Ryshkov domains of reductive algebraic groups

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Abstract

Let \( G \) be a connected reductive algebraic group defined over a number field \( k \). In this paper, we introduce the Ryshkov domain \( R \) for the arithmetical minimum function \( m_Q \) defined from a height function associated to a maximal \( k \)-parabolic subgroup \( Q \) of \( G \). The domain \( R \) is a \( Q(k) \)-invariant subset of the adele group \( G(A) \). We show that a fundamental domain \( \Omega \) for \( Q(k) \backslash R \) yields a fundamental domain for \( G(k) \backslash G(A) \). We also see that any local maximum of \( m_Q \) is attained in the boundary of \( \Omega \).

Introduction Let \( P_n \) be the cone of positive definite \( n \) by \( n \) real symmetric matrices, and let \( m(A) \) be the arithmetical minimum \( \min_{x \neq y \in \mathbb{Z}^n \setminus \{x \} : x A x} \) of \( A \in P_n \). The function \( f : A \mapsto m(A)/(\det A)^{1/n} \) on \( P_n \) is called the Hermite invariant. Since the maximum of \( f \) gives the Hermite constant \( \gamma_n \) for dimension \( n \), the determination of local maxima of \( f \) is a fundamental problem of lattice sphere packings in Euclidean spaces and arithmetic theory of quadratic forms. Voronoi’s theorem [15, Théorème 17] states that \( f \) attains a local maximum at a point \( A \) if and only if \( A \) is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi’s reduction theory of \( P_n \) with respect to the action of \( GL_n(\mathbb{Z}) \) (see, e.g., [8], [10]). In [9], Ryshkov introduced a locally finite polyhedron \( R(m) \) in \( P_n \) defined by the condition \( m(A) \geq 1 \). It is not difficult to show that \( A \) is perfect with \( m(A) = 1 \) if and only if \( A \) is a vertex of the boundary of \( R(m) \). In particular, any local maximum of the Hermite invariant \( f \) is attained in the boundary of \( R(m) \). In this sense, we can say that the Ryshkov polyhedron \( R(m) \) is well matched with \( f \).

Let \( G \) be a connected isotropic reductive algebraic group defined over a number field \( k \), and let \( Q \) be a maximal \( k \)-parabolic subgroup of \( G \). In previous papers [16] and [17], we investigated a constant \( \gamma(G, Q, k) \) as a generalization of Hermite’s constant \( \gamma_n \). Precisely, the constant \( \gamma(G, Q, k) \) is defined to be the maximum of the function \( m_Q(g) = \min_{x \in Q(k) \setminus G(k)} H_Q(xg) \) on \( G(k) \backslash G(A)^1 \), where \( H_Q \) denotes the height function associated to \( Q \). To prove the existence of the maximum of \( m_Q \), we used Borel and Harish-Chandra’s reduction theory for the adele group \( G(A) \) with respect to \( G(k) \). However, a Siegel set in \( G(A) \) is not well matched with \( m_Q \) in a sense that one can not obtain any information on locations of extreme points of \( m_Q \) in a Siegel set.

The purpose of this paper is to construct a fundamental domain of \( G(A)^1 \) with respect to \( G(k) \) which is well matched with \( m_Q \). We first consider an analog of the Ryshkov polyhedron. For a given \( g \in G(A)^1 \), we set \( X_Q(g) = \{ x \in Q(k) \setminus G(k) : m_Q(g) = H_Q(xg) \} \). This is a finite subset of \( Q(k) \backslash G(k) \) and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain \( R(m_Q) \) as follows:

\[
R(m_Q) = \{ g \in G(A)^1 : \pi \in X_Q(g) \}
\]
where \( \mathcal{T} \) denotes the trivial class \( Q(k) \in Q(k) \setminus G(k) \). The set \( R(m_Q) \) is a left \( Q(k) \)-invariant closed set with non-empty interior. The interior of \( R(m_Q) \) is just a subset \( R_1 \) consisting of \( g \in R(m_Q) \) such that \( X_Q(g) \) is the one-point set \( \{ \mathcal{T} \} \). We denote by \( R_1^- \) the closure of \( R_1 \) in \( G(A) \). Both \( R_1 \) and \( R_1^- \) are also left \( Q(k) \)-invariant. By Baer and Levi’s theorem [1, Satz 7], there exists an open fundamental domain \( \Omega_Q \) of \( R_1^- \) with respect to \( Q(k) \), i.e., \( \Omega_Q \) is a relatively open subset of \( R_1^- \) satisfying

- \( Q(k)\Omega_Q = R_1^- \), where \( \Omega_Q \) denotes the closure of \( \Omega_Q \) in \( R_1^- \), and
- \( \gamma \Omega_Q \cap \Omega_Q = \emptyset \) for any \( \gamma \in Q(k) \setminus \{ e \} \).

Let \( \Omega_Q^* \) denote the interior of \( \Omega_Q \) in \( G(A) \). Then our main theorem is stated as follows:

**Theorem.** The set \( \Omega_Q^* \) is an open fundamental domain of \( G(A) \) with respect to \( G(k) \). Any local maximum of \( m_Q \) is attained in the intersection of the boundary of \( \Omega_Q^* \) and the boundary of \( \Omega_Q^* \).

If we denote by \( r_G \) the \( k \)-rank of the commutator subgroup of \( G \), then \( G \) has \( r_G \) standard maximal \( k \)-parabolic subgroups. Since \( \Omega_Q \) depends on \( Q \), we obtain \( r_G \) different kinds of fundamental domains of \( G(A) \) with respect to \( G(k) \). The method to construct \( \Omega_Q \) may be viewed as a generalization of the highest point method (see [6], [13], §4,4]). For example, let \( k = Q \). \( G = GL_n \) and \( Q \) be a standard maximal \( Q \)-parabolic subgroup such that \( Q \) is a projective space. Then our construction gives a fundamental domain \( \Omega_Q \) whose Archimedean part is isomorphic with Grenier’s fundamental domain. If we choose another standard maximal \( Q \)-parabolic subgroup of \( GL_n \) as \( Q \), then the Archimedean part of \( \Omega_Q \) yields a new kind of fundamental domain of \( P_n \) with respect to \( GL_n(Z) \) (see Example 3 in §7).

**Notation** For a given ring \( \mathfrak{A} \), the set of all \( n \) by \( k \) matrices with entries in \( \mathfrak{A} \) is denoted by \( M_{n,k}(\mathfrak{A}) \). We write \( M_n(\mathfrak{A}) \) for \( M_{n,n}(\mathfrak{A}) \). The transpose of a given matrix \( a \in M_{n,k}(\mathfrak{A}) \) is denoted by ‘\( a^t \)’. In this paper, \( k \) denotes an algebraic number field of finite degree over \( Q \) and \( \mathcal{O} \) the ring of integers of \( k \). The sets of all infinite and finite places of \( k \) are denoted by \( \mathfrak{p}_\infty \) and \( \mathfrak{p}_f \), respectively. For \( \sigma \in \mathfrak{p}_\infty \cup \mathfrak{p}_f \), \( k_\sigma \) denotes the completion of \( k \) at \( \sigma \). For \( \sigma \in \mathfrak{p}_f \), \( k_\sigma \) denotes the closure of \( \sigma \) in \( k_\sigma \). The étale \( R \)-algebra \( k_\sigma = k \otimes_Q R \) is identified with \( \prod_{\sigma \in \mathfrak{p}_f} k_\sigma \). Let \( A \) and \( A^\times \) denote the adele ring and the idele group of \( k \), respectively. The idele norm of \( A^\times \) is denoted by \( | \cdot |_A \).

**1. Height functions**

Let \( G \) be a connected affine algebraic group defined over \( k \). For any \( k \)-algebra \( \mathfrak{A} \), \( G(\mathfrak{A}) \) stands for the set of \( \mathfrak{A} \)-rational points of \( G \). Let \( X^*(G)_k \) be the free \( Z \)-module consisting of all \( k \)-rational characters of \( G \). For each \( g \in G(\mathfrak{A}) \), we define the homomorphism \( \vartheta_G(g) : X^*(G)_k \to R_{\geq 0} \) by \( \vartheta_G(g)(\chi) = |\chi(g)|_A \) for \( \chi \in X^*(G)_k \). Then \( \vartheta_G \) is a homomorphism from \( G(\mathfrak{A}) \) into \( \text{Hom}_Z(X^*(G)_k, R_{\geq 0}) \). We write \( G(\mathfrak{A})^1 \) for the kernel of \( \vartheta_G \).

In the following, let \( G \) be a connected isotropic reductive group defined over \( k \). We fix a maximal \( k \)-split torus \( S \) of \( G \) and a minimal \( k \)-parabolic subgroup \( P_0 \) of \( G \) containing \( S \). Denote by \( \Phi_k \) and \( \Delta_k \) the relative root system of \( G \) with respect to \( S \) and the set of simple roots of \( \Phi_k \) corresponding to \( P_0 \), respectively. Let \( M_0 \) be the centralizer of \( S \) in \( G \). Then \( P_0 \) has a Levi decomposition \( P_0 = M_0U_0 \), where \( U_0 \) is the unipotent radical of \( P_0 \). A \( k \)-parabolic subgroup of \( G \) containing \( P_0 \) is called a standard \( k \)-parabolic subgroup of \( G \). Every standard \( k \)-parabolic subgroup \( R \) of \( G \) has a unique Levi subgroup \( M_R \) containing \( M_0 \). We
denote by $U_R$ the unipotent radical of $R$ and $Z_R$ the greatest central $k$-split torus in $M_R$.
Throughout this paper, we fix a maximal compact subgroup $K = \prod_{\sigma \in \varphi} K_{\sigma} \times \prod_{\sigma \in \varphi} K_{\sigma}$ of $G(A)$ satisfying the following property: For every standard $k$-parabolic subgroup $R$ of $G$, $K \cap M_R(A)$ is a maximal compact subgroup of $M_R(A)$ and $M_R(A)$ possesses an Iwasawa decomposition $(M_R(A) \cap U_0(A))M_0(A)(K \cap M_R(A))$.

Let $Q$ be a standard proper maximal $k$-parabolic subgroup of $G$. There is an only one simple root $\alpha_0 \in \Delta_k$ such that the restriction of $\alpha_0$ to $Z_Q$ is non-trivial. Let $n_Q$ be the positive integer such that $n_Q^{-1} a_0|Z_Q$ is a $Z$-basis of $X^*(Z_Q/Z_G)$. We write $\alpha_Q$ and $\tilde{\alpha}_Q$ for $n_Q^{-1} a_0|Z_Q$ and $\tilde{\alpha}_Q n_Q^{-1} a_0|Z_Q$, respectively, where $\tilde{\alpha}_Q = [X^*(Z_Q/Z_G)_{\kappa} : X^*(M_Q/Z_G)_{\kappa}]$. Then $\tilde{\alpha}_Q$ is a $Z$-basis of the submodule $X^*(M_Q/Z_G)_{\kappa}$ of $X^*(Z_Q/Z_G)_{\kappa}$. Define the map $z_Q : G(A) \rightarrow Z_G(A)M_Q(A)\backslash M_Q(A)$ by $z_Q(g) = Z_G(A)M_Q(A)^{\dagger}m$ if $g = umh$, $u \in U_Q(A), m \in M_Q(A)$ and $h \in K$. This is well defined and left $Z_G(A)Q(A)^{\dagger}$-invariant. Since $Z_G(A)^{\dagger} = Z_G(A) \cap G(A)^{\dagger} \subset M_Q(A)^{\dagger}$, $z_Q$ gives rise to a map from $Y_Q = Q(A)^{\dagger}\backslash Q(G)^{\dagger}$ to $M_Q(A)^{\dagger}\backslash (M_Q(A) \cap G(A)^{\dagger})$. Namely, we have the following commutative diagram:

$$
\begin{array}{ccc}
Y_Q & \rightarrow & M_Q(A)^{\dagger}\backslash (M_Q(A) \cap G(A)^{\dagger}) \\
\downarrow & & \downarrow \\
Z_G(A)Q(A)^{\dagger}\backslash G(A) & \rightarrow & Z_G(A)M_Q(A)^{\dagger}\backslash M_Q(A)
\end{array}
$$

Here both vertical arrows are natural maps. We define the height function $H_Q : G(A) \rightarrow R_{\geq 0}$ by $H_Q(g) = |\tilde{\alpha}_Q(z_Q(g))|^{-1}$ for $g \in G(A)$. We notice that the restriction of $H_Q$ to $M_Q(A)$ is a homomorphism from $M_Q(A)$ onto $R_{\geq 0}$.

**Example 1.** Let $G$ be a general linear group $GL_n$ defined over the rational number field $Q$, $P_0$ the group of upper triangular matrices in $G$ and $S$ the group of diagonal matrices in $G$. We fix an integer $k \in \{1, \ldots, n-1\}$, and let

$$Q(Q) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL_k(Q), b \in M_{k, n-k}(Q), d \in GL_{n-k}(Q) \right\}.$$ 

Then $Q$ is a standard maximal $Q$-parabolic subgroup of $G$. The rational character $\tilde{\alpha}_Q$ and the height $H_Q$ are given by

$$\tilde{\alpha}_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (\det a)^{(n-k)/r}(\det d)^{-k/r}$$

and

$$H_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = |\det a|^{- (n-k)/r}_{A} |\det d|^{k/r}_{A},$$

where $r$ denotes the greatest common divisor of $k$ and $n-k$. The height $H_Q$ has another expression. To explain this, let $Q^n$ be an $n$-dimensional column vector space over $Q$ with standard basis $e_1, \ldots, e_n$. The maximal parabolic subgroup $Q(Q)$ stabilizes the subspace spanned by $e_1, \ldots, e_k$. Let $V_{n,k}(Q) = \bigwedge^k Q^n$ be the $k$-th exterior product of $Q^n$. We set $V_{n,k}(A) = V_{n,k}(Q) \otimes_Q A$ and $V_{n,k}(Q_\sigma) = V_{n,k}(Q) \otimes_Q Q_\sigma$ for $\sigma \in \varphi_\infty \cup \varphi_f$. A $Q$-basis of $V_{n,k}(Q)$ is formed by the elements $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \ldots, n\}$. For a unique infinite place $\infty \in \varphi_\infty$, we define the local height $H_{\infty} : V_{n,k}(Q_\infty) \rightarrow R_{\geq 0}$ by

$$H_{\infty}(\sum_I a_I e_I) = \left( \sum_I |a_I|_{\infty}^2 \right)^{1/2},$$

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where \( | \cdot |_{\infty} \) denotes the usual absolute value of \( \mathbb{Q}_{\infty} = \mathbb{R} \). For each finite prime \( p \in \mathcal{P}_f \), we define the local height \( H_p : V_{n,k}(\mathbb{Q}_p) \rightarrow \mathbb{R}_{>0} \) by

\[
H_p \left( \sum_{i} a_i e_i \right) = \sup_{i} |a_i|_p,
\]

where \( | \cdot |_p \) denotes the \( p \)-adic absolute value of \( \mathbb{Q}_p \) normalized so that \( |p|_p = p^{-1} \). Then the global height \( H_{n,k} : V_{n,k}(\mathbb{Q}) \rightarrow \mathbb{R}_{>0} \) is defined to be a product of all local heights, i.e., 

\[
H_{n,k}(x) = \prod_{\sigma \in \mathcal{P}_{\infty} \cup \mathcal{P}_f} H_\sigma(x) \quad \text{for} \quad x \in V_{n,k}(\mathbb{Q}).
\]

This \( H_{n,k} \) is immediately extended to the subset \( \text{GL}(V_{n,k}(\mathbb{A}))V_{n,k}(\mathbb{Q}) \) of the adele space \( V_{n,k}(\mathbb{A}) \) by

\[
H_{n,k}(Ax) = \prod_{\sigma \in \mathcal{P}_{\infty} \cup \mathcal{P}_f} H_\sigma(A_{\sigma}x)
\]

for \( A = (A_{\sigma}) \in \text{GL}(V_{n,k}(\mathbb{A})) \) and \( x \in V_{n,k}(\mathbb{Q}) \). In particular, for \( g \in G(\mathbb{A}) = \text{GL}_n(\mathbb{A}) \), we can take the value \( H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \wedge ge_k) \). We choose a maximal compact subgroup \( K_{\infty} \) of \( \text{GL}(\mathbb{Q}_{\infty}) \) as \( \{ g \in \text{GL}(\mathbb{Q}_{\infty}) : g^{-1} = g \} \). Let \( K_f = \prod_{p \in \mathcal{P}_f} \text{GL}_n(\mathbb{Z}_p) \) and \( K = K_{\infty} \times K_f \). Then, by elementary computations, we have

\[
H_{n,k}(ge_1 \wedge ge_2 \wedge \cdots \wedge ge_k) = | \det a|_A \quad \text{if} \quad g = h \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)
\]

with \( h \in K, a \in \text{GL}_k(\mathbb{A}), b \in \mathbb{M}_{n-k}(\mathbb{A}) \) and \( d \in \text{GL}_{n-k}(\mathbb{A}) \). Therefore, if \( g \in G(\mathbb{A}) \), i.e., \( | \det g|_A = 1 \), then the equality

\[
H_Q(g) = H_{n,k}(g^{-1}e_1 \wedge g^{-1}e_2 \wedge \cdots \wedge g^{-1}e_k)^{n/r}
\]

holds.

2. Twisted height functions restricted to one parameter subgroups

Let \( N_G(S) \) be the normalizer of \( S \) in \( G \) and \( W_G = N_G(S)(k)/M_0(k) \) the Weyl group of \( G \) with respect to \( S \). For a simple root \( \alpha \in \Delta_k, s_{\alpha} \in W_G \) denotes the simple reflection corresponding to \( \alpha \). Then \( \{ s_{\alpha} \}_{\alpha \in \Delta_k} \) generates \( W_G \). We denote by \( W_G^Q \) the subgroup of \( W_G \) generated by \( \{ s_{\alpha} \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}} \). For each \( w \in W_G \), we use the same notation \( w \) for a representative of \( w \) in \( N_G(S)(k) \). The following cell decomposition of \( G(k) \) holds via Bruhat decomposition ([5, Proposition 4.10, Corollaire 5.20]);

\[
G(k) = \bigsqcup_{[w] \in W_G^Q \setminus W_G^Q / W_G^Q} Q(k)wQ(k),
\]

where \([w]\) stands for the class \( W_G^QwW_G^Q \) in \( W_G^Q \setminus W_G / W_G^Q \).

The Weyl group \( W_G \) acts on \( X^*(S)_k \) by \( \chi : t \mapsto \chi(w^{-1}tw) \) for \( w \in W_G \) and \( \chi \in X^*(S)_k \). We consider the restriction \( \hat{\alpha}_Q|_S \) of the rational character \( \hat{\alpha}_Q \) of \( M_Q \) to \( S \).

**Lemma 1.** The subgroup of \( W_G \) fixing \( \hat{\alpha}_Q|_S \) is equal to \( W_G^Q \).

**Proof.** Put \( W' = \{ w \in W_G : w \cdot \hat{\alpha}_Q|_S = \hat{\alpha}_Q|_S \} \). Since a representative of \( w \in W_G^Q \) is contained in \( M_Q(k) \), we have \( \hat{\alpha}_Q(w^{-1}tw) = \hat{\alpha}_Q(w)^{-1} \hat{\alpha}_Q(t) \hat{\alpha}_Q(w) = \hat{\alpha}_Q(t) \) for all \( t \in S \). Hence \( W_G^Q \) is contained in \( W' \). By [7, §1.12 Theorem (a) and (c) \], \( W' \) is generated by a subset \( W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k} \) of simple reflections. From \( W_G^Q \subset W' \), it follows \( \{ s_{\alpha} \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}} \subset W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k} \subset \{ s_{\alpha} \}_{\alpha \in \Delta_k} \). Since \( \hat{\alpha}_Q \) is non-trivial on \( S/Z_G \), \( W' \cap \{ s_{\alpha} \}_{\alpha \in \Delta_k} \) must be equal to \( \{ s_{\alpha} \}_{\alpha \in \Delta_k \setminus \{ \alpha_0 \}} \). Therefore \( W' \) coincides with \( W_G^Q \). \( \square \)
Let $X_*(S)_k$ be the free $Z$-module consisting of all $k$-rational cocharacters of $S$. A natural pairing

$$\langle \cdot, \cdot \rangle : X^*(S)_k \times X_*(S)_k \rightarrow Z$$

defined as in [4, §8.6] is a regular pairing over $Z$.

**Lemma 2.** Let $w_1$ and $w_2$ be elements of $W_G$ such that $w_1^{-1}W_G^Q \neq w_2^{-1}W_G^Q$. Then there exist a cocharacter $\xi = \xi_{w_1,w_2} \in X_*(S)_k$ such that $H_Q(w_1(\xi)w_1^{-1}) > H_Q(w_2(\xi)w_2^{-1})$ holds for all $\lambda \in A^{\times}_1$, where $A^{\times}_1$ denotes the set of $\lambda \in A^\times$ satisfying $|\lambda|_A > 1$.

Proof. Since $w_1^{-1} \cdot \hat{\xi}Q|_S - w_2^{-1} \cdot \hat{\xi}Q|_S \neq 0$ by Lemma 1, there is a $\xi \in X_*(S)_k$ such that

$$\langle w_1^{-1} \cdot \hat{\xi}Q|_S - w_2^{-1} \cdot \hat{\xi}Q|_S, \xi \rangle < 0.$$  

The value $\ell = \langle w_1^{-1} \cdot \hat{\xi}Q|_S - w_2^{-1} \cdot \hat{\xi}Q|_S, \xi \rangle$ is a negative integer. We have

$$\hat{\xi}Q(w_1(\lambda)w_1^{-1}) \cdot \hat{\xi}Q(w_2(\lambda)w_2^{-1})^{-1} = \lambda^\ell$$

for all $\lambda \in G_m$. Therefore,

$$H_Q(w_1(\lambda)w_1^{-1})H_Q(w_2(\lambda)w_2^{-1})^{-1} = |\lambda|_A^{-\ell} > 1$$

holds for all $\lambda \in A^{\times}_1$.

3. The Hermite function associated to $Q$ and minimal points

We set $X_Q = Q(k) \backslash G(k)$, which is regarded as a subset of $Y_Q = Q(A)^1 \backslash G(A)^1$. Let $\pi_X : G(k) \rightarrow X_Q$ be the natural quotient map. The symbol $\overline{\pi} = \pi_X(e) \in X_Q$ denotes the class of the unit element $e \in G(k)$. The Hermite function $m_Q : G(A)^1 \rightarrow R_{\geq 0}$ is defined to be

$$m_Q(g) = \min_{x \in X_Q} H_Q(xg).$$

By definition, $m_Q$ is a positive valued continuous function on $G(k) \backslash G(A)^1 / K$.

For each $g \in G(A)^1$, we put

$$X_Q(g) = \{ x \in X_Q : m_Q(g) = H_Q(xg) \},$$

which is a finite subset of $X_Q$. Thus we can define the counting function $n_Q(g) = |X_Q(g)|$.

**Lemma 3.** For any $g \in G(A)^1$, $\gamma \in G(k)$ and $h \in K$, one has $X_Q(\gamma gh) = X_Q(g)\gamma^{-1}$. Especially, the counting function $n_Q$ is left $G(k)$-invariant and right $K$-invariant.

The following Lemma is proved by the same method as in [18, Proof of Proposition 4.1].

**Lemma 4.** For $g \in G(A)^1$, there is a neighbourhood $\mathcal{U}$ of $g$ in $G(A)^1$ such that $X_Q(g') \subset X_Q(g)$ for all $g' \in \mathcal{U}$.

**Example 2.** Let $G$ be a general linear group $GL_n$ defined over $Q$. We keep notations used in Example 1. In this case, we can express $m_Q$ in terms of some minimum of positive definite symmetric matrices. Since $GL_n/Q$ is of class number one, $G(A)^1 = \{ g \in GL_n(A) : |\det g|_A = 1 \}$ has the following decomposition:

$$G(A)^1 = G(Q)(G(Q_{\infty})^1 \times K_f),$$

where $G(Q_{\infty})^1 = \{ g \in GL_n(Q_{\infty}) : \det g = \pm 1 \}$ and $K_f = \prod_{p \in p_f} GL_n(Z_p)$. We fix $g = \delta(g_\infty \times g_f) \in G(A)^1$ with $\delta \in G(Q)$, $g_\infty \in G(Q_{\infty})^1$ and $g_f \in K_f$. From the left $G(Q)$-invariance and the right $K$-invariance of $m_Q$, it follows that

$$m_Q(g) = m_Q(g_\infty) = \min_{x \in X_Q} H_Q(xg_\infty) = \min_{\gamma \in G(Q)} H_Q(\gamma g_\infty).$$

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Furthermore, since \( G(Q) = Q(Q)GL_n(Z) \) and \( H_Q \) is left \( Q(Q) \)-invariant, we have
\[
m_Q(g) = \min_{\gamma \in GL_n(Z)} H_Q(\gamma g_{\infty}).
\]
An elementary proof of the decomposition \( G(Q) = Q(Q)GL_n(Z) \) is found in [11, Theorem 3]. By Example 1,
\[
H_Q(\gamma g_{\infty}) = H_{n,k}(g_{\infty}^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_{\infty}^{-1} \gamma^{-1} e_k)^{n/r}
\]
\[
= H_{\infty}(g_{\infty}^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_{\infty}^{-1} \gamma^{-1} e_k)^{n/r} \times \prod_{p \in P_f} H_{p}(\gamma^{-1} e_1 \wedge \cdots \wedge \gamma^{-1} e_k)^{n/r}
\]
\[
= H_{\infty}(g_{\infty}^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_{\infty}^{-1} \gamma^{-1} e_k)^{n/r}.
\]
Here we notice that \( H_{p}(\gamma^{-1} e_1 \wedge \cdots \wedge \gamma^{-1} e_k) = 1 \) for all \( p \in P_f \) and \( \gamma \in GL_n(Z) \). For a given \( \gamma \in GL_n(Z) \), \( X_\gamma \) stands for the \( n \) by \( k \) matrix consisting of the first \( k \)-columns of \( \gamma \). Binet’s formula (see [2, Proposition 2.8.8]) yields
\[
H_{\infty}(g_{\infty}^{-1} \gamma^{-1} e_1 \wedge \cdots \wedge g_{\infty}^{-1} \gamma^{-1} e_k) = \det(tX_{\gamma}^{-1} g_{\infty}^{-1} g_{\infty}^{-1} X_{\gamma}^{-1})^{1/2}.
\]
As a consequence, we obtain
\[
m_Q(g) = \min_{X \in M_{n,k}(Z)^*} \det(tX_{\gamma}^{-1} g_{\infty}^{-1} g_{\infty}^{-1} X)^{n/2r},
\]
where \( M_{n,k}(Z)^* \) denotes the set of \( X \), for all \( \gamma \in GL_n(Z) \). In the case of \( k = 1 \), \( M_{n,1}(Z)^* \) is just the set of primitive vectors of the lattice \( Z^n \), and hence \( m_Q(g) \) coincides with the \( n/2 \) power of the arithmetical minimum of the positive definite symmetric matrix \( t g_{\infty}^{-1} g_{\infty}^{-1} \).

4. The Ryshkov domain of \( G \) associated to \( Q \)
We define the Ryshkov domain \( R = R(m_Q) \) of \( m_Q \) by
\[
R = R(m_Q) = \{ g \in G(A)^1 : m_Q(g)/H_Q(g) \geq 1 \}.
\]
Since \( m_Q(g) \leq H_Q(g) \) holds for all \( g \in G(A)^1 \), we have
\[
R = \{ g \in G(A)^1 : m_Q(g) = H_Q(g) \}
\]
\[
= \{ g \in G(A)^1 : \tau \in X_Q(g) \}. \tag{6}
\]
Since both \( H_Q \) and \( m_Q \) are continuous, \( R \) is a closed subset in \( G(A)^1 \).

Lemma 5. One has \( Q(k)RK = R \) and \( G(A)^1 = G(k)R \).

Proof. The first assertion is obvious by the definition of \( H_Q \). To prove the second assertion, we choose a minimal point \( x \in X_Q(g) \) for a given \( g \in G(A)^1 \). There is a \( \gamma \in G(k) \) such that \( x = \pi_X(\gamma) \). Then \( H_Q(xg) = H_Q(\gamma g) = m_Q(g) = m_Q(\gamma g) \) since \( m_Q \) is left \( G(k) \)-invariant. Therefore, we have \( \gamma g \in R \).

Lemma 6. Let \( C \) be an arbitrary subset of \( G(A)^1 \), and let \( g \in G(A)^1 \) and \( \gamma \in G(k) \). Then we have the following:
(1) \( \gamma g \in R \) if and only if \( \pi_X(\gamma) \in X_Q(g) \).
(2) \( X_Q(g) = \pi_X(\{ \gamma \in G(k) : \gamma g \in R \}) \).
(3) \( \gamma C \subset R \) if and only if \( \pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g) \).
(4) \( \bigcap_{g \in C} X_Q(g) = \{ \tau \} \).
(5) \( \gamma R \subset R \) if and only if \( \gamma \in Q(k) \).
Proof. By definition, \( \gamma g \in \mathbb{R} \) if and only if \( m_Q(\gamma g) = H_Q(\gamma g) \). This is equivalent with \( \pi_X(\gamma) \in X_Q(g) \) because of \( m_Q(\gamma g) = m_Q(g) \). Both (2) and (3) follow from (1). For a point \( x = \pi_X(\gamma) \in \bigcap_{g \in \mathbb{R}} X_Q(g) \), we have \( Q(k)R \subset \mathbb{R} \), in other words, \( xQ(k) \subset \bigcap_{g \in \mathbb{R}} X_Q(g) \). Since \( xQ(k) \) is an infinite set for \( x \neq \tau \) by Bruhat decomposition, we must have \( x = \tau \). This shows (4). (5) follows from (3) and (4).

Lemma 7. Let \( y_0 \in \mathbb{R} \) be an element such that \( n_Q(y_0) > 1 \) and \( x_0 \) be an arbitrary element in \( X_Q(y_0) \). Then, any neighbourhood \( U \) of \( y_0 \) in \( G(A)^1 \) contains a point \( g \) such that \( X_Q(g) \subset X_Q(y_0) \) and \( x_0 \notin X_Q(g) \).

Proof. We may assume that \( U \) satisfies \( X_Q(g) \subset X_Q(y_0) \) for all \( g \in U \) by Lemma 4. Since \( n_Q(y_0) > 1 \), there is an \( x \in X_Q(y_0) \) such that \( x \neq \tau \). This \( x \) is of the form \( \pi_X(w\gamma) \) with \( w \in W_G \setminus W_G^0 \) and \( \gamma \in Q(k) \). By Lemma 2, there is a cocharacter \( \xi = \xi_{w,e} \in X_*(S)_k \) such that \( H_Q(w\xi(\lambda)w^{-1}) > H_Q(\xi(\lambda)) \) holds for all \( \lambda \in A_{\geq 1}^\times \). Let \( \lambda \in A^\times \) be an element sufficiently close to 1 so that \( g_{\lambda} = \gamma^{-1}(\lambda)\gamma y_0 \) is contained in \( U \). We have

\[
H_Q(g_{\lambda}) = H_Q(\xi(\lambda)\gamma y_0) = H_Q(\xi(\lambda)H_Q(y_0)) = H_Q(\xi(\lambda)H_Q(g_0)) = H_Q(\xi(\lambda)m_Q(g_0))
\]

and

\[
H_Q(xg_{\lambda}) = H_Q(w\xi(\lambda)\gamma y_0) = H_Q(w\xi(\lambda)w^{-1})H_Q(w\gamma y_0) = H_Q(w\xi(\lambda)w^{-1}m_Q(g_0)).
\]

If \( x_0 = \tau \), then we choose \( \lambda \) sufficiently close to 1 satisfying \( \lambda^{-1} \in A_{\leq 1}^\times \). Since \( X_Q(g_{\lambda}) \subset X_Q(y_0) \) and \( m_Q(g_{\lambda}) \leq H_Q(xg_{\lambda}) < H_Q(g_{\lambda}) \), \( X_Q(g_{\lambda}) \) does not contain \( \tau \). If \( x_0 \neq \tau \), then we choose \( x \) as \( x_0 \) and \( \lambda \in A_{\geq 1}^\times \) sufficiently close to 1. Since \( m_Q(g_{\lambda}) \leq H_Q(g_{\lambda}) < H_Q(xg_{\lambda}) \), \( X_Q(g_{\lambda}) \) does not contain \( x_0 \).

Lemma 8. \( \min_{g \in G(A)} n_Q(g) = \min_{g \in \mathbb{R}} n_Q(g) = 1 \).

Proof. From Lemma 5 and the \( G(k) \)-invariance of \( n_Q \), it follows that \( \min_{g \in G(A)} n_Q(g) \) equals \( \min_{g \in \mathbb{R}} n_Q(g) \). If \( y_0 \in \mathbb{R} \) satisfies \( \min_{g \in \mathbb{R}} n_Q(g) = n_Q(g_0) > 1 \), then, by Lemmas 5 and 7, there exists a point \( g_1 \in G(A)^1 \) and \( g_1 \in G(k) \) such that \( n_Q(g_1) = n_Q(g_1) < n_Q(g_0) \) and \( g_1 \in \mathbb{R} \). This is a contradiction.

We define the subset \( R_1 \) of \( R \) by

\[
R_1 = \{ g \in \mathbb{R} : n_Q(g) = 1 \} = \{ g \in G(A)^1 : X_Q(g) = \{ \tau \} \}.
\]

Lemma 9. \( R_1 \) coincides with the interior \( R^o \) of \( R \) in \( G(A)^1 \).

Proof. For \( g \in R_1 \), we choose a neighbourhood \( U \) of \( g \) in \( G(A)^1 \) as in Lemma 4. Then \( U \subset R_1 \). Therefore, \( R_1 \) is open and is contained in \( R^o \). If there exists an element \( y_0 \in R^o \) such that \( n_Q(y_0) > 1 \), then, by Lemma 7, \( R^o \) contains an element \( g \) satisfying \( \tau \notin X_Q(g) \). This contradicts \( g \in R \).

It is obvious that \( G(k)R_1 = \{ g \in G(A)^1 : n_Q(g) = 1 \} \).

Lemma 10. \( G(k)R_1 \) is open and dense in \( G(A)^1 \).

Proof. Since \( R_1 \) is open in \( G(A)^1 \), so is \( G(k)R_1 \). We assume that \( G(A)^1 \setminus G(k)R_1 \) has an interior point \( y_0 \). Let \( U \) be a neighbourhood of \( y_0 \) in \( G(A)^1 \) so that \( U \cap G(k)R_1 = \emptyset \). By Lemma 5, we can take \( g_0 \in G(k) \) such that \( n_Q(g_0) > 1 \), by Lemmas 5 and 7, there exists \( g_1 \in g_0(U) \) and \( g_1 \in G(k) \) such that \( n_Q(g_1) < n_Q(g_0) \).
and $\gamma g_1 \in \mathbb{R}$. If $n_0(g_1) > 1$, then there exists $g_2 \in \gamma\gamma_0\mathcal{U}$ and $\gamma_2 \in G(k)$ such that $n_0(g_2) < n_0(g_1)$ and $\gamma_2 g_2 \in \mathbb{R}$. This process terminates after finitely many iterations. At the last step, we obtain an element $g_\ell \in \gamma_{\ell-1}\cdots\gamma_0\mathcal{U}$ such that $n_0(g_\ell) = 1$. Then $(\gamma_{\ell-1}\cdots\gamma_0)^{-1}g_\ell$ is contained in $\mathcal{U} \cap G(k)R_1$. This contradicts $\mathcal{U} \cap G(k)R_1 = \emptyset$. Therefore, $G(A)^1 \setminus G(k)R_1$ is nowhere dense in $G(A)^1$. \hfill \Box

Lemma 11. For $\gamma \in G(k)$, $R_1 \cap \gamma R \neq \emptyset$ if and only if $\gamma \in Q(k)$. 

Proof. If $R_1 \cap \gamma R$ has an element $g$, then $\pi_X(\gamma^{-1}) \in X_Q(g) = \{\pi\}$ by Lemma 6. \hfill \Box

Lemma 12. Let $R_1^-$ be the closure of $R_1$. Then we have the following subdivision of $G(A)^1$:

\[ G(A)^1 = \bigcup_{\gamma \in G(k)/Q(k)} \gamma R_1^- . \]

Proof. We fix an arbitrary $g \in G(A)^1$. By Lemma 10, there exists a sequence $\{g_\ell\} \subset G(k)R_1$ such that $\lim_{\ell \rightarrow \infty} g_\ell = g$. We take a neighbourhood $\mathcal{U}$ of $g$ as in Lemma 4 and may assume that $\{g_\ell\} \subset \mathcal{U}$. Since $g_\ell \in G(k)R_1$, $X_Q(g_\ell)$ consists of a single element $\pi_X(\gamma_\ell)$, where $\gamma_\ell \in G(k)$. From $g_\ell \in \mathcal{U}$, it follows that $\pi_X(\gamma_\ell) \in X_Q(g)$ for all $\ell$. Since $X_Q(g)$ is a finite set, we can take a subsequence $\{g_{\ell_\gamma}\}$ such that $\pi_X(\gamma_{\ell_\gamma}) = \pi_X(\gamma) \in X_Q(g)$ for all $\ell$. Then $\{g_{\ell_\gamma}\} \subset \gamma^{-1}R_1$, and $g$ is contained in the closure of $\gamma^{-1}R_1$. \hfill \Box

For $g \in G(A)^1$, we put

\[ S_Q(g) = \pi_X(\{\gamma \in G(k) : \gamma g \in R_1^-\}) . \]

By Lemmas 6 and 12, $S_Q(g)$ is a non-empty subset of $X_Q(g)$.

Lemma 13. For $g_0 \in G(A)^1$, there is a neighbourhood $\mathcal{U}$ of $g_0$ in $G(A)^1$ such that $S_Q(g) \subset S_Q(g_0)$ for all $g \in \mathcal{U}$.

Proof. Let $\mathcal{U}$ be a neighbourhood of $g_0$ such that $X_Q(g) \subset X_Q(g_0)$ for all $g \in \mathcal{U}$. Since $g_0 \notin \gamma^{-1}R_1^-$ for any $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$, we can take a sufficiently small $\mathcal{U}$ so that $\mathcal{U} \cap \gamma^{-1}R_1^- = \emptyset$ for all $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$. Then, for all $g \in \mathcal{U}$, $S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0)$ is empty, namely $S_Q(g)$ is contained in $S_Q(g_0)$. \hfill \Box

Remark. We do not know whether $R_1^- = \mathbb{R}$ holds or not in general. If $R_1^- = \mathbb{R}$ holds, then $S_Q(g) = X_Q(g)$ holds for all $g$.

5. A fundamental domain of $G(A)^1$ with respect to $G(k)$

Definition. Let $T$ be a locally compact Hausdorff space and $\Gamma$ be a discrete group acting on $T$ from the left. Assume that the action of $\Gamma$ on $T$ is properly discontinuous. An open subset $\Omega$ of $T$ is called an open fundamental domain of $T$ with respect to $\Gamma$ if $\Omega$ satisfies the following conditions:

(i) $T = \Gamma\Omega^-$, where $\Omega^-$ stands for the closure of $\Omega$ in $T$, and

(ii) $\Omega \cap \gamma\Omega^- = \emptyset$ if $\gamma \in \Gamma \setminus \{e\}$.

A subset $F$ of $T$ is called a fundamental domain of $T$ with respect to $\Gamma$ if there is an open fundamental domain $\Omega$ as above such that $\Omega \subset F \subset \Omega^-$. 

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Theorem 17. Let $Q$ be an open fundamental domain of $R^+_1$ with respect to $Q(k)$. Then one has $\Omega^+_Q = \Omega^+ \cap R_1$ and $\Omega^-_Q = (\Omega^+ \cap R_1)^-$.

**Proof.** Since $\Omega^+_Q$ is an open set in $R^+_1$ with respect to the relative topology, there is an open set $U$ in $G(A)^1$ such that $\Omega^+_Q = R^+_1 \cap U$. Therefore, $\Omega^+_Q \cap R_1 = U \cap R_1$ is open in $G(A)^1$, and hence $\Omega^-_Q = \Omega^+_Q \cap R_1$. Since $R_1$ is dense in $R^+_1$ and $\Omega^+_Q$ is relatively open in $R^+_1$, the closure of $\Omega^+_Q \cap R_1$ in $R^+_1$ contains $\Omega^+_Q$, i.e., $\Omega^+_Q \subset (\Omega^+ \cap R_1)^-$. Hence we have $\Omega^-_Q = (\Omega^+ \cap R_1)^-$.

**Theorem 15.** Let $Q$ be an open fundamental domain of $G(A)^1$ with respect to $Q(k)$. Then $\Omega^+_Q$ is an open fundamental domain of $G(A)^1$ with respect to $G(k)$.

**Proof.** From $R^+_1 = Q(k)\Omega^-_Q$ and Lemma 12, it follows $G(A)^1 = G(k)\Omega^-_Q$. For $\gamma \in G(k)$, we assume $\Omega^+_Q \cap \gamma \Omega^-_Q \neq \emptyset$. By Lemma 11, $\gamma$ is contained in $Q(k)$. Since $\Omega^+_Q$ is an open fundamental domain of $R^+_1$ with respect to $Q(k)$, $\gamma$ must be equal to $e$.

For a given subset $A$ of $G(A)^1$, we denote by $\partial A$ the boundary of $A$.

**Lemma 16.** If $g_0 \in R^+_1$ attains a local maximum of $m_Q$, then $g_0$ is contained in $\partial R^+_1$.

**Proof.** Suppose $g_0 \in R_1$. Since $R_1$ is open, $z g_0$ is contained in $R_1$ if $z \in Z_Q(A)$ is sufficiently close to $e$. Then we have

$$m_Q(z g_0) = H_Q(z) m_Q(g_0) = H_Q(z) H_Q(g_0).$$

Since $H_Q(z)$ can vary on the interval $(1 - \epsilon, 1 + \epsilon)$ for a sufficiently small $\epsilon > 0$, $m_Q(g_0)$ is not a local maximum of $m_Q$.

Since $(\Omega^-_Q)^e = \Omega^-_Q \subset R_1$, the following theorem immediately follows from Lemma 16.

**Theorem 17.** Let $\Omega_Q$ be the same as in Theorem 15. If $g_0 \in \Omega^-_Q$ attains a local maximum of $m_Q$, then $g_0$ is contained in $\partial \Omega^-_Q \cap \partial R^+_1$.

**Remark.** A point $g_0 \in G(A)^1$ is said to be extreme if $g_0$ attains a local maximum of $m_Q$. By Theorem 17, any extreme point is contained in $G(k)(\partial \Omega^-_Q \cap \partial R^+_1)$. A candidate of the notion analogous to perfect quadratic forms is the following: A point $g \in G(A)^1$ is said to be $Q$-perfect if there is a neighbourhood $U$ of $g$ such that

$$U \cap \bigcap_{\pi(\delta) \in \mathcal{S}_Q(g)} \delta^{-1} R^+_1 = \{g\}.$$

6. The case when $G$ is of class number one

We put $K_f = \prod_{\pi \in \Pi_f} K_{\pi}$, $G_{A, \infty} = G(k_{\infty}) \times K_f$, $G_{A, \infty} = G_{A, \infty} \cap G(A)^1$ and $G_0 = G(k) \cap G_{A, \infty}$. By identifying $G(k_{\infty})$ with the subgroup $\{(g_\sigma) \in G(A) : g_\sigma = e \text{ for all } \sigma \in \Pi_f\}$ of $G(A)$, we put $G(k_{\infty})^1 = G(k_{\infty}) \cap G(A)^1$. The number $n_k(G)$ of double cosets in $G(A)$
modular $G(k)$ and $G_{A,\infty}$ is called the class number of $G$. For example, $n_k(\text{GL}_n)$ is equal to the class number of $k$. If $G$ is almost k-simple, k-isotropic and simply connected, then $n_k(G) = 1$ by the strong approximation theorem. In this section, we assume that $n_k(G) = 1$. Then we have $G(A)^1 = G(k)G_{A,\infty}^1$. Let $h_Q$ be the number of double cosets of $G(k)$ modulo $Q(k)$ and $G_o$. By [3, Proposition 7.5], $h_Q$ is equal to the class number of $M_Q$. Let $\{\xi_i = e, \xi_2, \ldots, \xi_{h_Q}\}$ be a complete system of representatives of $Q(k)\backslash G(k)/G_o$. For each $\xi_i$, we define the subset $R_{\xi_i,\infty}$ of $G(k_\infty)^1$ as

$$\{g_\infty \in G(k_\infty)^1 : m_Q(g_\infty) = H_Q(\xi_i g_\infty)\}.$$ 

Since $G(k)$ is a disjoint union of $Q(k)\xi_i G_o$, $i = 1, \ldots, h_Q$, $m_Q(g_\infty)$ is equal to

$$\min_{1 \leq i \leq h_Q} \min_{\delta \in G_o} H_Q(\xi_i \delta g_\infty).$$

**Lemma 18.** One has

$$R = \bigsqcup_{i=1}^{h_Q} Q(k)\xi_i(R_{\xi_i,\infty} \times K_f).$$

**Proof.** For each $i$, $Q(k)\xi_i(R_{\xi_i,\infty} \times K_f) \subset R$ is trivial. Since

$$G(A)^1 = \bigsqcup_{i=1}^{h_Q} Q(k)\xi_i G_{A,\infty}^1$$

by [3, §7], a given $g \in R$ is represented as $g = \gamma \xi_i(g_\infty \times g_f)$ by some $i, \gamma \in Q(k)$ and $g_\infty \times g_f \in G_{A,\infty}^1$. Then $m_Q(g) = H_Q(g)$ implies $m_Q(g_\infty) = H_Q(\xi_i g_\infty)$. Therefore, $g_\infty \in R_{\xi_i,\infty}$. \(
\)

We write $Q_i$ for the conjugate $\xi_i^{-1}Q\xi_i$ of $Q$. This $Q_i$ is a maximal k-parabolic subgroup of $G$. We put $Q_{i,o} = Q_i(k) \cap G_{A,\infty}$.

**Lemma 19.** If $g(R_{\xi_i,\infty} \times K_f) \cap (R_{\xi_i,\infty} \times K_f)$ is non-empty for $g \in Q_i(k)$, then $g \in Q_{i,o}$.

**Proof.** If there is an $h \in R_{\xi_i,\infty} \times K_f$ such that $gh \in R_{\xi_i,\infty} \times K_f$, then $g \in (R_{\xi_i,\infty} \times K_f)h^{-1} \subset G_{A,\infty}$.

It is easy to prove that the group $Q_{i,o}$ stabilizes $R_{\xi_i,\infty} \times K_f$ by left multiplications. We fix a complete system $\{\gamma_{ij}\}$ of representatives of $Q_i(k)/Q_{i,o}$. It follows from Lemma 19 that $\gamma_{ij}(R_{\xi_i,\infty} \times K_f) \cap \gamma_{ik}(R_{\xi_i,\infty} \times K_f) = \emptyset$ if $j \neq k$. Therefore, we obtain the following subdivision of $R$:

$$R = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i \gamma_{ij}(R_{\xi_i,\infty} \times K_f).$$  \(1\)

Let $R_{\xi_i,\infty}^2$ be the interior of $R_{\xi_i,\infty}$ and $R_{\xi_i,\infty}^*$ be the closure of $R_{\xi_i,\infty}^2$ in $G(k_\infty)^1$. Since the union of (1) is disjoint, it is obvious that

$$R_{\xi_i,\infty}^i = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i \gamma_{ij}(R_{\xi_i,\infty}^* \times K_f).$$  \(2\)
Proposition 20. Let $\Omega_{i,\infty}$ be an open fundamental domain of $R^*_{\epsilon_i,\infty}$ with respect to $Q_{i,\sigma}$ for $i = 1, \cdots, h_Q$. Then the set

$$\Omega = \bigcup_{i=1}^{h_Q} \xi_i(\Omega_{i,\infty} \times K_f)$$

gives an open fundamental domain of $R^-_\infty$ with respect to $Q(k)$.

Proof. Let $\Omega^+_{i,\infty}$ denote the closure of $\Omega_{i,\infty}$ in $G(k_\infty)^1$. For $g \in Q(k)$, we assume $\Omega \cap g\Omega^+ \neq \emptyset$. Then, for some $i, j$, we have

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap g\xi_j(\Omega_{j,\infty} \times K_f) \neq \emptyset. \quad (3)$$

There exist $\gamma_{jk}$ and $\delta \in Q_{j,\sigma}$ such that $\xi_j^{-1}g\xi_j = \gamma_{jk}\delta$. Then (3) is the same as

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap \xi_j(\delta\Omega^+_{j,\infty} \times K_f) \neq \emptyset.$$

By (1), we have $i = j$, $\gamma_{jk} = e$ and $\Omega_{j,\infty} \cap \delta\Omega^+_{j,\infty} \neq \emptyset$. Since $\Omega_{j,\infty}$ is an open fundamental domain of $R^*_j,\infty$ with respect to $Q_{j,\sigma}$, $\delta$ must be equal to $e$. Therefore, $\Omega \cap g\Omega^- \neq \emptyset$ implies $g = e$. Finally, $Q(k)^- = R^-_1$ follows from (2) and $Q_{i,\sigma}\Omega_{i,\infty}^- = R^*_\infty,\infty$.

By Theorem 17, we obtain the following.

Corollary 21. If $g_0 \in \Omega^-$ attains a local maximum of $m_Q$, then $g_0$ is contained in the set

$$\bigcup_{i=1}^{h_Q} \xi_i((\partial\Omega^-_{i,\infty} \cap \partial R^*_{\epsilon_i,\infty}) \times K_f).$$

We consider the infinite part $\Omega_\infty$ of $\Omega$ given in Proposition 20, i.e.,

$$\Omega_\infty = \bigcup_{i=1}^{h_Q} \xi_i \Omega_{i,\infty}.$$

Let $\Omega^\circ_{\infty}$ and $\Omega^-_{\infty}$ be the interior and the closure of $\Omega_{\infty}$ in $G(k_\infty)^1$, respectively. The projection from $G(A)^1 = G(k)G^1_{\lambda,\infty}$ to the infinite component $G(k_\infty)^1$ gives an isomorphism $G(k)\backslash G(A)^1 / K_f \cong G_\delta \backslash G(k_\infty)^1$. Since $\Omega$ is a fundamental domain of $G(A)^1$ with respect to $G(k)$ by Theorem 15, we have $G_\delta\Omega^-_{\infty} = G(k_\infty)^1$.

Corollary 22. If $h_Q = 1$, then $\Omega_\infty$ is a fundamental domain of $G(k_\infty)^1$ with respect to $G_\sigma$.

Proof. Since $\Omega_{\infty} = \Omega_{1,\infty}$ is a relatively open set in $R^*_{\epsilon_1,\infty}$, we have $\Omega^\circ_{\infty} = \Omega_{\infty} \cap R^*_{\epsilon_1,\infty}$. Thus the closure of $\Omega^\circ_{\infty}$ coincides with $\Omega_{\infty}$. If $\Omega^\circ_{\infty} \cap g\Omega_{\infty}^\circ \neq \emptyset$ for $g \in G_\sigma$, then $(\Omega^\circ_{\infty} \times K_f) \cap g(\Omega^\circ_{\infty} \times K_f) \neq \emptyset$ because of $gK_f = K_f$. This implies $g = e$ since $\Omega^\circ_{\infty} \times K_f$ is an open fundamental domain of $G(A)^1$ with respect to $G(k)$.

7. Examples
Example 3. Let $G$ be a general linear group $\text{GL}_n$ defined over $\mathbb{Q}$. We continue an illustration given in Examples 1 and 2. We fix an integer $k \in \{1, \ldots, n-1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \text{GL}_k(\mathbb{Q}), b \in M_{k,n-k}(\mathbb{Q}), d \in \text{GL}_{n-k}(\mathbb{Q}) \right\}.$$ 

Since $h_\mathbb{Q} = 1$, we have $\xi_1 = e$ and $Q_1 = \mathbb{Q}$.

Let $P_n$ be the cone of positive definite $n \times n$ real symmetric matrices, and let $P^1_n$ be the intersection of $P_n$ and $\text{SL}_n(\mathbb{R})$. The group $G(\mathbb{Q}_\infty) = \text{GL}_n(\mathbb{R})$ acts on $P_n$ from the right by $A \mapsto A|_P = tAg$ for $(A, g) \in P_n \times G(\mathbb{Q}_\infty)$. The maximal compact subgroup $K_\infty$ of $G(\mathbb{Q}_\infty)$ defined as in Example 2 stabilizes the identity matrix $I_n \in P_n$. The map $\pi : g \mapsto t^{-1}g^{-1}t$ from $G(\mathbb{Q}_\infty)$ onto $P_n$ gives an isomorphism between $G(\mathbb{Q}_\infty)/K_\infty$ and $P_n$.

Since $G(\mathbb{Q}_\infty)^1 = \{ g \in G(\mathbb{Q}_\infty) : \det g = \pm 1 \}$, we have $G(\mathbb{Q}_\infty)^1/K_\infty \cong \pi(G(\mathbb{Q}_\infty)^1) = P_n^1$. An element $A \in P_n$ is written as

$$A = \begin{pmatrix} I_k & 0 & & \\ t & u & I_{n-k} & \\ 0 & 0 & w & \\ & & & 0 \end{pmatrix},$$

where $v \in P_k$, $w \in P_{n-k}$ and $u \in M_{k,n-k}(\mathbb{R})$. We write $u_A$, $A[k]$ and $A_{[n-k]}$ for $u$, $v$, and $w$, respectively.

By definition, $G_{\mathbb{Z}} = G(\mathbb{Q}) \cap G_{A,\infty}$ and $Q_{\mathbb{Z}} = Q(\mathbb{Q}) \cap G_{A,\infty}$ are just groups $\text{GL}_n(\mathbb{Z})$ and $Q(\mathbb{Q}) \cap \text{GL}_n(\mathbb{Z})$ of integral matrices in $G(\mathbb{Q})$ and $Q(\mathbb{Q})$, respectively. As in Example 2, $X_\gamma$ stands for the $n$ by $k$ matrix consisting of the first $k$-columns of $\gamma \in G_{\mathbb{Z}}$, and $M_{n,k}(\mathbb{Z})^*$ stands for the set of $X_\gamma$ for all $\gamma \in G_{\mathbb{Z}}$. We define the closed subset $F_{n,k}$ of $P_n$ as follows:

$$F_{n,k} = \{ A \in P_n : \text{det} A[k] \leq \text{det} (t^*AXA) \text{ for all } X \in M_{n,k}(\mathbb{Z})^* \}.$$ 

In Example 2, we showed

$$H_Q(\gamma g) = \det(t^*X_\gamma^{-1}\pi(g)X_\gamma^{-1})^{n/2r}$$

for any $\gamma \in G_{\mathbb{Z}}$ and $g \in G(\mathbb{Q}_\infty)^1$. Since $H_Q(g) = (\det(\pi(g))^{n/2r}$, we obtain

$$\mathbb{R}_{r,\infty}/K_\infty \cong \pi(\mathbb{R}_{r,\infty}) = F_{n,k} \cap \text{SL}_n(\mathbb{R}).$$

Therefore, $Q_{\mathbb{Z}}\mathbb{R}_{r,\infty}/K_\infty$ is isomorphic with $(F_{n,k} \cap \text{SL}_n(\mathbb{R}))/Q_{\mathbb{Z}}$. If $\gamma \in Q_{\mathbb{Z}}$ is of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a \in \text{GL}_k(\mathbb{Z})$, $d \in \text{GL}_{n-k}(\mathbb{Z})$ and $b \in M_{k,n-k}(\mathbb{Z})$, then components of $t^*\gamma A \gamma$ for $A \in P_n$ are given by

$$w_{t^*\gamma A} = a^{-1}(u_A + b), \quad (t^*\gamma A \gamma)[k] = tA[k]a, \quad (t^*\gamma A \gamma)[n-k] = tA[n-k]d.$$ 

Let $\mathcal{D}$ and $\mathcal{E}$ be arbitrary fundamental domains for $P_k/\text{GL}_k(\mathbb{Z})$ and $P_{n-k}/\text{GL}_{n-k}(\mathbb{Z})$, respectively. We define the subset $F_{n,k}(\mathcal{D}, \mathcal{E})$ of $F_{n,k}$ as

$$F_{n,k}(\mathcal{D}, \mathcal{E}) = \left\{ A \in F_{n,k} : A[k] \in \mathcal{D}, A_{[n-k]} \in \mathcal{E}, \quad u_A = (u_{ij}), -1/2 \leq u_{ij} \leq 1/2 \text{ for all } i, j, \text{ and } 0 \leq u_{11} \right\}.$$ 

Since $F_{n,k}(\mathcal{D}, \mathcal{E})$ is a fundamental domain of $F_{n,k}$ with respect to $Q_{\mathbb{Z}}$, the inverse image $\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R}))$ of $F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})$ gives a fundamental domain of $\mathbb{R}_{r,\infty}$ with respect to $Q_{\mathbb{Z}}$. As a consequence of Theorem 15 and Proposition 20, the set

$$\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})) \times K_f$$
gives a fundamental domain of \( G(A)^1 \) with respect to \( G(Q) \). Moreover, from Corollary 22, it follows that \( F_{n,k}(D, E) \) is a fundamental domain of \( P_n \) with respect to \( GL_n(Z) \).

In the case of \( k = 1 \), this gives an inductive construction of a fundamental domain \( \Omega_n \) of \( P_n \) with respect to \( GL_n(Z) \) as follows. First, put \( \Omega_2 = F_{2,1}(P_1, P_1) \). By definition, \( \Omega_2 \) is Minkowski’s fundamental domain of \( P_2 \). Then we define inductively \( \Omega_3 = F_{3,1}(P_1, \Omega_2), \ldots, \Omega_n = F_{n,1}(P_1, \Omega_{n-1}) \). The domain \( \Omega_n \) coincides with Grenier’s fundamental domain [6].

Finally, we show that, in the case of \( k = 1 \), \( R_{c,\infty}/K_{\infty} \) corresponds to a face of the Ryshkov polyhedron \( R(m) = \{ A \in P_n : m(A) = \min_{0 \neq x \in \mathbb{Z}^n} t^tAx \geq 1 \} \). For \( A \in P_n \), \( S(A) \) denotes the set of minimal integral vectors of \( A \), i.e., \( S(A) = \{ x \in \mathbb{Z}^n : m(A) = t^tAx \} \). We take \( e_1 = (1, 0, \cdots, 0) \in \mathbb{Z}^n \). It is obvious that the subset \( \{ A \in P_n : e_1 \in S(A) \} \) of \( P_n \) coincides with \( F_{n,1} \). As was shown in [18, Lemma 1.5], \( F_{\{e_1\}} = F_{n,1} \cap \partial R(m) = \{ A \in F_{n,1} : m(A) = 1 \} \) is a face of \( R(m) \). It is easy to see that the map \( A \mapsto m(A)^{-1}A \) gives a bijection from \( F_{n,1} \cap SL_n(R) \) onto \( F_{\{e_1\}} \). Therefore, \( R_{c,\infty}/K_{\infty} \cong \pi(R_{c,\infty}) \) corresponds to \( F_{\{e_1\}} \).

Example 4. Let \( k \) be a totally real number field of degree \( r \) and \( n = 2m \) be an even integer. We consider a symplectic group \( G(k) = Sp_n(k) = \{ g \in GL_{2m}(k) : t^g \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \end{array} \right) g = \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \end{array} \right) \} \).

For a fixed \( k \in \{1, 2, \cdots, m\} \), \( Q \) denotes the maximal parabolic subgroup of \( G \) given as follows:

\[
Q(k) = U(k)M(k),
\]

\[
M(k) = \left\{ \delta(a, b) = \left( \begin{array}{ccc} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & t_{a}^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{array} \right) : a \in GL_2(k), b = (b_{ij}) \in Sp_2(m-k)(k) \right\},
\]

\[
U(k) = \left\{ \left( \begin{array}{cccc} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{array} \right) \in G(k) \right\}.
\]

The module of \( k \)-rational characters \( X^*(M)_k \) of \( M \) is a free \( \mathbb{Z} \)-module of rank 1 generated by the character \( \hat{\alpha}_Q(\delta(a, b)) = \det a \).

The height \( H_Q : G(A) \to R_{>0} \) is given by \( H_Q(g) = |\det a|^A \) if \( g = u\delta(a, b)h \) with \( u \in U(A), \delta(a, b) \in M(A) \) and \( h \in K \).

We restrict ourselves to the case \( k = m \). An element of \( M(A) \) is denoted by

\[
\delta(a) = \left( \begin{array}{cc} a & 0 \\ 0 & t_{a}^{-1} \end{array} \right), \quad (a \in GL_m(A)).
\]

Let

\[
H_m = \{ Z \in M_m(C) : t^t Z = Z, \im Z \in P_m \}
\]

be the Siegel upper half space and \( H_m^r \) the direct product of \( r \) copies of \( H_m \). For \( Z = (Z_\sigma)_{\sigma \in \mathbb{P}_\infty} \in H_m^r \), \( \im Z \) and \( \det Z \) stand for \( \im(Z_\sigma)_{\sigma \in \mathbb{P}_\infty} \) and \( \det(Z_\sigma)_{\sigma \in \mathbb{P}_\infty} \), respectively. The group \( G(k_{\mathbb{Q}}) \) acts transitively on \( H_m^r \) by

\[
g(Z) = ((a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1})_{\sigma \in \mathbb{P}_\infty}
\]
for \( Z = (Z_{\sigma}) \in \mathbb{H}_m^r \) and

\[
g = (g_{\sigma}) = \left( \begin{array}{cc} a_{\sigma} & b_{\sigma} \\ c_{\sigma} & d_{\sigma} \end{array} \right)_{\sigma \in \mathbb{P}_m} \in G(k_\infty).
\]

The stabilizer \( K_\infty \) of \( Z_0 = (\sqrt{-1}I_m, \ldots, \sqrt{-1}I_m) \in \mathbb{H}_m^r \) in \( G(k_\infty) \) is a maximal compact subgroup of \( G(k_\infty) \). We choose \( K = K_\infty \times \prod_{\sigma \in \mathbb{P}_m} \text{Sp}_m(o_\sigma) \). The map \( \pi : g_\infty \mapsto g(Z_0) \) from \( G(k_\infty) \) onto \( \mathbb{H}_m^r \) give an isomorphism \( G(k_\infty)/K_\infty \cong \mathbb{H}_m^r \), and hence \( G(k)\backslash G(A)/K \cong G_\alpha \backslash \mathbb{H}_m^r \). Since \( \text{Im}\{(u\delta(a)h)(Z_0)\} = a'a \) holds for \( u \in U(k_\infty), a \in \text{GL}_m(k_\infty) \) and \( h \in K_\infty \), we have

\[
H_Q(g_\infty) = \text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}\{g_\infty(Z_0)\})^{-1/2} = \left( \prod_{\sigma \in \mathbb{P}_m} \det \text{Im}\{g_\sigma(\sqrt{-1}I_m)\} \right)^{-1/2}
\]

for any \( g_\infty = (g_{\sigma}) \in G(k_\infty) \), where \( \text{Nr}_{k_\infty/\mathbb{R}} \) denotes the norm of \( k_\infty \) over \( \mathbb{R} \).

The class number \( h_Q \) of \( M \cong \text{GL}_m \) defined over \( k \) is equal to the class number \( h_k \) of \( k \). We assume \( h_k = 1 \) for simplicity. Then we have \( G(k) = Q(k)G_\alpha \) and \( G(A) = Q(k)G_{A,\infty} \), and hence

\[
m_Q(g_\infty) = \min_{\gamma \in G_\alpha} H_Q(\gamma g_\infty).
\]

Since

\[
\text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}\{\gamma(Z)\}) = \prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_{\sigma} + \sigma(d))|^{-2} \text{Nr}_{k_\infty/\mathbb{R}}(\det \text{Im}Z)
\]

for \( Z = (Z_{\sigma}) \in \mathbb{H}_m^r \) and

\[
\gamma = \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in G_\alpha = \text{Sp}_m(o),
\]

the condition \( m_Q(g_\infty) = H_Q(g_\infty) \) of \( g_\infty \) is equivalent with the following condition of \( Z = g_\infty(Z_0) \):

\[
\prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_{\sigma} + \sigma(d))| \geq 1 \quad \text{for all} \quad \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in G_\alpha.
\]

Therefore, the domain \( R_{c,\infty} \) modulo \( K_\infty \) is isomorphic with

\[
F = \left\{ (Z_{\sigma}) \in \mathbb{H}_m^r : \prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_{\sigma} + \sigma(d))| \geq 1 \quad \text{for all} \quad \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in G_\alpha \right\}.
\]

Let \( \mathcal{C} \) be an arbitrary fundamental domain of the additive group \( M_m(k_\infty) \) with respect to \( M_m(o) \), and let \( D \) be an arbitrary fundamental domains of \( \mathbb{P}_m^r \) with respect to \( \text{GL}_m(o) \). It is easy to see that

\[
F(\mathcal{C}, D) = \{ Z \in F : \text{Re}Z \in \mathcal{C}, \text{Im}Z \in D \}
\]

is a fundamental domain of \( F \) with respect to \( Q_\infty \). By Corollary 22, \( F(\mathcal{C}, D) \) is a fundamental domain of \( \mathbb{H}_m^r \) with respect to \( G_\alpha \).

If \( k = \mathbb{Q} \) and \( D \) is Minkowski’s fundamental domain, then \( F(\mathcal{C}, D) \) coincides with Siegel’s fundamental domain [12].

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References


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