A NOTE ON LIE ALGEBRAS OF TYPE E_6 OVER ALGEBRAIC NUMBER FIELDS

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Abstract. Let $\mathfrak{g}$ be a Lie algebra of type E_6, C a Cayley algebra and J a cubic Jordan algebra of type A defined over an algebraic number field F. Ferrar proved that Tits' construction of exceptional Lie algebras yields a surjection ξ from $H^1(F, \mathrm{Aut}\, C) \times H^1(F, \mathrm{Aut}\, J)$ onto $H^1(F, \mathrm{Aut}\, \mathfrak{g})$. In this paper, we study the fiber of ξ and compute the class number of a given F-form of $\mathfrak{g}$.

Key words: Lie algebra of type E_6, Tits' construction, Galois cohomology.

Introduction. Let $\mathfrak{g}(F)$ be a split Lie algebra of type E_6 over an algebraic number field F. In a paper [1, II], Ferrar proved that any F-form of $\mathfrak{g}(F)$ is isomorphic to the Tits algebra $T(C_a(F), J_b(F))$ constructed from a suitable pair of a Cayley algebra $C_a(F)$ and a cubic Jordan algebra $J_b(F)$ of type A over F. The Cayley algebra $C_a(F)$ is obtained from the twist of a split Cayley algebra $C(F)$ by a 1-cocycle $a \in Z^1(\Gamma, \mathrm{Aut}\, C(\overline{F}))$ of the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Similarly, the Jordan algebra $J_b(F)$ is obtained from the twist of a split Jordan algebra $J(F) = M_3(F)^+$ by a 1-cocycle $b \in Z^1(\Gamma, \mathrm{Aut}\, J(\overline{F}))$. In terms of Galois cohomology, Ferrar's theorem is stated as the Tits construction gives rise to a surjection ξ from $H^1(F, \mathrm{Aut}\, C(\overline{F})) \times H^1(F, \mathrm{Aut}\, J(\overline{F}))$ onto $H^1(F, \mathrm{Aut}\, \mathfrak{g}(\overline{F}))$. In general, this ξ is not injective and, by [1, II, Proposition(7.1)], one knows only that $T(C_a(F), J_b(F)) \cong T(C_{a'}(F), J_{b'}(F))$ implies $C_a(F) \cong C_{a'}(F)$. A purpose of this paper is to give a refinement of this isomorphism condition. We will give, in Theorem 2 below, a necessary and sufficient condition for $T(C_a(F), J_b(F)) \cong T(C_{a'}(F), J_{b'}(F))$.

One reason of complication of a classification of F-forms of $\mathfrak{g}(F)$ is a failure of the Hasse principle of $H^1(F, \mathrm{Aut}\, \mathfrak{g}(\overline{F}))$. As was shown in [1, II, Theorem(5.1)], the Hasse principle holds only for $H^1(F, \mathrm{Int}\, \mathfrak{g}(\overline{F}))$. A failure of the Hasse principle allows us to define the class number $h(\mathfrak{g}_z(F))$ of a given F-form $\mathfrak{g}_z(F)$. We denote by $\Lambda(\mathfrak{g}_z(F))$ the set of F-isomorphism classes $[\mathfrak{g}_{z'}(F)]$ such that $\mathfrak{g}_z(F_v) \cong \mathfrak{g}_{z'}(F_v)$ for all places v of F. Then $h(\mathfrak{g}_z(F))$

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is defined to be the cardinal number of $\Lambda(\mathfrak{g}_z(F))$. Since the same situation occurs for F-forms of $J(F)$, the set $\Lambda(J_b(F))$ and the class number $h(J_b(F))$ are similarly defined. If a Cayley algebra $C_a(F)$ is fixed, one has two sets $\Lambda(J_b(F))$ and $\Lambda(T(C_a(F), J_b(F)))$. Then we will prove that the mapping $[J_{b'}(F)] \mapsto [T(C_a(F), J_{b'}(F))]$ yields a bijection from $\Lambda(J_b(F))$ to $\Lambda(T(C_a(F), J_b(F)))$. Therefore, one understands that a failure of the Hasse principle for forms of E_6 is "heredity" of that for forms of $M_3(F)^+$. The class number $h(J_b(F)) = h(T(C_a(F), J_b(F)))$ will be computed in Section 1.

Notation. For a given field F, its separable algebraic closure is denoted by $\overline{F}$ and the Galois group of $\overline{F}/F$ is denoted by Γ. If $A(F)$ is an F-algebra and E/F is an extension, then $A(E)$ stands for $A(F) \otimes_F E$. If $E = \overline{F}$, we will simply write A for $A(\overline{F})$. The group of the n-th root of unity in $\overline{F}$ is denoted by μ_n. This is regarded as an algebraic group defined over F. For a finite separable extension E/F, $R_{E/F}(\mu_n)$ stands for the restriction of scalars of μ_n/E to F. The kernel of the norm homomorphism from $R_{E/F}(\mu_n)$ to μ_n is denoted by $R^{(1)}_{E/F}(\mu_n)$. If $E = F$, $R^{(1)}_{E/F}(\mu_n)$ is equal to μ_n. For a finite set X, the cardinal number of X is denoted by $|X|$.

1. Class numbers of cubic Jordan algebras of type A.

Let $J(F)$ be the Jordan algebra consisting of 3 by 3 matrices with entries in a field F of characteristic 0. As usual, the Jordan algebra product of $J(F)$ is given by $x * y = (xy + yx)/2$ for $x, y \in J(F)$. The automorphism group $\operatorname{Aut} J$ is isomorphic to a semi-direct product of the group $PGL_3(\overline{F})$ of inner automorphisms and the cyclic group Θ generated by the transpose $\theta\colon x \mapsto {}^t x$. The F-forms of $J(F)$ is classified by the Galois cohomology set $H^1(F, \operatorname{Aut} J)$ as follows. For a given 1-cocycle $z \in Z^1(\Gamma, \operatorname{Aut} J)$, set $J_z(F) = \{x \in J\colon z(\gamma)\gamma x = x \ \text{ for all } \gamma \in \Gamma\}$. Then $J_z(F)$ is an F-form of $J(F)$ and its F-isomorphism class $[J_z(F)]$ depends only on the cohomology class $[z]$. The map $[z] \mapsto [J_z(F)]$ yields a bijection from $H^1(F, \operatorname{Aut} J)$ to the set of F-isomorphism classes of F-forms of $J(F)$. By the exact sequence

$$H^1(F, PGL_3(\overline{F})) \longrightarrow H^1(F, \operatorname{Aut} J) \xrightarrow{\ \varepsilon\ } H^1(F, \Theta) = \operatorname{Hom}(\Gamma, \Theta)\,,$$

$H^1(F, \operatorname{Aut} J)$ decomposes into a disjoint union of $\varepsilon^{-1}(\alpha)$, $\alpha \in \operatorname{Hom}(\Gamma, \Theta)$. We choose a splitting

$$H^1(F, \Theta) \longrightarrow H^1(F, \operatorname{Aut} J)\colon \alpha \mapsto [z_\alpha]$$

of ε as

$$z_\alpha(\gamma) = \begin{cases} \text{the identity} & (\gamma \in \operatorname{Ker} \alpha) \\ \operatorname{int}(w) \circ \theta & (\gamma \notin \operatorname{Ker} \alpha) \end{cases} \qquad \text{where } w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We write simply $J_\alpha(F)$ for $J_{z_\alpha}(F)$. Let E_α be the invariant field of $\operatorname{Ker}\alpha$ in $\overline{F}$, so that the degree of the extension E_α/F is of at most two. If α is non-trivial and $x \mapsto \overline{x}$ denotes the Galois involution of E_α/F, then $J_\alpha(F)$ is the reduced Freudenthal algebra $\{x \in J(E_\alpha) \colon w^t\overline{x}w^{-1} = x\}$. The automorphism group $\operatorname{Aut} J_\alpha$ is an algebraic group defined over F. Its identity component $(\operatorname{Aut} J_\alpha)^0$ is a projective linear group or a projective quasi-split unitary group of degree 3 according as α is trivial or non-trivial. The action $\operatorname{int}(g) \mapsto \theta\circ\operatorname{int}(g)\circ\theta^{-1} = \operatorname{int}({}^tg^{-1})$ on $(\operatorname{Aut} J_\alpha)^0$ induces the action of Θ on $H^1(F, (\operatorname{Aut} J_\alpha)^0)$. Then, by the diagram

$$\begin{array}{ccccc} & & H^1(F, \operatorname{Aut} J) & \xrightarrow{\ \varepsilon\ } & H^1(F, \Theta) \\ & & \downarrow & & \downarrow \\ H^1(F, (\operatorname{Aut} J_\alpha)^0) & \xrightarrow{\ i_\alpha\ } & H^1(F, \operatorname{Aut} J_\alpha) & \xrightarrow{\ \varepsilon_\alpha\ } & H^1(F, \Theta)\,, \end{array}$$

where vertical arrows are those natural bijections which transform $[z_\alpha]$ and α to trivial classes, one has

$$\varepsilon^{-1}(\alpha) \cong \operatorname{Ker} \epsilon_\alpha = \operatorname{Im} i_\alpha \cong \Theta\backslash H^1(F, (\operatorname{Aut} J_\alpha)^0)\,,$$

and hence

$$H^1(F, \operatorname{Aut} J) = \bigsqcup_{\alpha\in\operatorname{Hom}(\Gamma,\Theta)} \varepsilon^{-1}(\alpha) \cong \bigsqcup_{\alpha\in\operatorname{Hom}(\Gamma,\Theta)} \Theta\backslash H^1(F, (\operatorname{Aut} J_\alpha)^0)\,. \tag{1}$$

Let $J_{\alpha,c}(F)$ be the twist of $J_\alpha(F)$ by a 1-cocycle $c \in Z^1(\Gamma, (\operatorname{Aut} J_\alpha)^0)$. It follows from (1) that $J_{\alpha,c}(F) \cong J_{\alpha,c'}(F)$ if and only if $\Theta[c] = \Theta[c']$. Since the center of the universal covering group of $(\operatorname{Aut} J_\alpha)^0$ is $R^{(1)}_{E_\alpha/F}(\mu_3)$, there is a connection morphism from $H^1(F, (\operatorname{Aut} J_\alpha)^0)$ to $H^2(F, R^{(1)}_{E_\alpha/F}(\mu_3))$. The action of θ on $R^{(1)}_{E_\alpha/F}(\mu_3)$ is given by $\zeta \mapsto \zeta^{-1}$. This action and the connection morphism induces the morphism

$$\Theta\backslash H^1(F, (\operatorname{Aut} J_\alpha)^0) \longrightarrow \Theta\backslash H^2(F, R^{(1)}_{E_\alpha/F}(\mu_3))\,.$$

If F is a nonarchimedean local field, this morphism is bijective ([4, Corollary to Theorem 6.20]). For simplicity, we denote by $\widehat{H}^1(F, (\operatorname{Aut} J_\alpha)^0)$ and by $\widehat{H}^2(F, R^{(1)}_{E_\alpha/F}(\mu_3))$ the orbit spaces $\Theta\backslash H^1(F, (\operatorname{Aut} J_\alpha)^0)$ and $\Theta\backslash H^2(F, R^{(1)}_{E_\alpha/F}(\mu_3))$, respectively.

In the following, let F be an algebraic number field, V_f the set of all finite places and $V_{\infty,1}$ the set of all real places of F. If $v \in V_f$, there is the canonical isomorphism

$$\operatorname{inv}_v \colon H^2(F_v, \mu_3) \longrightarrow \frac{1}{3}\mathbb{Z}/\mathbb{Z}.$$

We fix an $\alpha \in \mathrm{Hom}(\Gamma, \Theta)$ and denote by $V_f^s(E_\alpha)$ (resp. $V_{\infty,1}^r(E_\alpha)$) the subset consisting of $v \in V_f$ (resp. $v \in V_{\infty,1}$) such that v splits (resp. does not split) in E_α. If $E_\alpha = F$, we regard $V_f^s(E_\alpha)$ (resp. $V_{\infty,1}^r(E_\alpha)$) as V_f (resp. the empty set) for convenience. By the Tate–Poitou duality, one has the isomorphism

$$H^2(F, R^{(1)}_{E_\alpha/F}(\mu_3)) \cong \begin{cases} \{(\beta_v) \in \prod_{v\in V_f} H^2(F_v, \mu_3) \colon \sum_{v\in V_f} \mathrm{inv}_v(\beta_v) = 0\} & (\alpha \equiv 1) \\ \prod_{v\in V_f^s(E_\alpha)} H^2(F_v, \mu_3) & (\alpha \not\equiv 1) \end{cases}$$

(cf. [4, Lemma 6.19]). We write $\mathcal{H}_\alpha$ for the group of the right hand side. Furthermore, we set

$$\widehat{\mathcal{H}}'_\alpha = \{\mathcal{O} \in \prod_{v\in V_f^s(E_\alpha)} \widehat{H}^2(F_v, \mu_3) \colon \mathcal{O}^{(1)} \neq \emptyset\}, \text{ where } \mathcal{O}^{(1)} = \left(\prod_{v\in V_f^s(E_\alpha)} \mathcal{O}_v\right) \cap \mathcal{H}_\alpha. \quad (2)$$

Note that $\widehat{\mathcal{H}}'_\alpha$ is distinct from $\widehat{\mathcal{H}}_\alpha = \Theta\backslash\mathcal{H}_\alpha \cong \widehat{H}^2(F, R^{(1)}_{E_\alpha/F}(\mu_3))$ and there is a natural quotient map $\widehat{\mathcal{H}}_\alpha \to \widehat{\mathcal{H}}'_\alpha$. The connection morphism and the Hasse map gives the following diagram.

$$\begin{array}{ccc} H^1(F, (\mathrm{Aut}\, J_\alpha)^0) & \longrightarrow & H^2(F, R^{(1)}_{E_\alpha/F}(\mu_3)) \cong \mathcal{H}_\alpha \\ \downarrow & & \\ \prod_{v\in V_{\infty,1}^r(E_\alpha)} H^1(F_v, (\mathrm{Aut}\, J_\alpha)^0) & & \end{array}$$

By the Hasse principle (cf. [5, Corollaire 4.5], [7]), one obtains the bijection

$$\lambda \colon H^1(F, (\mathrm{Aut}\, J_\alpha)^0) \xrightarrow{\cong} \mathcal{H}_\alpha \times \prod_{v\in V_{\infty,1}^r(E_\alpha)} H^1(F_v, (\mathrm{Aut}\, J_\alpha)^0). \quad (3)$$

This and the triviality of the action of Θ on $H^1(F_v, (\mathrm{Aut}\, J_\alpha)^0)$ for $v \in V_{\infty,1}^r(E_\alpha)$ give the surjection

$$\widehat{\lambda} \colon \widehat{H}^1(F, (\mathrm{Aut}\, J_\alpha)^0) \longrightarrow \widehat{\mathcal{H}}'_\alpha \times \prod_{v\in V_{\infty,1}^r(E_\alpha)} H^1(F_v, (\mathrm{Aut}\, J_\alpha)^0). \quad (4)$$

We fix a $[c] \in H^1(F, (\mathrm{Aut}\, J_\alpha)^0)$. The set $\Lambda(J_{\alpha,c}(F))$ of F-isomorphism classes in the genus of $J_{\alpha,c}(F)$ is defined by

$$\{[J_{\alpha,c'}(F)] \colon [c'] \in H^1(F, (\mathrm{Aut}\, J_\alpha)^0) \text{ and } J_{\alpha,c}(F_v) \cong J_{\alpha,c'}(F_v) \text{ for all } v \in V_f \cup V_{\infty,1}\}.$$

In the rest of this section, we compute the class number $h(J_{\alpha,c}(F)) = |\Lambda(J_{\alpha,c}(F))|$.

Let $\lambda_v([c])$ be the v-component of $\lambda([c])$ and $\widehat{\lambda}(\Theta[c])_f$ the $\widehat{H}'_\alpha$-component of $\widehat{\lambda}(\Theta[c])$. Define $\widehat{\lambda}(\Theta[c])_f^{(1)}$ as in (2). By (3) and (4), the correspondence $\Theta[c'] \mapsto [J_{\alpha,c'}(F)]$ gives a bijection

$$\widehat{\lambda}^{-1}(\widehat{\lambda}(\Theta[c])) = \Theta\backslash\lambda^{-1}(\widehat{\lambda}(\Theta[c])_f^{(1)} \times (\lambda_v[c])_{v\in V^r_{\infty,1}(E_\alpha)}) \xrightarrow{\cong} \Lambda(J_{\alpha,c}(F))\,.$$

We simply write $\mathrm{inv}_v(c)$ for the Hasse invariant of $\lambda_v([c])$. Let $V_f^s(E_\alpha)_c$ be the set of all $v \in V_f^s(E_\alpha)$ such that $\mathrm{inv}_v(c)$ is not zero. The set $\mathrm{Map}(V_f^s(E_\alpha)_c, \{\pm1\})$ of all mappings from $V_f^s(E_\alpha)_c$ to $\{\pm1\}$ is regarded as a finite abelian group by the product $(\sigma\tau)(v) = \sigma(v)\tau(v)$ for $\sigma, \tau \in \mathrm{Map}(V_f^s(E_\alpha)_c, \{\pm1\})$ and $v \in V_f^s(E_\alpha)_c$. If $V_f^s(E_\alpha)_c$ is non-empty, then we define the subset M_c^α of $\mathrm{Map}(V_f^s(E_\alpha)_c, \{\pm1\})$ as follows:

$$M_c^\alpha = \begin{cases} \{\sigma \in \mathrm{Map}(V_f^s(E_\alpha)_c, \{\pm1\})\colon \sum_{v\in V_f^s(E_\alpha)_c} \sigma(v)\mathrm{inv}_v(c) = 0\} & (\alpha \equiv 1) \\ \mathrm{Map}(V_f^s(E_\alpha)_c, \{\pm1\}) & (\alpha \not\equiv 1) \end{cases}$$

In the case of $V_f^s(E_\alpha)_c = \emptyset$, we set $M_c^\alpha = \{0\}$. For $\sigma \in M_c^\alpha$, we define the cohomology class $\sigma[c]$ by

$$\sigma[c] = \lambda^{-1}((\mathrm{inv}_v^{-1}(\sigma(v)\mathrm{inv}_v(c)))_{v\in V_f^s(E_\alpha)} \times (\lambda_v[c])_{v\in V^r_{\infty,1}(E_\alpha)})\,.$$

Then the mapping $\sigma \mapsto \sigma[c]$ yields a bijection from M_c^α to $\lambda^{-1}(\widehat{\lambda}(\Theta[c])_f^{(1)} \times (\lambda_v[c])_{v\in V^r_{\infty,1}(E_\alpha)})$. Since $\theta(\sigma[c]) = \sigma(\theta[c]) = (-\sigma)[c]$, one has

$$\{\pm1\}\backslash M_c^\alpha \cong \Theta\backslash\lambda^{-1}(\widehat{\lambda}(\Theta[c])_f^{(1)} \times (\lambda_v[c])_{v\in V^r_{\infty,1}(E_\alpha)}) \cong \Lambda(J_{\alpha,c}(F))\,.$$

Proposition. *The class number $h(J_{\alpha,c}(F))$ is equal to*

$$|\{\pm1\}\backslash M_c^\alpha| = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{j=0}^{q}\sum_{k=-[j/3]}^{[(p-j)/3]} \binom{q}{j}\binom{p}{3k+j} & (\alpha \equiv 1 \textit{ and } [c] \neq 0) \\ 2^{p+q-1} & (\alpha \not\equiv 1 \textit{ and } [c] \neq 0) \\ 1 & ([c] = 0) \end{cases} \tag{5}$$

Here we set

$$V_f^s(E_\alpha)_c^\pm = \{v \in V_f^s(E_\alpha)_c \colon \mathrm{inv}_v(c) = \pm\frac{1}{3} + \mathbb{Z}\}$$

and $p = \max(|V_f^s(E_\alpha)_c^+|, |V_f^s(E_\alpha)_c^-|)$, $q = \min(|V_f^s(E_\alpha)_c^+|, |V_f^s(E_\alpha)_c^-|)$. For a real number r, $[r]$ denotes the largest integer which is less than or equal to r.

Proof. The only non-trivial case is the one of $\alpha \equiv 1$ and $[c] \neq 0$. We may assume $p = |V_f^s(E_\alpha)_c^+|$ by replacing $[c]$ with $\theta[c]$ if necessary. From Hasse's product formula, $p - q \in 3\mathbb{Z}$ follows. For $\sigma \in M_c^\alpha$, if we set

$$i = |\sigma^{-1}(-1) \cap V_f^s(E_\alpha)_c^+|, \qquad j = |\sigma^{-1}(-1) \cap V_f^s(E_\alpha)_c^-|,$$

then $k = (i-j)/3$ must be an integer by the definition of M_c^α. Therefore, $|M_c^\alpha|$ is equal to the number of subsets $X \times Y$ of $V_f^s(E_\alpha)_c^+ \times V_f^s(E_\alpha)_c^-$ such that $(|X| - |Y|)/3$ is an integer, $0 \leq |Y| \leq q$ and $-[|Y|/3] \leq (|X| - |Y|)/3 \leq [(p - |Y|)/3]$. This is given by

$$\sum_{j=0}^{q} \sum_{k=-[j/3]}^{[(p-j)/3]} \binom{q}{j} \binom{p}{3k+j}. \qquad \square$$

2. Class numbers of Lie algebras of type E_6.

Let $\mathfrak{g}(F)$ be a split Lie algebra of type E_6 over a field F of characteristic 0. The automorphism group $\mathrm{Aut}\,\mathfrak{g}$ is a semidirect product of its connected component $(\mathrm{Aut}\,\mathfrak{g})^0$ and the group Θ' generated by the opposition involution of the Dynkin diagram of $\mathfrak{g}$. Since Θ' is isomorphic to Θ, we will identify Θ' with Θ in the following. Similarly as in Section 1, the F-isomorphism classes of F-forms of $\mathfrak{g}(F)$ is classified by the set $H^1(F, \mathrm{Aut}\,\mathfrak{g})$ and we have the exact sequence

$$H^1(F, (\mathrm{Aut}\,\mathfrak{g})^0) \longrightarrow H^1(F, \mathrm{Aut}\,\mathfrak{g}) \xrightarrow{\varepsilon'} H^1(F, \Theta) = \mathrm{Hom}(\Gamma, \Theta).$$

For $\alpha \in \mathrm{Hom}(\Gamma, \Theta)$, let $\mathfrak{g}_\alpha(F)$ be a quasi-split F-form of $\mathfrak{g}(F)$ corresponding to α and $(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0$ the identity connected component of the automorphism group of $\mathfrak{g}_\alpha$. By the same way as (1), $H^1(F, \mathrm{Aut}\,\mathfrak{g})$ decomposes into a disjoint union of $\widehat{H}^1(F, (\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) = \Theta \backslash H^1(F, (\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)$, $(\alpha \in \mathrm{Hom}(\Gamma, \Theta))$. If F is a local field, the classification theory by Satake and Tits concludes that elements of $\widehat{H}^1(F, (\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)$ bijectively correspond to Tits indices of type E_6 realized over F. So that one can identify $\widehat{H}^1(F, (\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)$ with the set of Tits indices as follows:

$$\widehat{H}^1(F, (\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) = \begin{cases} \{{}^1E_{6,6}^0,\ {}^1E_{6,2}^{16}\} & (F \text{ is nonarchimedean and } \alpha \equiv 1) \\ \{{}^2E_{6,4}^2\} & (F \text{ is nonarchimedean and } \alpha \not\equiv 1) \\ \{{}^1E_{6,6}^0,\ {}^1E_{6,2}^{28}\} & (F = \mathbb{R} \text{ and } \alpha \equiv 1) \\ \{{}^2E_{6,4}^2,\ {}^2E_{6,2}^{16\prime},\ {}^2E_{6,0}^{78}\} & (F = \mathbb{R} \text{ and } \alpha \not\equiv 1) \end{cases} \tag{6}$$

Let F be an algebraic number field. Since the center of the universal covering group of $(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0$ is isomorphic to $R^{(1)}_{E_\alpha/F}(\mu_3)$, the group $\mathcal{H}_\alpha$ is defined as in Section 1 and one has the commutative diagram

$$\begin{array}{ccc} H^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) & \xrightarrow[\cong]{\lambda} & \mathcal{H}_\alpha \times \prod_{v\in V^r_{\infty,1}(E_\alpha)} H^1(F_v,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) \\ \downarrow & & \downarrow \\ \widehat{H}^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) & \xrightarrow{\widehat{\lambda}} & \widehat{\mathcal{H}}'_\alpha \times \prod_{v\in V^r_{\infty,1}(E_\alpha)} H^1(F_v,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) \end{array}$$

Therefore, the situation is the same as in Section 1. For $[d]\in H^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)$, three sets $\Lambda(\mathfrak{g}_{\alpha,d}(F))$, $V^s_f(E_\alpha)_d$ and M^α_d are defined in the same way. The class number $h(\mathfrak{g}_{\alpha,d}(F)) = |\Lambda(\mathcal{H}_{\alpha,d}(F))|$ is equal to $|\{\pm 1\}\backslash M^\alpha_d|$ and it is given by the formula (5).

We recall Ferrar's result. Let $C(F)$ be a split Cayley algebra over a field F of characteristic 0. The automorphism group $\mathrm{Aut}\,C$ is an adjoint group of type G_2. The isomorphism classes of Cayley algebras over F is classified by $H^1(F,\mathrm{Aut}\,C)$. The twist of $C(F)$ by a 1-cocycle $a\in Z^1(\Gamma,\mathrm{Aut}\,C)$ is denoted by $C_a(F)$. For a Cayley algebra $C_a(F)$ and a cubic Jordan algebra $J_b(F)$ of type A, one obtains a Lie algebra $T(C_a(F),J_b(F))$ of type E_6 over F by Tits' construction. Namely, $T(C_a(F),J_b(F)) = \mathrm{Der}\,C_a(F) + C_a(F)^*\otimes_F J_b(F)^* + \mathrm{Der}\,J_b(F)$ as a set, where $\mathrm{Der}\,X$ is the Lie algebra of derivations of X and X^* the space of elements of generic trace 0 for $X = C_a(F), J_b(F)$, and the Lie bracket product is defined as in [1, II, §1]. We write $T_{a,b}(F)$ for $T(C_a(F),J_b(F))$ and denote by $\xi([a],[b])$ the cohomology class corresponding to the F-isomorphism class $[T_{a,b}(F)]$. Then we have the diagram

$$\begin{array}{ccc} H^1(F,\mathrm{Aut}\,C)\times H^1(F,\mathrm{Aut}\,J) & \xrightarrow{\xi} & H^1(F,\mathrm{Aut}\,\mathfrak{g}) \\ \downarrow & & \downarrow{\scriptstyle \varepsilon'} \\ \mathrm{Hom}(\Gamma,\Theta) & = & \mathrm{Hom}(\Gamma,\Theta) \end{array}$$

where the left vertical arrow is given by $[a]\times[b]\mapsto \varepsilon([b])$. Since this diagram is commutative ([1, II, Lemma(2.4), (i)]), for each $\alpha\in\mathrm{Hom}(\Gamma,\Theta)$, ξ induces the map

$$H^1(F,\mathrm{Aut}\,C)\times \widehat{H}^1(F,(\mathrm{Aut}\,J_\alpha)^0) \xrightarrow{\widehat{\xi}^\alpha} \widehat{H}^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)\,.$$

We write $T^\alpha_{a,c}(F)$ for the Lie algebra $T(C_a(F),J_{\alpha,c}(F))$. The cohomology class $\widehat{\xi}^\alpha([a],\Theta[c])$ corresponds to the F-isomorphism class $[T^\alpha_{a,c}(F)]$. Furthermore, by a relation of $\mathrm{Aut}\,C\times \mathrm{Aut}\,J_\alpha$ and $\mathrm{Aut}\,\mathfrak{g}_\alpha$ given by [1, II, Proposition(2.3) and Lemma(2.4)], it is known that the following diagram is commutative.

$$\begin{array}{ccc} H^1(F,\mathrm{Aut}\,C)\times \widehat{H}^1(F,(\mathrm{Aut}\,J_\alpha)^0) & \xrightarrow{\widehat{\xi}^\alpha} & \widehat{H}^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0) \\ \downarrow & & \downarrow \\ \widehat{H}^2(F,R^{(1)}_{E_\alpha/F}(\mu_3)) & = & \widehat{H}^2(F,R^{(1)}_{E_\alpha/F}(\mu_3)) \end{array} \qquad (7)$$

where vertical arrows are induced from connection morphisms on $H^1(F,(\mathrm{Aut}\,J_\alpha)^0)$ and $H^1(F,(\mathrm{Aut}\,\mathfrak{g}_\alpha)^0)$.

If F is a nonarchimedean local field, then ξ is bijective ([1, II, Lemmas (6.2) and (6.3)]). In this case, $H^1(F,\mathrm{Aut}\,C)$ is trivial. If $\alpha\equiv 1$, then $\widehat{H}^1(F,(\mathrm{Aut}\,J_\alpha)^0)$ consists of a trivial class $\Theta[0]$ and a non-trivial class $\Theta[1/3]$ of Hasse invariant $\pm 1/3$. By the identification of (6), one has $\widehat{\xi}^\alpha([0],\Theta[0])={}^1E^0_{6,6}$ and $\widehat{\xi}^\alpha([0],\Theta[1/3])={}^1E^{16}_{6,2}$. If $\alpha\not\equiv 1$, Then $\widehat{H}^1(F,(\mathrm{Aut}\,J_\alpha)^0)$ consists only of a trivial class $\Theta[0]$ and $\widehat{\xi}^\alpha([0],\Theta[0])$ corresponds to ${}^2E^2_{6,4}$.

If $F=\mathbb{R}$, then ξ is surjective, but not bijective. The set $H^1(F,\mathrm{Aut}\,C)$ has two elements, a trivial class $[0]$ and a nontrivial class $[1]$ corresponding to a division Cayley algebra. By [2, p.114], one obtains the following classification. If $\alpha\equiv 1$, $\widehat{H}^1(\mathbb{R},(\mathrm{Aut}\,J_\alpha)^0)$ consists only of a trivial class $\Theta[0]$ and one has $\widehat{\xi}^\alpha([0],\Theta[0])={}^1E^0_{6,6}$ and $\widehat{\xi}^\alpha([1],\Theta[0])={}^1E^{28}_{6,2}$. On the other hand, if $\alpha\not\equiv 1$, then $\widehat{H}^1(\mathbb{R},(\mathrm{Aut}\,J_\alpha)^0)$ consists of a trivial class $\Theta[0]$ and a non-trivial class $\Theta[1]$ corresponding to the reduced Freudenthal algebra $\{x\in M_3(\mathbb{C})\colon {}^t\overline{x}=x\}$. One has $\widehat{\xi}^\alpha([0],\Theta[0])=\widehat{\xi}^\alpha([0],\Theta[1])={}^2E^2_{6,4}$, $\widehat{\xi}^\alpha([1],\Theta[0])={}^2E^{16}_{6,2}{}'$ and $\widehat{\xi}^\alpha([1],\Theta[1])={}^2E^{78}_{6,0}$.

Let F be an algebraic number field. Ferrar proved the following

Theorem. ([1,II,Theorem(6.4),Proposition(7.1)]) *The map ξ is surjective and $\xi([a],[b])=\xi([a'],[b'])$ implies $[a]=[a']$. If $\alpha\equiv 1$, then $\widehat{\xi}^\alpha$ is bijective.*

By the surjectivity of ξ, any Lie algebra of type E_6 over F is F-isomorphic to some $T^\alpha_{b,c}(F)$. In what follow, we give a refinement of this theorem. We fix $[a]\in H^1(F,\mathrm{Aut}\,C)$, $\Theta[c]\in\widehat{H}^1(F,(\mathrm{Aut}\,J_\alpha)^0)$ and put $\Theta[d]=\widehat{\xi}^\alpha([a],\Theta[c])$. From the commutative diagram (7), it follows that $V^s_f(E_\alpha)_c=V^s_f(E_\alpha)_d$, $M^\alpha_c=M^\alpha_d$ and $\widehat{\xi}^\alpha([a],\sigma\Theta[c])=\sigma\widehat{\xi}^\alpha([a],\Theta[c])$ holds for any $\sigma\in M^\alpha_c$.

Theorem 1. *The map*

$$\Lambda(J_{\alpha,c}(F))\longrightarrow\Lambda(T^\alpha_{a,c}(F))\colon [J_{\alpha,c'}(F)]\mapsto[T(C_a(F),J_{\alpha,c'}(F))]$$

is bijective. In particular, one has $h(T^\alpha_{a,c}(F))=h(J_{\alpha,c}(F))$.

Proof. In Section 1 and the first paragraph of this section, we showed that the map $\sigma\mapsto\sigma\Theta[c]$ (resp. $\sigma\mapsto\sigma\Theta[d]$) gives rise to a bijection from $\{\pm1\}\backslash M^\alpha_c$ to $\widehat{\lambda}^{-1}(\widehat{\lambda}(\Theta[c]))$ (resp. $\widehat{\lambda}^{-1}(\widehat{\lambda}(\Theta[d]))$). $\square$

Next, we give an isomorphism condition for $T^\alpha_{a,c}(F)$ and $T^{\alpha'}_{a',c'}(F)$. Let $V^r_{\infty,1}(E_\alpha)_a$ be the subset consisting of $v\in V^r_{\infty,1}(E_\alpha)$ such that $C_a(F_v)\cong C(F_v)$. We say that $J_{\alpha,c'}(F)$ is congruent to $J_{\alpha,c}(F)$ modulo $V^r_{\infty,1}(E_\alpha)_a$ if $(\mathrm{inv}_v(c'))_{v\in V_f}=\epsilon(\mathrm{inv}_v(c))_{v\in V_f}$ with $\epsilon=\pm1$ and

$J_{\alpha,c'}(F_v) \cong J_{\alpha,c}(F_v)$ for all $v \in V^r_{\infty,1}(E_\alpha) - V^r_{\infty,1}(E_\alpha)_a$. We denote this case by $J_{\alpha,c}(F) \simeq J_{\alpha,c'}(F) \mod V^r_{\infty,1}(E_\alpha)_a$. Clearly, by (3), one has that $J_{\alpha,c}(F) \cong J_{\alpha,c'}(F)$ if and only if $J_{\alpha,c'}(F) \simeq J_{\alpha,c}(F) \mod V^r_{\infty,1}(E_\alpha)_a$ and $J_{\alpha,c}(F_v) \cong J_{\alpha,c'}(F_v)$ for all $v \in V^r_{\infty,1}(E_\alpha)_a$.

Theorem 2. *$T^\alpha_{a,c}(F) \cong T^{\alpha'}_{a',c'}(F)$ if and only if $\alpha' = \alpha$, $[a'] = [a]$ and $J_{\alpha,c'}(F) \simeq J_{\alpha,c}(F)$* mod $V^r_{\infty,1}(E_\alpha)_a$.

Proof. If $T^{\alpha'}_{a',c'}(F)$ is F-isomorphic to $T^\alpha_{a,c}(F)$, then one has obviously $\alpha' = \alpha$, and by Ferrar's theorem, $[a'] = [a]$. It follows from the diagram (7) that $(\mathrm{inv}_v(c'))_{v \in V_f} = \pm(\mathrm{inv}_v(c))_{v\in V_f}$. Let $v \in V^r_{\infty,1}(E_\alpha) - V^r_{\infty,1}(E_\alpha)_a$. Since $C_a(F_v)$ is a division Cayley algebra, $T^\alpha_{a,c'}(F_v) \cong T^\alpha_{a,c}(F_v)$ implies $J_{\alpha,c'}(F_v) \cong J_{\alpha,c}(F_v)$. Conversely, let $\alpha' = \alpha$, $[a'] = [a]$ and $J_{\alpha,c'}(F) \simeq J_{\alpha,c}(F) \mod V^r_{\infty,1}(E_\alpha)_a$. We may assume $a' = a$. If $v \in V^r_{\infty,1}(E_\alpha)_a$, then $T^\alpha_{a,c}(F_v)$ is always quasi-split and its F_v-isomorphism class is independent of $[c]$. By this and the definition of congruence, one has $T^\alpha_{a,c'}(F_v) \cong T^\alpha_{a,c}(F_v)$ for all $v \in V_{\infty,1}$ and $(\mathrm{inv}_v(c'))_{v\in V_f} = \pm(\mathrm{inv}_v(c))_{v \in V_f}$. This concludes $T^\alpha_{a,c'}(F) \cong T^\alpha_{a,c}(F)$. □

As a result, the cardinal number of the fiber $\xi^{-1}(\xi([a],[b]))$ is equal to $2^{|V^r_{\infty,1}(E_{\varepsilon(b)})_a|}$. Since $V^r_{\infty,1}(E_{\varepsilon(b)})_a$ is empty for all $[a]$, $[b]$ if and only if $V_{\infty,1}$ is empty, we obtain

Corollary. *The map ξ is bijective if and only if F is totally imaginary.*

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