On Voronoï's theorem and related problems

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Outline

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1 Simple generalization of Voronoï's theorem

1.1 Voronoï's theorem

Let $V_n = \{a \in \mathrm{M}_n(\mathbb{R}) \ : \ {}^t a = a\}$, $P_n = \{a \in V_n \ : \ a > 0\}$
Let $m(a) = \inf_{0 \neq x \in \mathbb{Z}^n} {}^t x a x$ for $a \in \overline{P}_n$, the closure of P_n.
The Hermite invariant $F \ : \ P_n \longrightarrow \mathbb{R}_{>0}$ is defined by

$$F(a) = \frac{m(a)}{(\det a)^{1/n}} .$$

Main Problem of Lattice Sphere Packings

Determine the actual value of the maximum $\gamma_n = \max_{a \in P_n} F(a)$.

Voronoï's theorem characterizes local maxima of F.

Let $S(a) = \{x \in \mathbb{Z}^n \ : \ a[x] = m(a)\}$, the set of minimal vectors.
For $x \in \mathbb{R}^n$, let $\varphi_x \ : \ a \mapsto a[x] = {}^t x a x$ be a linear form on V_n.

Definition

Let $\sqrt{a} \in P_n$ be the square root of $a \in P_n$.

- a is said to be perfect if $\{\varphi_{\sqrt{a}x}\}_{x \in S(a)}$ spanns V_n^*.
- a is said to be eutactic if ${}^\exists \rho_x > 0$, $x \in S(a)$, such that

$$\mathrm{Tr} = \sum_{x \in S(a)} \rho_x \varphi_{\sqrt{a}x} \, .$$

Theorem (Voronoï, 1908)

$F(a)$ is a local maximum if and only if a is perfect and eutactic.

$(F = m / \det^{1/n})$

1.2 Generalization to type one functions

Definition

A function $\phi : \overline{P}_n \longrightarrow \mathbb{R}_{\geq 0}$ is called a type one (class) function if

1. $\phi(\lambda a) = \lambda \phi(a)$ for $a \in \overline{P}_n$ and $\lambda \geq 0$.
2. $\phi(a+b) \geq \phi(a) + \phi(b)$ for $a, b \in \overline{P}_n$.
3. $\phi(a) > 0$ for $a \in P_n$.
4. ϕ is upper semicontinuous on $\overline{P}_n$.
5. $(\phi(a[g]) = \phi(a)$ for $a \in P_n$ and $g \in \mathrm{GL}_n(\mathbb{Z})$.)

Example

- Both m and $\det^{1/n}$ are type one class functions.
- If ϕ is a type one class function, then so is $\phi^\circ(a) := \inf_{b \in P_n} \frac{\mathrm{Tr}(ab)}{\phi(b)}$.

If ϕ is a type one function, then

- ϕ is continuous on P_n
- ϕ is log-concave, i.e.,

$$\log \phi(\lambda a + (1-\lambda) b) \geq \lambda \log \phi(a) + (1-\lambda) \log \phi(b)$$

holds for ${}^\forall a, b \in P_n$ and $0 < {}^\forall \lambda < 1$.

We say ϕ is strictly log-concave if this inequality is strict for all $a \neq b$.

We want to generalize Voronoï's theorem to $F_\phi := m/\phi$.

Assume ϕ is differentiable on P_n. Then

$$(\partial \log \phi)_a(v) = \lim_{t \to 0} \frac{\log \phi((\mathrm{I} + tv)[\sqrt{a}]) - \log \phi(a)}{t}$$

exists for $a \in P_n$ and $v \in V_n$.

Definition

$a \in P_n$ is said to be ϕ-eutactic if $^\exists \rho_x > 0$, $x \in S(a)$, such that

$$(\partial \log \phi)_a = \sum_{x \in S(a)} \rho_x \varphi_{\sqrt{a}x} .$$

Theorem (Sawatani–W., 2009)

Assume a type one function ϕ is differentiable and strictly log-concave. Then $\boldsymbol{F}_{\phi} = \boldsymbol{m}/\phi$ attains a local maximum on $\boldsymbol{a} \in \boldsymbol{P_n}$ if and only if $\boldsymbol{a}$ is perfect and ϕ-eutactic.

Question

Can we replace $\boldsymbol{m}$ with another type one function?

2 Geometry of perfect forms

2.1 Kernels

Definition

A subset $K \subset \overline{P}_n$ is called a kernel if

1. K is a closed convex subset.
2. $0 \notin K$.
3. $K = \mathbb{R}_{\geq 1} \cdot K$.
4. $P_n \subset \mathbb{R}_{\geq 0} \cdot K$.

If ϕ is a type one function, then

$$K_1(\phi) := \{a \in \overline{P}_n \ : \ \phi(a) \geq 1\}$$

is a kernel.

Conversely, if $\boldsymbol{K}$ is a kernel, then

$$\phi_{\boldsymbol{K}}(\boldsymbol{a}) := \max(\{\boldsymbol{\lambda} > 0 \ : \ \boldsymbol{a} \in \boldsymbol{\lambda K}\} \cup \{0\})$$

is a type one function.

These correspondences are inverse each other.

$$\boxed{\text{Type One Functions}} \underset{\phi_{\boldsymbol{K}} \leftarrow \boldsymbol{K}}{\overset{\phi \to \boldsymbol{K}_1(\phi)}{\longleftrightarrow}} \boxed{\text{Kernels}}$$

2.2 Ryshkov polyhedron

Recall $m(a) = \inf_{0 \neq x \in \mathbb{Z}^n} a[x]$ is a type one class function.
The kernel $K_1(m)$ is called the Ryshkov polyhedron.
We have

- $K_1(m) \subset P_n$.
- $K_1(m)$ is the intersection of affine half-spaces:

$$K_1(m) = \bigcap_{x \in \mathbb{Z}^n \setminus \{0\}} \{a \in V_n \ : \ a[x] \geq 1\}.$$

- $K_1(m)$ is a locally finite polyhedron, i.e., the intersection of $K_1(m)$ and an arbitrary polytope is a polytope.

Let $\partial K_1(m)$ be the boundary of $K_1(m)$ and

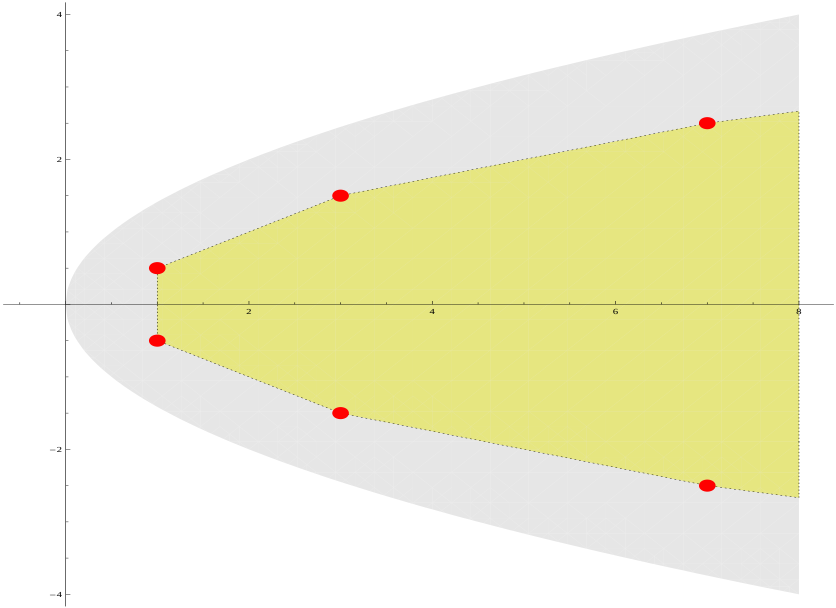
$$\mathcal{F}_{S(a)} := \{b \in \partial K_1(m) \ : \ S(a) \subset S(b)\} \quad \text{for } a \in P_n.$$

Theorem (Voronoï, Ryshkov, etal.)

- *$\mathcal{F}_{S(a)}$ is a face of $K_1(m)$. Any face of $K_1(m)$ is this form.*
- *$\mathcal{F}_{S(a)}$ is a vertex if and only if a is perfect.*
- *The set of all faces of $K_1(m)$ has finite $\mathbf{GL}_n(\mathbb{Z})$-orbits.*

Let $\partial^0 K_1(m)$ be the set of all vertices of $K_1(m)$.

- *$\mathbb{R}_{>0} \cdot \partial^0 K_1(m)$ equals the set of all perfect forms.*
- *$K_1(m)$ is the convex hull of $\partial^0 K_1(m)$.*
- *$\sharp(\partial^0 K_1(m)/\mathbf{GL}_n(\mathbb{Z}))$ is finite.*


4
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2.3 Local maximality of $S(a)$

Lemma

For $a \in P_n$, $^\exists$ nbd $O_a \subset P_n$ of a such that $S(b) \subset S(a)$ for $^\forall b \in O_a$.

We say $S(a)$ is locally maximal if

$$^\exists O_a \text{ such that } S(b) \subsetneq S(a) \text{ for } {}^\forall b \in O_a \setminus \mathbb{R}_{>0}a.$$

We can prove

$$a \text{ is perfect} \iff S(a) \text{ is locally maximal}$$

Conclusion

a is perfect $\iff$ $a \in \mathbb{R}_{>0} \cdot \partial^0 K_1(m)$ $\iff$ $S(a)$ is locally maximal

2.4 Existence of Hermite like constants

Let ϕ be a type one class function.

Since

1. $P_n \subset \mathbb{R}_{>0} \cdot K_1(m) = \mathbb{R}_{>0} \cdot \partial K_1(m)$ and
2. $K_1(m)$ is the convex hull of $\partial^0 K_1(m)$,

the Hermite like constant

$$\gamma_\phi := \sup_{a \in P_n} \frac{m(a)}{\phi(a)} = \sup_{a \in \partial K_1(m)} \frac{1}{\phi(a)}$$

$$= \sup_{a \in \partial^0 K_1(m)} \frac{1}{\phi(a)} = \max_{a \in \partial^0 K_1(m)/\mathrm{GL}_n(\mathbb{Z})} \frac{1}{\phi(a)}$$

exists.

Let $\phi^\circ(a) = \inf_{b \in P_n} \mathrm{Tr}(ab)/\phi(b)$, the dual of ϕ.

Put $\xi_\phi = \gamma_\phi \cdot \gamma_{\phi^\circ}$.

Example

- $\xi_{\det^{1/n}} = \gamma_n^2/n$.
- $\xi_m = \max_{(a,b)\in P_n \times P_n} \dfrac{m(a)m(b)}{\mathrm{Tr}(ab)}$.

Theorem (Sawatani–W., 2009)

$\xi_m \leq \xi_\phi$ *for any type one class function* ϕ.

3 Voronoï type theorem of the Rankin invariant

3.1 Rankin's constant

Fix $1 \leq j \leq n-1$.

Let

$$\mathrm{M}_{n,j}^*(\mathbb{Z}) = \{(x_1, \cdots, x_j) \in \mathrm{M}_{n,j}(\mathbb{Z}) \;:\; x_1 \wedge \cdots \wedge x_j \neq 0\}.$$

Define the function $m_j : \overline{P}_n \longrightarrow \mathbb{R}_{\geq 0}$ by

$$m_j(a) = \inf_{X \in \mathrm{M}_{n,j}^*(\mathbb{Z})} (\det a[X])^{1/j}.$$

m_j is a type one class function. The constant

$$\gamma_{n,j} = (\max_{a \in P_n} F_j(a))^n, \quad F_j(a) = \frac{m_j(a)}{(\det a)^{1/n}}$$

was introduced by Rankin(1953).

Explicit values (Rankin, 1953, Sawatani–W.–Okuda, 2008)

- $\gamma_{4,2} = 3/2$.
- $\gamma_{6,2} = 3^{2/3}$, $\gamma_{8,2} = 3$, $\gamma_{8,3} = \gamma_{8,4} = 4$.

Coulangeon characterized local maxima of $F_j = m_j / \det^{1/n}$.

Theorem (Coulangeon, 1996)

$F_j(a)$ *is a local maximum if and only if* a *is* j*-perfect and* j*-eutactic.*

3.2 j-perfection and j-eutaxy

Let $S_j^*(a) = \{X \in \mathrm{M}_{n,j}^*(\mathbb{Z}) \ : \ (\det a[X])^{1/j} = m_j(a)\}$.
Then $S_j(a) := S_j^*(a)/\mathrm{GL}_j(\mathbb{Z})$ is a finite set.
Define the linear form $\varphi_X \ : V_n \longrightarrow \mathbb{R}$ for $X \in \mathrm{M}_{n,j}^*(\mathbb{Z})$ by

$$\varphi_X(v) = \mathrm{Tr}(p_X \cdot v)\,,$$

where $p_X \ : \mathbb{R}^n \longrightarrow \mathrm{span}(x_1, \cdots, x_j)$ is an orthogonal projection.

Definition

- a is j-perfect if $\{\varphi_{\sqrt{a}X}\}_{[X]\in S_j(a)}$ spanns V_n^*.
- a is j-eutactic if $^\exists\rho_X > 0$, $[X] \in S_j(a)$, such that

$$\mathrm{Tr} = \sum_{[X]\in S_j(a)} \rho_X \varphi_{\sqrt{a}X}\,.$$

3.3 Some problems of j-perfect forms

Let $j \geq 2$. The kernel $K_1(m_j) = \{a \in P_n \,:\, m_j(a) \geq 1\}$ is bounded by hypersurfaces $\det(a[X]) = 1$, $X \in \mathrm{M}^*_{n,j}(\mathbb{Z})$.

Problem 1

Determine locations of j-perfect forms in $\partial K_1(m_j)$.

Lemma

For $a \in P_n$, $^\exists$ nbd $O_a \subset P_n$ of a s.t. $S_j(b) \subset S_j(a)$ for $^\forall b \in O_a$.

We can define the local maximality for $S_j(a)$.

Problem 2

a is j-perfect $\overset{?}{\Longleftrightarrow}$ $S_j(a)$ is locally maximal.

Let ϕ be a type one (class) function.

Problem 3

Characterize local maxima of m_j/ϕ as Voronoï's theorem.

Problem 4

When is the Rankin like constant $\displaystyle\sup_{a \in P_n} \frac{m_j(a)}{\phi(a)}$ finite ?

4 Generalizations of Voronoï's theorem

4.1 Arithmetic or geometric generalizations

There are several works:

- Extensions of a base field from $\mathbb{Q}$ to algebraic number fields were studied by Coulangeon, Icaza, Leibak and others.
- Ash(1977) generalized the domain $\boldsymbol{P_n}$ to an arbitrary self-dual homogeneous cone $\boldsymbol{\Omega}$. The function $\boldsymbol{F = m/\det^{1/n}}$ is replaced with a packing function of $\boldsymbol{\Omega}$.
- Bavard(1997, 2005) extended a geometric framework underlying Voronoï's theorem.

4.2 Toward Voronoï's theorem for height functions

Let $\mathbf{k}$ be a global field, $\mathbf{G}$ a connected reductive algebraic group $/\mathbf{k}$ and $\mathbf{P}$ a maximal $\mathbf{k}$-parabolic subgroup of $\mathbf{G}$.
Let $\mathbf{G}_{\mathbb{A}}$ be the adele of $\mathbf{G}$, $\mathbf{K}_{\mathbb{A}}$ a max. compact subgroup of $\mathbf{G}_{\mathbb{A}}$.
We define the height $H_{\mathbf{P}} : \mathbf{G}_{\mathbb{A}} \longrightarrow \mathbb{R}_{>0}$ by

$$H_{\mathbf{P}}(ph) = H_{\mathbf{P}}(p) = |\alpha_{\mathbf{P}}(p)|_{\mathbb{A}}^{-1}$$

for $p \in \mathbf{P}_{\mathbb{A}}$ and $h \in \mathbf{K}_{\mathbb{A}}$, where $\alpha_{\mathbf{P}}$ is a simple root associated with $\mathbf{P}$.
Define $F_{\mathbf{P}} : \mathbf{G}_{\mathbf{k}} \backslash \mathbf{G}_{\mathbb{A}} / \mathbf{K}_{\mathbb{A}} \longrightarrow \mathbb{R}_{>0}$ by

$$F_{\mathbf{P}}(g) = \min_{[v] \in \mathbf{P}_{\mathbf{k}} \backslash \mathbf{G}_{\mathbf{k}}} H_{\mathbf{P}}(vg)\,.$$

The maximum

$$\gamma_{\mathbf{G},\mathbf{P}} = \max_{[g]\in \mathbf{G}_{\mathbf{k}}\backslash \mathbf{G}_{\mathbb{A}}/\mathbf{K}_{\mathbb{A}}} F_{\mathbf{P}}(g)$$

is called a generalized Hermite constant.

Example

If $\mathbf{k} = \mathbb{Q}$, $\mathbf{G} = \mathbf{GL}_n$ and $\mathbf{P} = \left\{ \left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right) \,:\, a \in \mathbf{GL}_j,\ d \in \mathbf{GL}_{n-j} \right\}$, then

$$H_{\mathbf{P}}\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}\right) = |\det a|_{\mathbb{A}}^{(j-n)/\gcd(j,n-j)} |\det d|_{\mathbb{A}}^{j/\gcd(j,n-j)}$$

and

$$\gamma_{\mathbf{G},\mathbf{P}} = (\gamma_{n,j})^{\frac{n}{2\gcd(j,n-j)}} .$$

Problem 5

Characterize local maxima of $F_{\mathbf{P}}$ as Voronoï's theorem.

- If $\mathbf{k}$ is a number field, Bavard's theory applies to several cases, e.g., $\mathbf{G} = \mathbf{GL}_n, \mathbf{SO}_{n,1}$, etc., so the problem was solved in some cases.
- The set of minimal vectors of $g \in \mathbf{G}_{\mathbb{A}}$ is given by

$$S_{\mathbf{P}}(g) = \{[v] \in \mathbf{P}_{\mathbf{k}} \backslash \mathbf{G}_{\mathbf{k}} \ : \ H_{\mathbf{P}}(vg) = F_{\mathbf{P}}(g)\}.$$

This is a finite subset of $\mathbf{P}_{\mathbf{k}} \backslash \mathbf{G}_{\mathbf{k}}$. We have

$^{\exists}$ nbd $O_g \subset \mathbf{G}_{\mathbb{A}}$ of g such that $S_{\mathbf{P}}(g') \subset S_{\mathbf{P}}(g)$ for $^{\forall} g' \in O_g$

Thus we can define the local maximality of $S_{\mathbf{P}}(g)$.

4.3 Example of $\gamma_{G,P}$ in the case of $G = Sp_{2n}/\mathbb{Q}$

Let

$$G = \left\{ g \in GL_{2n} \ : \ {}^t g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \ : \ a \in GL_n \right\}.$$

The rational character $\alpha_P : P \longrightarrow GL_1$ is given by

$$\alpha_P \left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \right) = \det a \, .$$

Since $\mathbf{G}$ and $\mathbf{P}$ satisfy

1. $\mathbf{G}_{\mathbb{A}} = \mathbf{G}_{\mathbb{Q}} \cdot \mathbf{G}_{\mathbb{R}} \cdot \mathbf{K}_{\mathbb{A}}$ (strong approximation),
2. $\mathbf{G}_{\mathbb{Q}} = \mathbf{P}_{\mathbb{Q}} \cdot \mathbf{G}_{\mathbb{Z}}$,

one has

$$\begin{aligned}\gamma_{\mathbf{G},\mathbf{P}} &= \max_{[g]\in\mathbf{G}_{\mathbb{Q}}\backslash\mathbf{G}_{\mathbb{A}}/\mathbf{K}_{\mathbb{A}}} \min_{[v]\in\mathbf{P}_{\mathbb{Q}}\backslash\mathbf{G}_{\mathbb{Q}}} H_{\mathbf{P}}(vg) \\ &= \max_{[g]\in\mathbf{G}_{\mathbb{Z}}\backslash\mathbf{G}_{\mathbb{R}}/\mathbf{K}_{\infty}} \min_{\gamma\in\mathbf{G}_{\mathbb{Z}}} H_{\mathbf{P}}^{\infty}(\gamma g),\end{aligned}$$

where $H_{\mathbf{P}}^{\infty}(ph) = |\alpha_{\mathbf{P}}(p)|^{-1}$ for $p \in \mathbf{P}_{\mathbb{R}}$, $h \in \mathbf{K}_{\infty}$.

Let $\mathbf{H}_n = \{Z \in \mathrm{M}_n(\mathbb{C}) \ : \ \mathrm{Re}Z \in V_n,\ \mathrm{Im}Z \in P_n\}$.
The group $\mathbf{G}_{\mathbb{R}}$ acts on $\mathbf{H}_n$ by

$$g\langle Z\rangle = (aZ+b)(cZ+d)^{-1}, \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ Z \in \mathbf{H}_n\right).$$

Then we have

$$H_{\mathbf{P}}^{\infty}(g) = (\det \mathrm{Im}\{g\langle\sqrt{-1}\mathrm{I}\rangle\})^{-1/2} \qquad (g \in \mathbf{G}_{\mathbb{R}})$$

and

$$\gamma_{\mathbf{G},\mathbf{P}} = \max_{[g]\in \mathbf{G}_{\mathbb{Z}}\backslash\mathbf{G}_{\mathbb{R}}/\mathbf{K}_{\infty}} \min_{\gamma\in\mathbf{G}_{\mathbb{Z}}} (\det \mathrm{Im}\{\gamma g\langle\sqrt{-1}\mathrm{I}\rangle\})^{-1/2}.$$

Since $g\langle\sqrt{-1}\mathbf{I}\rangle$ runs over a fundamental domain of $\mathbf{G}_{\mathbb{Z}}\backslash\mathbf{H}_n$, we have

$$\gamma_{\mathbf{G},\mathbf{P}}^{-2} = \min_{[Z]\in\mathbf{G}_{\mathbb{Z}}\backslash\mathbf{H}_n} \max_{\gamma\in\mathbf{G}_{\mathbb{Z}}} \det \mathrm{Im}\{\gamma\langle Z\rangle\}$$
$$= \min_{Z\in\mathbf{S}_n} \max_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\mathbf{G}_{\mathbb{Z}}} \frac{\det \mathrm{Im} Z}{|\det(cZ+d)|^2},$$

where $\mathbf{S}_n$ is Siegel's fundamental domain:

$$\left\{ Z = X + \sqrt{-1}Y \ : \ \begin{array}{l} \bullet\ |\det(cZ+d)| \geq 1 \ \text{ for } {}^\forall \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathbf{G}_{\mathbb{Z}} \\ \bullet\ |x_{ij}| \leq 1/2, \quad Y \in \text{(Minkowski's domain)} \end{array} \right\}$$

From

$$Z \in \mathbf{S}_n \implies \max_{\gamma\in\mathbf{G}_{\mathbb{Z}}} \det \mathrm{Im}\{\gamma\langle Z\rangle\} = \det \mathrm{Im} Z,$$

it follows

$$\gamma_{\mathbf{G},\mathbf{P}}^{-2} = \min_{Z\in\mathbf{S}_n} \det \mathrm{Im} Z\,.$$

When $n = 1$, $\min_{Z\in S_1} \det \mathrm{Im} Z = \sqrt{3}/2$.
When $n = 2$, Takashi Kawamura determined $\min_{Z\in S_2} \det \mathrm{Im} Z$ by using Gottschling's description of S_2.

Theorem (Kawamura, 2009)

$\min_{Z\in S_2} \det \mathrm{Im} Z = 2/3$.
This minimum is attained only when $Z = Z_8$ or $-\overline{Z_8}$, where

$$Z_8 = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\sqrt{2}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\sqrt{-1}.$$

The domain S_2 is described by 28 polynomials in 6 real variables.
Hayata computed 0-dimensional cells of the boundary ∂S_2 of S_2.
There are at least 526 0-dimensional cells of ∂S_2.
Both Z_8 and $-\overline{Z_8}$ are contained in Hayata's list.

Appendix: Bavard's theory

We consider a quadruplet $\mathcal{E} = (V, \Gamma, C, \{f_s\})$:

V : Riemannian manifold,

Γ : discrete subgroup of the isometry group of V,

C : index set endowed with a right action of Γ,

$\{f_s\}$: family of C^1 functions $f_s : V \longrightarrow \mathbb{R}$ parametrized by $s \in C$.

Assume

1. $f_s \circ \gamma = f_{s\gamma}$ for ${}^\forall s \in C$ and ${}^\forall \gamma \in \Gamma$.
2. $\sharp\{s \in C \ : \ f_s(v) \leq \lambda\}$ is finite for ${}^\forall v \in V$ and ${}^\forall \lambda \in \mathbb{R}$.

What we do is to characterize local maxima of the function $F_{\mathcal{E}} : v \mapsto \min_{s \in C} f_s(v)$.

For $v \in V$, let

$$\begin{aligned} T_v &= \text{tangent space of } V \text{ at } v, \\ X_s(v) &= (\mathbf{grad} f_s)(v), \\ S_{\mathcal{E}}(v) &= \{s \in C \ : \ f_s(v) = F_{\mathcal{E}}(v)\}, \\ \mathbf{Conv}(v) &= \text{convex hull of } \{X_s(v)\}_{s \in S_{\mathcal{E}}(v)} \text{ in } T_v, \\ \mathbf{Aff}(v) &= \text{affine subspace spanned by } \{X_s(v)\}_{s \in S_{\mathcal{E}}(v)} \text{ in } T_v. \end{aligned}$$

Definition

- v is said to be perfect if $T_v = \mathbf{Aff}(v)$.
- v is said to be eutactic if $\mathbf{0} \in \mathbf{Conv}(v)$.

We say $\mathcal{E}$ has the Voronoï property if the equivalence

$$F_{\mathcal{E}} \text{ attains a local maximum on } v \iff v \text{ is perfect and eutactic.}$$

holds.

Theorem (Bavard)

Assume f_s is convex on any geodesic line on V for all s, i.e.,

$$f_s(\ell(\lambda\alpha + (1-\lambda)\beta)) \leq \lambda f_s(\ell(\alpha)) + (1-\lambda) f_s(\ell(\beta))$$

holds for any geodesic $\ell : [0, \epsilon) \to V$, $\alpha, \beta \in (0, \epsilon)$ and $0 < \lambda < 1$. Then $\mathcal{E}$ has the Voronoï property.

Example

Let $P_n^1 = \{a \in P_n \ : \ \det a = 1\} \cong SL_n(\mathbb{R})/SO_n(\mathbb{R})$.
$\mathcal{E} = (P_n^1, SL_n(\mathbb{Z}), \mathbb{Z}^n \setminus \{0\}, \{\varphi_x\})$ has the Voronoï property.
Here $\varphi_x(a) = a[x]$.

Example

Let G be a connected Lie subgroup of $SL_n(\mathbb{R})$ and $G \cdot \mathbf{I}$ be the G-orbit of $\mathbf{I}$ in P_n^1. If G is invariant by the transpose $g \mapsto {}^t g$, then
$\mathcal{E} = (G \cdot \mathbf{I}, G \cap SL_n(\mathbb{Z}), \mathbb{Z}^n \setminus \{0\}, \{\varphi_x|_{G\cdot\mathbf{I}}\})$ has the Voronoï property.