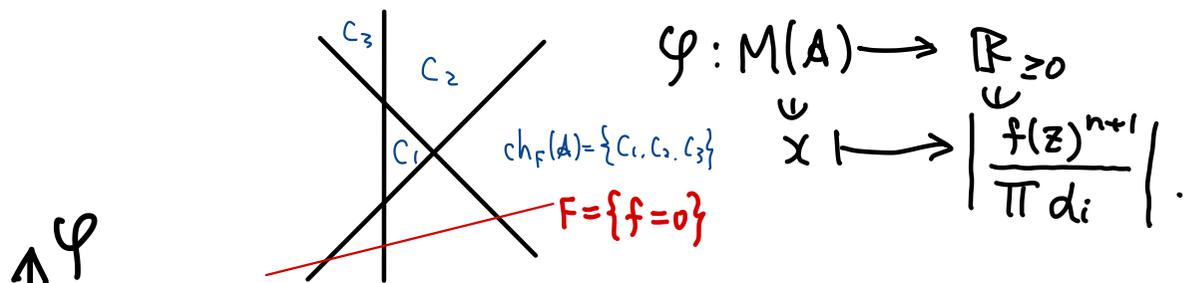


Retract  
by  
 $-\text{grad } \varphi$

$$\varphi^{-1}(0) = F \cap M$$

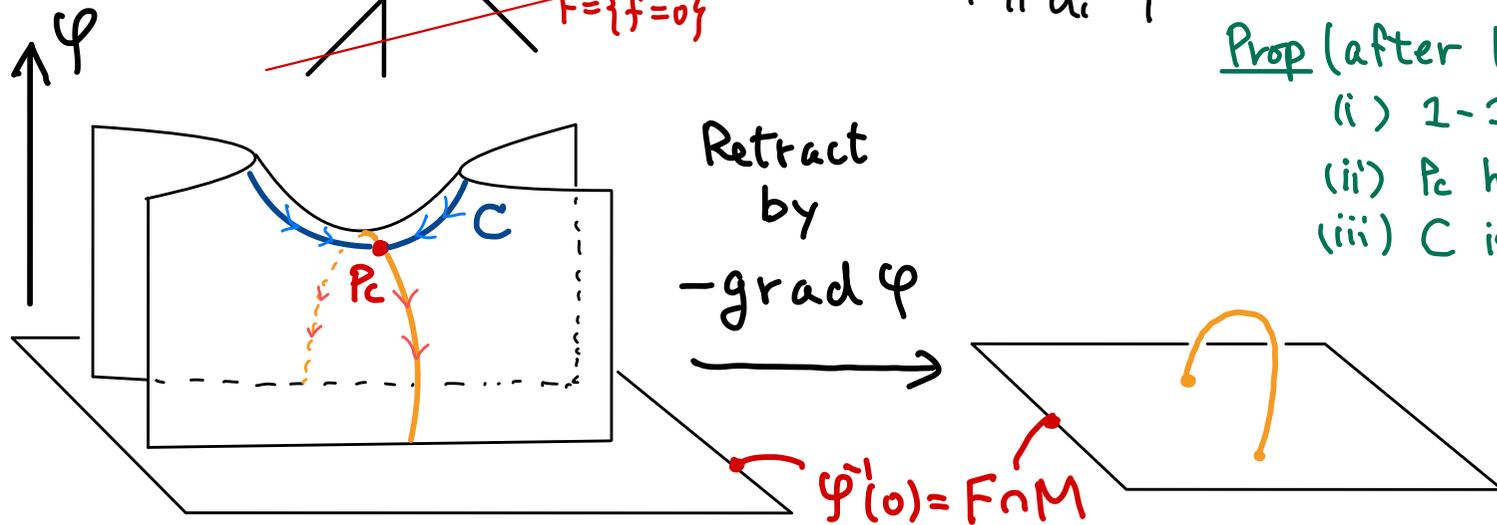


# 2. Topology of arrangements



$$\varphi^{-1}(0) = F \cap M(A)$$

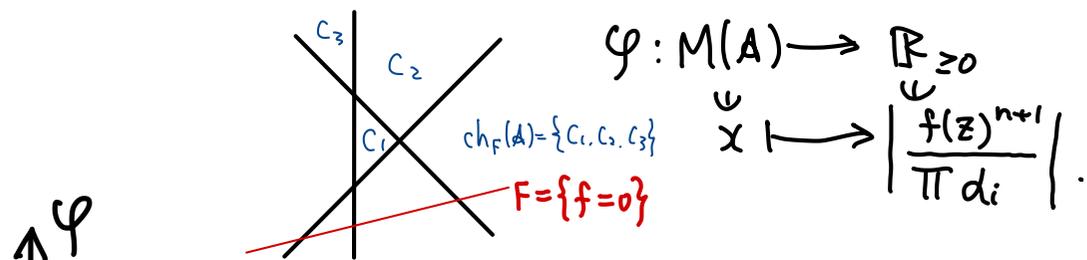
- Prop (after little perturbation) This gives
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The unstable cells (orange cells) describe the attaching of  $l$ -cells to  $F \cap M(A)$ .

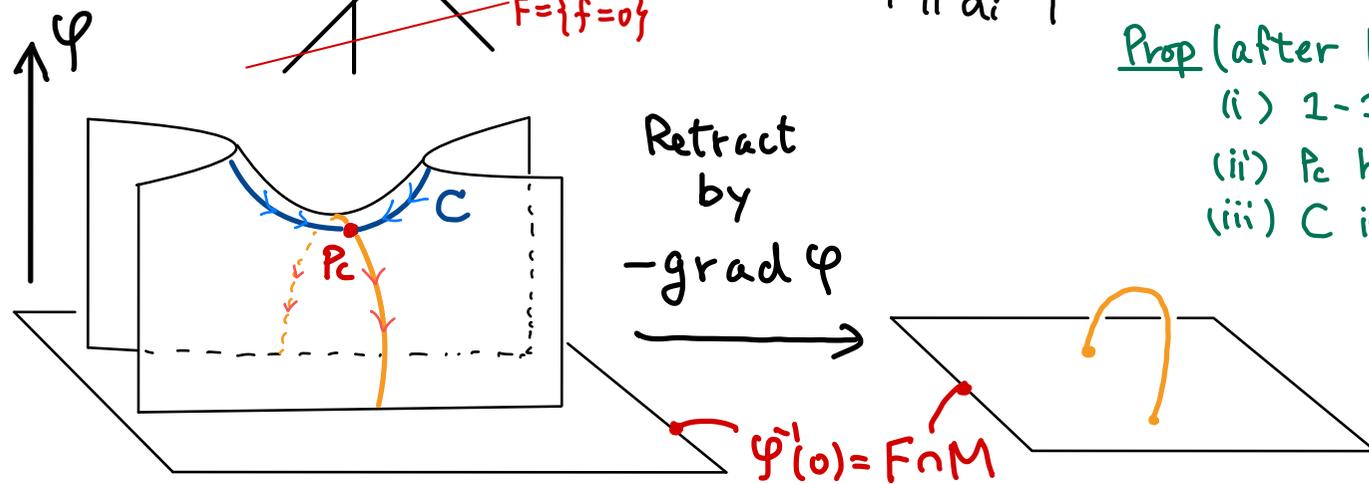
Since  $F \cap M(A)$  is a complement of  $A \cap F$ , we may assume  $F \cap M(A)$  is homotopic to an  $(l-1)$ dim minimal CW-cpx. Then  $|ch_{\varphi}(A)| = b_l$  concludes  $M$  is minimal. //

# 2. Topology of arrangements



$$\varphi^{-1}(0) = F \cap M(A)$$

- Prop (after little perturbation) This gives
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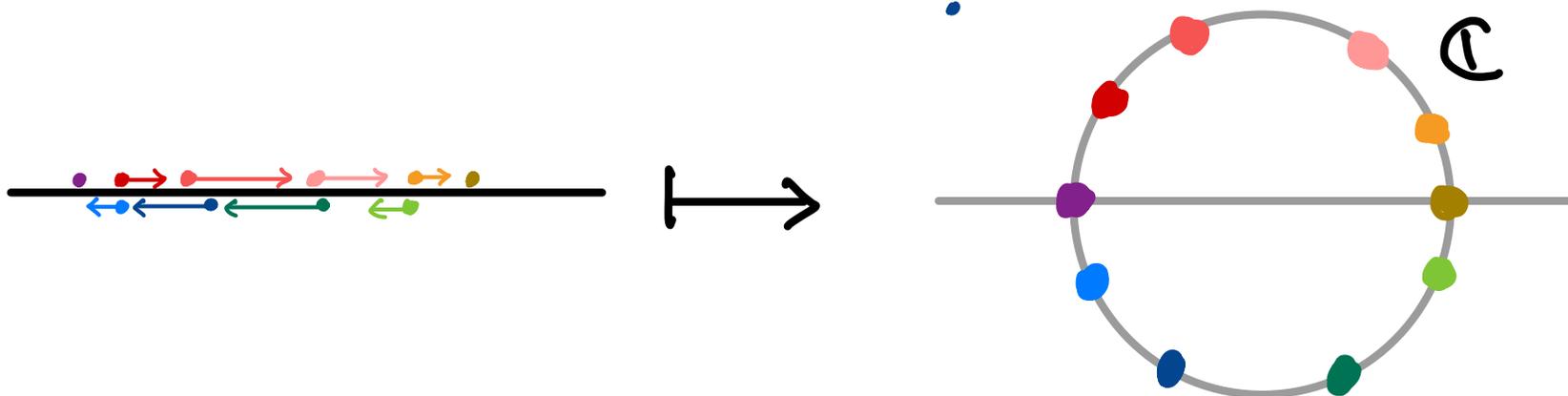
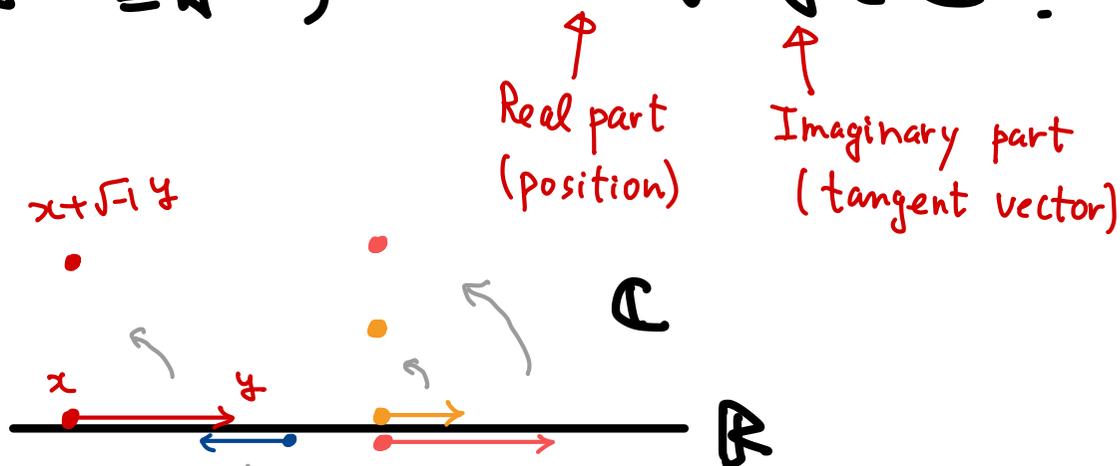
The attaching maps (up to homotopy) are explicitly described by using A'Campo's divide type description.

## 2. Topology of arrangements

The idea: Identify  $\mathbb{C}^l$  with the total space of the tangent bundle of  $\mathbb{R}^l$ .

$$(x \in \mathbb{R}^l, \gamma \in T_x \mathbb{R}^l \cong \mathbb{R}^l) \mapsto x + \sqrt{-1} \cdot \gamma \in \mathbb{C}^l.$$

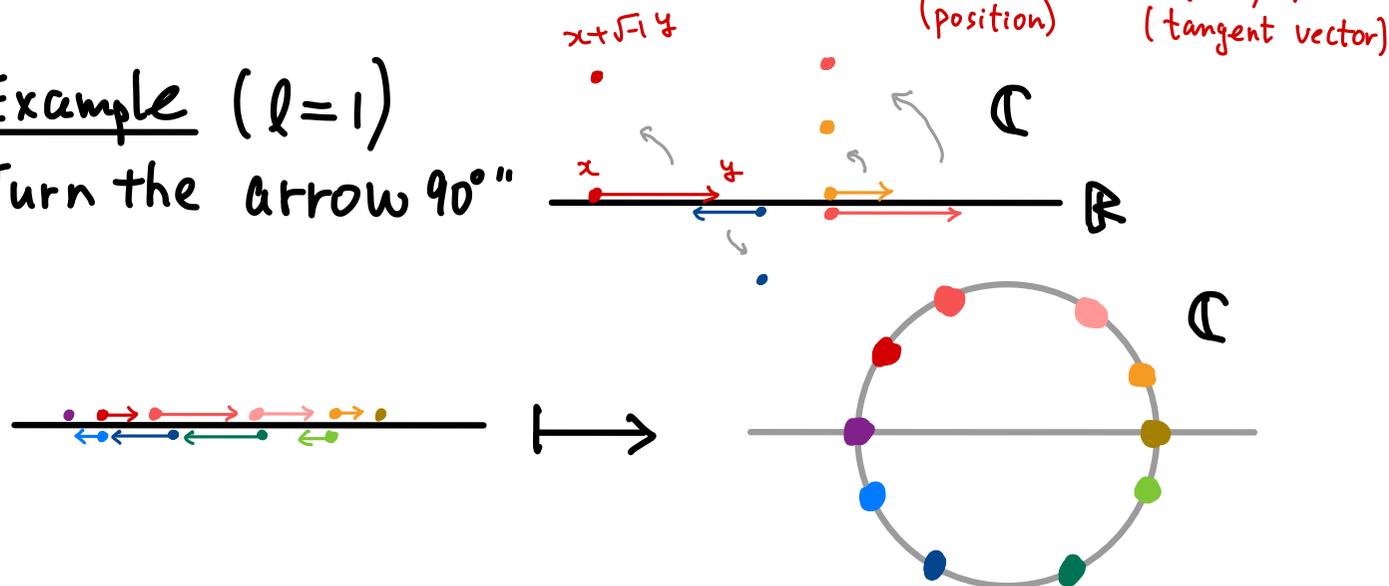
Example ( $l=1$ )  
"Turn the arrow 90°"



## 2. Topology of arrangements

$$(x \in \mathbb{R}^l, \gamma \in T_x \mathbb{R}^l \cong \mathbb{R}^l) \mapsto x + \sqrt{-1} \cdot \gamma \in \mathbb{C}^l.$$

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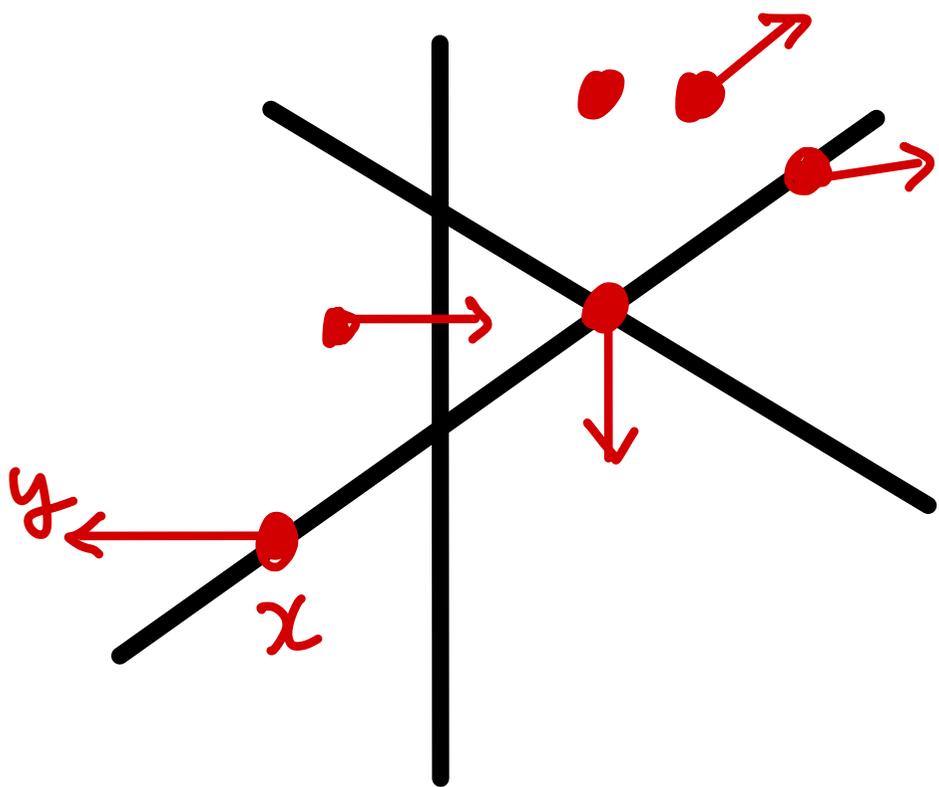
Using this identification, for  $\mathcal{A} = \{H_1, \dots, H_n\} / \mathbb{R}$ ,  
 $M(\mathcal{A})$  is described as

$$M(\mathcal{A}) = \{ (x, \gamma \in T_x \mathbb{R}^l) \mid \text{if } x \in H \ (H \in \mathcal{A}), \gamma \notin T_x H \}$$

## 2. Topology of arrangements

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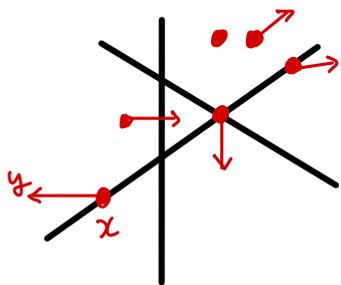


If the real part is on  $H$ , the imaginary part must be transversal to  $H$ .

## 2. Topology of arrangements

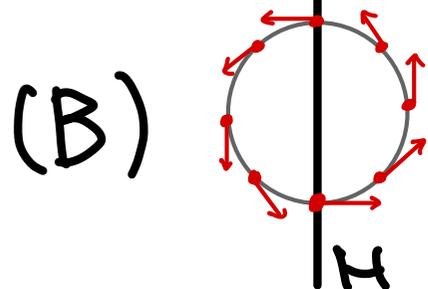
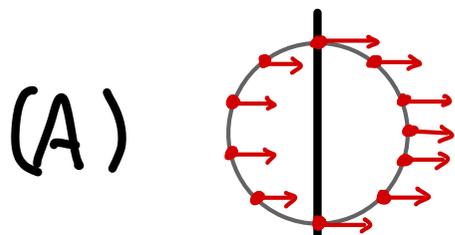
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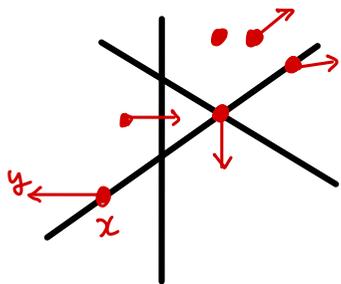
Quiz 1  $A = \{H\}$ . Which is linking?



## 2. Topology of arrangements

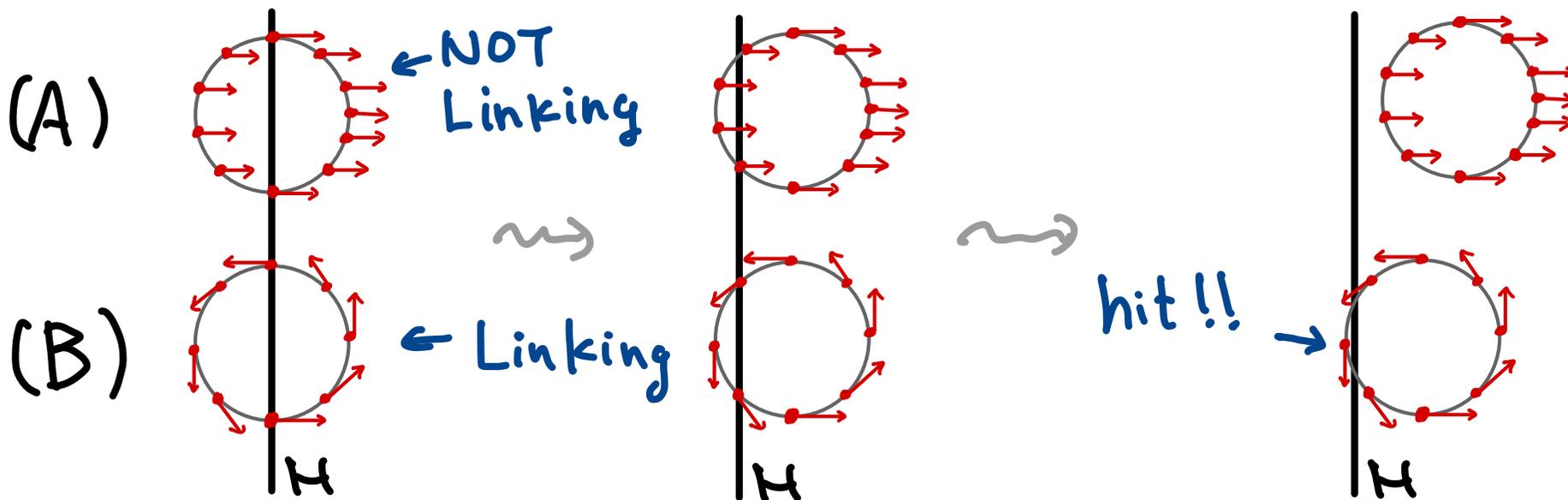
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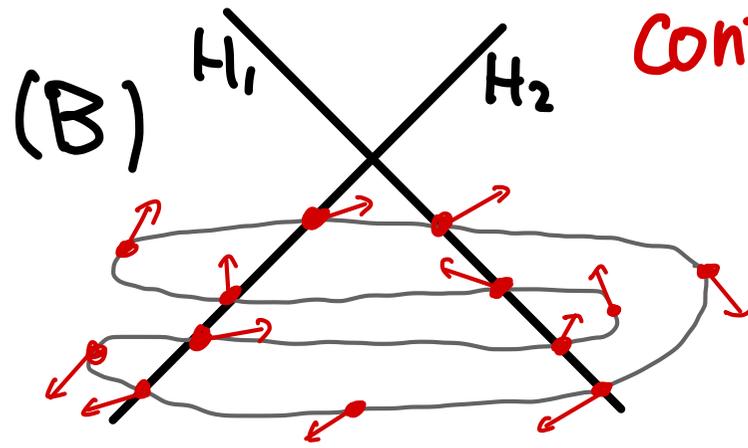
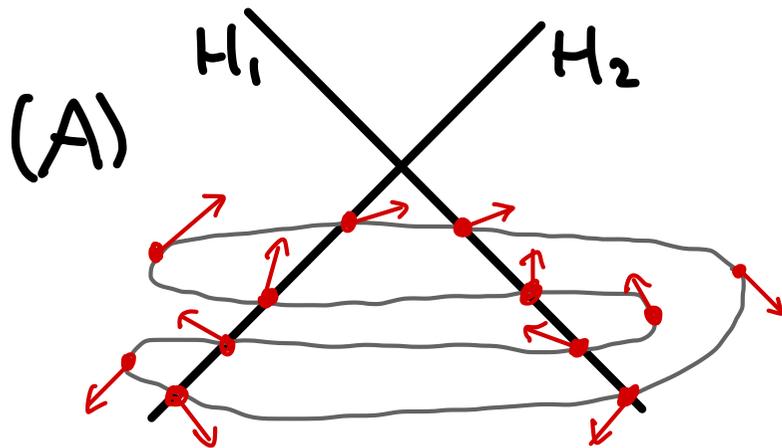
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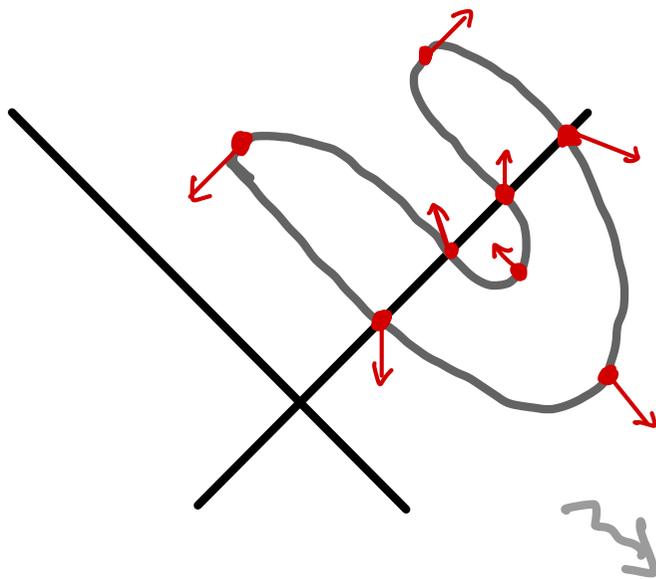
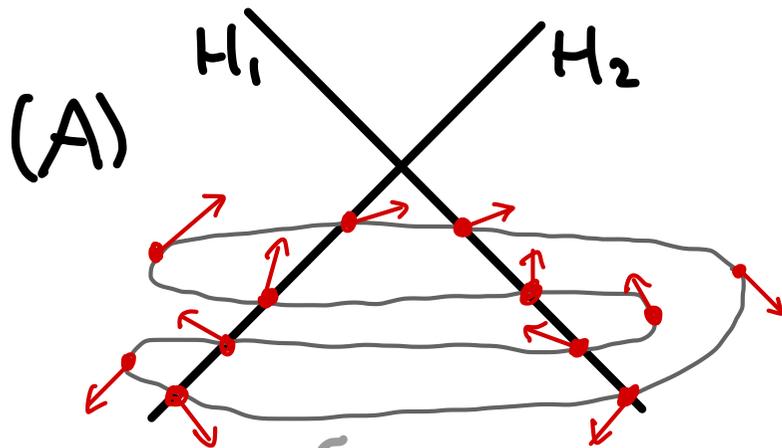
## 2. Topology of arrangements

Quiz 2 :  $A = \{H_1, H_2\}$ . Which is linking? (interpolate continuously)

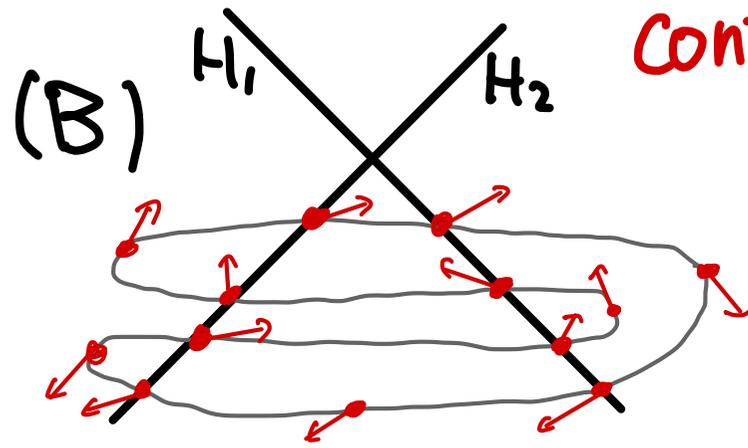


# 2. Topology of arrangements

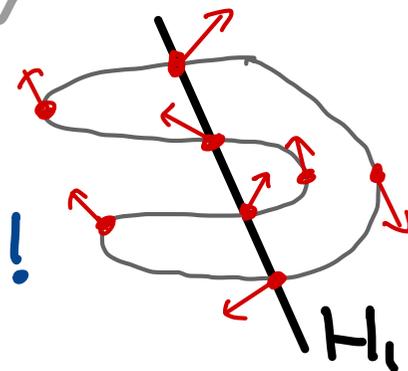
Quiz 2:  $A = \{H_1, H_2\}$ . Which is linking? (interpolate continuously)



NOT LINKING!



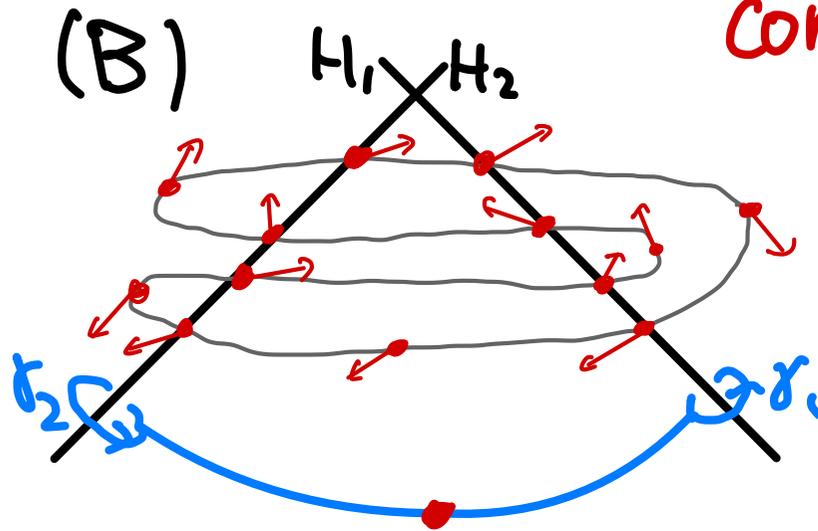
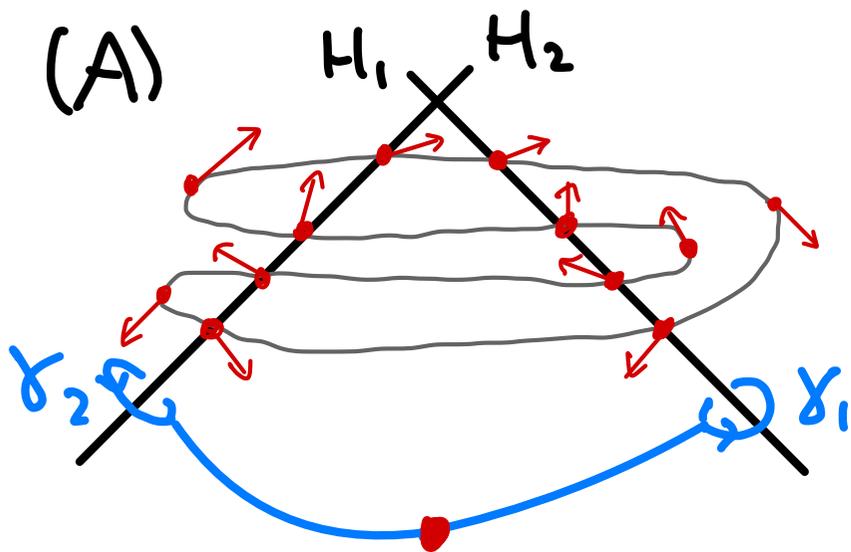
remove  $H_2$  ↓  
turning  $H_1$   
twice!!



LINKING!

## 2. Topology of arrangements

Quiz 2:  $A = \{H_1, H_2\}$ . Which is linking? (interpolate continuously)



Take meridians  $\gamma_1, \gamma_2$ . Corresponding words are

$$\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$$

$$\gamma_1 \gamma_2 \gamma_1 \gamma_2$$

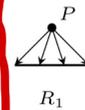
# 2. Topology of arrangements

Using these techniques,  
we can describe attaching  
maps (Y.07).

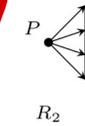
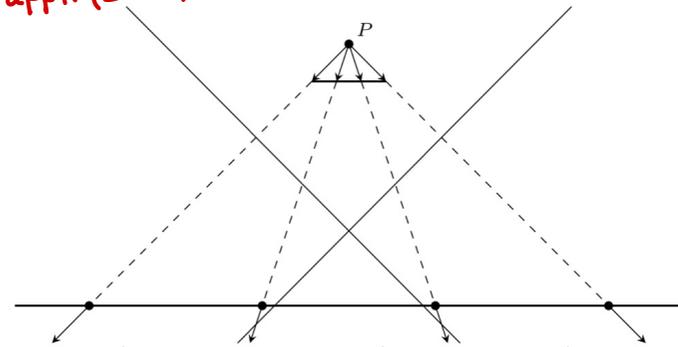
If  $l=2$ , Sugawara-Y. gave

PL description of the  
attaching map, which  
enables us to give Kirby  
diagram for  $M(A)$ .

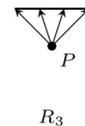
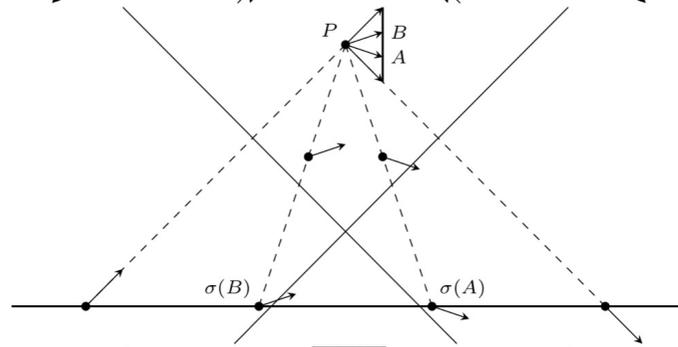
Sugawara-Y. Top. appl. (2022)



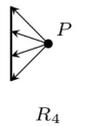
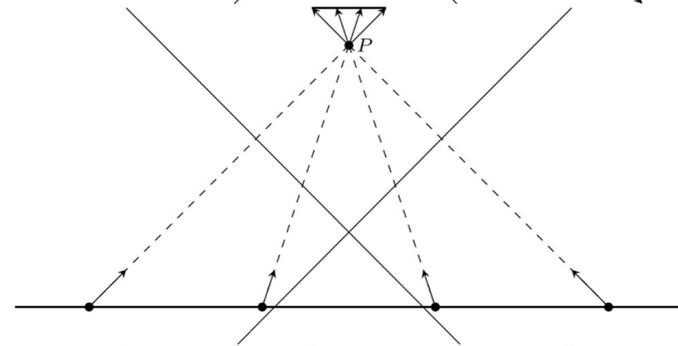
$\sigma \rightarrow$



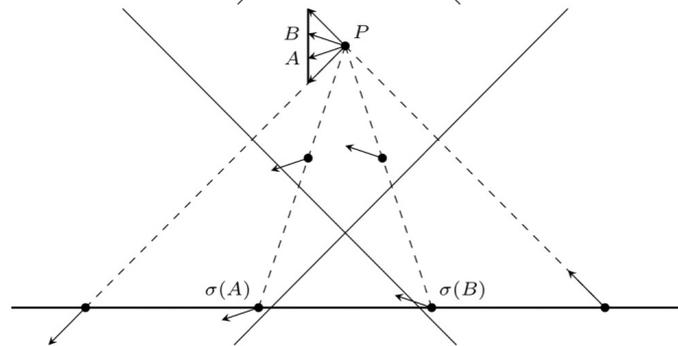
$\sigma \rightarrow$



$\sigma \rightarrow$

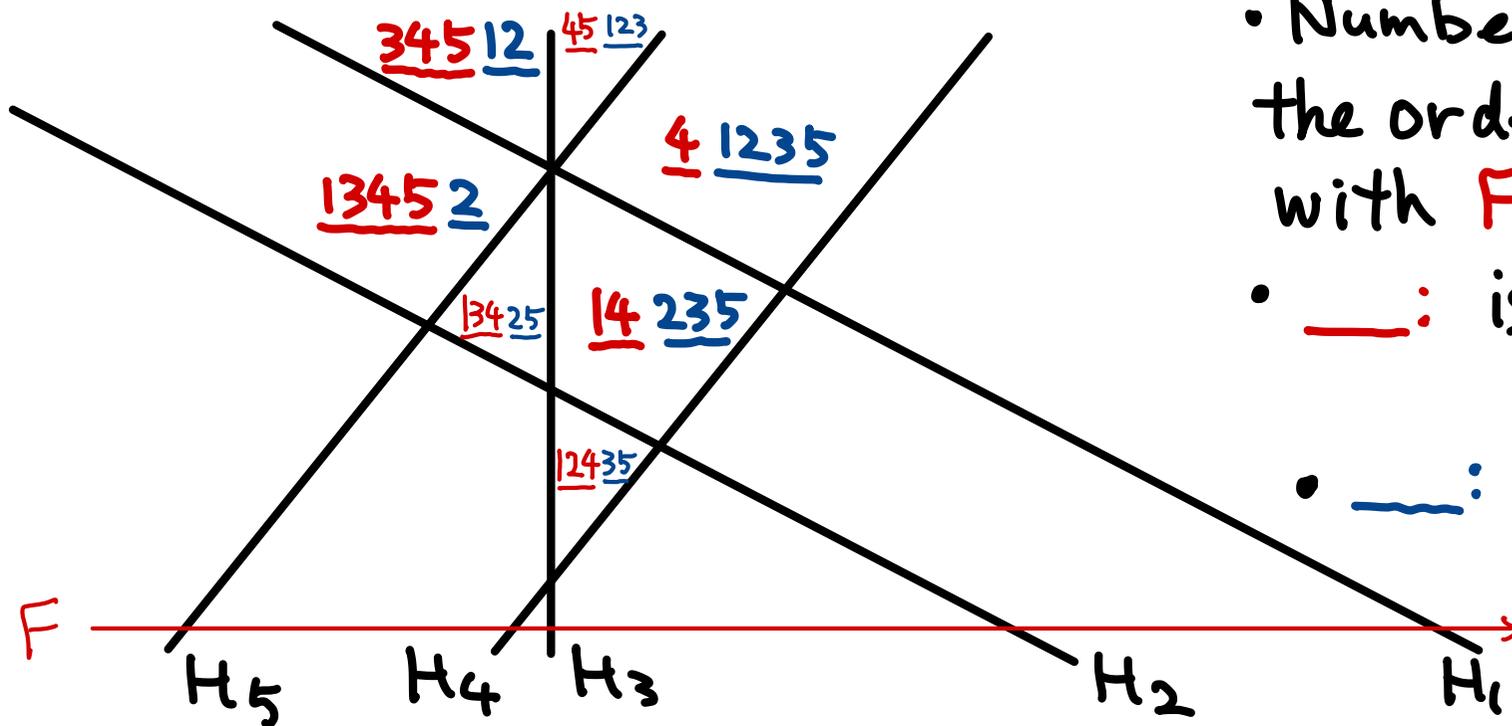


$\sigma \rightarrow$



## 2. Topology of arrangements

### A presentation and remark on $\pi_1(M)$



- Numbering according to the order of intersections with  $F$ .

- —: is the lines passing through right

- —: — left.

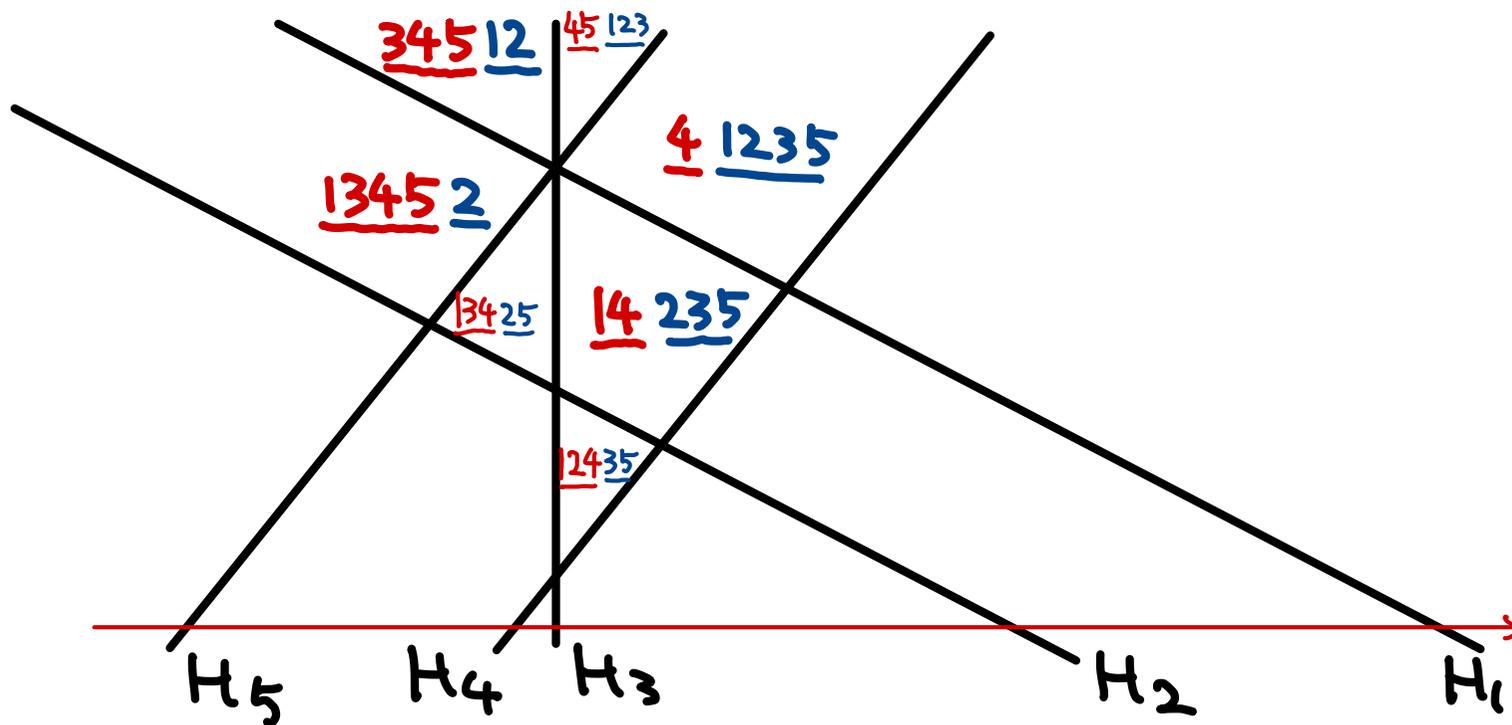
$$\pi_1(M(A)) \cong \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid \left. \begin{aligned} 12345 &= 12435 = 14235 = 13425 \\ &= 13452 = 34512 = 45123 \\ &= 41235 \end{aligned} \right\} \rangle$$

(Y. 2012)

## 2. Topology of arrangements

A presentation and remark on  $\pi_1(M)$

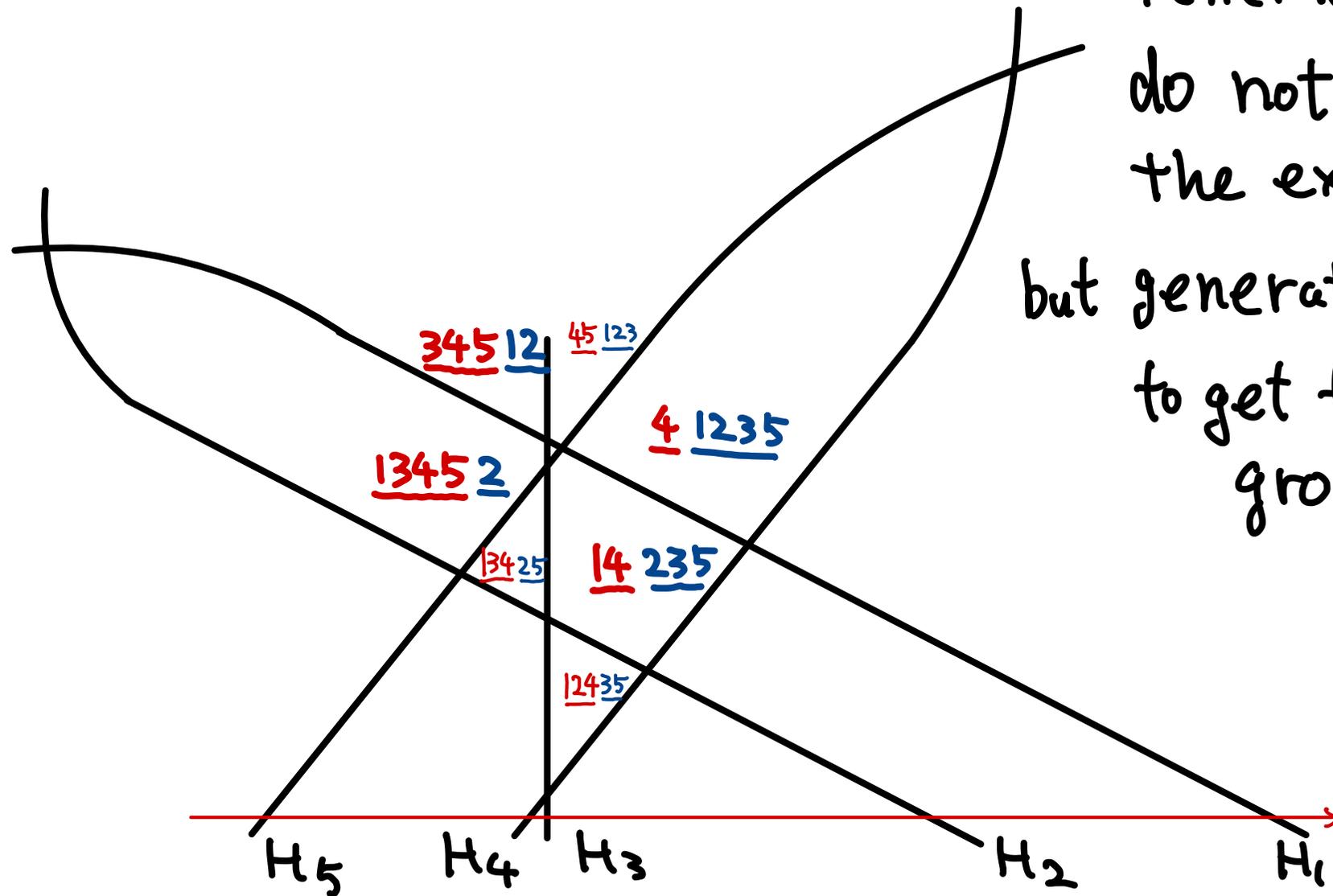
Generic perturbations  
do not affect on  
the existing relations.



## 2. Topology of arrangements

A presentation and remark on  $\pi_1(M)$

Generic perturbations do not affect on the existing relations, but generate new relations to get free abelian group  $\mathbb{Z}^{|\mathcal{A}|}$ .



## 2. Topology of arrangements

Thm. (Y.?) Let  $A = \{H_1, \dots, H_n\}$  be an arr. /  $\mathbb{R}$ .

Let  $b = b_2(M(A))$ . Then, there exist relations

$$r_1, r_2, \dots, r_b, r_{b+1}, \dots, r_{n(n-1)/2}$$

of  $\gamma_1, \gamma_2, \dots, \gamma_n$  s.t.

$$\pi_1(M(A)) \cong \langle \gamma_1, \dots, \gamma_n \mid r_1, \dots, r_b \rangle, \text{ and}$$

$$\mathbb{Z}^n \cong \langle \gamma_1, \dots, \gamma_n \mid r_1, \dots, r_{n(n-1)/2} \rangle.$$

## 2. Topology of arrangements

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$\mathbb{Z}^n \cong \langle \gamma_1, \dots, \gamma_n \mid r_1, \dots, r_{n(n-1)/2} \rangle$ .

(suggest to call "Abelian Extendable minimal presentation".)

### Questions

(i) How about general complex arr.  $A$ ?

(ii) Let  $\pi_1(M) \cong \langle \gamma'_1, \dots, \gamma'_n \mid r'_1, \dots, r'_b \rangle$  be another such presentation. Are the corresponding 2-dim CW complexes homotopy equivalent? If Yes, then the group  $\pi_1(M)$  determines the homotopy type of  $M = M(A)$  for line arr.  $A$ .

## 2. Topology of arrangements

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Rem. Let  $A_1, A_2$  be line arr. in  $\mathbb{C}^2$ . Randell proved if  $\pi_1(M(A_1)) \cong \pi_1(M(A_2))$ , then  $bi(M(A_1)) = bi(M(A_2))$ .  
Guerville-Ballé constructed examples s.t.  $M(A_1) \not\cong M(A_2)$ .

Are they always homotopy equivalent? ↑  
Non homeomorphic

i.e. does  $\pi_1(M(A_1)) \cong \pi_1(M(A_2))$  imply  $M(A_1) \cong M(A_2)$ ?  
↑  
homotopy equiv.

## 2. Topology of arrangements (2nd day)

Let  $A = \{H_1, \dots, H_n\}$  be an arr. in  $V = \mathbb{K}^d$ .

Def.

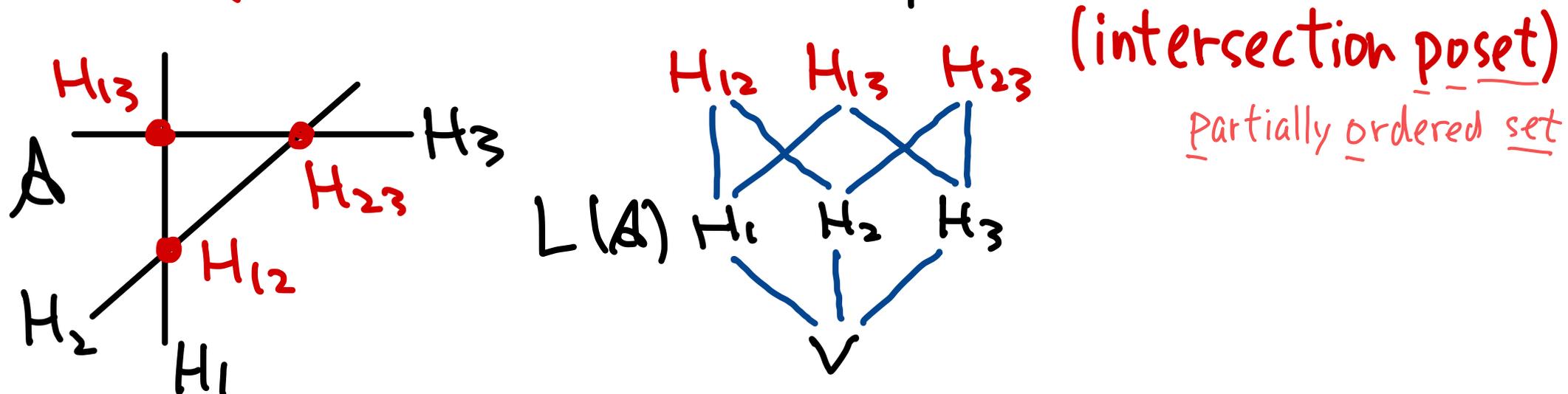
$$L(A) := \left\{ H_I \neq \emptyset \mid I \subset [n] \right\}$$

$\cap_{i \in I} H_i$        $\{1, 2, \dots, n\}$

is the set of non-empty intersections.

Rem •  $H_\emptyset = V$ .

• (Convention) Ordered by reverse inclusion.

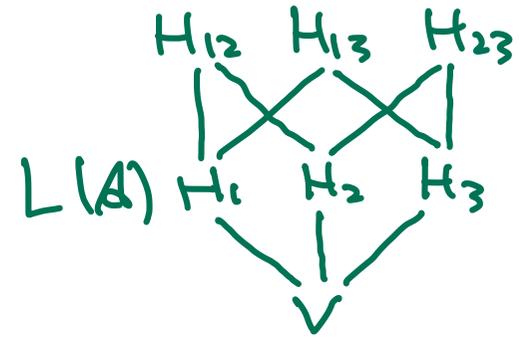
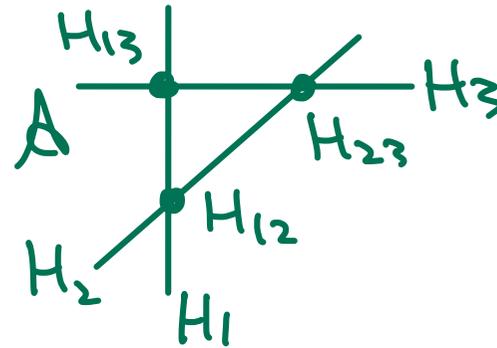


## 2. Topology of arrangements (2nd day)

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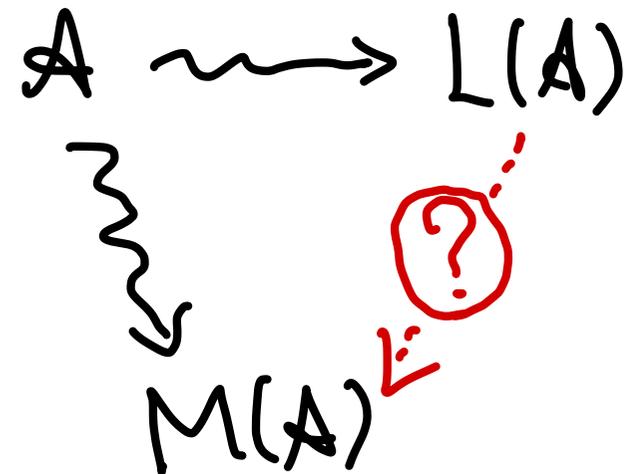
Rem •  $H_\emptyset = V$ .

• (Convention) Ordered by reverse inclusion. (intersection poset)



## Major Problem

What kind of properties of  $M(A)$  is determined by  $L(A)$ ?



## 2. Topology of arrangements

In general, posets potentially can encode lots of subtle structures.

Let  $\mathbb{K}_1, \mathbb{K}_2$  be fields,  $V_i$  be a fin. dim. vector space /  $\mathbb{K}_i$ .

Def  $L(V_i) := \{W \subset V_i \mid \text{linear subspace}\}$

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(proof)  $\dim_{\mathbb{K}_i} V_i = \max \{n \mid \exists W_0, W_1, \dots, W_n \in L(V_i)$

{the length of maximal chain}, s.t.  $W_0 \subset W_1 \subset \dots \subset W_n\}$ .

## 2. Topology of arrangements

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## 2. Topology of arrangements

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Prop Suppose  $\dim V_i \geq 3$ . If  $L(V_1) \cong L(V_2)$  as posets, then  $\mathbb{K}_1 \cong \mathbb{K}_2$  as fields.

Idea of the proof  $\mathbb{K}$ : a field. One can recover  $\mathbb{K}$  from the incidence relation of points and lines on  $\mathbb{K}\mathbb{P}^2$ .

① Choose a line  $L \subset \mathbb{K}\mathbb{P}^2$ . Let  $\mathcal{P} := \{P \in \mathbb{K}\mathbb{P}^2 \mid P \in L\}$ . (projective plane)

② Choose three points  $P_0, P_1, P_\infty \in \mathcal{P}$ . (von Staudt construction)

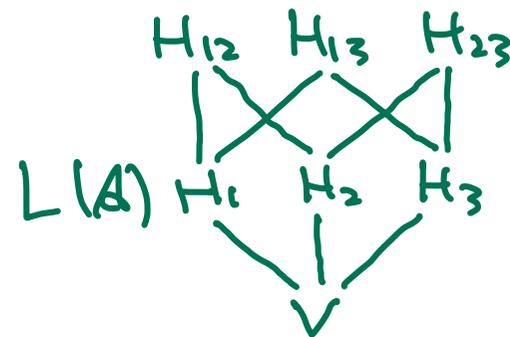
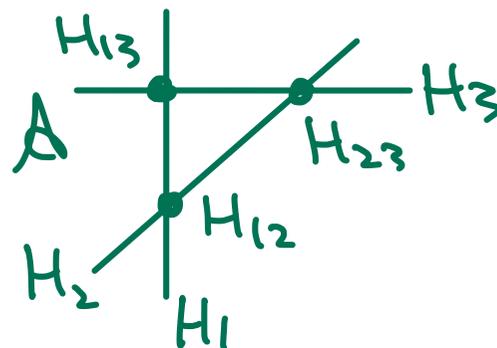
③ Define "+", "x" on  $\mathcal{P} \setminus \{P_\infty\}$ , which is isomorphic to  $\mathbb{K}$ .

## 2. Topology of arrangements (2nd day)

Def.  $L(A) := \{H_I \neq \emptyset \mid I \subset [n]\}$  is the set of non-empty intersections.

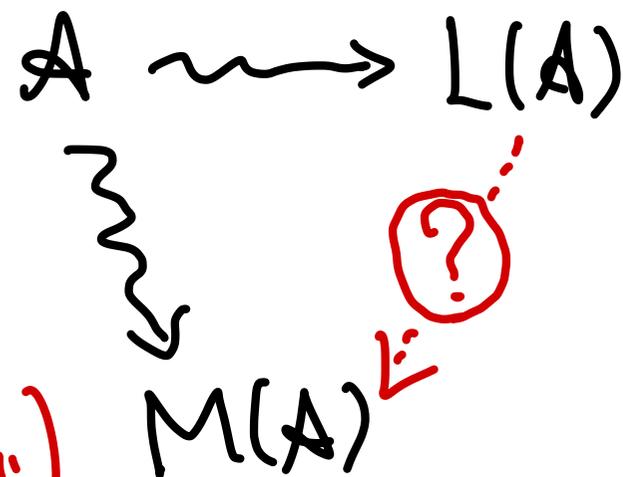
Rem •  $H_\emptyset = V$ .

- (Convention) Ordered by reverse inclusion. (intersection poset)



## Major Problem

What kind of properties of  $M(A)$  is determined by  $L(A)$ ? ("Combinatorially determined")

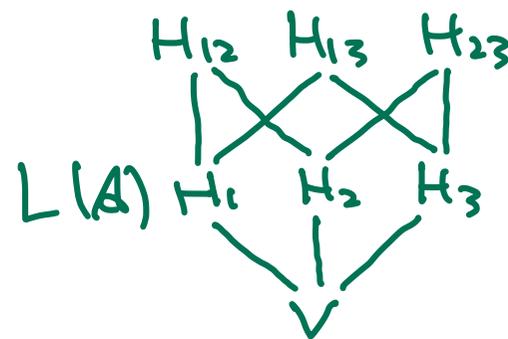
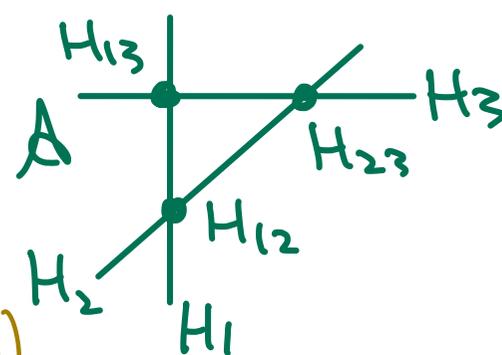


## 2. Topology of arrangements (2nd day)

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### Major Problem

What kind of properties of  $M(A)$  is determined by  $L(A)$ ? ("Combinatorially determined")



Since  $\dim H_I$  is recovered from  $L(A)$  (and  $\dim V$ ),

$$\chi(A, t) = \sum_{\substack{I \subset [n] \\ H_I \neq \emptyset}} (-1)^{|I|} t^{\dim H_I}$$

is combinatorial.

Cor.  $|ch(A)|$  ( $= |\chi(A, -1)|$ ),  $|bch(A)|$  ( $= |\chi(A, 1)|$ )  $A/\mathbb{R}$ ,  
 $|M(A)|$  ( $= \chi(A, 0)$ )  $A/\mathbb{F}_2$ ,  $\text{Poin}(M(A), t)$  ( $= (-t)^2 \chi(A, -\frac{1}{t})$ )  $A/\mathbb{C}$   
 are combinatorial.

## 2. Topology of arrangements

Prop.  $A = \{H_1, \dots, H_n\} : \text{arr.} / \mathbb{C}$ . Then the cohomology ring of the complement  $H^*(M(A), \mathbb{Z})$  is combin.

(proof)

$$H^*(M(A), \mathbb{Z}) \cong \Lambda E$$

$$E = \bigoplus_{i=1}^n \mathbb{Z} \cdot e_i$$

Orlik-Solomon alg.  $\rightarrow$   
is combinatorial.

$$\left( \begin{array}{l} e_I := e_{i_1} \cdots e_{i_p}, \text{ for } I = \{i_1, \dots, i_p\} \subset [n] \\ \text{s.t. } H_I = \emptyset \\ \\ \sum_{s=1}^p (-1)^{s-1} e_{i_1} \cdots \hat{e}_{i_s} \cdots e_{i_p}, \text{ for } I \subset [n] \\ \text{s.t. } \text{codim } H_I < |I| \end{array} \right)$$

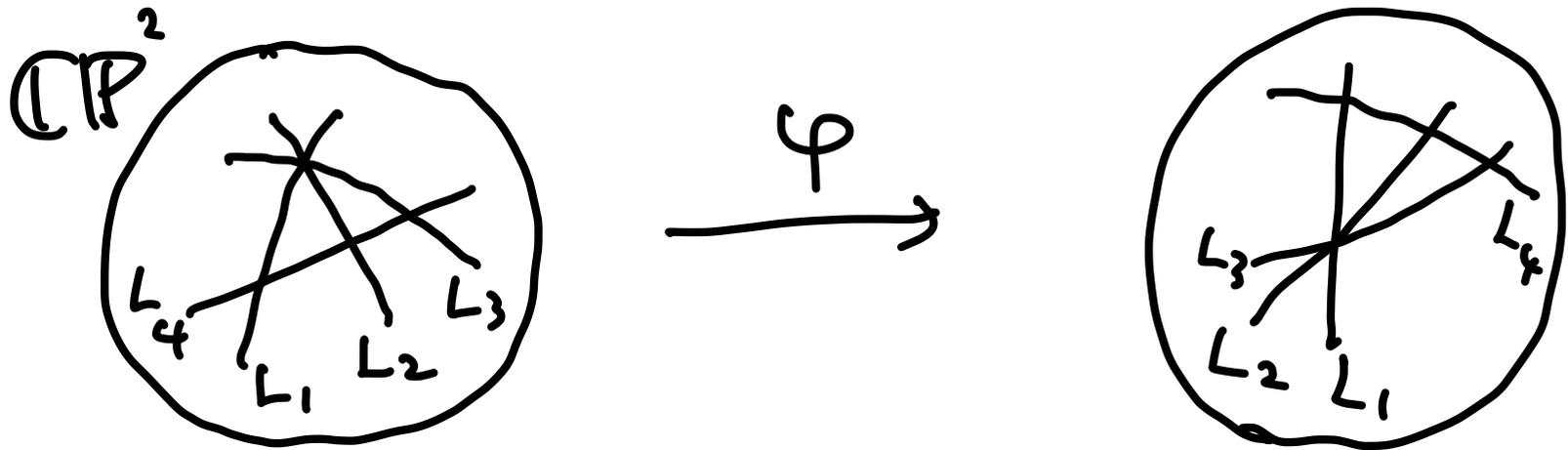
On the other hand, there are several  
"non combinatorial" properties.

## 2. Topology of arrangements

Let  $\mathcal{L} = \{L_1, L_2, \dots, L_n\}$ ,  $\mathcal{L}' = \{L'_1, \dots, L'_n\}$  be line arr. on  $\mathbb{C}P^2$ .

Def  $\mathcal{L}$  and  $\mathcal{L}'$  have same **embedded type** if

$\exists \varphi: \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$  homeomorphism s.t.  $\varphi(L_i) = L'_i$ .



Def  $(\mathcal{L}, \mathcal{L}')$  is a **Zariski pair** if they are of different embedded type.

## 2. Topology of arrangements

### Rem.

1998: Rybnikov constructed Zariski pair  $(\mathcal{L}_1, \mathcal{L}_2)$   
s.t.  $\pi_1(\mathbb{C}P^2 \setminus \cup \mathcal{L}_1) \not\cong \pi_1(\mathbb{C}P^2 \setminus \cup \mathcal{L}_2)$ .

2003: E. Artal, J. Carmona, J.I. Cogolludo, M. Marco  
confirmed Rybnikov's construction and  
provided new (smaller) Zariski pairs of 11 lines.

~ : B. Guerville-Ballé, V. Florens, J. Vieu Sos ...

2010: Nazir -Y., F. Ye  $\not\cong$  Zariski up to 9 lines.

Open Problem  $\exists$  or  $\not\cong$  Zariski pairs of 10 lines.

(cf. M. Amram, M. Teicher, F. Ye : realizations)

## 2. Topology of arrangements

### A Remark on moduli space

Consider  $\mathcal{L} = \{L_1, \dots, L_n\} \subset (\mathbb{C}\mathbb{P}^2)^* \times \mathbb{N}$ .

$\mathcal{M} := \{ \mathcal{L} \mid \mathcal{L} \text{ has the prescribed incidence} \}$

is called the **realization space (moduli space)** of line arrangements with prescribed incidence.

#### Fact

If  $\mathcal{L}, \mathcal{L}' \in \mathcal{M}$  are on the same connected component, then  $\mathcal{L}$  and  $\mathcal{L}'$  have same embedded type.

$$\text{i.e. } \exists (\mathbb{C}\mathbb{P}^2, \mathcal{L}) \xrightarrow{\sim} (\mathbb{C}\mathbb{P}^2, \mathcal{L}')$$

## 2. Topology of arrangements

### Combinatorial

$|\text{ch}(A)|, A/\mathbb{R}$

$H^*(M(A), \mathbb{Z}) \quad A/\mathbb{C}$

### Not combinatorial

Emb. type  $(\mathbb{C}P^2, \mathcal{L})$

$\pi_1(M(A))$ , homotopy type  
of  $M(A)$

### Unknown

- rank of local system (co)homology  $H_i(M(A), \mathbb{L})$ .
- Betti #'s of covering spaces  $X \rightarrow M(A)$ ,
- Betti #'s of Milnor fiber.

## 2. Topology of arrangements

### Double covering

Let  $X$  be a CW cpx. Let  $\varphi: \pi_1(X) \rightarrow G$  be a group surjective.

$\tilde{X} / \ker \varphi \xrightarrow{p} X$  is a  $G$ -covering.  
*universal covering*

double covering  $p: X \rightarrow W \iff \omega: \pi_1(X) \rightarrow \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$

$$\iff \omega: H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}_2$$

$$\iff \omega \in H^1(X, \mathbb{Z}_2)$$

For any  $\omega \in H^1(X, \mathbb{Z}_2)$  ( $\omega \neq 0$ ), we denote the associated double cover  $p_\omega: X^\omega \rightarrow X$ .

## 2. Topology of arrangements

double covering  $p: X \rightarrow W \iff W: \pi_1(X) \rightarrow \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$

$$\iff W: H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}_2$$

$$\iff W \in H^1(X, \mathbb{Z}_2)$$

For any  $W \in H^1(X, \mathbb{Z}_2)$ , associated double cover  $p_W: X^W \rightarrow X$ .

Thm (Y. 2020) Let  $A = \{H_1, \dots, H_n\}$  be an arr. in  $\mathbb{C}^d$ .

Let  $W \in A^1_{\mathbb{Z}_2}(A) = H^1(M(A), \mathbb{Z}_2)$ . Then,  $H^1(M, \mathbb{Z}_2)$   
 $\cong$  Aomoto cpx.

$$\text{rank}_{\mathbb{Z}_2} H^k(M(A)^W, \mathbb{Z}_2) = \underbrace{b_k(M(A))}_{\text{combin.}} + \text{rank}_{\mathbb{Z}_2} \underbrace{H^k(A^{\circ}_{\mathbb{Z}_2}(A), W^1)}_{\text{combin.}}.$$

In particular, the mod 2 Betti # of the double cover is combinatorial.

Rem. Suciu generalized to CW complexes.

## 2. Topology of arrangements

### Combinatorial

$$|\text{ch}(A)|, A/\mathbb{R}$$

$$H^*(M(A), \mathbb{Z}) \quad A/\mathbb{C}$$

$$\text{rank}_{\mathbb{Z}_2}(M(A)^\omega, \mathbb{Z}_2) \\ (\omega \in H^1(M, \mathbb{Z}_2)).$$

### Not combinatorial

$$\text{Emb. type } (\mathbb{C}P^2, \mathcal{L})$$

$$\pi_1(M(A)), \text{ homotopy type} \\ \text{of } M(A).$$

### Unknown

- rank of local system (co)homology  $H_i(M(A), \mathbb{L})$ .
- Betti #'s of covering spaces  $X \rightarrow M(A)$ ,
- Betti #'s of Milnor fiber.
- $b_2(M(A)^\omega)$  ( $\omega \in H^1(M, \mathbb{Z}_2)$ )
- Cohomology ring  $H^*(M(A)^\omega, \mathbb{Z}_2)$ .

## 2. Topology of arrangements

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## 2. Topology of arrangements

Milnor fiber Setting:  $A = \{H_1, \dots, H_n\}$  a central arr.

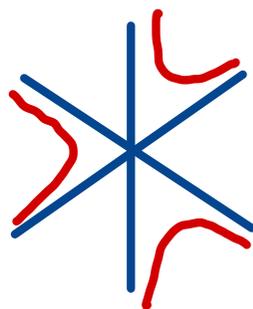
$H_i = \{d_i = 0\}$ ,  $d_i$ : linear form.

$\forall H_i \ni 0.$

$$Q := \prod_{i=1}^n d_i \quad (\text{homogeneous})$$

Arr. on  $\mathbb{P}^{l-1}$

$$\bar{A} = Q^{-1}(0) = \bigcup \bar{H}_i \text{ in } \mathbb{C}\mathbb{P}^{l-1}$$



Milnor fiber

$$F_A := Q^{-1}(1) \ni \text{monodromy action}$$

$\mathbb{Z}_n$ -cover  $(x \mapsto e^{2\pi i J/n})$

(Put  $\bar{H}_n$  at infinity)

affine arr.

$$A = \{H_1, \dots, H_{n-1}\} \text{ in } \mathbb{C}^{l-1}$$

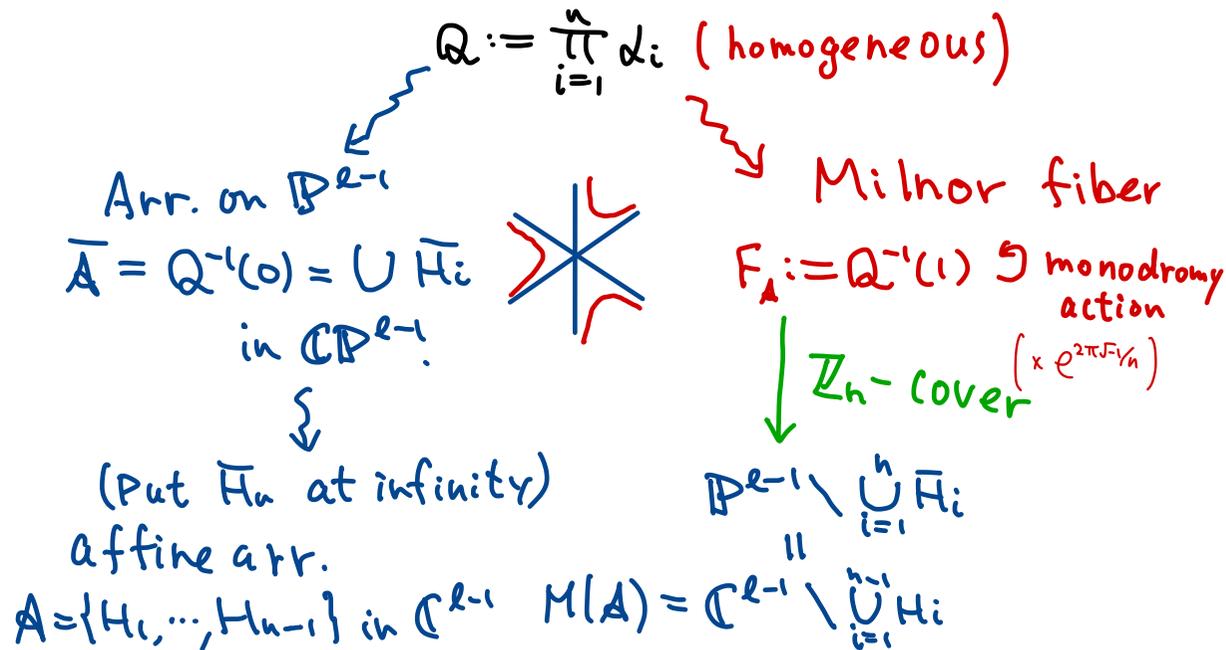
$$M(A) = \mathbb{C}^{l-1} \setminus \bigcup_{i=1}^{n-1} H_i$$

$$\mathbb{P}^{l-1} \setminus \bigcup_{i=1}^n \bar{H}_i$$

$\parallel$

# 2. Topology of arrangements

Milnor fiber Setting:  $A = \{H_1, \dots, H_n\}$  a central arr.  
 $H_i = \{d_i = 0\}$ ,  $d_i$ : linear form.  $\forall H_i \ni 0.$



Question  $H^k(F, \mathbb{C}) = ?$

Basic strategy: monodromy eigen space decomposition.

$$H^k(F, \mathbb{C}) \cong \bigoplus_{\lambda^n = 1} H^k(F)_\lambda = \underbrace{H^k(F)_1}_{\cong H^k(M)} \oplus \underbrace{H^k(F)_{\neq 1}}_{\cong \bigoplus_{\substack{\lambda \neq 1 \\ \lambda^n = 1}} H^k(M, \mathcal{L}_\lambda)}$$

$\uparrow$   
 $\lambda$ -eigen space

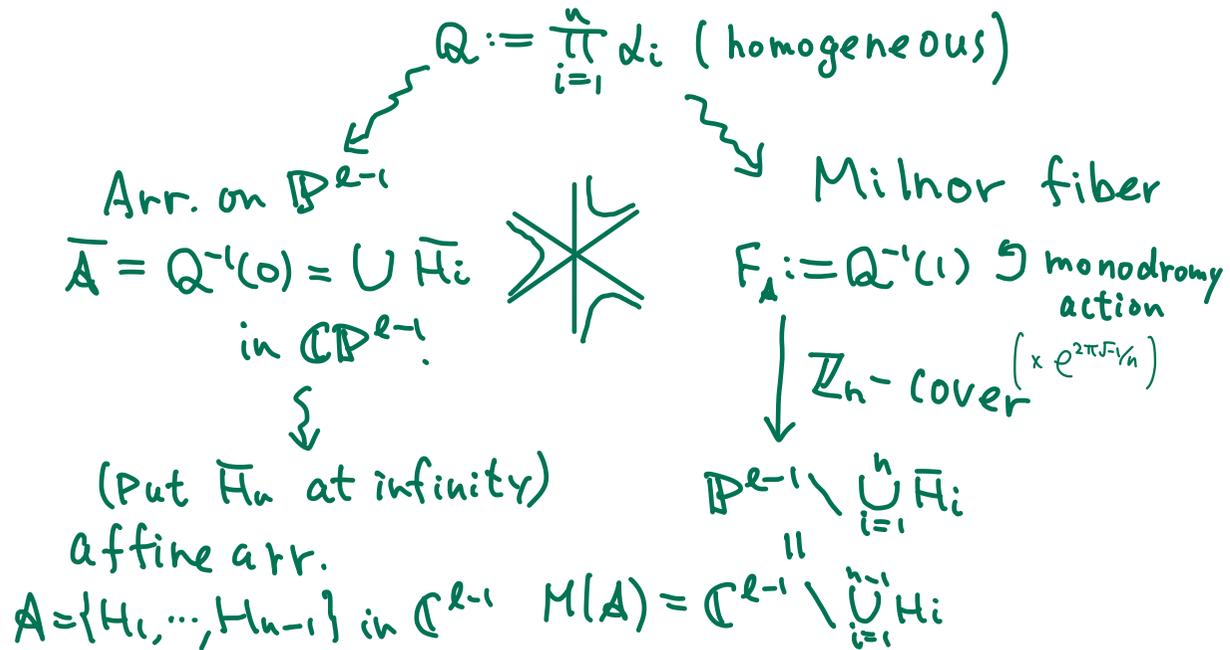
local sys. on  $M$

↓

$\rightarrow H_i$   
 $\times \lambda$

# 2. Topology of arrangements

Milnor fiber Setting:  $A = \{H_1, \dots, H_n\}$  a central arr.  
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Basic strategy

$$\begin{aligned} H^k(F, \mathbb{C}) &\cong \bigoplus_{\lambda^n = 1} H^k(F)_\lambda \\ &= \underbrace{H^k(F)_1}_{\cong H^k(M)} \oplus \underbrace{H^k(F)_{\neq 1}}_{\cong \bigoplus_{\substack{\lambda \neq 1 \\ \lambda^n = 1}} H^k(M, \mathbb{Z}_\lambda)} \end{aligned}$$

Today, we focus on

- $l=3$ ,  $F \rightarrow \mathbb{C}^2 \setminus \bigcup H_i$ ,  $H^k(F, \mathbb{C}) = ?$
- When  $H^k(F)_{\neq 1}$  appears?
- Is  $\dim H^k(F)_\lambda$  ( $\lambda \neq 1$ ) combinatorial?

# 2. Topology of arrangements

Milnor fiber Setting:  $A = \{H_1, \dots, H_n\}$  a central arr.  
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$$Q := \prod_{i=1}^n d_i \text{ (homogeneous)}$$

Arr. on  $\mathbb{P}^{l-1}$   
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 in  $\mathbb{C}\mathbb{P}^{l-1}$ .



Milnor fiber

$F_A := Q^{-1}(1) \ni$  monodromy action  
 $\downarrow \mathbb{Z}_n$ -cover  $(x \mapsto e^{2\pi i/n} x)$

$$\mathbb{P}^{l-1} \setminus \bigcup_{i=1}^n H_i$$

(Put  $H_n$  at infinity)  
 affine arr.  
 $A = \{H_1, \dots, H_{n-1}\}$  in  $\mathbb{C}^{l-1}$   $M(A) = \mathbb{C}^{l-1} \setminus \bigcup_{i=1}^{n-1} H_i$

Question  $H^k(F, \mathbb{C}) = ?$

Basic strategy

$$H^k(F, \mathbb{C}) \cong \bigoplus_{\lambda^n = 1} H^k(F)_\lambda$$

$$= \underbrace{H^k(F)_1} \oplus \underbrace{H^k(F)_{\neq 1}}$$

$$\cong H^k(M) \cong \bigoplus_{\substack{\lambda \neq 1 \\ \lambda^n = 1}} H^k(M, \mathbb{Z}_\lambda)$$

Today, we focus on

- $l=3$ ,  $F \rightarrow \mathbb{C}^2 \setminus \bigcup H_i$ ,  $H^1(F, \mathbb{C}) = ?$
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- Is  $\dim H^1(F)_\lambda$  ( $\lambda \neq 1$ ) combinatorial?

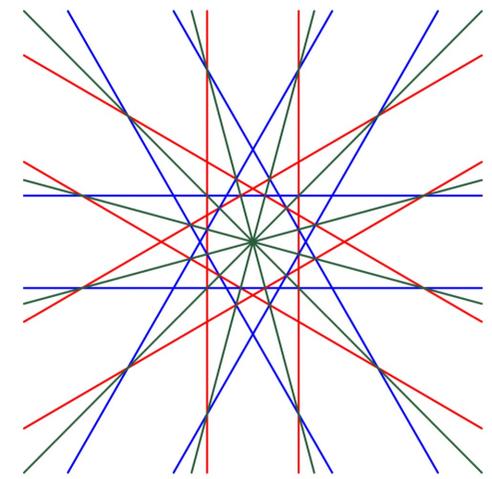
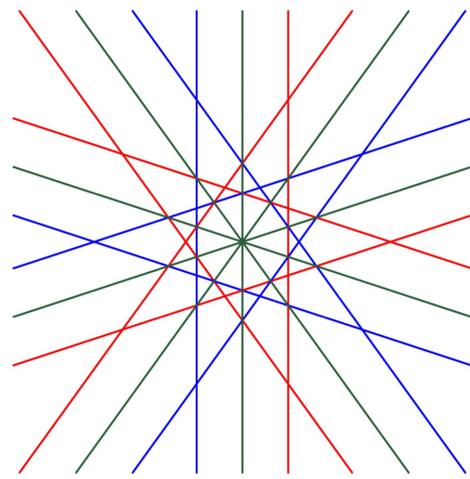
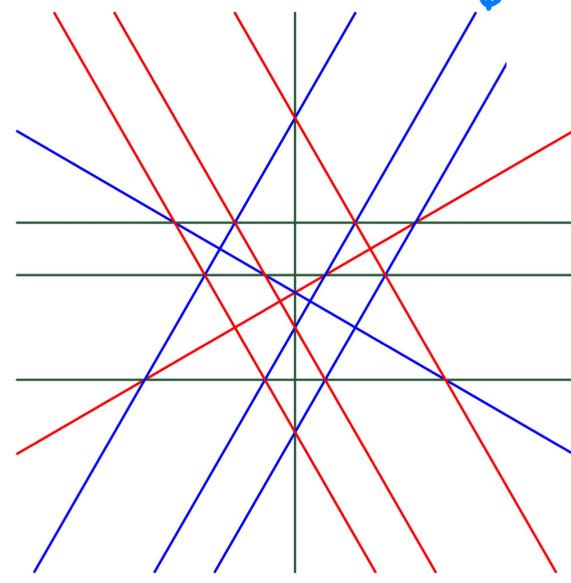
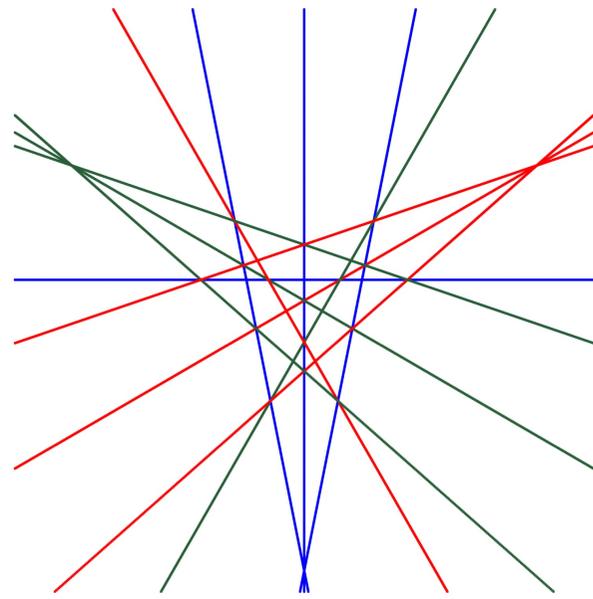
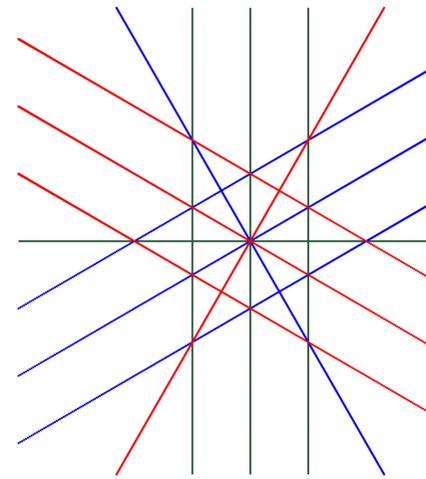
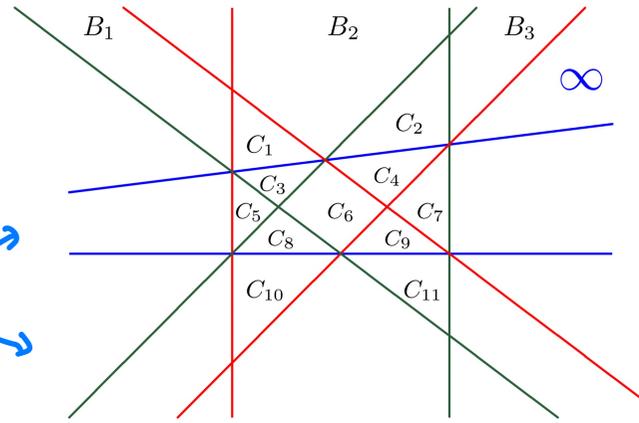
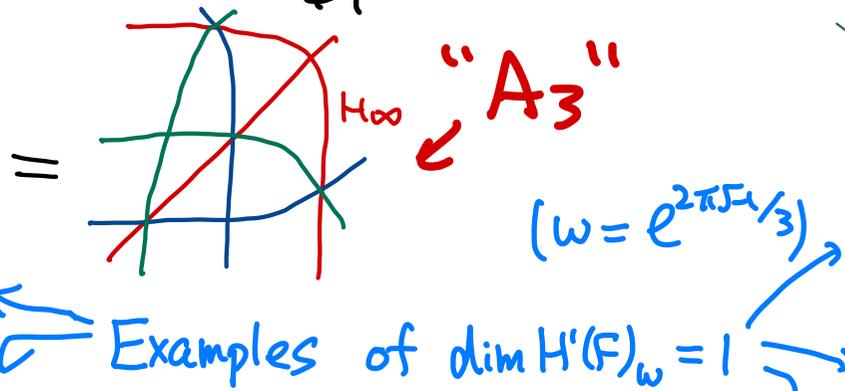
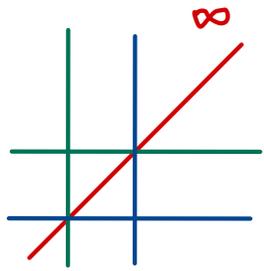
## Classically known fact

If  $\bar{A}$  is generic (normal crossing), then  $H^1(F)_{\neq 1} = 0$ .

Philosophy: line arr. with  $H^1(F)_{\neq 1} \neq 0$  is rare (and interesting).

# 2. Topology of arrangements

Examples of  $H^1(F)_{\neq 1} \neq 0$ :

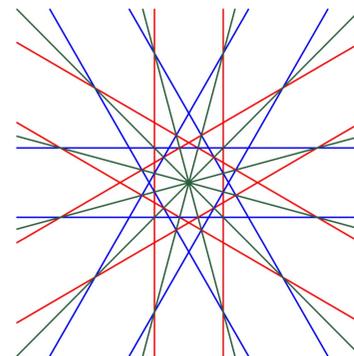
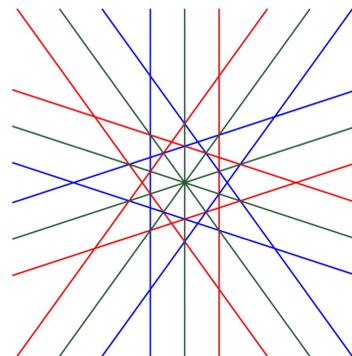
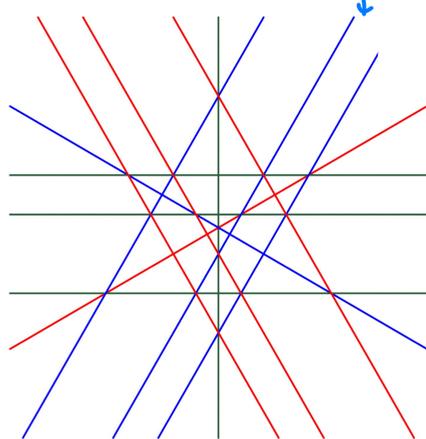
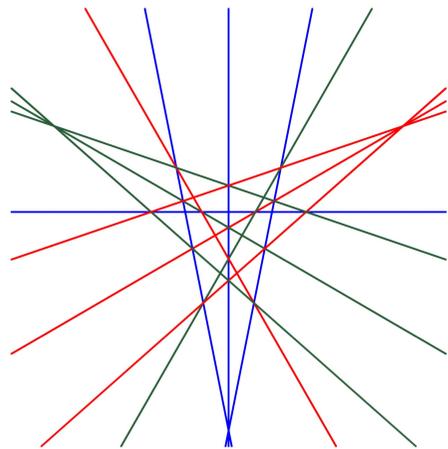
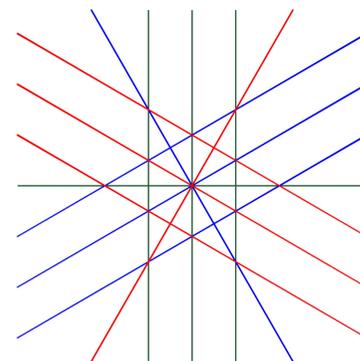
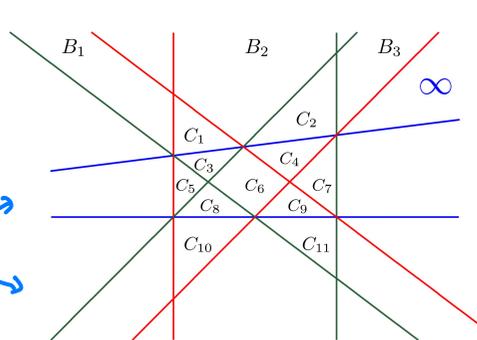
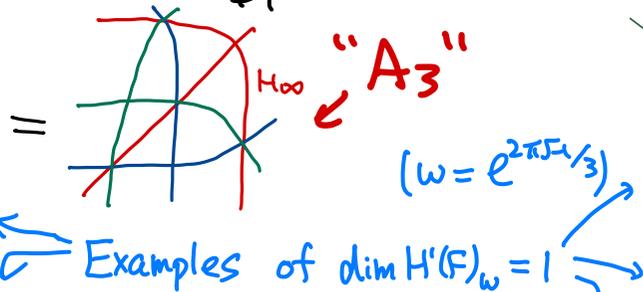
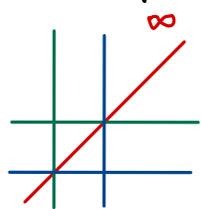


Fact (Libgober 2002) If  $H^1(F)_{e^{2\pi i / k}} \neq 0$ , then  $\forall \bar{H}_i \in \bar{A}$  has at least one point with multiplicity divisible by  $k$ .  
 ( $\bar{A} : / \mathbb{C}$ )

Fact (Y. 2013) If  $n = |\bar{A}| > 6$ , and  $H^1(F)_{e^{2\pi i / k}} \neq 0$ , then  $\forall \bar{H}_i \in \bar{A}$  has at least **three** points with multiplicity divisible by  $k$ .  
 ( $\bar{A} : / \mathbb{R}$ )

# 2. Topology of arrangements

Examples of  $H^1(F)_{\neq 1} \neq 0$ :



Fact (Y. 2013) If  $n = |\bar{A}| > 6$ , and  $H^1(F)_{e^{2\pi i / k}} \neq 0$ , then  $\forall \bar{H}_i \in \bar{A}$  has at least **three** points with multiplicity divisible by  $k$ .  
 ( $\bar{A} : / \mathbb{F}$ )

Question Does the above result hold for any line arr /  $\mathbb{C}$  ?

## 2. Topology of arrangements

Let  $d = e^{2\pi\sqrt{-1}/3}$ . Papadima and Suciu discovered that  $H^*(F)_d$  is closely related to  $\mathbb{F}_3$ -Aomoto cpx.

Thm (Papadima-Suciu 2017) Let  $\mathcal{A} = \{H_1, \dots, H_{n-1}\}$  in  $\mathbb{C}^2$ .

(i) If all multiplicities of intersections are in  $\{2, 3, 4, 5, \cancel{6}, 7, 8, \cancel{9}, 10, \dots\}$  (delete 3,  $\mathbb{Z}_{2,2}$ ), then

$$\dim_{\mathbb{C}} H^*(F)_d = \dim_{\mathbb{C}} H^*(F)_{d^2} = \dim_{\mathbb{F}_3} H^*(A_{\mathbb{F}_3}(\mathcal{A}), \omega),$$

where  $\omega = e_1 + e_2 + \dots + e_{n-1}$ .

(ii) If all multiplicities are in  $\{2, 3\}$ , then  $\dim_{\mathbb{C}} H^*(F)$  is combinatorially determined.

## 2. Topology of arrangements

Thm (Papadima-Suciu 2017) Let  $\mathcal{A} = \{H_1, \dots, H_{n-1}\}$  in  $\mathbb{C}^2$ .  $d = e^{2\pi i \sqrt{-1}/3}$ .

(i) If all multiplicities of intersections are in  $\{2, 3, 4, 5, \cancel{6}, 7, 8, \cancel{9}, 10, \dots\}$ , then

$$\dim_{\mathbb{C}} H^1(F)_{\lambda} = \dim_{\mathbb{C}} H^1(F)_{\lambda^2} = \dim_{\mathbb{F}_3} H^1(A_{\mathbb{F}_3}^{\bullet}(\mathcal{A}), \omega),$$

where  $\omega = e_1 + e_2 + \dots + e_{n-1}$ .

(ii) If all multiplicities are in  $\{2, 3\}$ , then  $\dim_{\mathbb{C}} H^1(F)$  is combin.

Conjecture (Papadima-Suciu 2017)

①  $\lambda = e^{2\pi i \sqrt{-1}/k}$  with  $k \geq 5 \implies H^1(F)_{\lambda} = 0$

②  $\lambda = e^{2\pi i \sqrt{-1}/3} \implies \dim_{\mathbb{C}} H^1(F)_{\lambda} = \dim_{\mathbb{C}} H^1(F)_{\lambda^2} = \dim_{\mathbb{F}_3} H^1(A_{\mathbb{F}_3}^{\bullet}(\mathcal{A}), \omega_n)$

③  $\dim H^1(F)_{-1} = \dim H^1(F)_{\pm \sqrt{-1}} = \dim_{\mathbb{F}_2} H^1(A_{\mathbb{F}_2}^{\bullet}(\mathcal{A}), \omega_n)$

Another (long-standing) Question : Is  $H_1(F, \mathbb{Z})$  torsion-free?

# 2. Topology of arrangements

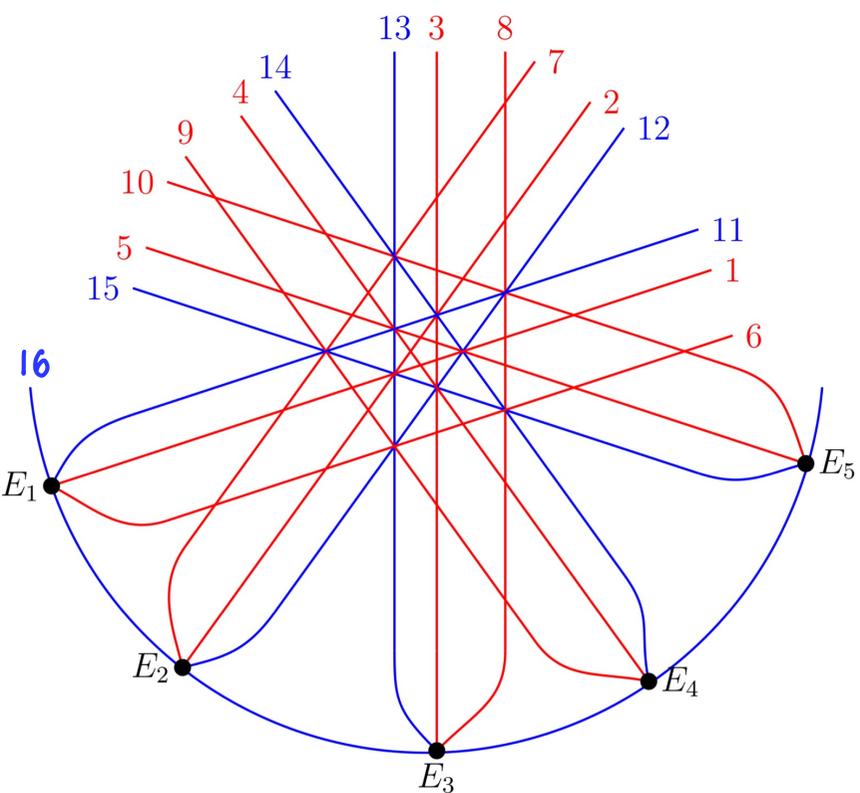
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③  $\dim H^1(F)_{-1} = \dim H^1(F)_{\pm\sqrt{-1}} \stackrel{\triangle}{=} \dim_{\mathbb{F}_2} H^1(A_{\mathbb{F}_2}(A), \omega_n)$

Another (long-standing) Question : Is  $H_1(F, \mathbb{Z})$  torsion-free?



Recently (Y. 2020)

→ The icosidodecahedral arr.  $\bar{A}_{\text{ID}} = \{\bar{H}_1, \dots, \bar{H}_{16}\}$

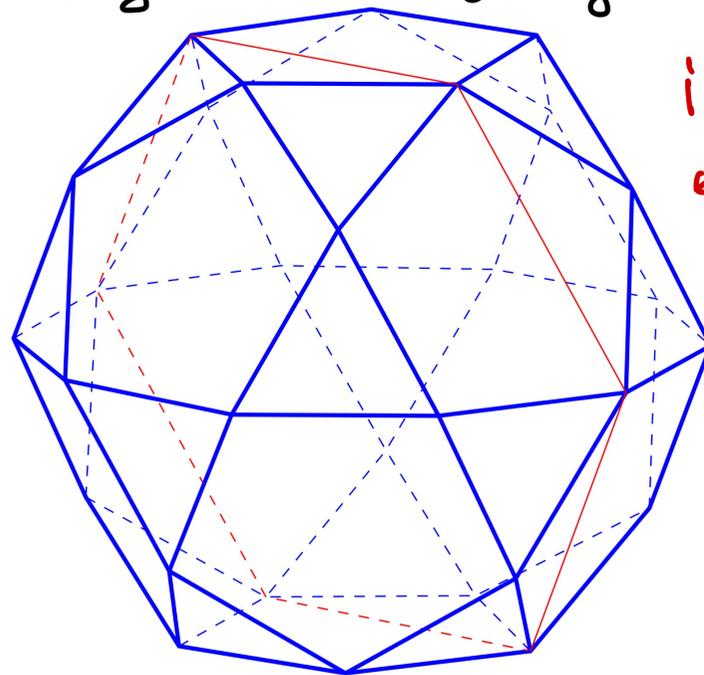
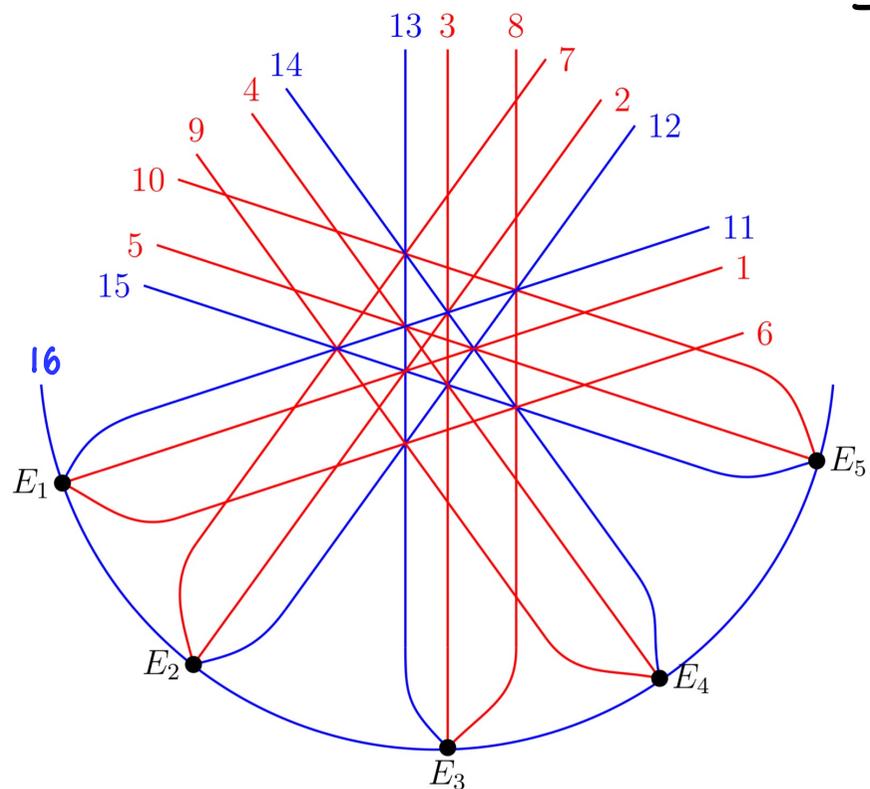
→ breaks a part of ③

$\dim H^1(F)_{-1} = \dim H^1(F)_{\pm\sqrt{-1}} = 0 < 1 = \dim_{\mathbb{F}_2} H^1(A_{\mathbb{F}_2}(A), \omega)$

→  $H_1(F_{\bar{A}_{\text{ID}}})$  has 2-torsion.

# 2. Topology of arrangements

The icosidodecahedral arr is realized using edges and diagonals of the



icosidodecahedron



Crucial property:  $\exists$  bi-coloring s.t. each intersection is either monocolour or even red and even blue

Rev D. Munkacsi (Hannover) 's computer search: such arrangement is rare.

Only 3 (up to  $\mathbb{F}_{241}$ ?) are known:

- Hessian arr. (12 lines /  $\mathbb{Q}(\sqrt{-3})$ )
- Icosidodecahedral arr. (16 lines /  $\mathbb{Q}(\sqrt{5})$ )
- Munkacsi's arr. (16 lines /  $\mathbb{F}_7$ )

# 2. Topology of arrangements

## Milnor fiber and $q$ -deformed Aomoto complex

Def. For  $n \in \mathbb{Z}$ ,  $[n] = [n]_q := \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{n-1}{2}} + q^{\frac{n-3}{2}} + \dots + q^{-\frac{n-1}{2}}$  ( $q$ -integers)

Example

$$[1] = 1$$

$$[2] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$$

$$[3] = q + 1 + q^{-1}$$

$$[4] = q^{3/2} + q^{1/2} + q^{-1/2} + q^{-3/2}$$

$$[5] = q^2 + q + 1 + q^{-1} + q^{-2}$$

$$[6] = q^{5/2} + q^{3/2} + q^{1/2} + q^{-1/2} + q^{-3/2} + q^{-5/2}$$

Facts

$$(1) \lim_{q \rightarrow 1} [n]_q = n$$

$$(2) [1]_q + [1]_q \neq [2]_q, [2]_q \cdot [2]_q \neq [4]_q.$$

However, there are many nice formulas,

$$\text{e.g. } [2]_q [3]_q = [4]_q + [2]_q \quad (\because \text{LHS} = (q^{1/2} + q^{-1/2})(q + 1 + q^{-1}) = q^{3/2} + 2q^{1/2} + 2q^{-1/2} + q^{3/2})$$

## 2. Topology of arrangements

$\mathfrak{g}$ -deformation of a chain complex:

$$\mathbb{Z} \xrightarrow{(1,1,1)} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 2 & & \\ -1 & & \\ -1 & & -2 \end{pmatrix}} \mathbb{Z}^2$$

$\rightsquigarrow$   $\mathfrak{g}$ -deformation is not a chain complex  
 $(\because [2] - [1] - [1] \neq 0)$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{(1,1,1)} & \mathbb{Z}^3 & \xrightarrow{\begin{pmatrix} 2 & & \\ -1 & & \\ -1 & & -2 \end{pmatrix}} & \mathbb{Z}^2 \\ \parallel & & \uparrow \times & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \parallel \\ \mathbb{Z} & \xrightarrow{(1,2,3)} & \mathbb{Z}^3 & \xrightarrow{\begin{pmatrix} 3 & & \\ 1 & & 0 \\ 1 & & -2 \end{pmatrix}} & \mathbb{Z}^2 \end{array}$$

$\rightsquigarrow$   $\mathfrak{g}$ -deformation is also a complex.

Choices of bases matter.

## 2. Topology of arrangements

Setting:  $\bar{A} = \{\bar{H}_1, \dots, \bar{H}_n\}$  lines in  $\mathbb{C}P^2$ ,  $\cup H_i = \{Q(x_1, x_2, x_3) = 0\}$ .

$A = \{H_1, \dots, H_{n-1}\}$  lines in  $\mathbb{C}^2 = \mathbb{C}P^2 \setminus \bar{H}_n$

$F = Q^{-1}(1) \xrightarrow{\mathbb{Z}_n\text{-cover}} M = M(A) = \mathbb{C}^2 \setminus \cup H_i.$

$$A_{\mathbb{Z}}^*(A) \cong H^*(M, \mathbb{Z}) \cong \Lambda \langle e_1, \dots, e_{n-1} \rangle_{\mathbb{Z}} / \text{OS-ideal}$$

$$\underbrace{\omega = e_1 + e_2 + \dots + e_{n-1}}$$

Consider the  $q$ -deformation of the Aomoto complex:

$$\dots \rightarrow A_{\mathbb{Z}}^i(A) \xrightarrow[\underbrace{(\alpha_{ij})_{ij}}_{\omega \wedge}]{} A_{\mathbb{Z}}^{i+1}(A) \rightarrow \dots$$

Once we fix a basis,

we have a **matrix presentation**. Define the  $q$ -deformation

$$[\omega \wedge]_q := ([\alpha_{ij}]_q)_{i,j} : A^i \otimes \mathbb{Z}(q^{1/2}) \rightarrow A^{i+1} \otimes \mathbb{Z}(q^{1/2}).$$

Is the  $q$ -deformation  $(A_{\mathbb{Z}}^*(A) \otimes \mathbb{Z}(q^{1/2}), [\omega \wedge]_q)$  again a complex?

# 2. Topology of arrangements

Setting:  $\bar{A} = \{\bar{H}_1, \dots, \bar{H}_n\}$  lines in  $\mathbb{C}\mathbb{P}^2$ ,  $\cup H_i = \{Q(x_1, x_2, x_3) = 0\}$ .

$A = \{H_1, \dots, H_n\}$  lines in  $\mathbb{C}^2 = \mathbb{C}\mathbb{P}^2 \setminus \bar{H}_n$

$F = Q^{-1}(1) \xrightarrow{\mathbb{Z}_n\text{-cover}} M = M(A) = \mathbb{C}^2 \setminus \cup H_i$ .

$$A_{\mathbb{Z}}^*(A) \cong H^*(M, \mathbb{Z}) \cong \Lambda \langle e_1, \dots, e_n \rangle_{\mathbb{Z}} / \text{OS-ideal}$$

$$\cup \\ \omega = e_1 + e_2 + \dots + e_n$$

Consider the  $q$ -deformation of the Aomoto complex:

$$\dots \rightarrow A_{\mathbb{Z}}^i(A) \xrightarrow{\omega_{\wedge}} A_{\mathbb{Z}}^{i+1}(A) \rightarrow \dots$$

$(a_{ij})_{i,j}$

Once we fix a basis, we have a **matrix presentation**. Define the  $q$ -deformation

$$[\omega_{\wedge}]_q := ([a_{ij}]_q)_{i,j} : A^i \otimes \mathbb{Z}(q^{1/2}) \rightarrow A^{i+1} \otimes \mathbb{Z}(q^{1/2}).$$

Is the  $q$ -deformation  $(A_{\mathbb{Z}}^{\bullet}(A) \otimes \mathbb{Z}(q^{1/2}), [\omega_{\wedge}]_q)$  again a complex?

Thm (Y. 20???) Suppose  $\bar{A}$  is defined over  $\mathbb{R}$ .

Then  $\exists$  basis of  $A_{\mathbb{Z}}^i(A)$  s.t.

(i)  $(A_{\mathbb{Z}[q^{1/2}]}^{\bullet}(A), [\omega_{\wedge}]_q)$  is a cochain complex.

(ii) Specialization at  $\sqrt[n]{\lambda}$  computes **monodromy eigenspace** of Milnor fiber.

$$H^i(A_{\mathbb{C}}^{\bullet}(A), [\omega_{\wedge}]_{q=\lambda}) \cong H^i(F)_{\lambda}, \text{ for } \lambda \in \mathbb{C}, \lambda^n = 1, \lambda \neq 1.$$

# 2. Topology of arrangements

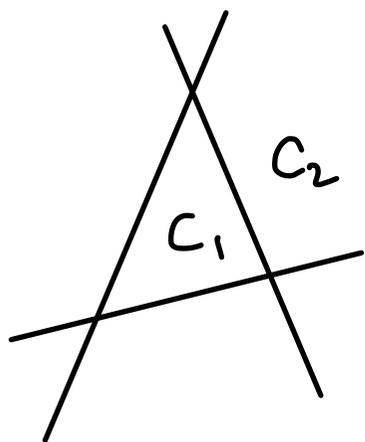
Thm (Y. 20??) Suppose  $\bar{A}$  is defined over  $\mathbb{R}$ . Then  $\exists$  basis of  $A_{\mathbb{Z}}^*(A)$  s.t.

(i)  $(A_{\mathbb{Z}(q^{1/2})}^*(A), [\omega^*]_q)$  is a cochain complex.

(ii) Specialization at  $\sqrt[n]{1}$  computes monodromy eigenspace of Milnor fiber.

$$H^i(A_{\mathbb{C}}^*(A), [\omega^*]_{z=\lambda}) \cong H^i(F)_{\lambda}, \text{ for } \lambda \in \mathbb{C}, \lambda^n = 1, \lambda \neq 1.$$

How to choose basis?



Chambers determine closed 2-dim submanifolds of  $M = M(d)$

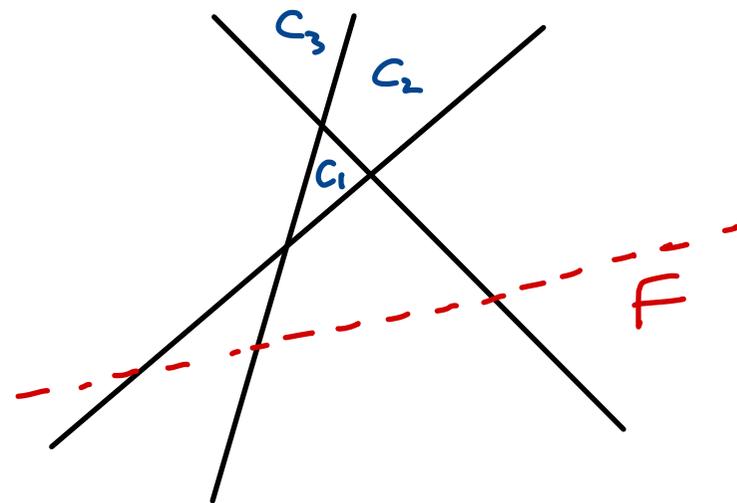
$$\rightsquigarrow [C] \in H_2^{BM}(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z})$$

Fact (Y.) Let  $F$  be a generic line.

Then  $\{C : \text{chamber} \mid C \cap F = \emptyset\}$

forms a basis of  $H_2^{BM}(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z})$ .

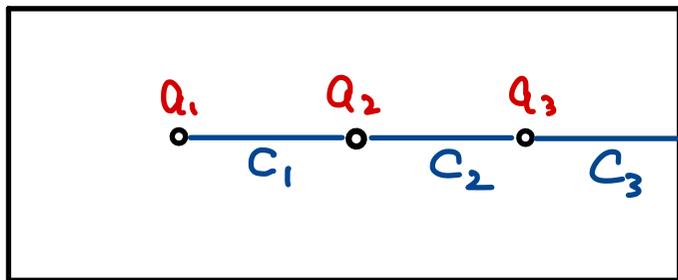
$\uparrow$  Use "chamber basis"



# 2. Topology of arrangements

How  $q$ -analogue appears?

Example  $A = \{a_1, a_2, a_3\} \subset \mathbb{R} \subset \mathbb{C}$ ,  $a_1 < a_2 < a_3$ ,  $e_i = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z-a_i}$



$$[c_1] = e_1 - e_2$$

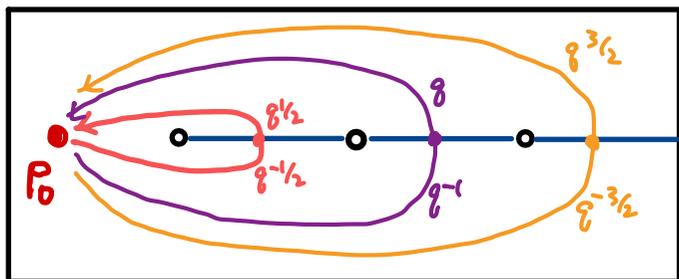
$$[c_2] = e_2 - e_3$$

$$[c_3] = e_3$$

$$\omega = e_1 + e_2 + e_3 = (e_1 - e_2) + 2(e_2 - e_3) + 3e_3$$

$$= [c_1] + 2 \cdot [c_2] + 3 \cdot [c_3].$$

On the other hand,



$$q^{3/2} - q^{-3/2} = (q^{1/2} - q^{-1/2}) \cdot [3]_q.$$

$$q^1 - q^{-1} = (q^{1/2} - q^{-1/2}) \cdot [2]_q$$

$$q^{1/2} - q^{-1/2} = (q^{1/2} - q^{-1/2}) \cdot [1]_q$$

$q$ -analogue  
(when  $q \neq 1$ )

Twisted cochain map:  $[p_0] \mapsto (q^{1/2} - q^{-1/2}) \cdot \{[1]_q \cdot [c_1] + [2]_q \cdot [c_2] + [3]_q \cdot [c_3]\}$

## 2. Topology of arrangements

### Summary (and Remarks)

- $H_1(F, \mathbb{C}) = \bigoplus_{\lambda^n=1} H_1(F)_\lambda = H_1(F)_1 \oplus H_1(F)_{\neq 1}$
- $H_1(F)_\alpha$  ( $\alpha = e^{2\pi i/3}$ ) is related to  $\mathbb{F}_3$ -Aomoto complex.

- $(A_{\mathbb{Z}}(A), \omega_n)$ : Aomoto complex

{  $q$ -deform  
↓ (with chamber basis)

$(A_{\mathbb{Z}[q^{\pm 1/2}]}(A), [\omega_n]_q)$

$q$ -deformed Aomoto complex computes  $H_1(F)_\lambda$ .

- “ $q$ -deformed Aomoto complex” can be defined for real hyperplane arr. (beyond line arr.). However, we need a tricky deform. e.g.  $6 \rightsquigarrow [3]_q + [3]_q$  (NOT  $[6]_q$ ).
- (?) Clebsch - Gordan rule for  $U_q(\mathfrak{sl}_2)$  (?)

# 3 Geometry behind Enumerative poly.

Characteristic quasi polynomial of arr.

Def.  $g: \mathbb{Z}$  (or  $\mathbb{Z}_{>0}$ )  $\rightarrow \mathbb{C}$  is a **quasi-polynomial**

$\Leftrightarrow \exists p > 0, \exists f_1, f_2, \dots, f_p \in \mathbb{C}[t]$

s.t.

$$g(n) = \begin{cases} f_1(n) & \text{if } n \equiv 1 \pmod{p} \\ f_2(n) & \text{if } n \equiv 2 \pmod{p} \\ \vdots & \\ f_p(n) & \text{if } n \equiv p \pmod{p}. \end{cases}$$

Rem. The number  $p$  is called the **period**, and

the polynomials  $f_1(t), \dots, f_p(t)$  are called **constituents** of  $g$ .

# 3 Geometry behind Enumerative poly.

Def.  $g: \mathbb{Z}$  (or  $\mathbb{Z}_{>0}$ )  $\rightarrow \mathbb{C}$  is a quasi-polynomial

$\Leftrightarrow \exists p > 0$  (period),  $\exists f_1, f_2, \dots, f_p \in \mathbb{C}[t]$  s.t.  $g(n) = \begin{cases} f_1(n) & \text{if } n \equiv 1 \pmod{p} \\ f_2(n) & \text{if } n \equiv 2 \pmod{p} \\ \vdots \\ f_p(n) & \text{if } n \equiv p \pmod{p}. \end{cases}$

Example

$$\lfloor \frac{n}{10} \rfloor = \begin{cases} \frac{n-1}{10} & \text{if } n \equiv 1 \pmod{10} \\ \frac{n-2}{10} & \text{if } n \equiv 2 \pmod{10} \\ \vdots \\ \frac{n-9}{10} & \text{if } n \equiv 9 \pmod{10} \\ \frac{n}{10} & \text{if } n \equiv 10 \pmod{10} \end{cases}$$

# 3 Geometry behind Enumerative poly.

Let  $A = \{H_1, \dots, H_n\}$  be an arr.  $/\mathbb{Z}$  i.e.

$$H_i = \{ (x_1, \dots, x_n) \mid a_{i1}x_1 + \dots + a_{in}x_n = b_i \}$$

with  $a_{ij}, b_i \in \mathbb{Z}$ .

$\downarrow \oplus \mathbb{Z}/\mathfrak{g}\mathbb{Z}$

$$\bar{H}_i := \{ (x_1, \dots, x_n) \in (\mathbb{Z}/\mathfrak{g}\mathbb{Z})^n \mid a_{i1}x_1 + \dots + a_{in}x_n \equiv b_i \pmod{\mathfrak{g}} \}$$

Then  $\mathfrak{g} \mapsto \# \left[ (\mathbb{Z}/\mathfrak{g}\mathbb{Z})^n \setminus \bigcup_{H \in A} \bar{H} \right]$  is a quasi-poly.

More precisely, ...

# 3 Geometry behind Enumerative poly.

Let  $A = \{H_1, \dots, H_n\}$  be an arr. /  $\mathbb{Z}$  i.e.

$$H_i = \{ (x_1, \dots, x_e) \mid a_{i1}x_1 + \dots + a_{ie}x_e = b_i \} \text{ with } a_{ij}, b_i \in \mathbb{Z}.$$

$$\bar{H}_i := \{ (x_1, \dots, x_e) \in (\mathbb{Z}/g\mathbb{Z})^e \mid a_{i1}x_1 + \dots + a_{ie}x_e = b_i \pmod{g} \}$$

Thm (Kamiya-Takemura-Terao 2007)

(1)  $\exists p > 0, \exists f_1, \dots, f_p \in \mathbb{Z}[t]$  s.t.

$$\chi_{\text{quasi}}(A, g) := \# \left[ (\mathbb{Z}/g\mathbb{Z})^e \setminus \bigcup_{H \in A} \bar{H} \right] = \begin{cases} f_1(g) & \text{if } g \equiv 1 \pmod{p} \\ \vdots \\ f_p(g) & \text{if } g \equiv p \pmod{p}. \end{cases}$$

(2) (GCD-property)  $\gcd(p, i) = \gcd(p, j) \Rightarrow f_i(t) = f_j(t).$

(3)  $f_1(t) = \chi(A, t).$

# 3 Geometry behind Enumerative poly.

$$\bar{H}_i := \{ (x_1, \dots, x_n) \in (\mathbb{Z}/q\mathbb{Z})^n \mid a_{i1}x_1 + \dots + a_{in}x_n = b_i \pmod{q} \}$$

Thm (Kamiya-Takemura-Terao 2007)

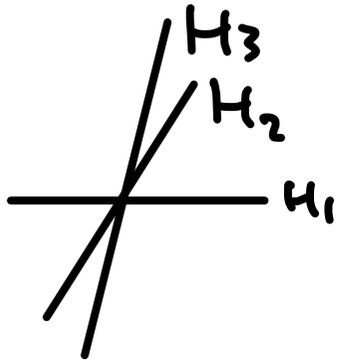
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$$(3) f_1(t) = \chi(A, t).$$

Example  $H_1 = \{y=0\}$ ,  $H_2 = \{y=2x\}$ ,  $H_3 = \{y=3x\}$ . Then



$$\chi_{\text{quasi}}(A, q) = \begin{cases} q^2 - 3q + 2, & q \equiv 1 \text{ or } 5 \pmod{6}, \\ q^2 - 3q + 3, & q \equiv 2 \text{ or } 4 \\ q^2 - 3q + 4, & q \equiv 3 \\ q^2 - 3q + 5, & q \equiv 6 \end{cases}$$

# 3 Geometry behind Enumerative poly.

$$\bar{H}_i := \{ (x_1, \dots, x_n) \in (\mathbb{Z}/g\mathbb{Z})^n \mid a_{i1}x_1 + \dots + a_{in}x_n = b_i \pmod{g} \}$$

Thm (Kamiya-Takemura-Terao 2007)

$$\chi_{\text{quasi}}(A, g) :=$$

$$(1) \exists p > 0, \exists f_1, \dots, f_p \in \mathbb{Z}[t] \text{ s.t. } \#[(\mathbb{Z}/g\mathbb{Z})^n \setminus \bigcup_{H \in \mathcal{A}} \bar{H}] = \begin{cases} f_1(g) & \text{if } g \equiv 1 \pmod{p} \\ \vdots \\ f_p(g) & \text{if } g \equiv p \pmod{p} \end{cases}$$

$$(2) (\text{GCD-property}) \gcd(p, i) = \gcd(p, j) \Rightarrow f_i(t) = f_j(t).$$

$$(3) \boxed{f_1(t) = \chi(A, t)}$$

Q. What are other constituents?

A. They are related to  $\otimes \mathbb{C}^n$  (torification?)

$$H = \{ (x_1, \dots, x_n) \mid a_1x_1 + \dots + a_nx_n = 0 \}$$

$$\downarrow \otimes \mathbb{C}^n$$

$$\tilde{H} = \{ (t_1, \dots, t_n) \in (\mathbb{C}^n) \mid t_1^{a_1} \cdot t_2^{a_2} \cdots t_n^{a_n} = 1 \}$$

# 3 Geometry behind Enumerative poly.

Thm (Kamiya-Takemura-Terao 2007)

$$\chi_{\text{quasi}}(A, \mathfrak{f}) := \begin{cases} f_i(\mathfrak{f}) & \text{if } \mathfrak{f} \equiv 1 \pmod{p} \\ \vdots \\ f_p(\mathfrak{f}) & \text{if } \mathfrak{f} \equiv p \pmod{p} \end{cases}$$

$$(1) \exists p > 0, \exists f_1, \dots, f_p \in \mathbb{Z}[t] \text{ s.t. } \#[(\mathbb{Z}/p\mathbb{Z})^r \setminus \bigcup_{H \in A} \tilde{H}] =$$

$$(2) (\text{GCD-property}) \gcd(p, i) = \gcd(p, j) \Rightarrow f_i(t) = f_j(t).$$

$$(3) f_i(t) = \chi(A, t)$$

Q. What are other constituents?

A. They are related to  $\otimes \mathbb{C}^*$  (torification?)

$$H = \{ (x_1, \dots, x_r) \mid a_1 x_1 + \dots + a_r x_r = 0 \}$$

$$\downarrow \otimes \mathbb{C}^*$$

$$\tilde{H} = \{ (t_1, \dots, t_r) \in (\mathbb{C}^*)^r \mid t_1^{a_1} \cdot t_2^{a_2} \cdot \dots \cdot t_r^{a_r} = 1 \}$$

Thm (Ye Liu, Tan Nhat Tran, M.Y. 2021)

$f_p(t)$  is the char. poly. of torus arr.  $\{ \tilde{H} \mid H \in A \}$  on  $(\mathbb{C}^*)^r$ .

Furthermore, the Poincaré poly. of the complement is

$$\sum_{\mathfrak{k}} b_{\mathfrak{k}}((\mathbb{C}^*)^r \setminus \bigcup \tilde{H}) t^{\mathfrak{k}} = (-t)^r \cdot f_p\left(-\frac{1+t}{t}\right).$$

# 3 Geometry behind Enumerative poly.

What is " $-Q$ " for a poset  $Q$ ?

... Ans.  $Q \times \mathbb{R}$  with lex. order

S. Schanuel "Negative sets ..." (1990)

{finite sets}  $\dashrightarrow$   $\textcircled{?}$



# 3 Geometry behind Enumerative poly.

Combinatorial reciprocity  $[n] = \{1, 2, \dots, n\}$

Example  $\mathcal{O}_3^<(n) := \#\{(x_1, x_2, x_3) \mid x_i \in [n], x_1 < x_2 < x_3\}$   
 $= \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

$$\mathcal{O}_3^{\leq}(n) := \#\{(x_1, x_2, x_3) \mid x_i \in [n], x_1 \leq x_2 \leq x_3\}$$
$$= \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$$

$$\mathcal{O}_3^{\leq}(n) = (-1)^3 \cdot \mathcal{O}_3^<(-n) \quad \text{"reciprocity"}$$

# 3 Geometry behind Enumerative poly.

$$\mathcal{O}_3^<(n) := \#\{(x_1, x_2, x_3) \mid x_i \in [n], x_1 < x_2 < x_3\}$$

$$\mathcal{O}_3^{\leq}(n) := \#\{(x_1, x_2, x_3) \mid x_i \in [n], x_1 \leq x_2 \leq x_3\}$$

## Stanley's generalization

$P, Q$ : posets

$$\text{Hom}^{\leq}(P, Q) = \left\{ f: P \rightarrow Q \mid x_1 <_P x_2 \Rightarrow f(x_1) \leq_Q f(x_2) \right\}$$

$$\text{Hom}^{<}(P, Q) = \left\{ f: P \rightarrow Q \mid x_1 <_P x_2 \Rightarrow f(x_1) <_Q f(x_2) \right\}$$

Example

$$\mathcal{O}_3^<(n) = \#\text{Hom}^{<}([3], [n])$$

$$\begin{array}{c} \psi \\ f \longleftrightarrow (f(1) < f(2) < f(3)) \end{array}$$

$$\mathcal{O}_3^{\leq}(n) = \#\text{Hom}^{\leq}([3], [n])$$

# 3 Geometry behind Enumerative poly.

$$\text{Hom}^{\leq}(P, Q) = \{ f: P \rightarrow Q \mid x_1 <_P x_2 \Rightarrow f(x_1) \leq_Q f(x_2) \}$$

$$\text{Hom}^{<}(P, Q) = \{ f: P \rightarrow Q \mid x_1 <_P x_2 \Rightarrow f(x_1) <_Q f(x_2) \}$$

Thm (R. Stanley 1970)

$P$ : finite poset. Then

①  $\exists \mathcal{O}_P^{\leq}(t), \mathcal{O}_P^{<}(t) \in \mathbb{Q}[t]$  s.t.

$$\# \text{Hom}^{<}(P, [n]) = \mathcal{O}_P^{\leq}(n)$$

$$\# \text{Hom}^{\leq}(P, [n]) = \mathcal{O}_P^{<}(n)$$

② (reciprocity)  $\mathcal{O}_P^{\leq}(t) = (-1)^{|P|} \cdot \mathcal{O}_P^{<}(-t)$

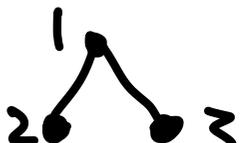
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- ② (reciprocity)  $\mathcal{O}_P^{\leq}(t) = (-1)^{|P|} \cdot \mathcal{O}_P^<(-t)$

Example  $P$ :  ( $1 > 2, 1 > 3$ ).

$$\begin{aligned} \# \text{Hom}^{\leq}(P, [n]) &= \# \{ (x_1, x_2, x_3) \mid x_i \in [n], x_1 \geq x_2, x_1 \geq x_3 \} \\ &= \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

$$\# \text{Hom}^<(P, [n]) = \sum_{k=1}^n (k-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

↕  $n \leftrightarrow -n$

# 3 Geometry behind Enumerative poly.

$$\text{Hom}^{\leq}(P, Q) = \{ f: P \rightarrow Q \mid x_1 <_P x_2 \Rightarrow f(x_1) <_Q f(x_2) \}$$

Thm (R. Stanley 1970)

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- ② (reciprocity)  $\mathcal{O}_P^{\leq}(t) = (-1)^{|P|} \cdot \mathcal{O}_P^{\leq}(-t)$

The reciprocity looks like

$$\# \text{Hom}^{\leq}(P, [n]) = (-1)^{\#P} \cdot \text{Hom}^{\leq}(P, [-n])$$

We want to define " $-Q$ " to justify

$$\# \text{Hom}^{\leq}(P, Q) = (-1)^{\#P} \cdot \# \text{Hom}^{\leq}(P, -Q)$$

# 3 Geometry behind Enumerative poly.

Hint



$$\sigma_d := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq \dots \leq x_d \leq 1\}$$

closed d-simplex



$$\sigma_d^\circ := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \dots < x_d \leq 1\}$$

open d-simplex.

Borel-Moore / cpt support Euler char.

$$e_{\text{BM}}(\sigma_d) = 1, \quad e_{\text{BM}}(\sigma_d^\circ) = (-1)^d.$$

$$e_{\text{BM}}(\sigma_d) = (-1)^d \cdot e_{\text{BM}}(\sigma_d^\circ) \quad \text{reciprocity?}$$

→ Use semialgebraic sets and semialgebraic Euler char.

# 3 Geometry behind Enumerative poly.

Def  $\mathbb{P}$  is a **semialgebraic poset**

$\iff$   
def

$\mathbb{P}$  has a semialgebraic structure  
s.t.  $\{ (x, y) \in \mathbb{P} \times \mathbb{P} \mid x < y \}$

is also semialgebraic. (**sapset**)

Example ① finite posets are semialgebraic

②  $\mathbb{R}^n$  with lexicographic order is sapset.

③  $\mathbb{R}^n$  with product order is semialg.

$$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \iff \forall i (x_i \leq y_i)$$

④ Any semialg. subset of these are sapsets.

# 3 Geometry behind Enumerative poly.

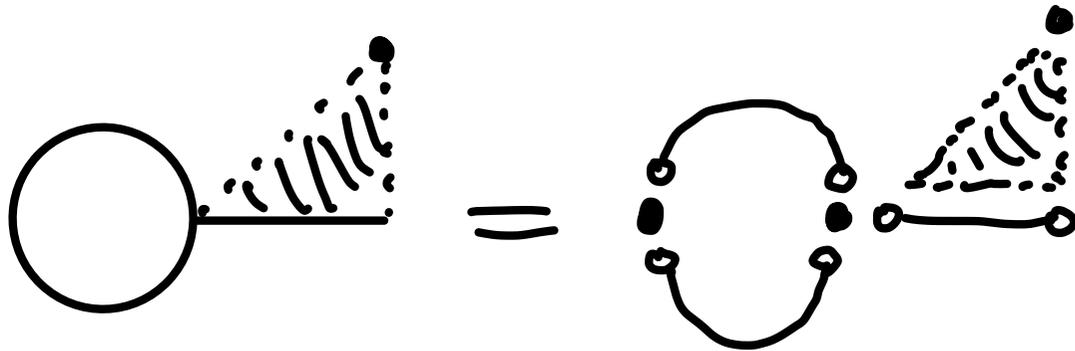
Rem For any semi alg set  $X$ ,

$\exists$  finite partition

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda \quad (X_\lambda: \text{semialgebraic})$$

s.t.  $X_\lambda \approx \mathbb{P}^{d_\lambda}$  for some  $d_\lambda \geq 0$ .

Ex



Fact  $e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda}$  is well defined  
(semialgebraic Euler char.)

# 3 Geometry behind Enumerative poly.

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda \quad \text{s.t.} \quad X_\lambda \approx \sigma_{d_\lambda}^0 \text{ for some } d_\lambda \geq 0. \quad e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda}$$

## Basic property

classical Euler  
char

① If  $X$  is compact,  $e(X) = e_{\text{top}}(X)$

② If  $X$  is locally cpt,  $e(X) = e_{\text{BM}}(X)$ .

③  $X \approx Y \Rightarrow e(X) = e(Y)$

④  $X$ : finite set  $\Rightarrow e(X) = |X|$

⑤  $e(X \sqcup Y) = e(X) + e(Y)$

⑥  $e(X \times Y) = e(X) \cdot e(Y)$

⑦  $e(\mathbb{R}) = -1$

# 3 Geometry behind Enumerative poly.

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda \quad \text{s.t.} \quad X_\lambda \approx \mathbb{P}_{d_\lambda}^0 \text{ for some } d_\lambda \geq 0. \quad e(X) := \sum_{\lambda \in \Lambda} (-1)^{d_\lambda}$$

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⑦  $e(\mathbb{R}) = -1$

Def. For a semialgebraic poset  $\mathcal{Q}$ , define

$$-\mathcal{Q} := \mathcal{Q} \times \mathbb{R}$$

with lex. order

$$(\theta_1, t_1) < (\theta_2, t_2) \iff \begin{cases} \theta_1 < \theta_2 \text{ or} \\ \theta_1 = \theta_2, t_1 < t_2 \end{cases}$$

Rem  $-(-\mathcal{Q}) \neq \mathcal{Q}$

# 3 Geometry behind Enumerative poly.

Def. For a semialgebraic poset  $Q$ , define

$$-Q := Q \times \mathbb{R} \text{ with } \underline{\text{lex. order}} \quad (q_1, t_1) < (q_2, t_2) \iff \begin{cases} q_1 < q_2 \text{ or} \\ q_1 = q_2, t_1 < t_2 \end{cases}$$

Thm (Hasebe, Miyatani, Y. 2017)

$P$ : finite poset,  $Q$ : saposet. Then

$$\textcircled{1} \quad e(\text{Hom}^<(P, Q)) = (-1)^{\#P} \cdot e(\text{Hom}^{\leq}(P, -Q))$$

$$\textcircled{2} \quad e(\text{Hom}^{\leq}(P, Q)) = (-1)^{\#P} \cdot e(\text{Hom}^<(P, -Q))$$

$\textcircled{3}$  If  $Q$  is totally ordered, then

$$e(\text{Hom}^{<(\leq)}(P, Q)) = \mathcal{O}_P^{<(\leq)}(e(Q)).$$

# 3 Geometry behind Enumerative poly.

Def. For a semialgebraic poset  $Q$ , define

$$-Q := Q \times \mathbb{R} \text{ with } \underline{\text{lex. order}} \quad (q_1, t_1) < (q_2, t_2) \iff \begin{cases} q_1 < q_2 \text{ or} \\ q_1 = q_2, t_1 < t_2 \end{cases}$$

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③ If  $Q$  is totally ordered, then  $e(\text{Hom}^{<(\leq)}(P, Q)) = \bigcup_P^{<(\leq)}(e(Q))$ .

② is recently refined

Thm. (Yoshida, Y. 2022)

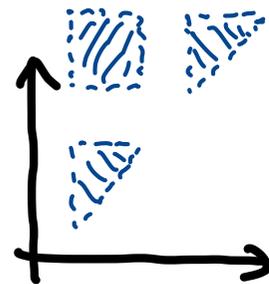
$$\text{Hom}^{\leq}(P, Q) \times \mathbb{R}^{|P|} \underset{\text{homeo}}{\approx} \text{Hom}^<(P, -Q).$$

Example  $P = Q = [2]$

$$-Q = [2] \times (0, 1)$$

$$\text{Hom}^{\leq}(P, Q) = \left\{ \begin{array}{c} \bullet \\ \bullet \end{array} \right\}$$

$$\text{Hom}^<(P, -Q) =$$



# 3 Geometry behind Enumerative poly.

Log concavity

# 3 Geometry behind Enumerative poly.

$$\chi(A, t) = t^l - a_1 t^{l-1} + a_2 t^{l-2} - \dots + (-1)^l a_l.$$

Thm (Huh, et.al.)

$$a_i^2 \geq a_{i-1} \cdot a_{i+1}.$$

Timeline :

- (i) 2012 Huh, for  $A / \mathbb{C}$
- (ii) 2012 Huh-Katz, for  $A / \mathbb{K}$  any field.
- (iii) 2018 Adiprasito-Huh-Katz, any matroid.

The proof is reduced to ...

- (i) log-concavity of mixed multiplicity (by Tessier)
- (ii) Khovanski-Tessier's inequality.
- (iii) extension of Hodge-Riemann rel. to matroids.

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Plan of today : Proof of (i) along the idea of (iii).

# 3 Geometry behind Enumerative poly.

Let  $X$  be a smooth projective variety/ $\mathbb{C}$ , of  $\dim_{\mathbb{C}} = d$

$A^*(X)$ : the Chow ring =  $A^0 \oplus A^1 \oplus \dots \oplus A^d$   
(an algebra generated by algebraic cycles with intersection product)

Intersection product:  $Z_1, Z_2 \subset X$  smooth subvariety,  
Assume  $Z_1 \cap Z_2$ . Then  $[Z_1] \cdot [Z_2] = [Z_1 \cap Z_2]$ .

Today we only looked at  $X$  such that

$$A^k(X) \xrightarrow{\cong} H^{2k}(X, \mathbb{R}), \text{ and } H^{\text{odd}}(X) = 0.$$

Example The following  $X$  satisfies the assumption:  
 $X = \mathbb{P}^d$ , Grassmann variety, flag variety,  
smooth projective toric variety, etc.

# 3 Geometry behind Enumerative poly.

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## Basic notions

- The integration determines  $\int: H^{2d}(X) \xrightarrow{\cong} \mathbb{R}$ .  
We also call this map **deg**:  $A^d(X) \xrightarrow{\cong} \mathbb{R}$ .
- (**Poincaré duality**)  $A^k(X) \times A^{d-k}(X) \rightarrow A^d(X) \xrightarrow{\text{deg}} \mathbb{R}$   
is a non degenerate pairing.
- A hypersurface  $S \subset X$  determines  $[S] \in A^1(X)$ ,  
and a curve  $C \subset X$  determines  $[C] \in A^{d-1}(X)$ .

# 3 Geometry behind Enumerative poly.

## Basic notions

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- A hypersurface  $S \subset X$  determines  $[S] \in A^1(X)$ ,  
and a curve  $C \subset X$  determines  $[C] \in A^{d-1}(X)$ .
- For  $f: X \rightarrow Y$ , we have  $f^*: A^i(Y) \rightarrow A^i(X)$ .
- Let  $H \subset \mathbb{P}^d$  be a hyperplane. Then  $A^i(\mathbb{P}^d)$   
is generated by  $d = [H] \in A^1(\mathbb{P}^d)$ . More  
precisely,  $A^i(\mathbb{P}^d) = \mathbb{R}[d]/(d^{d+1})$ .

# 3 Geometry behind Enumerative poly.

## Basic notions

- For  $f: X \rightarrow Y$ , we have  $f^*: A^*(Y) \rightarrow A^*(X)$ .
- Let  $H \subseteq \mathbb{P}^d$  be a hyperplane. Then  $A^*(\mathbb{P}^d)$  is generated by  $d = [H] \in A^1(\mathbb{P}^d)$ . More precisely,  $A^*(\mathbb{P}^d) = \mathbb{R}[d]/(d^{d+1})$ .
- $\alpha \in A^*(X)$  is called **ample**, if
$$\exists f: X \hookrightarrow \mathbb{P}^N \quad \text{s.t.} \quad \alpha = f^* d.$$
- The cone generated by ample elements is called the **ample cone**  $C_{\text{amp}}$ .
- **Kleiman's criterion**:
$$C_{\text{amp}} = \{ \alpha \in A^*(X) \mid \deg(C \cdot \alpha) > 0, \text{ for any curve } C \subseteq X \}$$

# 3 Geometry behind Enumerative poly.

## Basic notions

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•  $C_{\text{nef}} := \overline{C_{\text{amp}}} = \{x \in A^1(X) \mid \deg(C \cdot x) \geq 0 \text{ for } \forall \text{ curve } C \subseteq X\}$   
 $\uparrow$  numerically effective

• Let  $\alpha \in C_{\text{amp}}$ . Define the **Hodge-Riemann bilinear form** by  $Q: A^1(X) \times A^1(X) \rightarrow \mathbb{R}$

$$(x, y) \mapsto \deg(x \cdot y \alpha^{d-2})$$

# 3 Geometry behind Enumerative poly.

## Basic notions

◦ The cone generated by ample elements is called the **ample cone**  $C_{amp}$ .

◦ **Kleiman's criterion:**

$$C_{amp} = \{x \in A'(X) \mid \deg(C \cdot x) > 0, \text{ for any curve } C \subseteq X\}$$

◦  $C_{nef} := \bar{C}_{amp} = \{x \in A'(X) \mid \deg(C \cdot x) \geq 0 \text{ for } \forall \text{ curve } C \subseteq X\}$

◦ Let  $\alpha \in C_{amp}$ . Define the **Hodge-Riemann bilinear form**  $Q: A'(X) \times A'(X) \rightarrow \mathbb{R}$

$$(x, y) \mapsto \deg(x \cdot y \alpha^{d-2})$$

◦ (**Hodge-Riemann relation: HR**)

(\*)  $Q$  is positive definite on  $\langle \alpha \rangle = \mathbb{R} \cdot \alpha$

(\*\*\*)  $Q$  is negative definite on  $\alpha^\perp = \{x \in A'(X) \mid Q(\alpha, x) = 0\}$

# 3 Geometry behind Enumerative poly.

Prop (Keel) Let  $S_1, \dots, S_g \subset X$  be hypersurfaces  
 s.t.  $-S_i$ : Smooth

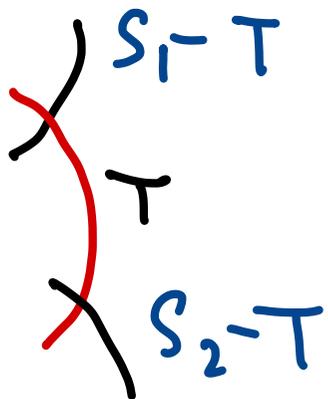
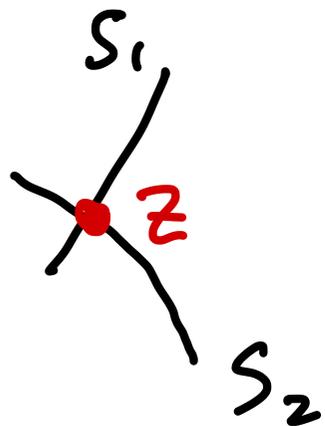
$-Z = S_1 \cap \dots \cap S_g$  is smooth,  $\dim Z = d - g$ ,  
 and  $H^{2*}(Z) = A^*(Z)$ .

Then

$$A^*(B\mathbb{P}_Z X) \cong \frac{A^*(X)[\tau]}{((\tau - [S_1]) \cdots (\tau - [S_g]), \tau \cdot \ker[Z])},$$

(and  $H^{\text{odd}}(X) = 0$ )

where  $\ker[Z] = \{x \in A^*(X) \mid x \cdot [Z] = 0\}$



strict transforms  
 do not intersect.  
 $([S_1] - T)([S_2] - T) = 0$

# 3 Geometry behind Enumerative poly.

Prop (Keel) Let  $S_1, \dots, S_g \subset X$  be hypersurfaces

s.t. —  $S_i$ : Smooth

—  $Z = S_1 \cap \dots \cap S_g$  is smooth,  $\dim Z = d - g$ , and  $H^{2*}(Z) = A^*(Z)$ .

$$\text{Then } A^*(\text{Bl}_Z X) \cong \frac{A^*(X)[\tau]}{((\tau - [S_1]) \cdots (\tau - [S_g]), \tau \cdot \text{ker}[Z])},$$

where  $\text{ker}[Z] = \{x \in A^*(X) \mid x \cdot [Z] = 0\}$

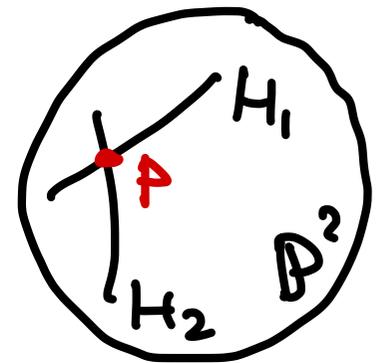
Example  $\text{Bl}_P \mathbb{P}^2$ .  $P \in \mathbb{P}^2$  is expressed as

$P = H_1 \cap H_2$ , ( $H_i \subseteq \mathbb{P}^2$  line). Since  $[H_i] = \alpha$ ,

$[P] = \alpha^2$ , and  $\text{ker}[P] = (\alpha)$ ,

$$A^*(\text{Bl}_P \mathbb{P}^2) = \frac{A^*(\mathbb{P}^2)[\tau]}{((\tau - \alpha)^2, \alpha \cdot \tau)}$$

$$\cong \frac{\mathbb{R}[\alpha, \tau]/(\alpha^3)}{(\alpha^3, \alpha \tau, \tau^2 + \alpha^2)}$$



Note:  $\deg(\tau^2) = -\deg(\alpha^2) = -1$ . “(-1)-curve”.

# 3 Geometry behind Enumerative poly.

Setting  $A = \{H_1, \dots, H_n\}$  : arr. in  $\mathbb{C}^l$ .

$$\chi(A, t) = t^l - a_1 t^{l-1} + a_2 t^{l-2} - \dots + (-1)^l a_l.$$

$\bar{A} := \{\bar{H}_1, \bar{H}_2, \dots, \bar{H}_n, \bar{H}_{\infty}\}$  : projective closure  
(arr. in  $\mathbb{C}P^l$ ).

Assume  $\bar{A}$  is essential, i.e.  $\exists$  0-dim intersections.

Def  $L_k$  is the set of all  $k$ -dim intersections.

Def (De Concini-Procesi's Wonderful compactification)

$$\mathbb{P}^l \leftarrow \text{Bl}_{L_0} \mathbb{P}^l \leftarrow \text{Bl}_{\bar{L}_1} (\text{Bl}_{L_0} \mathbb{P}^l) \leftarrow \text{Bl}_{\bar{L}_2} (\dots) \leftarrow Y_A$$

$\uparrow$   
Strict transform  
of 1-dim intersections

$$Y_A = \text{Bl}_{\bar{L}_{l-2}} (\text{Bl}_{\bar{L}_{l-3}} (\dots \text{Bl}_{L_0} \mathbb{P}^l) \dots).$$

# 3 Geometry behind Enumerative poly.

Setting  $A = \{H_1, \dots, H_n\}$  : arr. in  $\mathbb{C}^2$ .

$$\chi(A, t) = t^2 - a_1 t^{2-1} + a_2 t^{2-2} - \dots + (-1)^2 a_2.$$

$\bar{A} := \{\bar{H}_1, \bar{H}_2, \dots, \bar{H}_n, \bar{H}_\infty\}$  : projective closure (arr. in  $\mathbb{C}P^2$ ).

$\bar{H}_\infty$

Assume  $\bar{A}$  is essential, i.e.  $\exists$  0-dim intersections.

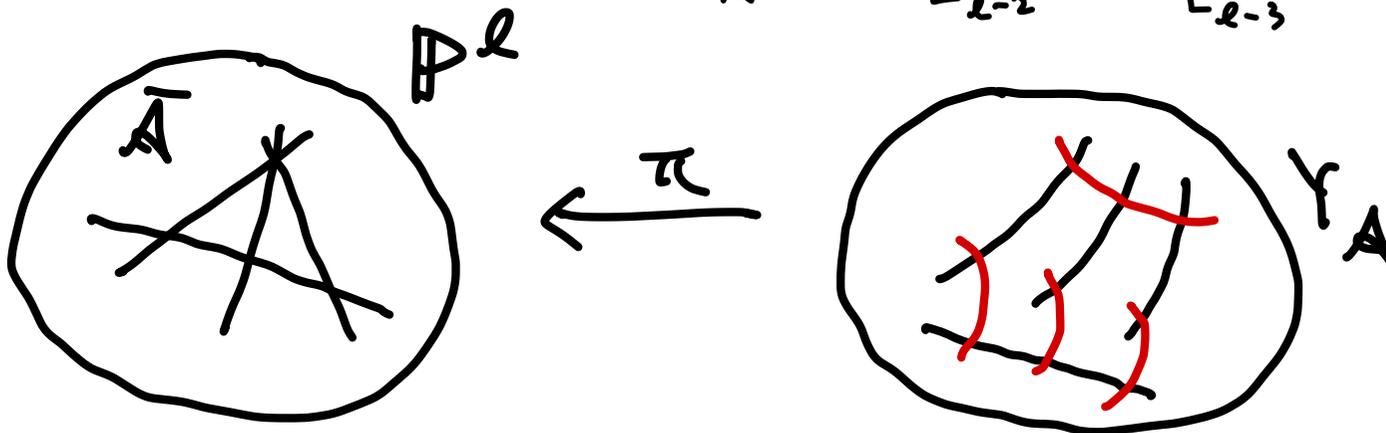
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$$\mathbb{P}^2 \leftarrow \text{Bl}_{L_0} \mathbb{P}^2 \leftarrow \text{Bl}_{L_1} (\text{Bl}_{L_0} \mathbb{P}^2) \leftarrow \text{Bl}_{L_2} (-) \dots \leftarrow Y_A$$

Strict transform  
of 1-dim intersections

$$Y_A = \text{Bl}_{L_{e-2}} (\text{Bl}_{L_{e-3}} (\dots \text{Bl}_{L_0} \mathbb{P}^2) \dots).$$



# 3 Geometry behind Enumerative poly.

Assume  $\bar{A}$  is essential, i.e.  $\exists 0$ -dim intersections.

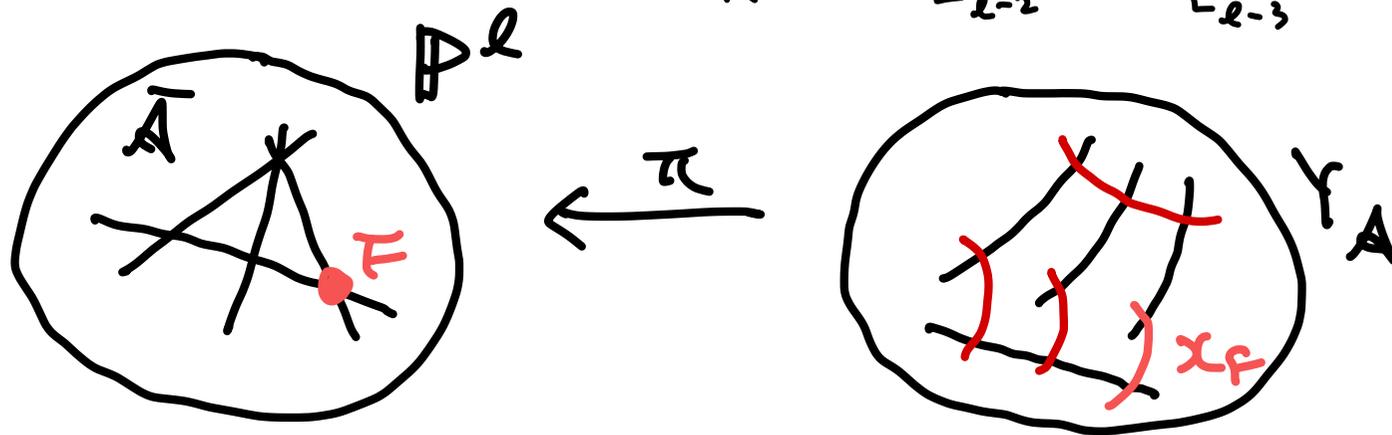
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Strict transform  $\nearrow$   
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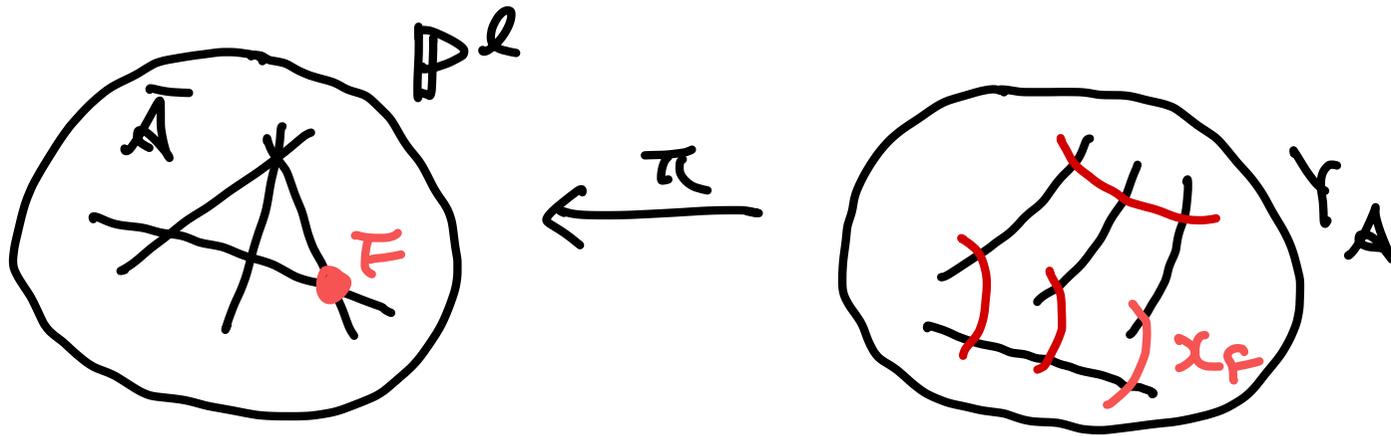
$$Y_A = \text{Bl}_{L_{l-2}} (\text{Bl}_{L_{l-3}} (\dots \text{Bl}_{L_0} \mathbb{P}^l) \dots)$$



Each intersection  $F \in L(\bar{A}) = L_0 \cup L_1 \cup \dots \cup L_{l-1}$  has corresponding hypersurface (strict transform)  $x_F$ .

# 3 Geometry behind Enumerative poly.

Def (Wonderful compactification)  $Y_A = \text{Bl}_{L_{\ell-2}}(\text{Bl}_{L_{\ell-3}}(\dots \text{Bl}_{L_0} \mathbb{P}^\ell) \dots)$ .

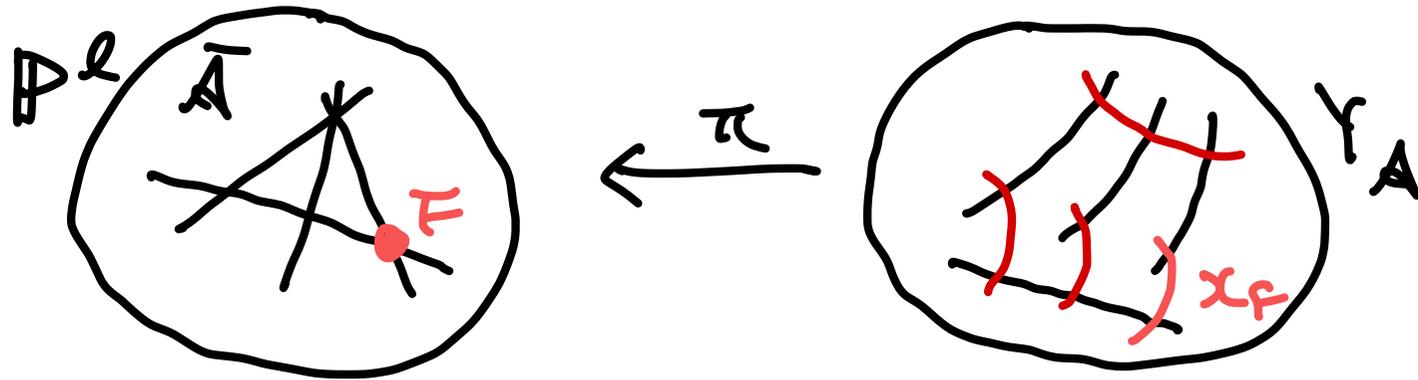


Each intersection  $F \in L(\bar{A}) = L_0 \cup L_1 \cup \dots \cup L_{\ell-1}$  has corresponding hypersurface (strict transform)  $X_F$ .

For each hyperplane  $H_i$ , let  $d_i := \sum_{F \subseteq H_i} X_F$ .

$d_i$  is a total transform of  $H_i$ . We have  $d_1 = d_2 = \dots = d_{n+1} (= \alpha)$ .

# 3 Geometry behind Enumerative poly.



Each intersection  $F \in L(\bar{A}) = L_0 \cup L_1 \cup \dots \cup L_{r-1}$  has corresponding hypersurface (strict transform)  $X_F$ .

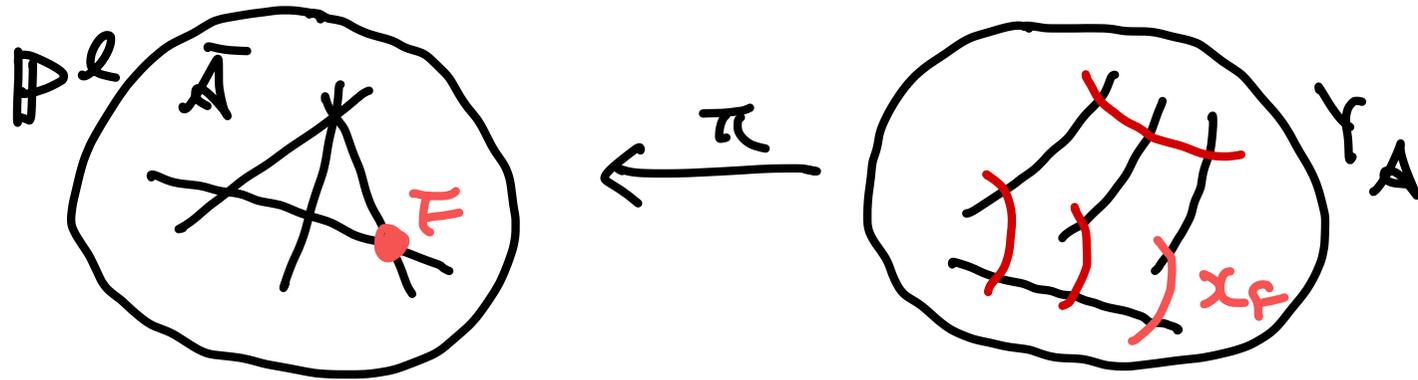
$$d_i := \sum_{F \in H_i} \chi_F. \quad d_1 = d_2 = \dots = d_{n-1} (= \alpha).$$

Thm (Feichtner-Yuzvinsky)

$$A^*(Y_A) \cong \mathbb{R}[\chi_F \mid F \in L(\bar{A})] / \left( \begin{array}{l} \chi_{F_1} \chi_{F_2} : F_1 \not\subseteq F_2 \\ d_i - d_j : 1 \leq i < j \leq n \end{array} \right)$$

Def  $\beta_i := \sum_{F \in H_i} \chi_F. \quad (\beta_1 = \beta_2 = \dots = \beta_n =: \beta)$

# 3 Geometry behind Enumerative poly.



Each intersection  $F \in L(\bar{A}) = L_0 \cup L_1 \cup \dots \cup L_{n-1}$  has corresponding hypersurface (strict transform)  $\chi_F$ .

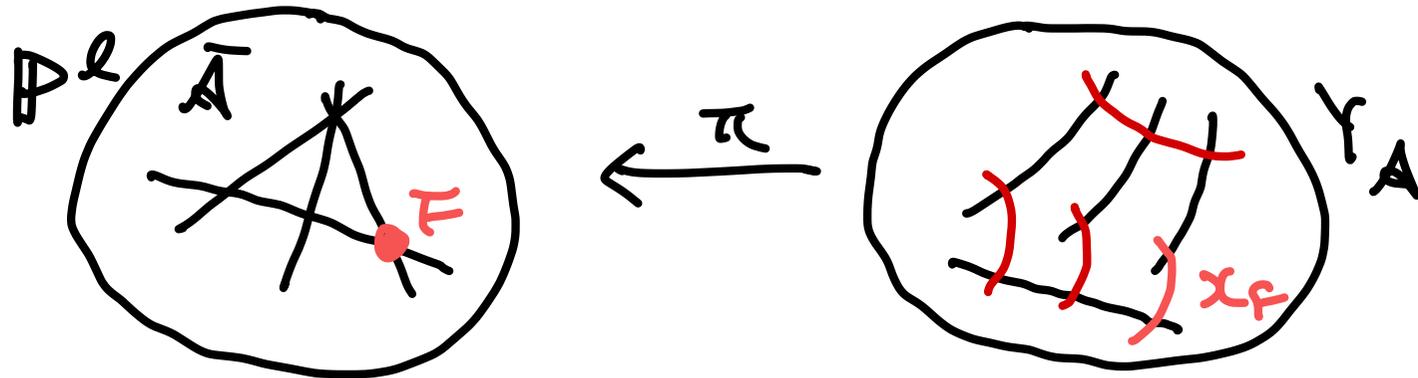
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# 3 Geometry behind Enumerative poly.



Each intersection  $F \in L(\bar{A}) = L_0 \cup L_1 \cup \dots \cup L_{n-1}$  has corresponding hypersurface (strict transform)  $\chi_F$ .

$$d_i := \sum_{F \in H_i} \chi_F. \quad d_1 = d_2 = \dots = d_{n-1} (= \alpha).$$

$$\beta_i := \sum_{F \notin H_i} \chi_F. \quad (\beta_1 = \beta_2 = \dots = \beta_n =: \beta)$$

Thm  $A^*(Y_A) \cong \mathbb{R}[\chi_F \mid F \in L(\bar{A})] / \left( \begin{array}{l} \chi_{F_1} \chi_{F_2} : F_1 \not\subseteq F_2 \\ d_i - d_j : 1 \leq i < j \leq n \end{array} \right)$

Coefficients of  $\chi(d, t)$  can be expressed as intersection #s.

Thm (Adiprasito-Huh-Katz Prop 9.5)

$$a_i = \deg(\alpha^{l-i} \cdot \beta^i)$$

# 3 Geometry behind Enumerative poly.

Thm (Adiprasito-Huh-Katz Prop 9.5)  $a_i = \deg(\alpha^{l-i} \cdot \beta^i)$

Thm (Huh)  $a_i^2 \geq a_{i+1} a_{i-1}$ .

(Proof) The case  $i=l-1$  ( $a_{l-1}^2 \geq a_{l-2} \cdot a_l$ ) is enough.

(Other cases are reduced to this case by Lefschetz hyperplane section type argument.)

Claim  $\deg(\alpha \cdot \beta^{l-1})^2 \geq \deg(\alpha^2 \cdot \beta^{l-2}) \cdot \deg(\beta^l)$  (\*)

Prop  $d, \beta \in C_{\text{nef}}$

Kleiman:  $C_{\text{amp}} = \{x \in A^1(X) \mid \deg(C \cdot x) > 0, \text{ for any curve } C \subseteq X\}$

$C_{\text{nef}} := \overline{C_{\text{amp}}} = \{x \in A^1(X) \mid \deg(C \cdot x) \geq 0 \text{ for } \forall \text{ curve } C \subseteq X\}$

By Kleiman's criterion,  $d, \beta$  are limit of sequences  $A_i, B_i$  ( $A_i, B_i \in C_{\text{amp}}$ ).

# 3 Geometry behind Enumerative poly.

Claim  $\deg(\alpha \cdot \beta^{l-1})^2 \geq \deg(\alpha^2 \cdot \beta^{l-2}) \cdot \deg(\beta^l)$  (\*)

$d = \lim A_i, \quad \beta = \lim B_i, \quad A_i, B_i \in \text{Camp.}$

Recall

- Let  $d \in \text{Camp.}$  Define the Hodge-Riemann bilinear form  $Q: A^l(X) \times A^l(X) \rightarrow \mathbb{R}$   
 $(x, y) \mapsto \deg(x \cdot y \cdot d^{d-2})$
- (Hodge-Riemann relation: HR)

(\*)  $Q$  is positive definite on  $\langle d \rangle = \mathbb{R} \cdot d$ .

(\*\*)  $Q$  is negative definite on  $d^\perp = \{x \in A^l(X) \mid Q(d, x) = 0\}$

Use  $B_i$  to define  $Q(x, y) = \deg(x \cdot y \cdot B_i^{l-2})$ .

$Q$  is pos. definite on  $\langle B_i \rangle$ , and

neg. definite on  $\langle B_i^\perp \rangle \cap \langle A_i, B_i \rangle$ .  
l-dim.

$\rightsquigarrow Q$  is indefinite on  $\langle A_i, B_i \rangle$ .

# 3 Geometry behind Enumerative poly.

Claim  $\deg(\alpha \cdot \beta^{\ell-1})^2 \geq \deg(\alpha^2 \cdot \beta^{\ell-2}) \cdot \deg(\beta^\ell)$  (\*)

$d = \lim A_i$ ,  $\beta = \lim B_i$ ,  $A_i, B_i \in \text{Camp}$ .

Use  $B_i$  to define  $Q(x, y) = \deg(x \cdot y \cdot B_i^{\ell-2})$ .

$Q$  is pos. definite on  $\langle B_i \rangle$ , and

neg. definite on  $\underbrace{\langle B_i^\perp \rangle \cap \langle A_i, B_i \rangle}_{1\text{-dim}}$ .

$\rightsquigarrow Q$  is indefinite on  $\langle A_i, B_i \rangle$ .

Hence

$$\det \begin{pmatrix} Q(A_i, A_i) & Q(A_i, B_i) \\ Q(A_i, B_i) & Q(B_i, B_i) \end{pmatrix} = \det \begin{pmatrix} A_i^2 \cdot B_i^{\ell-2} & A_i B_i^{\ell-1} \\ A_i B_i^{\ell-1} & B_i^\ell \end{pmatrix} \leq 0.$$

Take the limit  $i \rightarrow \infty$ ,  $a_{\ell-2} a_\ell - a_{\ell-1}^2 \leq 0$ .

(Q.E.D.)