Maximal compact subgroups of $\mathsf{Sp}_4(\mathsf{k})$ and $\mathsf{GSp}_4(\mathsf{k})$

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Contents

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1 Maximal compact subgroups

Let k be a locally compact field of characteristic 0, G a connected linear algebraic group defined over k and G_{k} the locally compact group of k-rational points of G.

The following result can be found in [Bruhat] and [Satake].

Theorem 1.1. *G_{k} has a maximal compact subgroup if and only if G is reductive. In this case, any compact subgroup of G_{k} is contained in a maximal compact subgroup.*

We assume G is reductive, and let

$S =$ maximal k-split torus of G,
$Z =$ centralizer of S in G,
$P =$ minimal parabolic subgroup of G over k of a Levi subgroup Z,
$U =$ unipotent radical of P.

In the case of $\mathsf{k} = \mathbf{R}$, the following result is well known.

Theorem 1.2. *If $\mathsf{k} = \mathbf{R}$, then two maximal compact subgroups of G_{k} are conjugate by an inner automorphism. If K is a maximal compact subgroup of G_{k}, then one has the following decompositions:*

$$\begin{aligned} G_{\mathsf{k}} &= K \cdot Z_{\mathsf{k}} \cdot U_{\mathsf{k}} \quad (\textit{Iwasawa decomposition}) \\ &= K \cdot Z_{\mathsf{k}} \cdot K \quad (\textit{Cartan decomposition}) \end{aligned}$$

This theorem does not true if k is a p-adic field.
Let k be a p-adic field. The following problems occurred in the early of 1960's.

- How many maximal compact subgroups of G_{k} up to conjugacy are there?
- Does G_{k} possess a maximal compact subgroup satisfying both Iwasawa and Cartan decompositions?

These problems were studied by many authors:

1960 – 1964 Shimura, Tsukamoto, Bruhat, Hijikata in classical groups

1965 Iwahori and Matsumoto in Chevalley groups

1966 – 1987 Bruhat and Tits in full generality

The main results of Bruhat–Tits theory are stated as follows.

Theorem 1.3. *Let $\mathcal{B}$ be the Bruhat–Tits building associated with G_{k}. For a point $\boldsymbol{x} \in \mathcal{B}$, $G_{\mathsf{k}}^{\boldsymbol{x}}$ denotes the stabilizer of $\boldsymbol{x}$ in G_{k}.*

(1) *For a maximal compact subgroup K of G_{k}, there is a point $\boldsymbol{x} \in \mathcal{B}$ such that $K = G_{\mathsf{k}}^{\boldsymbol{x}}$.*

(2) *If $\boldsymbol{x} \in \mathcal{B}$ is a point contained in a facet of minimal dimension, then $G_{\mathsf{k}}^{\boldsymbol{x}}$ is a maximal compact subgroup of G_{k}.*

(3) *The number $m(G_{\mathsf{k}})$ of maximal compact subgroups of G_{k} up to conjugacy is finite.*

(4) *If G is simply connected, then every maximal compact subgroup of G_{k} is the stabilizer of a vertex (= 0-dimensional facet) of $\mathcal{B}$, and $m(G_{\mathsf{k}})$ is equal to the number of vertices of a chamber in $\mathcal{B}$. Precisely,*

$$m(G_{\mathsf{k}}) = \prod_{i=1}^{\ell} (\mathrm{rank}_{\mathsf{k}}(G_i) + 1)$$

where $G_1, \cdots, G_\ell$ are k-simple factors of G.

(5) *$\mathcal{B}$ has special points. The stabilizer of every special point of $\mathcal{B}$ is a maximal compact subgroup, which is called a special maximal compact subgroup. Every special maximal compact subgroup satisfies both Iwasawa and Cartan decompositions.*

Remark If G is semisimple but not simply connected, then it is possible to happen a case where $\boldsymbol{x} \in \mathcal{B}$ is not a vertex but $G_{\mathsf{k}}^{\boldsymbol{x}}$ is a maximal compact subgroup of G_{k}. For example, in the case of PGL_n, every chamber has n vertices. Stabilizers of vertices are maximal compact subgroups and they are mutually conjugate in $\mathsf{PGL}_n(\mathsf{k})$. However, $m(\mathsf{PGL}_\mathsf{n}(\mathsf{k}))$ is equal to the number of divisors of n.

Remark The building $\mathcal{B}$ is a union of translations of an apartment A by the action of G_{k} , i.e.,

$$\mathcal{B} = \bigcup_{g \in G_{\mathsf{k}}} gA \,.$$

Let $\overline{C}$ be a closed chamber in A. For a given point $\boldsymbol{x} \in \mathcal{B}$, there is a $g \in G_{\mathsf{k}}$ such that $g^{-1}\boldsymbol{x} \in \overline{C}$. Then $G_{\mathsf{k}}^{\boldsymbol{x}}$ and $G_{\mathsf{k}}^{g^{-1}\boldsymbol{x}}$ are conjugate. To classify conjugacy classes of maximal compact subgroups, it is sufficient to consider only stabilizers of points in $\overline{C}$.

2 Bruhat–Tits theory of $\mathsf{Sp}_4(\mathsf{k})$

Let k be a p-adic field, o the maximal compact subring of k and p the maximal ideal of o.

2.1 $\mathsf{Sp_4}$ and its minimal parabolic subgroup

Let $G = \mathsf{Sp_4}$ be a symplectic group, i.e.,

$$G_{\mathsf{k}} = \left\{ g \in \mathsf{GL}_4(\mathsf{k}) \ : \ {}^t g \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \right\}.$$

We fix a maximal split torus S and a maximal unipotent subgroup U as follows:

$$S_{\mathsf{k}} = Z_{\mathsf{k}} = \left\{ h(s,t) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix} \ : \ s,t \in \mathsf{k}^\times \right\}$$

$$U_{\mathsf{k}} = \left\{ \begin{pmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ : \ w,x,y,z \in \mathsf{k} \right\}$$

Then $P = SU$ is a minimal parabolic subgroup of G over k.

2.2 Rational characters and cocharacters of S

Define k-rational characters $\mathsf{e}_1, \mathsf{e}_2 \ : \ S \longrightarrow \mathsf{G_m}$ by

$$\mathsf{e}_1(h(s,t)) = s, \qquad \mathsf{e}_2(h(s,t)) = t\,.$$

Then $\{\mathsf{e}_1, \mathsf{e}_2\}$ is a basis of $\mathsf{X}^*(S) = \mathrm{Hom}_{\mathsf{k}}(S, \mathsf{G_m})$.
Cocharacters $\mathsf{e}_1^\vee, \mathsf{e}_2^\vee \ : \ \mathsf{G_m} \longrightarrow S$ are defined by

$$\mathsf{e}_1^\vee(s) = h(s,1), \qquad \mathsf{e}_2^\vee(s) = h(1,s)\,,$$

which give the dual basis of $\{\mathsf{e}_1, \mathsf{e}_2\}$ in $\mathsf{X}_*(S) = \mathrm{Hom}_k(\mathsf{G_m}, S)$.

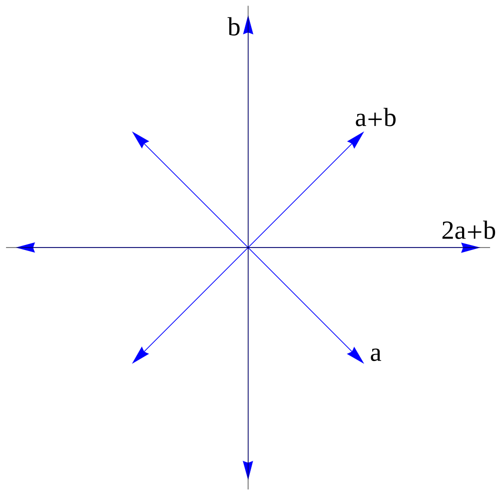
2.3 Affine root system and root subgroups

Define $\mathsf{a}, \mathsf{b} \in \mathsf{X}^*(S)$ by

$$\mathsf{a} = \mathsf{e}_1 - \mathsf{e}_2, \qquad \mathsf{b} = 2\mathsf{e}_2 .$$

The relative root system Φ and the affine root system Φ_{aff} of (G, S) over k are given by

$$\Phi = \{\pm\mathsf{a},\ \pm\mathsf{b},\ \pm(\mathsf{a}+\mathsf{b}),\ \pm(2\mathsf{a}+\mathsf{b})\}, \qquad \Phi_{\text{aff}} = \Phi \times \mathbf{Z} .$$



We fix a one-parameter subgroup $u_{\mathsf{c}} : \mathsf{k} \longrightarrow G_{\mathsf{k}}$ for each $\mathsf{c} \in \Phi$ such that

$$h \cdot u_{\mathsf{c}}(x) \cdot h^{-1} = u_{\mathsf{c}}(\mathsf{c}(h)x) \quad \text{for } h \in S_{\mathsf{k}} ,$$

e.g., for positive roots,

$$u_{\mathsf{a}}(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix}, \qquad u_{\mathsf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$u_{\mathsf{a}+\mathsf{b}}(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u_{2\mathsf{a}+\mathsf{b}}(x) = \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For each affine root $\delta = (\mathsf{c}, n) \in \Phi_{\text{aff}}$, the root subgroup X_δ is defined to be

$$X_\delta = u_{\mathsf{c}}(\mathsf{p}^n) .$$

2.4 Apartment and chambers

The apartment A is an affine space under the real vector space $\mathsf{X}_*(S) \otimes_{\mathbf{Z}} \mathbf{R}$. By $\mathbf{R}$-linear extension of the natural pairing

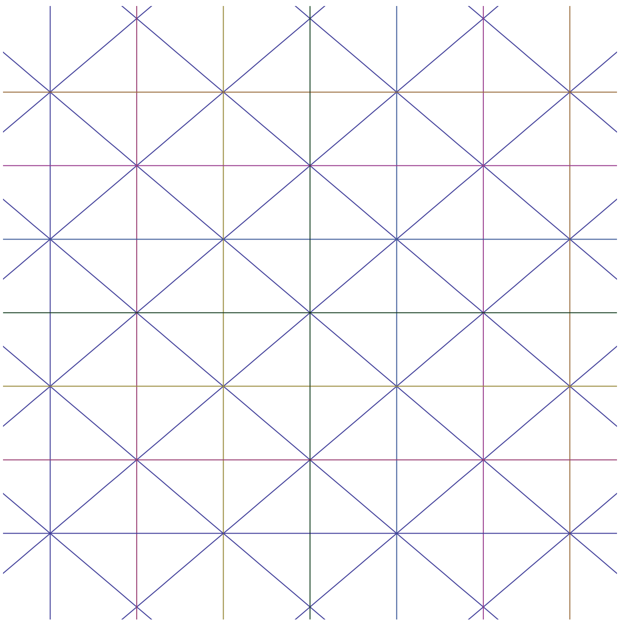
$$\langle \cdot, \cdot \rangle \,:\, \mathsf{X}^*(S) \times \mathsf{X}_*(S) \longrightarrow \mathbf{Z}$$

each affine root $\delta = (\mathsf{c}, n) \in \Phi_{\mathrm{aff}}$ defines an affine function:

$$\delta \,:\, A \longrightarrow \mathbf{R} \,:\, \delta(\boldsymbol{x}) = \langle \mathsf{c}, \boldsymbol{x} \rangle + n\,.$$

The null set $\delta^{-1}(0)$ is an affine hyperplane of A. In our case, $\dim A = 2$ and $\delta^{-1}(0)$ is a line of the form:

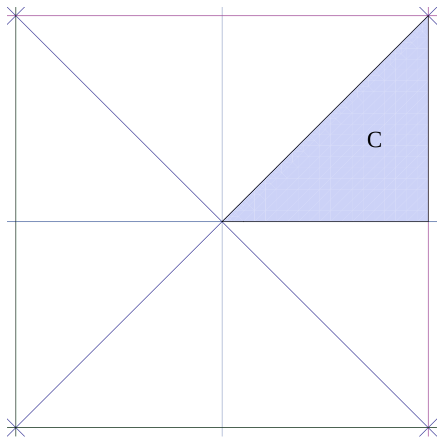
$$\begin{aligned}
\delta = (\mathsf{a}, n) \quad &: \quad \delta(x_1 \mathsf{e}_1^\vee + x_2 \mathsf{e}_2^\vee) = x_1 - x_2 + n = 0 \\
\delta = (\mathsf{b}, n) \quad &: \quad 2x_2 + n = 0 \\
\delta = (\mathsf{a}+\mathsf{b}, n) \quad &: \quad x_1 + x_2 + n = 0 \\
\delta = (2\mathsf{a}+\mathsf{b}, n) &: \quad 2x_1 + n = 0
\end{aligned}$$



A connected component C of the set

$$A - \bigcup_{\delta \in \Phi_{\text{aff}}} \delta^{-1}(0)$$

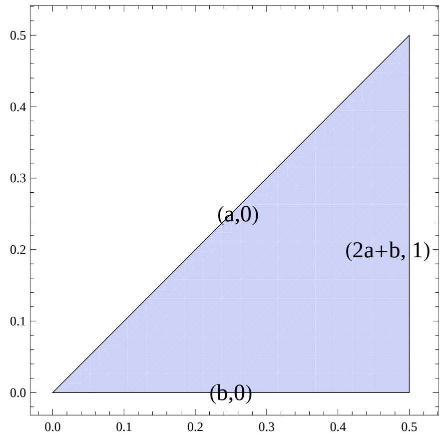
is called a chamber, which is a polytope.



Define the subset $\Delta_{\text{aff}}(C)$ of Φ_{aff} by

$$\Delta_{\text{aff}}(C) = \{\delta \in \Phi_{\text{aff}} \,:\, \delta/2 \notin \Phi_{\text{aff}} \text{ and } \delta^{-1}(0) \cap \partial C \neq \emptyset\}\,.$$

For example, if C is chosen as follows



then

$$\Delta_{\text{aff}}(C) = \{(\mathsf{a}, 0),\ (\mathsf{b}, 0),\ (2\mathsf{a} + \mathsf{b}, 1)\}\,.$$

$\Delta_{\mathrm{aff}}(C)$ is displayed by the affine Dynkin diagram:

$$\underset{\mathsf{2a+b}}{\circ} \Longrightarrow \underset{\mathsf{a}}{\circ} \Longleftarrow \underset{\mathsf{b}}{\circ}$$

There is the homomorphism $\nu : S_{\mathsf{k}} \longrightarrow \mathsf{X}_*(S) \otimes_{\mathbf{Z}} \mathbf{R}$ so that

$$h^{-1} X_{(\mathsf{c},n)} h = X_{(\mathsf{c},\langle \mathsf{c},\nu(h)\rangle + n)}$$

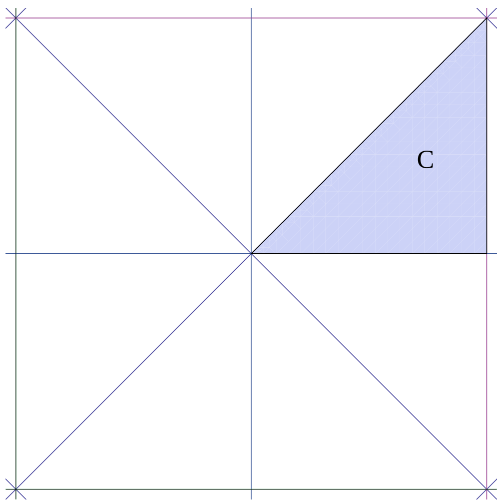
holds for every $(\mathsf{c}, n) \in \Phi_{\mathrm{aff}}$ and $h \in S_{\mathsf{k}}$. Precisely, ν is given by

$$\nu(h(s,t)) = -\mathrm{ord}_{\mathsf{p}}(s)\mathsf{e}_1^\vee - \mathrm{ord}_{\mathsf{p}}(t)\mathsf{e}_2^\vee .$$

The kernel of ν is the group S_{o} of o rational points of S. The translation of A induced by $\nu(h)$ defines the action of S_{k} on A.

Let N be the normalizer of S in G. The Weyl group $W = N/S = N_{\mathsf{k}}/S_{\mathsf{k}}$ of Φ acts on A by reflections as usual. The affine Weyl group $W_{\mathrm{aff}} = N_{\mathsf{k}}/S_{\mathsf{o}}$ is isomorphic with $S_{\mathsf{k}}/S_{\mathsf{o}} \rtimes W$. Both N_{k} and W_{aff} act on A by affine transformations.

Remark In our case, the closed chamber $\overline{C} = C \cup \partial C$ of A is a fundamental domain of $A/N_{\mathsf{k}} = A/W_{\mathrm{aff}}$. This is not true in general.



For $\delta \in \Delta_{\mathrm{aff}}(C)$, w_δ denotes the orthogonal reflection of A with respect to the affine hyperplane $\delta^{-1}(0)$. The set $W_{\mathrm{aff}}(C) = \{\mathsf{w}_\delta\}_{\delta \in \Delta_{\mathrm{aff}}(C)}$ is a subset of W_{aff}.

2.5 Tits system

For a subset $F \subset \overline{C}$ and a root $\mathsf{c} \in \Phi$, we set

$$n_F(\mathsf{c}) = \inf\{n \in \mathbf{Z} \ : \ \langle \mathsf{c}, \boldsymbol{x} \rangle + n \geq 0 \text{ for all } \boldsymbol{x} \in F\} .$$

Define unipotent subgroups X_F^+ and X_F^- of G_{k} by

$$\begin{aligned} X_F^+ &= \prod_{0<\mathsf{c}\in\Phi} X_{(\mathsf{c},n_F(\mathsf{c}))} = \prod_{0<\mathsf{c}\in\Phi} u_{\mathsf{c}}(\mathsf{p}^{n_F(\mathsf{c})}), \\ X_F^- &= \prod_{0>\mathsf{c}\in\Phi} X_{(\mathsf{c},n_F(\mathsf{c}))} = \prod_{0>\mathsf{c}\in\Phi} u_{\mathsf{c}}(\mathsf{p}^{n_F(\mathsf{c})}) . \end{aligned}$$

If $F = C$, then the product

$$B_C = X_C^- \cdot S_{\mathsf{o}} \cdot X_C^+$$

is a subgroup of G_{k}, which is called the Iwahori subgroup of G_{k} corresponding to C. The following is a fundamental result due to Iwahori–Matsumoto.

Theorem 2.1. *The quadruple $(G_{\mathsf{k}}, B_C, N_{\mathsf{k}}, W_{\mathrm{aff}}(C))$ is a Tits system, i.e., this satisfies*

(T1) *$B_C \cup N_{\mathsf{k}}$ generates G_{k} and $B_C \cap N_{\mathsf{k}} = S_{\mathsf{o}}$ is a normal subgroup of N_{k},*

(T2) *$W_{\mathrm{aff}}(C)$ generates W_{aff} and every element in $W_{\mathrm{aff}}(C)$ is of order 2,*

(T3) *$\mathsf{s}B_C\mathsf{s} \neq B_C$ for each $\mathsf{s} \in W_{\mathrm{aff}}(C)$,*

(T4) *$\mathsf{s}B_C\mathsf{w} \subset B_C\mathsf{w}B_C \cup B_C\mathsf{sw}B_C$ for each $\mathsf{s} \in W_{\mathrm{aff}}(C)$ and $\mathsf{w} \in W_{\mathrm{aff}}$.*

As a consequence of the theory of Tits systems, we obtain the following double coset decomposition of G_{k}:

$$G_{\mathsf{k}} = \bigsqcup_{\mathsf{w}\in W_{\mathrm{aff}}} B_C\mathsf{w}B_C \qquad \text{(Bruhat decomposition)}$$

For $\boldsymbol{x} \in \overline{C}$, $W_{\mathrm{aff}}^{\boldsymbol{x}}$ stands for the stabilizer of $\boldsymbol{x}$ in W_{aff}. Then

$$G_{\mathsf{k}}^{\boldsymbol{x}} = \bigsqcup_{\mathsf{w}\in W_{\mathrm{aff}}^{\boldsymbol{x}}} B_C\mathsf{w}B_C$$

is a subgroup of G_{k}.

2.6 Building

Since G_{k} does not act on A, we need to build an enlargement of A on which G_{k} acts. Since $\overline{C}$ is a fundamental domain of A/W_{aff}, the apartment A is identified with the quotient space

$$(W_{\text{aff}} \times \overline{C})/\sim\,,$$

where $(\mathsf{w}, \boldsymbol{x}) \sim (\mathsf{w}', \boldsymbol{x}')$ if $\boldsymbol{x} = \boldsymbol{x}'$ and $\mathsf{w}^{-1}\mathsf{w}' \in W_{\text{aff}}^{\boldsymbol{x}}$. We extend the equivalent relation $\sim$ to $G_{\mathsf{k}} \times \overline{C}$ by

$$(g, \boldsymbol{x}) \sim (g', \boldsymbol{x}') \text{ if } \boldsymbol{x} = \boldsymbol{x}' \text{ and } g^{-1}g' \in G_{\mathsf{k}}^{\boldsymbol{x}}\,.$$

Then the quotient space

$$\mathcal{B} = \mathcal{B}(G_{\mathsf{k}}) = (G_{\mathsf{k}} \times \overline{C})/\sim$$

gives the building of G_{k}. Let n_{w} be an arbitrary representative in N_{k} of $\mathsf{w} \in W_{\text{aff}}$. Then, by the map $(\mathsf{w}, \boldsymbol{x}) \mapsto (n_{\mathsf{w}}, \boldsymbol{x})$, the apartment $A = (W_{\text{aff}} \times \overline{C})/\sim$ is embedded in $\mathcal{B}$. The group G_{k} acts on $\mathcal{B}$ by $g(h, \boldsymbol{x}) = (gh, \boldsymbol{x})$.

Theorem 2.2 (Tits, §2.1)**.** *The building $\mathcal{B}$ is uniquely characterized as a G_{k}-set satisfying the following properties:*

- $\mathcal{B} = \bigcup_{g \in G_{\mathsf{k}}} gA$*,*
- *N_{k} stabilizes A and operates on it by the same way defined as in* §2.4*,*
- *for any $\delta \in \Phi_{\text{aff}}$, the root subgroup X_δ fixes $\delta^{-1}([0, \infty))$ pointwise.*

Remark For every $\boldsymbol{x} \in A$, there is an $n \in N_{\mathsf{k}}$ such that $n\boldsymbol{x} \in \overline{C}$. Then we define $G_{\mathsf{k}}^{\boldsymbol{x}}$ by $n^{-1}G_{\mathsf{k}}^{n\boldsymbol{x}}n$. Another definition of $\mathcal{B}$ is given by

$$\mathcal{B} = (G_{\mathsf{k}} \times A)/\sim\,,$$

where

$$(g, \boldsymbol{x}) \sim (g', \boldsymbol{x}') \text{ if } {}^{\exists}n \in N_{\mathsf{k}} \text{ such that } \boldsymbol{x}' = n\boldsymbol{x} \text{ and } g^{-1}g'n \in G_{\mathsf{k}}^{\boldsymbol{x}}$$

This definition does not need to assume that $\overline{C}$ is a fundamental domain of A/W_{aff}.

2.7 Stabilizers

For a subset $F \subset \mathcal{B}$, the pointwise stabilizer of F in G_{k} is denoted by G_{k}^F, i.e.,

$$G_{\mathsf{k}}^F = \{g \in G_{\mathsf{k}} \ : \ g\boldsymbol{x} = \boldsymbol{x} \ \text{ for all } \boldsymbol{x} \in F\}\,.$$

The structure of G_{k}^F is determined by

Theorem 2.3 (Bruhat–Tits Proposition 2.4.13, Tits §3.1.1)**.** *Let $F \subset \mathcal{B}$ be a bounded subset.*

(1) *If $F^\dagger$ denotes the closed convex closure of F in $\mathcal{B}$, then $G_{\mathsf{k}}^F = G_{\mathsf{k}}^{F^\dagger}$.*

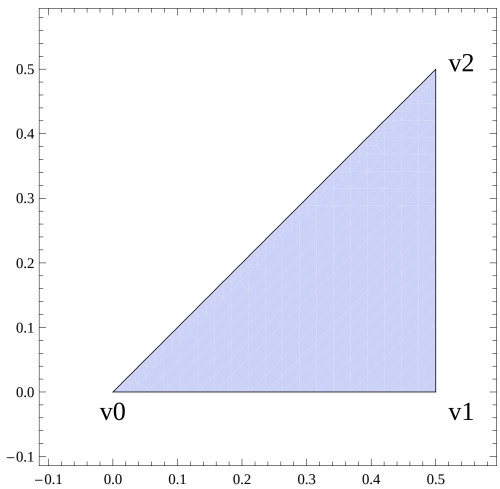
(2) *If $F \subset \overline{C}$ and N_{k}^F denotes the pointwise stabilizer of F in N_{k}, then*

$$G_{\mathsf{k}}^F = X_F^- \cdot N_{\mathsf{k}}^F \cdot X_F^+\,.$$

If $F = \{\boldsymbol{x}\} \subset \overline{C}$ is a one point, then $G_{\mathsf{k}}^{\{\boldsymbol{x}\}}$ is coincides with $G_{\mathsf{k}}^{\boldsymbol{x}}$ defined in §2.5, i.e.,

$$G_{\mathsf{k}}^{\{\boldsymbol{x}\}} = G_{\mathsf{k}}^{\boldsymbol{x}} = B_C \cdot W_{\text{aff}}^{\boldsymbol{x}} \cdot B_C$$

We write $\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2$ for vertices of a chamber C as follows.



The chamber C has 7 facets:

$$\boldsymbol{v}_0, \quad \boldsymbol{v}_1, \quad \boldsymbol{v}_2, \quad \overline{\boldsymbol{v}_0\boldsymbol{v}_1}, \quad \overline{\boldsymbol{v}_1\boldsymbol{v}_2}, \quad \overline{\boldsymbol{v}_2\boldsymbol{v}_0}, \quad C\,.$$

The stabilizers of vertices are

$$G_{\mathsf{k}}^{\boldsymbol{v}_0} = X_{\boldsymbol{v}_0}^{-} \cdot S_{\mathsf{o}} \cdot W \cdot X_{\boldsymbol{v}_0}^{+} = \mathsf{Sp}_4(\mathsf{o})$$

$$G_{\mathsf{k}}^{\boldsymbol{v}_1} = \begin{pmatrix} 1 & & & \\ \mathsf{p} & 1 & & \\ \mathsf{p} & \mathsf{p} & 1 & \mathsf{p} \\ \mathsf{p} & \mathsf{o} & & 1 \end{pmatrix} \cdot S_{\mathsf{o}} \cdot \begin{pmatrix} 1 & \mathsf{o} & \mathsf{p}^{-1} & \mathsf{o} \\ & 1 & \mathsf{o} & \mathsf{o} \\ & & 1 & \\ & & \mathsf{o} & 1 \end{pmatrix}$$

$$G_{\mathsf{k}}^{\boldsymbol{v}_2} = \begin{pmatrix} 1 & & & \\ \mathsf{o} & 1 & & \\ \mathsf{p} & \mathsf{p} & 1 & \mathsf{o} \\ \mathsf{p} & \mathsf{p} & & 1 \end{pmatrix} \cdot S_{\mathsf{o}} \cdot \{I_4, \mathsf{w}_0\} \cdot \begin{pmatrix} 1 & \mathsf{o} & \mathsf{p}^{-1} & \mathsf{p}^{-1} \\ & 1 & \mathsf{p}^{-1} & \mathsf{p}^{-1} \\ & & 1 & \\ & & \mathsf{o} & 1 \end{pmatrix},$$

where $\mathsf{w}_0 = \mathsf{w}_{(\mathsf{a},0)}$. By Theorem 2.3 (1), we have

$$G_{\mathsf{k}}^{\overline{\boldsymbol{v}_i\boldsymbol{v}_j}} = G_{\mathsf{k}}^{\boldsymbol{v}_i} \cap G_{\mathsf{k}}^{\boldsymbol{v}_j}\,, \qquad G_{\mathsf{k}}^{C} = G_{\mathsf{k}}^{\boldsymbol{v}_0} \cap G_{\mathsf{k}}^{\boldsymbol{v}_1} \cap G_{\mathsf{k}}^{\boldsymbol{v}_2} = B_C\,.$$

For each facet F of C, G_{k}^{F} has the double coset decomposition:

$$G_{\mathsf{k}}^{F} = \bigsqcup_{\mathsf{w} \in W_{\mathrm{aff}}^{F}} B_C \mathsf{w} B_C\,,$$

where W_{aff}^{F} is the subgroup of W_{aff} generated by $\{\mathsf{w}_\delta \,:\, F \subset \delta^{-1}(0)\}$.

By Theorem 1.3 (4), $G_{\mathsf{k}}^{\boldsymbol{v}_0}, G_{\mathsf{k}}^{\boldsymbol{v}_1}$ and $G_{\mathsf{k}}^{\boldsymbol{v}_2}$ are maximal compact subgroups of G_{k} and they are not conjugate in G_{k} each other. Every maximal compact subgroup of G_{k} is conjugate to one of $G_{\mathsf{k}}^{\boldsymbol{v}_i}$s.

2.8 Special maximal compact subgroups

Let $\boldsymbol{x} \in \overline{C}$. We define subsets of $\Phi_{\text{aff}} = \Phi \times \mathbf{Z}$ by

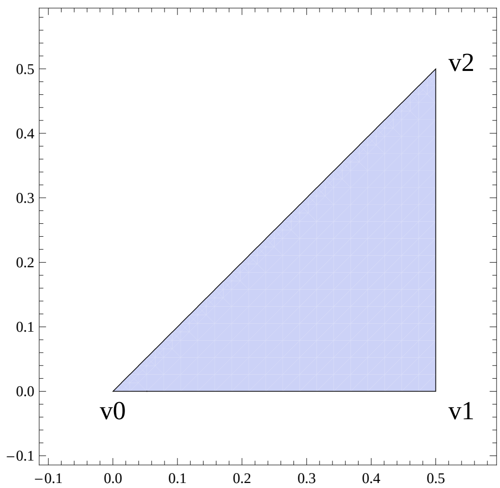
$$\Phi_{\text{aff}}(\boldsymbol{x}) = \{\delta \in \Phi_{\text{aff}} \ : \ \delta(\boldsymbol{x}) = 0\}, \qquad \Phi(\boldsymbol{x}) = \Phi\text{-part of } \Phi_{\text{aff}}(\boldsymbol{x})$$

and

$$I_{\boldsymbol{x}} = \{\delta \in \Delta_{\text{aff}}(C) \ : \ \delta(\boldsymbol{x}) \neq 0\}.$$

Then $\Phi(\boldsymbol{x})$ is a subroot system of Φ. If $\boldsymbol{x}$ is a point in the interior of C, then $\Phi_{\text{aff}}(\boldsymbol{x}) = \emptyset$ and $I_{\boldsymbol{x}} = \Delta_{\text{aff}}(C)$.

The point $\boldsymbol{x}$ is called special if every element of Φ is proportional to some element of $\Phi(\boldsymbol{x})$.



For example,

- if $\boldsymbol{x} = \boldsymbol{v}_0$, then $\Phi(\boldsymbol{v}_0) = \Phi$ and $I_{\boldsymbol{v}_0} = \{(2\mathsf{a} + \mathsf{b}, 1)\}$,
- if $\boldsymbol{x} = \boldsymbol{v}_1$, then $\Phi(\boldsymbol{v}_1) = \{\mathsf{b}, 2\mathsf{a} + \mathsf{b}\}$ and $I_{\boldsymbol{v}_1} = \{(\mathsf{a}, 0)\}$,
- if $\boldsymbol{x} = \boldsymbol{v}_2$, then $\Phi(\boldsymbol{v}_2) = \Phi$ and $I_{\boldsymbol{v}_2} = \{(\mathsf{b}, 0)\}$.

Hence both $\boldsymbol{v}_0$ and $\boldsymbol{v}_2$ are special, but not $\boldsymbol{v}_1$.

The stabilizer of a special point is a special maximal compact subgroup. Both $G_{\mathsf{k}}^{\boldsymbol{v}_0}$ and $G_{\mathsf{k}}^{\boldsymbol{v}_2}$ are special maximal compact subgroups, but not $G_{\mathsf{k}}^{\boldsymbol{v}_1}$.

2.9 o-models of G

Let $J : \mathsf{k}^4 \times \mathsf{k}^4 \longrightarrow \mathsf{k}$ be the symplectic form defining G, and let e_1, e_2, e'_1, e'_2 be the canonical basis. We define o-lattices L_0, L_1, L_2 as follows:

$$\begin{aligned} L_0 &= \mathsf{o}e_1 + \mathsf{o}e_2 + \mathsf{o}e'_1 + \mathsf{o}e'_2, \\ L_1 &= \mathsf{o}e_1 + \mathsf{o}e_2 + \mathsf{p}e'_1 + \mathsf{o}e'_2, \\ L_2 &= \mathsf{o}e_1 + \mathsf{o}e_2 + \mathsf{p}e'_1 + \mathsf{p}e'_2 \,. \end{aligned}$$

The stabilizer of L_i in G_{k} is $G_{\mathsf{k}}^{\boldsymbol{v}_i}$ for $i = 0, 1, 2$.

Since $J(L_i, L_i) \subset \mathsf{o}$, (L_i, J) gives an o-structure of the symplectic space (k^4, J), and hence an o-model $\mathcal{G}^i$ of G. We have $\mathcal{G}^i_{\mathsf{o}} = G_{\mathsf{k}}^{\boldsymbol{v}_i}$. One of the main results of Bruhat–Tits theory is:

Theorem 2.4 (Tits §3.4.1)**.** *Let F be a non-empty bounded subset of $\mathcal{B}$. Then there exists a unique smooth group o-scheme $\mathcal{G}^F$ satisfying*

- $\mathcal{G}^F \times_{\mathsf{o}} \mathsf{k} = G$,
- $\mathcal{G}^F_{\mathsf{o}'} = G^F_{\mathsf{k}'}$ *for any unramified extension* k'/k.

We put $\mathcal{G}^i_{\mathsf{f}} = (\mathcal{G}^i \times_{\mathsf{o}} \mathsf{f})_{\mathsf{f}}$, where $\mathsf{f} = \mathsf{o}/\mathsf{p}$ is the residue field of k. It is easy to see that $\mathcal{G}^0_{\mathsf{f}} = \mathcal{G}^2_{\mathsf{f}} = \mathsf{Sp}_4(\mathsf{f})$. We determine $\mathcal{G}^1_{\mathsf{f}}$. Let π be a prime element of o. Since $J(L_1, L_1) = \mathsf{o}$, the bilinear form

$$\overline{J} : L_1/\pi L_1 \times L_1/\pi L_1 \longrightarrow \mathsf{f} \,.$$

over f is defined from J. For $x \in L$, $[x]$ denotes $x \mod \pi L$. Then $[e_1], [e_2], [\pi e'_1], [e'_2]$ is a basis of $L_1/\pi L_1$ over f. Since the radical $R_{\overline{J}}$ of $\overline{J}$ is spanned by $[e_1], [\pi e'_1]$, the automorphism group of $(L_1/\pi L_1, \overline{J})$ is isomorphic with $\mathsf{M}_2(\mathsf{f}) \rtimes (\mathsf{GL}_2(\mathsf{f}) \times \mathsf{SL}_2(\mathsf{f}))$. Hence $\mathcal{G}^1_{\mathsf{f}} \cong \mathsf{M}_2(\mathsf{f}) \rtimes (\mathsf{GL}_2(\mathsf{f}) \times \mathsf{SL}_2(\mathsf{f}))$.

3 Maximal compact subgroups of $\mathsf{GSp}_4(\mathsf{k})$

Let GSp_4 be the symplectic group of similitude. There is the following exact sequence:

$$1 \longrightarrow \mathsf{Sp}_4 \longrightarrow \mathsf{GSp}_4 \xrightarrow{\chi} \mathsf{G}_m \longrightarrow 1$$

The similitude character χ has a splitting:

$$s \mapsto d(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, GSp_4 is isomorphic with $\mathsf{Sp}_4 \rtimes d(\mathsf{G}_m)$.

3.1 Bruhat–Tits theory of $\mathsf{GSp}_4(\mathsf{k})$

- In general, the apartment of a connected reductive group H over k is identified with the apartment of $Z(H) \times [H, H]$, where $Z(H)$ denotes the maximal central k-split torus of H. Since $Z(\mathsf{GSp}_4)$ is a one-dimensional split torus and $[\mathsf{GSp}_4, \mathsf{GSp}_4] = \mathsf{Sp}_4$, the apartment $\widetilde{A}$ of $\mathsf{GSp}_4(\mathsf{k})$ is identified with $\mathbf{R} \times A$.

- $T = S \cdot d(\mathsf{G}_m)$ is a maximal k-split torus of GSp_4. The root system of GSp_4 with respect to T is the same as Φ. For $\delta \in \Phi_{\mathsf{aff}}$, the affine function $\delta : A \longrightarrow \mathbf{R}$ is trivially extended to $\widetilde{A}$ by composition with the projection $\widetilde{A} \longrightarrow A$. Therefore, $\widetilde{C} = \mathbf{R} \times C$ is a chamber of $\widetilde{A}$.

- For a vertex $\boldsymbol{v}_i$ of C, we denote by $\widetilde{\boldsymbol{v}}_i$ the one-dimensional facet $\mathbf{R} \times \boldsymbol{v}_i$ of $\widetilde{C}$. Since $\widetilde{\boldsymbol{v}}_i$ is a facet of minimal dimension, its stabilizer $K_i = \mathsf{GSp}_4(\mathsf{k})^{\widetilde{\boldsymbol{v}}_i}$ is a maximal compact subgroup of $\mathsf{GSp}_4(\mathsf{k})$.

- The homomorphism $\nu : S_{\mathsf{k}} \longrightarrow \mathsf{X}_*(S) \otimes_{\mathbf{Z}} \mathbf{R}$ is extended to T_{k} by

$$\nu(d(s)) = -\frac{\mathrm{ord}_{\mathsf{p}}(s)}{2}(\mathsf{e}_1^\vee + \mathsf{e}_2^\vee)\,.$$

  Then T_{k} acts on $\widetilde{A}$ by the translation: $(r, \boldsymbol{x}) \mapsto (r, \boldsymbol{x} + \nu(h))$ for $(r, \boldsymbol{x}) \in \widetilde{A}$ and $h \in T_{\mathsf{k}}$. The facet $\widetilde{\boldsymbol{v}}_0$ transforms to the facet $\widetilde{\boldsymbol{v}}_2$ by the action of $d(\pi^{-1})$. Hence K_0 and K_2 are conjugate in $\mathsf{GSp}_4(\mathsf{k})$.

- K_0 is not conjugate to K_1 because that $\boldsymbol{v}_0$ is a special point but not $\boldsymbol{v}_1$.

3.2 Conjugacy of stabilizers of facets

The subset $\Delta_{\mathrm{aff}}(\widetilde{C}) = \Delta_{\mathrm{aff}}(C)$ of Φ_{aff} is the local Dynkin diagram of $\mathsf{GSp}_4(\mathsf{k})$:

$$\underset{\mathsf{2a+b}}{\circ} \Longrightarrow \underset{\mathsf{a}}{\circ} \Longleftarrow \underset{\mathsf{b}}{\circ}$$

The local Dynkin diagram does not depend, up to canonical isomorphism, on the choice of the chamber $\widetilde{C}$. The torus T_{k} acts on the set of chambers in $\widetilde{A}$, and hence on $\Delta_{\mathrm{aff}}(\widetilde{C})$. We denote by $\Xi(\mathsf{GSp}_4)$, or simply Ξ, the image of the homomorphism $T_{\mathsf{k}} \longrightarrow \mathrm{Aut}(\Delta_{\mathrm{aff}}(\widetilde{C}))$

Theorem 3.1 (Tits §2.5)**.** *Ξ is isomorphic with $T_{\mathsf{k}}/T_{\mathsf{o}}S_{\mathsf{k}}Z(\mathsf{GSp}_4)_{\mathsf{k}}$, in particular $\Xi = \mathrm{Aut}(\Delta_{\mathrm{aff}}(\widetilde{C}))$.*

For every facet $\widetilde{F}$ of the chamber $\widetilde{C}$, we define the subset $I_{\widetilde{F}}$ of $\Delta_{\mathrm{aff}}(\widetilde{C})$ by

$$I_{\widetilde{F}} = \{\delta \in \Delta_{\mathrm{aff}}(\widetilde{C}) \ : \ \delta|_{\widetilde{F}} \neq 0\}$$

Obviously, we have $I_{\widetilde{\boldsymbol{v}}_i} = I_{\boldsymbol{v}_i}$.

Theorem 3.2 (Tits §2.5)**.** *Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be facets of $\widetilde{C}$. Then $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{F}_1}$ and $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{F}_2}$ are conjugate in $\mathsf{GSp}_4(\mathsf{k})$ if and only if $I_{\widetilde{F}_1}$ and $I_{\widetilde{F}_2}$ are in the same orbit of Ξ.*

4 Conclusions

- $G_{\mathsf{k}}^{\boldsymbol{v}_0}, G_{\mathsf{k}}^{\boldsymbol{v}_1}$ and $G_{\mathsf{k}}^{\boldsymbol{v}_2}$ are are representatives of conjugacy classes of maximal compact subgroups of $G_{\mathsf{k}} = \mathsf{Sp}_4(\mathsf{k})$.

- Both $G_{\mathsf{k}}^{\boldsymbol{v}_0}$ and $G_{\mathsf{k}}^{\boldsymbol{v}_2}$ are special maximal compact subgroups of G_{k}. They satisfy both Iwasawa and Cartan decompositions. The reduction mod p of each of them is isomorphic with $\mathsf{Sp}_4(\mathsf{f})$.

- $G_{\mathsf{k}}^{\boldsymbol{v}_1}$ is not a special maximal compact subgroup. The reduction mod p of $G_{\mathsf{k}}^{\boldsymbol{v}_1}$ is isomorphic with $\mathsf{M}_2(\mathsf{f}) \rtimes (\mathsf{GL}_2(\mathsf{f}) \times \mathsf{SL}_2(\mathsf{f}))$.

- $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{\boldsymbol{v}}_0}$ and $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{\boldsymbol{v}}_1}$ are representatives of conjugacy classes of maximal compact subgroups of $\mathsf{GSp}_4(\mathsf{k})$.

- $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{\boldsymbol{v}}_0}$ is a special maximal compact subgroup.

- $\mathsf{GSp}_4(\mathsf{k})^{\widetilde{\boldsymbol{v}}_1}$ is not a special maximal compact subgroup.