FINITENESS OF BOUNDARY CONDITIONS OF SOME FUNDAMENTAL DOMAIN FOR $\mathbf{P}_4/\mathrm{GL}_4(\mathbf{Z})$

TAKAO WATANABE

Abstract. The general linear group over the integers $\Gamma_n = \mathrm{GL}_n(\mathbf{Z})$ acts on the space $\mathbf{P}_n$ of $n \times n$ positive-definite real symmetric matrices. In a previous work, the author constructed a new family of fundamental domains for $\mathbf{P}_n/\Gamma_n$. Although these domains are initially defined by an infinite set of inequalities, a natural question arises as to whether a finite subset of these inequalities is sufficient to determine this domain. In this paper, we give an affirmative answer to this problem for the specific case $n = 4$.

Introduction

Let $\mathrm{M}_{n,k}(\mathbf{R})$ be the set of all $n \times k$ matrices with real entries and $\mathbf{P}_n$ be the set of all $n \times n$ positive-definite real symmetric matrices. The group of invertible elements in the ring $\mathrm{M}_n(\mathbf{Z})$ of $n \times n$ matrices with integer entries is denoted by Γ_n; that is, $\Gamma_n = \mathrm{GL}_n(\mathbf{Z})$. For $A \in \mathbf{P}_n$ and $X \in \mathrm{M}_{n,k}(\mathbf{R})$, we define

$$A[X] := {}^{\mathrm{t}}XAX\,,$$

where ${}^{\mathrm{t}}X$ denotes the transpose matrix of X. The group Γ_n acts on $\mathbf{P}_n$ from the right via the map

$$\mathbf{P}_n \times \Gamma_n \to \mathbf{P}_n \,:\, (A, X) \mapsto A[X]\,.$$

A closed subset $\Omega \subset \mathbf{P}_n$ is called a fundamental domain for $\mathbf{P}_n/\Gamma_n$ if it satisfies the following conditions:

- $\mathbf{P}_n = \bigcup_{X \in \Gamma_n} {}^{\mathrm{t}}X\Omega X$;
- For $A, B \in \Omega$, if $B = A[X]$ for some $X \in \Gamma_n$ with $X \neq \pm E_n$, then $A \in \partial\Omega$.

There are several fundamental domains for $\mathbf{P}_n/\Gamma_n$. The most famous one is Minkowski's fundamental domain. For each $k \in \{1, 2, \ldots, n\}$, we define a subset $\mathbf{Z}_{(n,k)}$ of $\mathbf{Z}^n$ by

$$\mathbf{Z}_{(n,k)} := \{z = {}^{\mathrm{t}}(z_1, \ldots, z_n) \in \mathbf{Z}^n \mid \gcd(z_k, \ldots, z_n) = 1\}\,.$$

Then Minkowski's fundamental domain $\mathbf{M}_n$ is defined as follows:

$$\mathbf{M}_n := \left\{ A = (a_{ij}) \in \mathbf{P}_n \;\middle|\; \begin{array}{ll} \text{(M1)}\ A[z] \geq a_{kk} & (\forall z \in \mathbf{Z}_{(n,k)},\ \forall k \in \{1, 2, \ldots, n\}) \\ \text{(M2)}\ a_{k,k+1} \geq 0 & (\forall k \in \{1, 2, \ldots, n-1\}) \end{array} \right\}.$$

Theorem (Minkowski)**.** $\mathbf{M}_n$ *is a fundamental domain for* $\mathbf{P}_n/\Gamma_n$*. Furthermore, there exists a finite subset* $S_{n,k} \subset \mathbf{Z}_{(n,k)}$ *such that condition* (M1) *can be replaced by*

$$A[z] \geq a_{kk}\ \ (\forall z \in S_{n,k},\ \forall k \in \{1, 2, \ldots, n\})\,.$$

2020 *Mathematics Subject Classification.* Primary 11H55.

Key words and phrases. Reduction Theory.

The latter statement of this theorem is known as Minkowski's second finiteness theorem ([1, Lecture XIII, §1]). In the case $n=2$, it becomes

$$\mathbf{M}_2 = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathbf{P}_2 \;\middle|\; 0 \le 2b \le a \le d \right\}.$$

In [3, Example 3], we constructed a new family of fundamental domains for $\mathbf{P}_n/\Gamma_n$. Fix $k \in \{1,\dots,n-1\}$. When the j-th column of the identity matrix E_n is denoted by $\boldsymbol{e}_j$, we define

$$\Gamma_{n,k} := \{XE_{n,k} \mid X \in \Gamma_n\}, \qquad E_{n,k} := (\boldsymbol{e}_1,\dots,\boldsymbol{e}_k) \in \mathrm{M}_{n,k}(\mathbf{R}).$$

The group $\Gamma_k = \mathrm{GL}_k(\mathbf{Z})$ acts naturally on $\Gamma_{n,k}$ by right multiplication. Let $\Delta_{n,k}$ be a complete system of representatives of the coset space $\Gamma_{n,k}/\Gamma_k$, and assume $E_{n,k} \in \Delta_{n,k}$. Any $A \in \mathbf{P}_n$ can be uniquely expressed by the Jacobi decomposition as

$$A = \begin{pmatrix} E_k & O \\ {}^{\mathrm{t}}u & E_{n-k} \end{pmatrix} \begin{pmatrix} v & O \\ O & w \end{pmatrix} \begin{pmatrix} E_k & u \\ O & E_{n-k} \end{pmatrix}, \qquad u \in \mathrm{M}_{k,n-k}(\mathbf{R}),\ v \in \mathbf{P}_k,\ w \in \mathbf{P}_{n-k}.$$

In this case, we denote

$$u = u_A, \qquad v = A^{[k]}, \qquad w = A_{[n-k]}.$$

For a non-empty subset $S \subset \Gamma_{n,k}$, we define a closed subset $\mathrm{R}(S)$ of $\mathbf{P}_n$ by

$$\mathrm{R}(S) := \{A \in \mathbf{P}_n \mid \det A^{[k]} \le \det(A[X]) \quad (\forall X \in S)\}.$$

In particular, we set $\mathrm{R}_{n,k} = \mathrm{R}(\Delta_{n,k})$. We define a subset $\mathrm{M}_{k,n-k}(\mathbf{R})_{1/2}$ of $\mathrm{M}_{k,n-k}(\mathbf{R})$ by

$$\mathrm{M}_{k,n-k}(\mathbf{R})_{1/2} := \{(u_{ij}) \in \mathrm{M}_{k,n-k}(\mathbf{R}) \mid |u_{ij}| \le 1/2 \quad (\forall i,j), \quad u_{11} \ge 0\}.$$

Then, by defining

$$\mathrm{F}_{n,k} := \left\{ A \in \mathbf{P}_n \mid A^{[k]} \in \mathbf{M}_k,\ A_{[n-k]} \in \mathbf{M}_{n-k},\ u_A \in \mathrm{M}_{k,n-k}(\mathbf{R})_{1/2} \right\},$$

the following holds:

Theorem ([3, Example 3])**.** *$\mathrm{R}_{n,k} \cap \mathrm{F}_{n,k}$ is a fundamental domain for $\mathbf{P}_n/\Gamma_n$.*

There are infinitely many inequalities defining $\mathrm{R}_{n,k}$. The following problem corresponds to Minkowski's second finiteness theorem.

Problem. *Does there exist a finite subset $S \subset \Delta_{n,k}$ such that $\mathrm{R}_{n,k} \cap \mathrm{F}_{n,k} = \mathrm{R}(S) \cap \mathrm{F}_{n,k}$?*

In this paper, we consider this problem in the case $(n,k) = (4,2)$, and prove the following result:

Theorem. *There exists a finite subset $S \subset \Delta_{4,2}$ such that $\mathrm{R}_{4,2} \cap \mathrm{F}_{4,2} = \mathrm{R}(S) \cap \mathrm{F}_{4,2}$.*

1. Inequalities in Minkowski's Fundamental Domain

Let $\mathbf{P}_n^-$ be the set of all $n \times n$ semi-positive definite symmetric matrices. For $A, B \in \mathbf{P}_n^-$, we write $A \le B$ or $B \ge A$ if $B - A \in \mathbf{P}_n^-$. We also write $A < B$ or $B > A$ if $B - A \in \mathbf{P}_n$.

Lemma 1. *Let $A, B \in \mathbf{P}_n^-$.*

(1) *If $A \le B$, then $\det A \le \det B$.*
(2) *If $A \le B$, then $A[U] \le B[U]$ for any $U \in \mathrm{M}_{n,k}(\mathbf{R})$.*
(3) *For any $V, W \in \mathrm{M}_{n,k}(\mathbf{R})$, the inequality $A[V+W] \le 2A[V] + 2A[W]$ holds.*

Proof. (1) If $B \in \mathbf{P}_n$, there exists $P \in \mathrm{GL}_n(\mathbf{R})$ such that $B = {}^{\mathrm{t}}PP$. Since $E_n - {}^{\mathrm{t}}P^{-1}AP^{-1} \in \mathbf{P}_n^-$, all eigenvalues of ${}^{\mathrm{t}}P^{-1}AP^{-1}$ are between 0 and 1 inclusive. Therefore, $\det A/\det B \leq 1$ holds. If $B \notin \mathbf{P}_n$, we can choose a sequence $\{B_m\} \subset \mathbf{P}_n$ such that $A \leq B_m$ and $B_m \to B$ as $m \to \infty$. Since $\det A \leq \det B_m$, taking the limit yields $\det A \leq \det B$.

(2) For any $\boldsymbol{x} \in \mathbf{R}^n$, $A[\boldsymbol{x}] \leq B[\boldsymbol{x}]$ holds. For $\boldsymbol{y} \in \mathbf{R}^k$, setting $\boldsymbol{x} = U\boldsymbol{y}$ gives $A[U\boldsymbol{y}] \leq B[U\boldsymbol{y}]$. Thus, $A[U][\boldsymbol{y}] \leq B[U][\boldsymbol{y}]$ holds for all $\boldsymbol{y} \in \mathbf{R}^k$, which implies $A[U] \leq B[U]$.

(3) follows from $O \leq A[V-W] = 2A[V] + 2A[W] - A[V+W]$. □

Lemma 2. *Let $A = (a_{ij}) \in \mathbf{M}_n$.*

(1) *If we define the constant K_n by*

$$K_n = \left(\frac{4}{\pi}\right)^n \left(\frac{3}{2}\right)^{n(n-1)} \left(\Gamma\left(\frac{n}{2}+1\right)\right)^2 ,$$

then

$$a_{11}a_{22}\cdots a_{nn} \leq K_n \det A$$

holds. If $n = 2$, then $a_{11}a_{22} \leq (4/3)\det A$ holds.

(2) *For all $\boldsymbol{x} = {}^{\mathrm{t}}(x_1, \ldots, x_n) \in \mathbf{R}^n \backslash \{\mathbf{0}\}$,*

$$\frac{n^{1-n}}{K_n} \leq \frac{A[\boldsymbol{x}]}{a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2} \leq n$$

holds.

For the proof, see [1, Lecture XI §4 (14), Lecture XIII §3 (9) and Lemma 1], or [2, Chapter IV Proposition 1].

Lemma 3. *If $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathbf{M}_2$, then $(a/2)E_2 \leq A$ holds.*

Proof. Let λ be the smallest eigenvalue of A. Since $4b^2 \leq a^2$, we have

$$\lambda = \frac{a+d-\sqrt{(a-d)^2+4b^2}}{2} \geq \frac{a+d-\sqrt{d^2-2ad+2a^2}}{2} .$$

Letting $h(t) = a + t - \sqrt{t^2 - 2at + 2a^2}$, we have

$$h'(t) = 1 - \frac{t-a}{\sqrt{(t-a)^2+a^2}} ,$$

which implies $h'(t) > 0$ for $t > a$. Therefore,

$$\lambda \geq \frac{h(a)}{2} = \frac{a}{2}$$

holds. Since all eigenvalues of $A - (a/2)E_2$ are non-negative, we obtain $(a/2)E_2 \leq A$. □

2. Finiteness of Boundary Conditions

The Jacobi decomposition of $A \in \mathbf{P}_4$ is expressed as

$$A = \begin{pmatrix} E_2 & O \\ {}^{\mathrm{t}}U & E_2 \end{pmatrix} \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} E_2 & U \\ O & E_2 \end{pmatrix}, \qquad U \in \mathrm{M}_2(\mathbf{R})\,,\ A_1, A_2 \in \mathbf{P}_2\,. \tag{2.1}$$

If we write $X \in \mathrm{M}_{4,2}(\mathbf{Z})$ as

$$X = i(Y,Z) := \begin{pmatrix} Y \\ Z \end{pmatrix}, \qquad Y, Z \in \mathrm{M}_2(\mathbf{Z})\,,$$

then

$$\det(A[X]) = \det(A_1[Y + UZ] + A_2[Z])\,.$$

Thus, for $A \in \mathbf{P}_4$,

$$A \in \mathrm{R}_{4,2} \iff \det A_1 \le \det(A_1[Y+UZ] + A_2[Z])\,, \quad \forall i(Y,Z) \in \Gamma_{4,2}\,.$$

We choose $X_0 = i(Y_0, Z_0) \in \Gamma_{4,2}$ as

$$Y_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proposition 1. *If $A \in \mathrm{R}(\{X_0\}) \cap \mathrm{F}_{4,2}$, then*

$$A_1[U] \le \frac{64}{9} A_2 \quad \text{and} \quad A_1 \le \frac{64}{9} A_2$$

hold.

Proof. Let $A_1, A_2 \in \mathbf{M}_2$ and $U \in \mathrm{M}_2(\mathbf{R})_{1/2}$ be

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ b_2 & d_2 \end{pmatrix}, \qquad U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

In this case,

$$\det(A[X_0]) = u_3^2 \det A_1 + a_1 a_2\,.$$

From $|u_3| \le 1/2$, we have

$$\det A_1 \le \det(A[X_0]) \le \frac{1}{4} \det A_1 + a_1 a_2\,.$$

Since $3a_1 d_1/4 \le \det A_1$ by Lemma 2(1), this gives

$$d_1 \le \frac{16}{9} a_2\,. \tag{2.2}$$

It follows from Lemma 3 that

$$\frac{a_2}{2} ||\boldsymbol{y}||^2 \le A_2[\boldsymbol{y}] \tag{2.3}$$

for any $\boldsymbol{y} \in \mathbf{R}^2$. Since $U \in \mathrm{M}_2(\mathbf{R})_{1/2}$, the operator norm of U satisfies

$$\max_{\boldsymbol{y} \ne \boldsymbol{0}} \frac{||U\boldsymbol{y}||}{||\boldsymbol{y}||} \le (u_1^2 + u_2^2 + u_3^2 + u_4^2)^{1/2} \le 1\,.$$

By Lemma 2(2) and (2.2), (2.3),

$$A_1[U][\boldsymbol{y}] = A_1[U\boldsymbol{y}] \le 2d_1 ||U\boldsymbol{y}||^2 \le 2d_1 ||\boldsymbol{y}||^2 \le \frac{32}{9} a_2 ||\boldsymbol{y}||^2 \le \frac{64}{9} A_2[\boldsymbol{y}]$$

holds for any $\boldsymbol{y} \in \mathbf{R}^2$. Similarly,

$$A_1[\boldsymbol{y}] \le 2d_1 ||\boldsymbol{y}||^2 \le \frac{32}{9} a_2 ||\boldsymbol{y}||^2 \le \frac{64}{9} A_2[\boldsymbol{y}]$$

holds for any $\boldsymbol{y} \in \mathbf{R}^2$. $\square$

When the Jacobi decomposition of $A \in \mathbf{P}_4$ is given by (2.1), we define

$$A^\circ := \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}.$$

Proposition 2. *Let $A \in \mathrm{R}(\{X_0\}) \cap \mathrm{F}_{4,2}$. Setting $\mu = 137/9$, we have*

$$\mu^{-1} A^\circ \le A \le \mu A^\circ\,.$$

In particular, this implies that

$$\mu^{-2} \det A^\circ[X] \le \det A[X] \le \mu^2 \det A^\circ[X]$$

for all $X \in \mathrm{M}_{4,2}(\mathbf{R})$.

Proof. If we write $\boldsymbol{x} \in \mathbf{R}^4$ as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}, \qquad \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R}^2\,,$$

then

$$A[\boldsymbol{x}] = A_1[\boldsymbol{y} + U\boldsymbol{z}] + A_2[\boldsymbol{z}]\,, \qquad A^\circ[\boldsymbol{x}] = A_1[\boldsymbol{y}] + A_2[\boldsymbol{z}]\,.$$

Here, from Lemma 1(3) and Proposition 1,

$$A_1[\boldsymbol{y} + U\boldsymbol{z}] \le 2A_1[\boldsymbol{y}] + 2A_1[U\boldsymbol{z}] \le 2A_1[\boldsymbol{y}] + \frac{128}{9} A_2[\boldsymbol{z}]\,.$$

Thus,

$$A[\boldsymbol{x}] \le 2A_1[\boldsymbol{y}] + \frac{128}{9} A_2[\boldsymbol{z}] + A_2[\boldsymbol{z}] \le 2A_1[\boldsymbol{y}] + \frac{137}{9} A_2[\boldsymbol{z}] \le \frac{137}{9} A^\circ[\boldsymbol{x}]$$

holds for any $\boldsymbol{x} \in \mathbf{R}^4$, and hence

$$A \le \frac{137}{9} A^\circ\,.$$

From Lemma 1 (1) and (2),

$$\det A[X] \le \left(\frac{137}{9}\right)^2 \det A^\circ[X]$$

holds for any $X \in \mathrm{M}_{4,2}(\mathbf{R})$. Also, from Lemma 1(3),

$$A_1[\boldsymbol{y}] = A_1[(\boldsymbol{y} + U\boldsymbol{z}) - U\boldsymbol{z}] \le 2A_1[\boldsymbol{y} + U\boldsymbol{z}] + 2A_1[U\boldsymbol{z}]\,.$$

Combining this with Proposition 1 gives

$$\begin{aligned} A^\circ[\boldsymbol{x}] = A_1[\boldsymbol{y}] + A_2[\boldsymbol{z}] &\le 2A_1[\boldsymbol{y} + U\boldsymbol{z}] + 2A_1[U\boldsymbol{z}] + A_2[\boldsymbol{z}] \\ &\le 2A_1[\boldsymbol{y} + U\boldsymbol{z}] + \frac{137}{9} A_2[\boldsymbol{z}] \\ &\le \frac{137}{9} A[\boldsymbol{x}] \end{aligned}$$

for any $\boldsymbol{x} \in \mathbf{R}^4$, which implies

$$A^\circ \le \frac{137}{9} A\,.$$

$\square$

Letting $\mu = 137/9$ and $\nu = 64/9$, we define

$$\rho := \mu\sqrt{1+\nu+\nu^2} = \frac{959\sqrt{97}}{81} = 116.606\cdots .$$

We consider the standard inner product with respect to the basis $\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j \mid 1 \leq i < j \leq 4\}$ of $\bigwedge^2 \mathbf{R}^4$. Let B_ρ be the ball of radius ρ centered at the origin in $\bigwedge^2 \mathbf{R}^4$. If we define a subset S_0 of $\Delta_{4,2}$ by

$$S_0 := \{X \in \Delta_{4,2} \mid \text{there exists } A \in \mathrm{R}(\{X_0\}) \cap \mathrm{F}_{4,2} \text{ such that } \det A[X] < \det A_1\},$$

then the following holds:

Proposition 3. *For $X = (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \Gamma_{4,2}$, let $w_X = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \in \bigwedge^2 \mathbf{Z}^4$. If $X \in S_0$, then $w_X \in B_\rho$.*

Proof. Let $X \in S_0$. By definition, there exists some $A \in \mathrm{R}(\{X_0\}) \cap \mathrm{F}_{4,2}$ such that

$$\det A[X] < \det A_1 .$$

From Proposition 2,

$$\mu^{-2} \det A^\circ[X] < \det A_1 \tag{2.4}$$

holds. Assuming A has the Jacobi decomposition (2.1), we use the same notation as in the proof of Proposition 1. Writing $X = i(Y, Z)$, we have

$$A^\circ[X] = A_1[Y] + A_2[Z] .$$

Since $A_2[Z] \leq A^\circ[X]$, Lemma 1(1) implies

$$\det A_2 \cdot (\det Z)^2 = \det A_2[Z] \leq \det A^\circ[X] .$$

By (2.4),

$$\det A_2 \cdot (\det Z)^2 < \mu^2 \det A_1 .$$

Since $\det A_1 \leq \nu^2 \det A_2$ by Proposition 1, we have

$$\det A_2 \cdot (\det Z)^2 < \mu^2 \nu^2 \det A_2 .$$

Thus,

$$(\det Z)^2 < \mu^2 \nu^2 \tag{2.5}$$

holds. Similarly, since $A_1[Y] \leq A^\circ[X]$, Lemma 1(1) yields

$$\det A_1 \cdot (\det Y)^2 = \det A_1[Y] \leq \det A^\circ[X] < \mu^2 \det A_1 .$$

Hence,

$$(\det Y)^2 < \mu^2 \tag{2.6}$$

holds. From Lemma 3, we have $A_1 \geq (a_1/2)E_2$ and $A_2 \geq (a_2/2)E_2$, so

$$A^\circ[X] = A_1[Y] + A_2[Z] \geq \frac{a_1}{2}{}^{\mathrm{t}}YY + \frac{a_2}{2}{}^{\mathrm{t}}ZZ .$$

Taking the determinants of both sides, Lemma 1(1) yields

$$\det A^\circ[X] \geq \det\left(\frac{a_1}{2}{}^{\mathrm{t}}YY + \frac{a_2}{2}{}^{\mathrm{t}}ZZ\right) .$$

Here, if we write $Y = (\boldsymbol{y}_1, \boldsymbol{y}_2) = (y_{ij})$ and $Z = (\boldsymbol{z}_1, \boldsymbol{z}_2) = (z_{ij})$, then

$$\begin{aligned}
&\det\left(\frac{a_1}{2}{}^{\mathrm{t}}YY + \frac{a_2}{2}{}^{\mathrm{t}}ZZ\right)\\
&= \left(\frac{a_1}{2}||\boldsymbol{y}_1||^2 + \frac{a_2}{2}||\boldsymbol{z}_1||^2\right)\left(\frac{a_1}{2}||\boldsymbol{y}_2||^2 + \frac{a_2}{2}||\boldsymbol{z}_2||^2\right) - \left(\frac{a_1}{2}\boldsymbol{y}_1\cdot\boldsymbol{y}_2 + \frac{a_2}{2}\boldsymbol{z}_1\cdot\boldsymbol{z}_2\right)^2\\
&= \frac{a_1^2}{4}\left(||\boldsymbol{y}_1||^2||\boldsymbol{y}_2||^2 - (\boldsymbol{y}_1\cdot\boldsymbol{y}_2)^2\right) + \frac{a_2^2}{4}\left(||\boldsymbol{z}_1||^2||\boldsymbol{z}_2||^2 - (\boldsymbol{z}_1\cdot\boldsymbol{z}_2)^2\right)\\
&\quad + \frac{a_1a_2}{4}\left(||\boldsymbol{y}_1||^2||\boldsymbol{z}_2||^2 + ||\boldsymbol{y}_2||^2||\boldsymbol{z}_1||^2 - 2(\boldsymbol{y}_1\cdot\boldsymbol{y}_2)(\boldsymbol{z}_1\cdot\boldsymbol{z}_2)\right)\\
&= \frac{a_1^2}{4}(\det Y)^2 + \frac{a_2^2}{4}(\det Z)^2 + \frac{a_1a_2}{4}\sum_{i,j=1}^{2}(y_{i1}z_{j2} - y_{i2}z_{j1})^2\,.
\end{aligned}$$

On the other hand, using (2.2), (2.4), and $\det A_1 \le a_1d_1$ from $A_1 \in \mathbf{M}_2$, we have

$$\det A^\circ[X] < \mu^2 \det A_1 \le \mu^2 a_1 d_1 \le \frac{16}{9}\mu^2 a_1 a_2\,.$$

Therefore,

$$\frac{a_1^2}{4}(\det Y)^2 + \frac{a_2^2}{4}(\det Z)^2 + \frac{a_1a_2}{4}\sum_{i,j=1}^{2}(y_{i1}z_{j2} - y_{i2}z_{j1})^2 < \frac{16}{9}\mu^2 a_1a_2\,.$$

From this, we obtain

$$\sum_{i,j=1}^{2}(y_{i1}z_{j2} - y_{i2}z_{j1})^2 < \frac{64}{9}\mu^2 = \mu^2\nu\,. \tag{2.7}$$

Since the wedge product corresponding to $X = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ is

$$w_X = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 = \begin{pmatrix}\boldsymbol{y}_1\\ \boldsymbol{z}_1\end{pmatrix} \wedge \begin{pmatrix}\boldsymbol{y}_2\\ \boldsymbol{z}_2\end{pmatrix} \in \bigwedge^2 \mathbf{Z}^4\,,$$

we have from (2.5), (2.6), and (2.7) that

$$||w_X||^2 = (\det Y)^2 + (\det Z)^2 + \sum_{i,j=1}^{2}(y_{i1}z_{j2} - y_{i2}z_{j1})^2 < \mu^2 + \mu^2\nu^2 + \mu^2\nu\,.$$

This shows that $w_X \in B_\rho$ when $X \in S_0$. □

We prove the main theorem. Since $\bigwedge^2 \mathbf{Z}^4$ is a lattice in $\bigwedge^2 \mathbf{R}^4$, the intersection $\bigwedge^2 \mathbf{Z}^4 \cap B_\rho$ is a finite set. Since the map

$$\Gamma_{4,2}/\Gamma_2 \longrightarrow \bigwedge^2 \mathbf{Z}^4/\{\pm 1\} \;:\; X\Gamma_2 \mapsto \pm w_X$$

is injective, the set

$$\{X \in \Delta_{4,2} \mid w_X \in B_\rho\}$$

is finite. Thus, S_0 is a finite set. If we set $S = S_0 \cup \{X_0\}$, then $\mathrm{R}_{4,2} \cap \mathrm{F}_{4,2} = \mathrm{R}(S) \cap \mathrm{F}_{4,2}$ holds.

References

[1] C. L. Siegel, Lectures on the Geometry of Numbers, Notes by B. Friedman, Rewritten by Komaravolu Chandrasekharan with the Assistance of Rudolf Suter, Springer-Verlag, 1989.
[2] A. Terras, Harmonic Analysis on Symmetric Spaces and Applications II, Springer-Verlag, 1988.
[3] T. Watanabe, Ryshkov domains of reductive algebraic groups, Pacific J. Math. 270 (2014) 237 - 255.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan

Email address: twatanabe@math.sci.osaka-u.ac.jp