Maximal compact subgroups of $\text{Sp}_4(k)$ and $\text{GSp}_4(k)$

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1 Maximal compact subgroups

Let $k$ be a locally compact field of characteristic 0, $G$ a connected linear algebraic group defined over $k$ and $G_k$ the locally compact group of $k$-rational points of $G$.

The following result can be found in [Bruhat] and [Satake].

**Theorem 1.1.** $G_k$ has a maximal compact subgroup if and only if $G$ is reductive. In this case, any compact subgroup of $G_k$ is contained in a maximal compact subgroup.

We assume $G$ is reductive, and let

- $S = \text{maximal } k\text{-split torus of } G$,
- $Z = \text{centralizer of } S \text{ in } G$,
- $P = \text{minimal parabolic subgroup of } G \text{ over } k \text{ of a Levi subgroup } Z$,
- $U = \text{unipotent radical of } P$.

In the case of $k = \mathbb{R}$, the following result is well known.

**Theorem 1.2.** If $k = \mathbb{R}$, then two maximal compact subgroups of $G_k$ are conjugate by an inner automorphism. If $K$ is a maximal compact subgroup of $G_k$, then one has the following decompositions:

$$G_k = K \cdot Z_k \cdot U_k \quad (\text{Iwasawa decomposition})$$
$$= K \cdot Z_k \cdot K \quad (\text{Cartan decomposition})$$

This theorem does not true if $k$ is a p-adic field.

Let $k$ be a p-adic field. The following problems occurred in the early of 1960’s.

- How many maximal compact subgroups of $G_k$ up to conjugacy are there?
- Does $G_k$ possess a maximal compact subgroup satisfying both Iwasawa and Cartan decompositions?
These problems were studied by many authors:

1960 – 1964 Shimura, Tsukamoto, Bruhat, Hijikata in classical groups
1965 Iwahori and Matsumoto in Chevalley groups
1966 – 1987 Bruhat and Tits in full generality

The main results of Bruhat–Tits theory are stated as follows.

**Theorem 1.3.** Let \( B \) be the Bruhat–Tits building associated with \( G_k \). For a point \( x \in B \), \( G_k^x \) denotes the stabilizer of \( x \) in \( G_k \).

1. For a maximal compact subgroup \( K \) of \( G_k \), there is a point \( x \in B \) such that \( K = G_k^x \).
2. If \( x \in B \) is a point contained in a facet of minimal dimension, then \( G_k^x \) is a maximal compact subgroup of \( G_k \).
3. The number \( m(G_k) \) of maximal compact subgroups of \( G_k \) up to conjugacy is finite.
4. If \( G \) is simply connected, then every maximal compact subgroup of \( G_k \) is the stabilizer of a vertex (= 0-dimensional facet) of \( B \), and \( m(G_k) \) is equal to the number of vertices of a chamber in \( B \). Precisely,

\[
m(G_k) = \prod_{i=1}^{\ell} (\text{rank}_k(G_i) + 1)
\]

where \( G_1, \cdots, G_\ell \) are \( k \)-simple factors of \( G \).

5. \( B \) has special points. The stabilizer of every special point of \( B \) is a maximal compact subgroup, which is called a special maximal compact subgroup. Every special maximal compact subgroup satisfies both Iwasawa and Cartan decompositions.
**Remark** If $G$ is semisimple but not simply connected, then it is possible to happen a case where $x \in B$ is not a vertex but $G^x_k$ is a maximal compact subgroup of $G_k$. For example, in the case of $\text{PGL}_n$, every chamber has $n$ vertices. Stabilizers of vertices are maximal compact subgroups and they are mutually conjugate in $\text{PGL}_n(k)$. However, $m(\text{PGL}_n(k))$ is equal to the number of divisors of $n$.

**Remark** The building $B$ is a union of translations of an apartment $A$ by the action of $G_k$, i.e.,

$$B = \bigcup_{g \in G_k} gA.$$  

Let $\overline{C}$ be a closed chamber in $A$. For a given point $x \in B$, there is a $g \in G_k$ such that $g^{-1}x \in \overline{C}$. Then $G^x_k$ and $G^{g^{-1}x}_k$ are conjugate. To classify conjugacy classes of maximal compact subgroups, it is sufficient to consider only stabilizers of points in $\overline{C}$. 


2 Bruhat–Tits theory of Sp\(_4(k)\)

Let \(k\) be a p-adic field, \(\mathfrak{o}\) the maximal compact subring of \(k\) and \(\mathfrak{p}\) the maximal ideal of \(\mathfrak{o}\).

2.1 \(\text{Sp}\_4\) and its minimal parabolic subgroup

Let \(G = \text{Sp}\_4\) be a symplectic group, i.e.,

\[
G_k = \left\{ g \in \text{GL}_4(k) : t^g \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \right\}.
\]

We fix a maximal split torus \(S\) and a maximal unipotent subgroup \(U\) as follows:

\[
S_k = Z_k = \left\{ h(s, t) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix} : s, t \in k^\times \right\}
\]

\[
U_k = \left\{ \begin{pmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : w, x, y, z \in k \right\}
\]

Then \(P = SU\) is a minimal parabolic subgroup of \(G\) over \(k\).

2.2 Rational characters and cocharacters of \(S\)

Define \(k\)-rational characters \(e_1, e_2 : S \rightarrow \mathbb{G}_m\) by

\[
e_1(h(s, t)) = s, \quad e_2(h(s, t)) = t.
\]

Then \(\{e_1, e_2\}\) is a basis of \(X^*(S) = \text{Hom}_k(S, \mathbb{G}_m)\).

Cocharacters \(e_1^\vee, e_2^\vee : \mathbb{G}_m \rightarrow S\) are defined by

\[
e_1^\vee(s) = h(s, 1), \quad e_2^\vee(s) = h(1, s),
\]

which give the dual basis of \(\{e_1, e_2\}\) in \(X_*(S) = \text{Hom}_k(\mathbb{G}_m, S)\).
2.3 Affine root system and root subgroups

Define \( a, b \in X^\ast(S) \) by
\[
  a = e_1 - e_2, \quad b = 2e_2.
\]

The relative root system \( \Phi \) and the affine root system \( \Phi_{aff} \) of \((G, S)\) over \( k \) are given by
\[
  \Phi = \{ \pm a, \pm b, \pm (a + b), \pm (2a + b) \}, \quad \Phi_{aff} = \Phi \times \mathbb{Z}.
\]

We fix a one-parameter subgroup \( u_c : k \rightarrow G_k \) for each \( c \in \Phi \) such that
\[
h \cdot u_c(x) \cdot h^{-1} = u_c(c(h)x) \text{ for } h \in S_k,
\]
e.g., for positive roots,
\[
u_a(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix}, \quad u_b(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
u_{a+b}(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u_{2a+b}(x) = \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

For each affine root \( \delta = (c, n) \in \Phi_{aff} \), the root subgroup \( X_\delta \) is defined to be
\[
  X_\delta = u_c(p^n).
\]
2.4 Apartment and chambers

The apartment $A$ is an affine space under the real vector space $X_*(S) \otimes \mathbb{Z}$. By $\mathbb{R}$-linear extension of the natural pairing

$$\langle \cdot, \cdot \rangle : X^*(S) \times X^*(S) \to \mathbb{Z}$$

each affine root $\delta = (c, n) \in \Phi_{\text{aff}}$ defines an affine function:

$$\delta : A \to \mathbb{R} : \delta(x) = \langle c, x \rangle + n.$$

The null set $\delta^{-1}(0)$ is an affine hyperplane of $A$. In our case, $\dim A = 2$ and $\delta^{-1}(0)$ is a line of the form:

- $\delta = (a, n) : \delta(x_1e_1^\vee + x_2e_2^\vee) = x_1 - x_2 + n = 0$
- $\delta = (b, n) : 2x_2 + n = 0$
- $\delta = (a + b, n) : x_1 + x_2 + n = 0$
- $\delta = (2a + b, n) : 2x_1 + n = 0$
A connected component $C$ of the set
\[ A - \bigcup_{\delta \in \Phi_{\text{aff}}} \delta^{-1}(0) \]
is called a chamber, which is a polytope.

Define the subset $\Delta_{\text{aff}}(C)$ of $\Phi_{\text{aff}}$ by
\[ \Delta_{\text{aff}}(C) = \{ \delta \in \Phi_{\text{aff}} : \delta/2 \notin \Phi_{\text{aff}} \text{ and } \delta^{-1}(0) \cap \partial C \neq \emptyset \} \, . \]

For example, if $C$ is chosen as follows

then
\[ \Delta_{\text{aff}}(C) = \{ (a, 0), (b, 0), (2a + b, 1) \} \, . \]
$\Delta_{\text{aff}}(C)$ is displayed by the affine Dynkin diagram:

\[
\begin{array}{c}
\circ \\
\circ \overset{2a+b}{\implies} \circ \\
\circ \overset{a}{\iff} \circ \\
\circ \overset{b}{\iff} \circ 
\end{array}
\]

There is the homomorphism \( \nu : S_k \longrightarrow X_*(S) \otimes \mathbb{Z} \mathbb{R} \) so that

\[
h^{-1}X_{(c,n)}h = X_{(c,(c,\nu(h)))+n}
\]

holds for every \((c,n) \in \Phi_{\text{aff}}\) and \(h \in S_k\). Precisely, \(\nu\) is given by

\[
\nu(h(s,t)) = -\text{ord}_p(s)e_1^\nu - \text{ord}_p(t)e_2^\nu.
\]

The kernel of \(\nu\) is the group \(S_\circ\) of \(\circ\) rational points of \(S\). The translation of \(A\) induced by \(\nu(h)\) defines the action of \(S_k\) on \(A\).

Let \(N\) be the normalizer of \(S\) in \(G\). The Weyl group \(W = N/S = N_k/S_k\) of \(\Phi\) acts on \(A\) by reflections as usual. The affine Weyl group \(W_{\text{aff}} = N_k/S_\circ\) is isomorphic with \(S_k/S_\circ \rtimes W\). Both \(N_k\) and \(W_{\text{aff}}\) act on \(A\) by affine transformations.

**Remark** In our case, the closed chamber \(\overline{C} = C \cup \partial C\) of \(A\) is a fundamental domain of \(A/N_k = A/W_{\text{aff}}\). This is not true in general.

For \(\delta \in \Delta_{\text{aff}}(C)\), \(w_\delta\) denotes the orthogonal reflection of \(A\) with respect to the affine hyperplane \(\delta^{-1}(0)\). The set \(W_{\text{aff}}(C) = \{w_\delta\}_{\delta \in \Delta_{\text{aff}}(C)}\) is a subset of \(W_{\text{aff}}\).
2.5 Tits system

For a subset $F \subset C$ and a root $c \in \Phi$, we set

$$n_F(c) = \inf \{ n \in \mathbb{Z} : \langle c, x \rangle + n \geq 0 \text{ for all } x \in F \}.$$

Define unipotent subgroups $X^+_F$ and $X^-_F$ of $G_k$ by

$$X^+_F = \prod_{0 < c \in \Phi} X_{(c, n_F(c))} = \prod_{0 < c \in \Phi} u_c(p^{n_F(c)}),$$

$$X^-_F = \prod_{0 > c \in \Phi} X_{(c, n_F(c))} = \prod_{0 > c \in \Phi} u_c(p^{n_F(c)}).$$

If $F = C$, then the product

$$B_C = X^-_C \cdot S_0 \cdot X^+_C$$

is a subgroup of $G_k$, which is called the Iwahori subgroup of $G_k$ corresponding to $C$. The following is a fundamental result due to Iwahori–Matsumoto.

**Theorem 2.1.** The quadruple $(G_k, B_C, N_k, W_{\text{aff}}(C))$ is a Tits system, i.e., this satisfies

(T1) $B_C \cup N_k$ generates $G_k$ and $B_C \cap N_k = S_0$ is a normal subgroup of $N_k$,

(T2) $W_{\text{aff}}(C)$ generates $W_{\text{aff}}$ and every element in $W_{\text{aff}}(C)$ is of order 2,

(T3) $sB_Cs \neq B_C$ for each $s \in W_{\text{aff}}(C)$,

(T4) $sB_Cw \subset B_CwB_C \cup B_CswB_C$ for each $s \in W_{\text{aff}}(C)$ and $w \in W_{\text{aff}}$.

As a consequence of the theory of Tits systems, we obtain the following double coset decomposition of $G_k$:

$$G_k = \bigsqcup_{w \in W_{\text{aff}}} B_CwB_C \quad \text{(Bruhat decomposition)}$$

For $x \in C$, $W_{\text{aff}}^x$ stands for the stabilizer of $x$ in $W_{\text{aff}}$. Then

$$G^x_k = \bigsqcup_{w \in W_{\text{aff}}^x} B_CwB_C$$

is a subgroup of $G_k$. 


2.6 Building

Since $G_k$ does not act on $A$, we need to build an enlargement of $A$ on which $G_k$ acts. Since $\overline{C}$ is a fundamental domain of $A/W_{\text{aff}}$, the apartment $A$ is identified with the quotient space

$$(W_{\text{aff}} \times \overline{C})/ \sim,$$

where $(w, x) \sim (w', x')$ if $x = x'$ and $w^{-1}w' \in W_{\text{aff}}^x$. We extend the equivalent relation $\sim$ to $G_k \times \overline{C}$ by

$$(g, x) \sim (g', x') \text{ if } x = x' \text{ and } g^{-1}g' \in G_k^x.$$ 

Then the quotient space

$$\mathcal{B} = \mathcal{B}(G_k) = (G_k \times \overline{C})/ \sim$$

gives the building of $G_k$. Let $n_w$ be an arbitrary representative in $N_k$ of $w \in W_{\text{aff}}$. Then, by the map $(w, x) \mapsto (n_w, x)$, the apartment $A = (W_{\text{aff}} \times \overline{C})/ \sim$ is embedded in $\mathcal{B}$. The group $G_k$ acts on $\mathcal{B}$ by $g(h, x) = (gh, x)$.

**Theorem 2.2** (Tits, §2.1). The building $\mathcal{B}$ is uniquely characterized as a $G_k$-set satisfying the following properties:

- $\mathcal{B} = \bigcup_{g \in G_k} gA$,
- $N_k$ stabilizes $A$ and operates on it by the same way defined as in §2.4,
- for any $\delta \in \Phi_{\text{aff}}$, the root subgroup $X_\delta$ fixes $\delta^{-1}([0, \infty))$ pointwise.

**Remark** For every $x \in A$, there is an $n \in N_k$ such that $nx \in \overline{C}$. Then we define $G_k^x$ by $n^{-1}G_k^x n$. Another definition of $\mathcal{B}$ is given by

$$\mathcal{B} = (G_k \times A)/ \sim,$$

where

$$(g, x) \sim (g', x') \text{ if } \exists n \in N_k \text{ such that } x' = nx \text{ and } g^{-1}g'n \in G_k^x.$$ 

This definition does not need to assume that $\overline{C}$ is a fundamental domain of $A/W_{\text{aff}}$. 

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2.7 Stabilizers

For a subset $F \subset B$, the pointwise stabilizer of $F$ in $G_k$ is denoted by $G_k^F$, i.e.,

$$G_k^F = \{ g \in G_k : gx = x \text{ for all } x \in F \}.$$

The structure of $G_k^F$ is determined by

Theorem 2.3 (Bruhat–Tits Proposition 2.4.13, Tits §3.1.1). Let $F \subset B$ be a bounded subset.

1. If $F^\dagger$ denotes the closed convex closure of $F$ in $B$, then $G_k^F = G_k^{F^\dagger}$.
2. If $F \subset C$ and $N_k^F$ denotes the pointwise stabilizer of $F$ in $N_k$, then

$$G_k^F = X_F^- \cdot N_k^F \cdot X_F^+.$$

If $F = \{x\} \subset C$ is a one point, then $G_k^{\{x\}}$ is coincides with $G_k^x$ defined in §2.5, i.e.,

$$G_k^{\{x\}} = G_k^x = B_C \cdot W_{aff} \cdot B_C.$$

We write $v_0, v_1, v_2$ for vertices of a chamber $C$ as follows.

The chamber $C$ has 7 facets:

$$v_0, v_1, v_2, v_0v_1, v_1v_2, v_2v_0, C.$$
The stabilizers of vertices are

\[
G_{v_0}^k = X_{v_0}^- \cdot S_0 \cdot W \cdot X_{v_0}^+ = Sp_4(o)
\]

\[
G_{v_1}^k = \begin{pmatrix} 1 & p & 1 & p \\ p & 1 & 1 & p \\ p & p & 1 & p \\ p & 0 & 1 \end{pmatrix} \cdot S_0 \cdot \begin{pmatrix} 1 & o & p^{-1} & o \\ 1 & o & o & 1 \\ 0 & 1 \end{pmatrix}
\]

\[
G_{v_2}^k = \begin{pmatrix} 1 & o & 1 & p & p & 1 & o & p & p \\ p & 1 & 1 & o & p & p & 1 & o & p \\ p & p & 1 & o & p & p & 1 & o & p \end{pmatrix} \cdot S_0 \cdot \{I_4, w_0\} \cdot \begin{pmatrix} 1 & o & p^{-1} & p^{-1} & p^{-1} & 1 \\ 1 & p^{-1} & 1 & p^{-1} & 1 & p^{-1} \end{pmatrix},
\]

where \(w_0 = w_{(i,0)}\). By Theorem 2.3 (1), we have

\[
G_{v_i}^k \cap G_{v_j}^k = G_{v_0}^k \cap G_{v_1}^k \cap G_{v_2}^k = B_C.
\]

For each facet \(F\) of \(C\), \(G_F^k\) has the double coset decomposition:

\[
G_F^k = \bigsqcup_{w \in W_{aff}^F} B_C w B_C,
\]

where \(W_{aff}^F\) is the subgroup of \(W_{aff}\) generated by \(\{w_\delta : F \subset \delta^{-1}(0)\}\).

By Theorem 1.3 (4), \(G_{v_0}^k\), \(G_{v_1}^k\) and \(G_{v_2}^k\) are maximal compact subgroups of \(G_k\) and they are not conjugate in \(G_k\) each other. Every maximal compact subgroup of \(G_k\) is conjugate to one of \(G_{v_i}^k\)s.
2.8 Special maximal compact subgroups

Let \( x \in \overline{C} \). We define subsets of \( \Phi_{\text{aff}} = \Phi \times \mathbb{Z} \) by

\[
\Phi_{\text{aff}}(x) = \{ \delta \in \Phi_{\text{aff}} : \delta(x) = 0 \}, \quad \Phi(x) = \Phi\text{-part of } \Phi_{\text{aff}}(x)
\]

and

\[
I_x = \{ \delta \in \Delta_{\text{aff}}(C) : \delta(x) \neq 0 \}.
\]

Then \( \Phi(x) \) is a subroot system of \( \Phi \). If \( x \) is a point in the interior of \( C \), then \( \Phi_{\text{aff}}(x) = \emptyset \) and \( I_x = \Delta_{\text{aff}}(C) \).

The point \( x \) is called special if every element of \( \Phi \) is proportional to some element of \( \Phi(x) \).

For example,

- if \( x = v_0 \), then \( \Phi(v_0) = \Phi \) and \( I_{v_0} = \{(2a + b, 1)\} \),
- if \( x = v_1 \), then \( \Phi(v_1) = \{b, 2a + b\} \) and \( I_{v_1} = \{(a, 0)\} \),
- if \( x = v_2 \), then \( \Phi(v_2) = \Phi \) and \( I_{v_2} = \{(b, 0)\} \).

Hence both \( v_0 \) and \( v_2 \) are special, but not \( v_1 \).

The stabilizer of a special point is a special maximal compact subgroup. Both \( G^{v_0}_k \) and \( G^{v_2}_k \) are special maximal compact subgroups, but not \( G^{v_1}_k \).
2.9 \( \mathfrak{o} \)-models of \( G \)

Let \( J : k^4 \times k^4 \rightarrow k \) be the symplectic form defining \( G \), and let \( e_1, e_2, e_1', e_2' \) be the canonical basis. We define \( \mathfrak{o} \)-lattices \( L_0, L_1, L_2 \) as follows:

\[
\begin{align*}
L_0 &= \mathfrak{o}e_1 + \mathfrak{o}e_2 + \mathfrak{o}e_1' + \mathfrak{o}e_2', \\
L_1 &= \mathfrak{o}e_1 + \mathfrak{o}e_2 + \mathfrak{p}e_1' + \mathfrak{o}e_2', \\
L_2 &= \mathfrak{o}e_1 + \mathfrak{o}e_2 + \mathfrak{p}e_1' + \mathfrak{p}e_2'.
\end{align*}
\]

The stabilizer of \( L_i \) in \( G_k \) is \( G_{k_i}^o \) for \( i = 0, 1, 2 \).

Since \( J(L_i, L_i) \subset \mathfrak{o} \), \( (L_i, J) \) gives an \( \mathfrak{o} \)-structure of the symplectic space \( (k^4, J) \), and hence an \( \mathfrak{o} \)-model \( G_i \) of \( G \). We have \( G_i^o = G_{k_i}^o \). One of the main results of Bruhat–Tits theory is:

**Theorem 2.4** (Tits §3.4.1). Let \( F \) be a non-empty bounded subset of \( B \). Then there exists a unique smooth group \( \mathfrak{o} \)-scheme \( G^F \) satisfying

- \( G^F \times_o k = G \),
- \( G^F_{o'} = G_{k'}^o \) for any unramified extension \( k'/k \).

We put \( G_i^f = (G^f \times_o f)_f \), where \( f = o/p \) is the residue field of \( k \). It is easy to see that \( G_i^f = G_i^f = \text{Sp}_4(f) \). We determine \( G_i^f \). Let \( \pi \) be a prime element of \( o \). Since \( J(L_1, L_1) = o \), the bilinear form

\[ \overline{J} : L_1/\pi L_1 \times L_1/\pi L_1 \rightarrow f. \]

over \( f \) is defined from \( J \). For \( x \in L \), \( [x] \) denotes \( x \mod \pi L \). Then \( [e_1], [e_2], [\pi e_1'], [\pi e_2'] \) is a basis of \( L_1/\pi L_1 \) over \( f \). Since the radical \( R_{\overline{J}} \) of \( \overline{J} \) is spanned by \( [e_1], [\pi e_1'] \), the automorphism group of \( (L_1/\pi L_1, \overline{J}) \) is isomorphic with \( M_2(f) \times (GL_2(f) \times SL_2(f)) \). Hence \( G_1^f \cong M_2(f) \times (GL_2(f) \times SL_2(f)) \).
3 Maximal compact subgroups of GSp$_4$(k)

Let GSp$_4$ be the symplectic group of similitude. There is the following exact sequence:

$$1 \longrightarrow \text{Sp}_4 \longrightarrow \text{GSp}_4 \longrightarrow \chi \longrightarrow \text{Gm} \longrightarrow 1$$

The similitude character $\chi$ has a splitting:

$$s \mapsto d(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, GSp$_4$ is isomorphic with Sp$_4 \rtimes d(G_m)$.

3.1 Bruhat–Tits theory of GSp$_4$(k)

- In general, the apartment of a connected reductive group $H$ over $k$ is identified with the apartment of $Z(H) \times [H, H]$, where $Z(H)$ denotes the maximal central $k$-split torus of $H$. Since $Z(\text{GSp}_4)$ is a one-dimensional split torus and $[\text{GSp}_4, \text{GSp}_4] = \text{Sp}_4$, the apartment $\tilde{A}$ of $\text{GSp}_4(k)$ is identified with $R \times A$.

- $T = S \cdot d(G_m)$ is a maximal $k$-split torus of GSp$_4$. The root system of GSp$_4$ with respect to $T$ is the same as $\Phi$. For $\delta \in \Phi_{\text{aff}}$, the affine function $\delta : A \longrightarrow R$ is trivially extended to $\tilde{A}$ by composition with the projection $\tilde{A} \longrightarrow A$. Therefore, $\tilde{C} = R \times C$ is a chamber of $\tilde{A}$.

- For a vertex $v_i$ of $C$, we denote by $\tilde{v}_i$ the one-dimensional facet $R \times v_i$ of $\tilde{C}$. Since $\tilde{v}_i$ is a facet of minimal dimension, its stabilizer $K_i = \text{GSp}_4(k)^{\tilde{v}_i}$ is a maximal compact subgroup of GSp$_4(k)$.

- The homomorphism $\nu : S_k \longrightarrow X_*(S) \otimes Z R$ is extended to $T_k$ by

$$\nu(d(s)) = -\frac{\text{ord}_p(s)}{2}(e'_1 + e'_2).$$

Then $T_k$ acts on $\tilde{A}$ by the translation: $(r, x) \mapsto (r, x + \nu(h))$ for $(r, x) \in \tilde{A}$ and $h \in T_k$. The facet $\tilde{v}_0$ transforms to the facet $\tilde{v}_2$ by the action of $d(\pi^{-1})$. Hence $K_0$ and $K_2$ are conjugate in GSp$_4(k)$.

- $K_0$ is not conjugate to $K_1$ because that $v_0$ is a special point but not $v_1$. 
3.2 Conjugacy of stabilizers of facets

The subset $\Delta_{\text{aff}}(\tilde{C}) = \Delta_{\text{aff}}(C)$ of $\Phi_{\text{aff}}$ is the local Dynkin diagram of $\text{GSp}_4(k)$:

$$\circ \xrightarrow{2a+b} \circ \leftrightarrow \circ \rightleftharpoons \circ$$

The local Dynkin diagram does not depend, up to canonical isomorphism, on the choice of the chamber $\tilde{C}$. The torus $T_k$ acts on the set of chambers in $\tilde{A}$, and hence on $\Delta_{\text{aff}}(\tilde{C})$. We denote by $\Xi(\text{GSp}_4)$, or simply $\Xi$, the image of the homomorphism $T_k \rightarrow \text{Aut}(\Delta_{\text{aff}}(\tilde{C}))$.

**Theorem 3.1** (Tits §2.5). $\Xi$ is isomorphic with $T_k/T_0S_kZ(\text{GSp}_4)_k$, in particular $\Xi = \text{Aut}(\Delta_{\text{aff}}(\tilde{C}))$.

For every facet $\tilde{F}$ of the chamber $\tilde{C}$, we define the subset $I_{\tilde{F}}$ of $\Delta_{\text{aff}}(\tilde{C})$ by

$$I_{\tilde{F}} = \{ \delta \in \Delta_{\text{aff}}(\tilde{C}) : \delta|_{\tilde{F}} \neq 0 \}$$

Obviously, we have $I_{\tilde{v}_i} = I_{v_i}$.

**Theorem 3.2** (Tits §2.5). Let $\tilde{F}_1$ and $\tilde{F}_2$ be facets of $\tilde{C}$. Then $\text{GSp}_4(k)^{\tilde{F}_1}$ and $\text{GSp}_4(k)^{\tilde{F}_2}$ are conjugate in $\text{GSp}_4(k)$ if and only if $I_{\tilde{F}_1}$ and $I_{\tilde{F}_2}$ are in the same orbit of $\Xi$. 

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4 Conclusions

- $G_k^{\nu_0}, G_k^{\nu_1}$ and $G_k^{\nu_2}$ are representatives of conjugacy classes of maximal compact subgroups of $G_k = \text{Sp}_4(k)$.

- Both $G_k^{\nu_0}$ and $G_k^{\nu_2}$ are special maximal compact subgroups of $G_k$. They satisfy both Iwasawa and Cartan decompositions. The reduction mod $p$ of each of them is isomorphic with $\text{Sp}_4(f)$.

- $G_k^{\nu_1}$ is not a special maximal compact subgroup. The reduction mod $p$ of $G_k^{\nu_1}$ is isomorphic with $M_2(f) \times (\text{GL}_2(f) \times \text{SL}_2(f))$.

- $\text{GSp}_4(k)^{\bar{\nu}_0}$ and $\text{GSp}_4(k)^{\bar{\nu}_1}$ are representatives of conjugacy classes of maximal compact subgroups of $\text{GSp}_4(k)$.

- $\text{GSp}_4(k)^{\bar{\nu}_0}$ is a special maximal compact subgroup.

- $\text{GSp}_4(k)^{\bar{\nu}_1}$ is not a special maximal compact subgroup.