CONTAINING BEST RULED FOOLSCAP
SPECIAL NOTEBOOK

Quadratic Forms

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§ 1. Definitions.

R : commutative ring with 1, $R \ni \alpha, \beta, \cdots, \xi, \eta, \cdots$

V : (right) R-module with finite basis, $V \ni a, b, \cdots x, y, \cdots$

$$V = e_1 R + \cdots + e_n R$$

$$x = \sum e_i \xi_i , \quad \xi_i \in R$$

$$V \ni x \longleftrightarrow \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in R^n$$

M_x または x で表わす.

<u>Def.</u> $B : V \times V \longrightarrow R$ bilinear form

$$B(x, y) = B(\sum e_i \xi_i , \sum e_i \eta_i)$$

$$= \sum_{i,j} B(e_i, e_j) \xi_i \eta_j$$

$(B(e_i, e_j))_{i,j}$ で定まる

M_B または B で表わす

$$B(x, y) = {}^t M_x M_B M_y \quad \text{or} \quad {}^t x B y$$

<u>Def.</u>

$B(x, y) = B(y, x)$ symmetric

$B(x, y) = - B(y, x)$ skew-symmetric

$\Uparrow$ ($\Downarrow$ if 2 : non zero divisor)

$B(x, x) = 0$ alternating

<u>Def.</u> $Q : V \to R$ quadratic form

1) $Q(x\alpha) = Q(x) \alpha^2$

2) $Q(x+y) - Q(x) - Q(y) = B(x, y)$ bilinear form symmetric になる.

$$B(x, x) = 2 Q(x)$$

$$Q(e_i) = \alpha_i, \quad B(e_i, e_j) = \beta_{ij}, \quad \beta_{ii} = 2\alpha_i$$

$$Q(x) = \sum_i \alpha_i \xi_i^2 + \sum_{i<j} \beta_{ij} \xi_i \xi_j$$

$$B(x, y) = \sum_i 2\alpha_i \xi_i \eta_i + \sum_{i \neq j} \beta_{ij} \xi_i \eta_j$$

$B(x, y)$ … bilinear form associated with $Q(x)$ という.

$$Q(x + y\lambda) = Q(x) + B(x, y)\lambda + Q(y)\lambda^2$$

$$\therefore \quad \left[\frac{d}{d\lambda} Q(x + y\lambda)\right]_{\lambda=0} = B(x, y)$$

$$\left[\frac{\partial^2}{\partial \xi_i \partial \xi_j} Q(x)\right]_{x=0} = B(e_i, e_j)$$

- $A(x, y)$: 任意の bil. f. $\Longrightarrow$ $Q(x) = A(x, x)$: q.f.
  $\rightarrow B(x, y) = A(x, y) + A(y, x)$

  $\forall Q, \ \exists A, \quad Q(x) = A(x, x)$

  $$A(x, y) = \sum_i \alpha_i \xi_i \eta_i + \sum_{i<j} \beta_{ij} \xi_i \eta_j$$ とおけばよい

- $2^{-1} \in R$ のとき

  q.f. $Q \rightleftarrows B$ sym. bil. f.

  $$Q(x) = \frac{1}{2} B(x, x)$$

  $$\mathcal{Q} \cong \mathcal{S}$$

- ch. $= 2$ のとき

  $$0 \to \mathcal{A} \to \mathcal{S} \to \mathcal{Q} \to \mathcal{A} \to 0$$

  $Q \to B$: alternating

- isomorphism

(V, Q) : quadratic R-module
metric ——

$(V, Q) \cong (V', Q')$

1) $x \longleftrightarrow x'$ isomorphism of R-module

2) $Q(x) = Q(x')$ $\therefore$ $B(x, y) = B(x', y')$

それぞれ basis を fix すれば, Q, Q' : homogeneous poly. of degree 2

$$(R^n, Q) \overset{P}{\cong} (R^n, Q') \qquad Q \sim Q' \text{ equivalent といい}$$

class を invariant の system によって決定すること.

$x' = Px$, $y' = Py$

$$ {}^t x B y = {}^t x' B' y' = {}^t x\, {}^t P B' P y$$

$$\therefore \quad \begin{cases} B = {}^t P B' P, & \det P \in R^* \\ \alpha_i \cdots \end{cases}$$

特に (V, Q) の automorphism

$$\begin{cases} {}^t P B P = B, & \det P \in R^* \\ \cdots \end{cases}$$

その全体 $O(V, Q)$ or $O(n, R, Q)$ orthogonal group

- homomorphism

$(V, Q) \longrightarrow (V', Q')$

1) $x \longrightarrow x'$ hom.

2) $Q(x) = Q(x')$

$$(R^n, Q) \xrightarrow{P^{(m,n)}} (R^m, Q')$$

$$\begin{cases} B = {}^t P B' P \\ \cdots \end{cases} \qquad Q' : \text{represents } Q \text{ という}$$

特に $P^{(n,1)} \neq 0$, $Q(P) = 0$ のとき, Q : represents 0 という.

- direct sum

$$(V_1, Q_1) \oplus (V_2, Q_2)$$

$V = V_1 \oplus V_2$ direct sum of R-module

$x = x_1 + x_2$

$Q(x) = Q_1(x_1) + Q_2(x_2)$

- orthogonality

B : bil. f. (sym. or skew-sym.)

$x \perp y \overset{def}{\Longleftrightarrow} B(x, y) = 0$

$(B(y, x) = 0)$

W submodule に対し, $W^{\perp} = \{ y \in V \mid x \perp y \text{ for } \forall x \in W \}$ annihilator

B : non-degenerate $\overset{def.}{\Longleftrightarrow} V^{\perp} = \{0\}$

一般に

$B(x, y)$ depends only on the classes of x, y modulo $V^{\perp}$

$\therefore \bar{B}(\bar{x}, \bar{y}) \overset{def.}{=} B(x, y) \quad \bar{x}, \bar{y} \in V / V^{\perp}$

$\bar{B}$: non-deg. bil. f. on $V / V^{\perp}$

(しかし, 一般に $V / V^{\perp}$ has basis は ?)

R : field, V : vector space over R のとき

B : non-degenerate $\Longleftrightarrow \det B \neq 0$

(一般に rank $B = n - \dim V^{\perp}$)

B : non-deg. のとき

$\dim W^{\perp} = n - \dim W$

$W^{\perp\perp} = W$

$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

$(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

orthogonal complement という.

$B \mid W \times W$: non-deg. $\Rightarrow W \cap W^{\perp} = 0 \Rightarrow V = W \oplus W^{\perp}$

$\Rightarrow B \mid W^{\perp} \times W^{\perp}$ も non-deg.

Q : q.f. $\longrightarrow$ B : sym. bil. f.

$x \perp y$, $W^{\perp}$, … は B に関していう。

$$V_* = \{ x \in V \mid Q(x) = 0 , \quad B(x, y) = 0 \text{ for } \forall y \in V \} \subset V^{\perp}$$

radical 核 of (V, Q)

Q : non-degenerate $\overset{\text{def}}{\Longleftrightarrow}$ $V_* = \{0\}$

一般に

$Q(x)$ $(x \in V)$ depends only on the class of x mod V_*

$\parallel$

$\bar{Q}(\bar{x})$ $\quad \bar{x}$ = class of $x \in V/V_*$

q.f. on V/V_*, non-deg.

$(V, Q) \longrightarrow (V/V_*, \bar{Q})$ homomorphism

逆に

$(V, Q) \longrightarrow (V', Q')$ hom.

$\Rightarrow$ kernel $\subset V_*$

R : field, V : vector sp. over R のとき

$V = V_1 + V_*$, $\quad Q_1 = Q \mid V_1$ とすれば

V_1 : 任意の complement,

$$(V, Q) = (V_1, Q_1) \oplus (V_*, 0)$$

<u>Theorem</u> 1. R : field of char. $\neq 2$

$$(V, Q) = \sum_{i=1}^{n} \{e_i'\}_R$$

i.e. $\exists\ P$, $\det P \neq 0$

$$^{t}PQP = \begin{pmatrix} \alpha_1' & & 0 \\ & \ddots & \\ 0 & & \alpha_n' \end{pmatrix}$$

$\because$) Q : non-degenerate $r\,(r \leq n)$. $\therefore$ B : non-deg.

$\exists\ e_1'$, $Q(e_1') \neq 0$ $\therefore$ $B(e_1', e_1') \neq 0$

$$\therefore \quad V = \{e_1'\} \oplus \{e_1'\}^{\perp}$$

$\ell > r$ induction.

§2. Witt's theorem.

$R = K$: field

V : vector space $/ K$

Q : q.f. , B : bil. f. ass. with Q

<u>Def</u> $x \in V$ isotropic $\overset{\text{def}}{\iff}$ $B(x, x) = 0$

$W \subset V$ " $\iff$ $W \cap W^{\perp} \neq \{0\}$

(i.e. $B|W \times W$: degenerate)

$W \subset V$ totally isotropic $\iff$ $W \subset W^{\perp}$

(i.e. $B|W \times W = 0$)

<u>Def.</u> $x \in V$ singular $\iff$ $Q(x) = 0$

$W \subset V$ " $\iff$ $\exists x$ sing $\in W \cap W^{\perp}$

(i.e. $Q|W$: deg.)

$W \subset V$ totally singular $\iff$ $\forall x \in W$ sing

(i.e. $Q|W = 0$)

x, W : singular $\rightarrow$ isotropic

W : t. sing. $\Rightarrow$ t. isotropic

ch. $\neq 2$ 逆も正.

ch. $= 2$ $\forall x$ isotropic

dim $V^\perp$ … defect という

以下 Q : non-degenerate とする. ($V^\perp$ no ~~isotropic~~ singular vector)

<u>Theorem</u> 1 $\quad V/K$, $\quad Q$: non-deg.

1) $\quad V = V_0 + W + W'$ ((直和))

$$\begin{cases} V_0 \perp (W + W') \\ W, W' : \text{totally singular}, \ (\dim W = \dim W' = \nu) \\ V_0 : \text{no } \cancel{\text{isotropic}}\ \text{singular vector}, \quad V_0 \supset V^\perp \end{cases}$$

2) uniqueness $\quad (V_0, Q_0)$, ν

$W^\perp = V_0 + W \quad \therefore \quad B \mid W \times W'$: non-deg.

$(e_1, \cdots, e_\nu)$ basis of $W \quad \exists \ (e_1', \cdots, e_\nu')$ basis of W'

$$B(e_i, e_j') = \delta_{ij}$$

$$V \ni x = x_0 + \sum e_i \xi_i + \sum e_i' \xi_i'$$

$$Q(x) = Q_0(x_0) + \sum_{i=1}^{\nu} \xi_i \xi_i' \qquad (\text{Witt decomposition})$$

reduced form という $\qquad \nu = $ index of Q

$$(Q_0(x_0) = 0 \Rightarrow x_0 = 0)$$

<u>Ex.</u> $\quad K = \mathbb{R} \qquad p \geq q = \nu$

$$Q \sim \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{p} & \\ & \overbrace{-1 \ \ddots \ -1}^{q} \end{pmatrix} \sim \left(\begin{array}{c|c|c} 1_{p-q} & & \\ \hline & & 1_\nu \\ \hline & 1_\nu & \end{array}\right)$$

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \sim \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

<u>Lemma</u> 1　$Q(e)=0$,　$B(e,x)\neq 0$

$\Rightarrow$　$\exists\ e' = e\lambda + x\mu$

$Q(e') = 0$,　$B(e,e') = 1$

$\because$)　$B(e,x) = 1$ としてよい.

$Q(e\lambda+x,\ e\lambda+x) = \lambda + Q(x)$　　$\lambda = -Q(x)$

<u>Lemma</u> 2　$V \supset W$: t. sing. of dim ν

$\Rightarrow$　$\exists\ W'$: t. sing. of dim ν

$W^{\perp} \cap W' = \{0\}$

(その とき. $W + W'$: non-isotropic

$\forall$ $(e_1, \cdots, e_\nu)$ basis of W, $\exists$ $(e_1', \cdots, e_\nu')$ basis of W'

s.t.　$B(e_i, e_j') = \delta_{ij}$)

$\because$) induction　$e_1', \cdots, e_{\mu-1}'$　　$W_{\mu-1}' = \{e_1', \cdots, e_{\mu-1}'\}$ t. sing.

$B(e_i, e_j') = \delta_{ij}$　$(1 \leq i \leq \nu,\ 1 \leq j \leq \mu-1)$

$e_1, \cdots, e_\nu, e_1', \cdots, e_{\mu-1}'$　lin. indep. mod $V^{\perp}$

$\therefore\ \exists\ x$

$B(e_i, x) = \delta_{i\mu}$,　$B(e_i', x) = 0$

$(1 \leq i \leq \nu)$　　$(1 \leq i \leq \mu-1)$

$e_\mu' = x + e_\mu\lambda$　　$Q(e_\mu') = 0$

最後に　$W' = \{e_1', \cdots, e_\nu'\}_K$　とおけばよい.

Proof of 1)　W maximal totally singular subsp. をとり

W' as in Lem. 2,

$V_0 = (W + W')^{\perp}$　とおけばよい.

Proof of 2)　次の Witt の Th から.

<u>Theorem</u> 2. (Witt) $W_1, W_2 \subset V$, $Q_i = Q|W_i$

$$(W_1, Q_1) \overset{P}{\cong} (W_2, Q_2)$$

$$\Longrightarrow \quad \exists\, \tilde{P} \in O(V, Q), \quad \tilde{P}|W_1 = P$$

∵) (Q_1 : non-isotropic としてよい (by Lem. 2)
ch. $\neq 2$ の場合. B : non-deg.

$$\left.\begin{array}{l} Q = Q_1 + Q_1' = Q_2 + Q_2' \\ Q_1 \sim Q_2 \end{array}\right) \Longrightarrow Q_1' \sim Q_2'$$

or

$$Q_1 + Q' \sim Q_1 + Q'' \Longrightarrow Q' \sim Q''$$

§1. Th. 1 により $\dim W_1 = 1$ としてよい.

$$Q(x) = Q(y) \neq 0 \Longrightarrow \{x\}^\perp \cong \{y\}^\perp$$ をいえばよい.

$\dim \{x, y\}_K = 1$ のとき, trivial.

$\quad$ 〃 $\quad = 2$ とす.

$\{x, y\}_K$ non-isotropic のとき

$$\{x, y\}_K = \{x\}_K + W_1' = \{y\}_K + W_2' \quad \text{(orth. sum)}$$

$Q(x) = Q(y)$ から $W_1' \cong W_2'$

$$\therefore \quad \{x\}_K^\perp = \{x, y\}_K^\perp + W_1' \cong \{x, y\}_K^\perp + W_2' = \{y\}_K^\perp.$$

$\{x, y\}_K$ isotropic のとき,

$\exists$ basis u, v $\qquad Q \sim \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$

$\exists\, w$, (u, v, w) に対し

$$Q \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & a & * \\ 1 & * & * \end{pmatrix}$$

$$\{u, v, w\}_K = \{x\}_K + W_1' = \{y\}_K + W_2' \quad \text{(orth. sum)}$$

Lem. 1 から $W_1' \cong W_2'$ 以下 上と同様.

一般の場合 (Chevalley)

1. induction on $\dim W_1 = r$

$W_1 \supset U \quad \dim r-1$

$P|U = S \longrightarrow \tilde{S} \in O(V, Q)$

$\tilde{S}^{-1} P | U = id$

よって $P|U = id$ としてよい。

2. $0 \to U \to W_1 \xrightarrow{P-1} D \to 0$

$\dim D = 1$

(*) $B(Px, Py - y) = B(Px, Py) - B(Px, y)$

$= B(x - Px, y) \qquad x, y \in W_1$

$\therefore \quad U \subset D^{\perp}$

$\exists\ W' \subset D^{\perp}, \quad W' \cap W_1 = W' \cap W_2 = 0$ とすれば

$B(Px, y) = B(x, y) \qquad x \in W_1,\ y \in W'$

よって P は

$W_1 + W' \longrightarrow W_2 + W'$

$x + y \longrightarrow Px + y$

に ext. される。

3. $W_1 \not\subset D^{\perp}$ (*)から $PW_1 \not\subset D^{\perp}$

$\therefore \quad W_1 \cap D^{\perp} = PW_1 \cap D^{\perp} = U$

W' … complement of U in $D^{\perp}$ は上の条件をみたす。

$W_1 + W' = W_1 + D^{\perp} = V$

4. $W_1 \subset D^{\perp}$ (*)から $PW_1 \subset D^{\perp}$

$\therefore \quad D \subset D^{\perp}$ 実は singular

$$Q(Px - x) = Q(Px) - B(Px, x) + Q(x) = 2Q(x) - B(x, x) = 0$$

$\exists\ W'$: complement of W_1 in $D^{\perp}$

〃 PW_1 〃

∵) $W_1 \neq PW_1$ としてよい　$W_1 = U + \{x\}_K$

$PW_1 = U + \{y\}_K$

そのとき　$x + y \notin W_1,\ PW_1$

よって　W'' : complement of $W_1 + PW_1$ in $D^\perp$

$W' = W'' + \{x+y\}_K$

とおけばよい．

∴　$W_1 = W_2 = D^\perp$ の場合に帰着．

5.　$W_1 = W_2 = D^\perp$

$W_1 \supset V^\perp$.　$W_1 = U' + D$，$U' \supset V^\perp$ とすれば

$U'^\perp \not\subset D^\perp$　∴　$\exists\ D'$　t. sing. of dim 1

dim $2/V^\perp$

$V = U' + (D + D')$ (orth. sum)

∴ $\begin{cases} D \cdots \text{radical of } W_1 \\ W_1^\perp = D + V^\perp \end{cases}$

P : extendable $\iff V^\perp \subset U$

$(\Rightarrow)$　$V^\perp$ invariant.　$V^\perp \ni u + \overset{\neq 0}{z} \xrightarrow{P} u + z\alpha$　$(\alpha \neq 1)$

$z(\alpha - 1) \in V^\perp \cap D$　矛盾 (Q : non-deg.)

$V \not\supset D$ のとき．

$(\Leftarrow)$　$V = U + D + D'$

$\tilde{P}$: $u + z + z' \longrightarrow u + z\alpha + z'\alpha^{-1}$ とすればよい

$Q(u + z + z')$

$= Q(u) + B(z, z')$

⊙ ~~A sufficient cond.　$W_1 \cap V^\perp = W_2 \cap V^\perp = \{0\}$~~

~~そのとき，上の証[4]において　$W' \supset V^\perp$ ととりうる．~~

~~∴　$U + W' \supset V^\perp$~~

~~よって 5 において　$U \supset V^\perp$ としてよい．~~

Nec. & Suf. Cond.　$P \mid W_1 \cap V^\perp = id$.

$P^{-1} \mid W_2 \cap V^\perp = id$

Cor. 1. W_1, W_2 : t. sing. of dim μ
$\exists\, T \in O(\overline{V}, Q)$, $TW_1 = W_2$

$(\overline{W}_1, \overline{W}_1')$, $(\overline{W}_2, \overline{W}_2')$ についても同様.

Cor. 2. dim of max. t. sing. subsp. は一定

Cor. 3. $Q_1 + Q_1' \sim Q_2 + Q_2'$ (non-deg.)
$Q_1 \sim Q_2$ (strongly non-deg.)
$\Rightarrow Q_1' \sim Q_2'$

$(\overline{V}, Q) \cong (\overline{V}', Q')$ のとき
$\overset{\cup}{W} \xrightarrow{P} \overset{\cup}{W'}$
は extendable

$\tilde{P} \in O(\overline{V}, Q) \Rightarrow \tilde{P} \mid \overline{V}^{\perp} = id.$

(∵) $x \in \overline{V}^{\perp}$ $\quad \tilde{P}x \in \overline{V}^{\perp}$
$Q(x - \tilde{P}x) = 2Q(x) = 0 \quad \therefore\ x - \tilde{P}x = 0$

($P \mid W_1 \cap \overline{V}^{\perp} = id.$ のとき. 上の注 4 において, $\underset{\overset{\|}{W_1 \cap \overline{V}^{\perp}}}{PW_1 \cap \overline{V}^{\perp}} \subset \overline{U}$
$P^{-1} \mid W_2 \cap \overline{V}^{\perp} = id$, $W' \supset$ compl. of $W_1 \cap \overline{V}^{\perp}$ in $\overline{V}^{\perp}$ をとる
$\therefore\ U + W' \supset \overline{V}^{\perp}$
よって 5 において $U \supset \overline{V}^{\perp}$ としてよい...

$\overline{U} \supset D$ のとき. $D = \{e\}$, $D' = \{e'\}$
$\to$ $e' \longrightarrow v + e\lambda + e' = e''$

$B(e', u) = B(e'', u)\ (\forall u \in U) \Longrightarrow v \in U^{\perp}$
$B(e', x) = B(e'', y) \Longrightarrow B(v, y) = B(e', \underset{\overset{\|}{-e}}{x - y}) = -1$
$Q(e'') = 0 \Longrightarrow Q(v + e') + \lambda = 0$

Extension 1. $\tilde{G} = \tilde{O}(V, Q)$ group of similitude

$T \in \tilde{G}$ $Q(Tx) = \mu(T)\, Q(x)$

multiplicator

$$M(Q) = \{ \mu(T) \mid T \in \tilde{G} \}$$

Th. 2'. $\exists\, T \in \tilde{G}, \quad T W_1 = W_2 \qquad W_i \cap V^{\perp} = \{0\}$

$\Leftrightarrow \exists\, \mu \in M(Q), \quad \mu Q_1 \sim Q_2$

2. Hermitian form, etc.

K : division ring (not. nec. commutative)

V : right vector space / K

involution $\alpha \to \bar{\alpha}$ (not id.)

$$\begin{cases} \overline{\xi + \eta} = \bar{\xi} + \bar{\eta} \\ \overline{\xi \eta} = \bar{\eta}\, \bar{\xi} \\ \bar{\bar{\xi}} = \xi \end{cases}$$

Def. $\Phi : V \times V \longrightarrow K$

hermitian sesquilinear form
(skew-hermitian)

1) $\Phi(x, y + y') = \Phi(x, y) + \Phi(x, y')$

$\Phi(x, y\alpha) = \Phi(x, y)\alpha$

2) $\Phi(x, y) = \pm \overline{\Phi(y, x)}$

1') $\cdots, \Phi(x\alpha, y) = \bar{\alpha}\, \Phi(x, y)$

$$\Phi(x, y) = \Phi\left(\sum e_i \xi_i, \sum e_i \eta_i\right)$$

$$= \sum_{i,j} \bar{\xi}_i\, \Phi(e_i, e_j)\, \eta_j$$

$\left(\Phi(e_i, e_j)\right)$ matrix of Φ

Def. $H : V \to K$

(skew) hermitian form $\qquad \overline{H(x)} = \pm H(x)$

1) $H(x\alpha) = \bar{\alpha} H(x) \alpha$ $\qquad \pm \overline{\Phi(x,y)}$

2) $H(x+y) - H(x) - H(y) = \Phi(x,y)$ sesquilinear

associated sesquilinear f.

$\Phi(x,x) = H(x)$ if ch. $\neq 2$.

特に $- = id.$ のとき, K : comm.

herm. f. $\to$ quad. f.

(skew) herm. s. f. $\to$ (skew) sym. bil. f.

alternating bil. f.

Def. H : condition (T)

$\forall x \in V, \quad \exists \lambda \in K,$

$H(x, x) = \lambda \pm \bar{\lambda}$

e.g. ~~(skew) herm. s. f. associated with H~~

sym. b. f. $\qquad$ " $\qquad Q$

alternating b. f. $\qquad$ non-deg. $\quad n = 2\nu$

∘ Th. 1, 2 は H 又は Φ sat. cond. (T) について も成立.

∘ $\exists$ orth. basis, except Φ : alt., $\neq 0$.

3. herm. f. over a lattice

- Type of quad. sp.

$$N_\nu = \{e_1, \cdots e_\nu, e_1', \cdots, e_\nu'\}_K = \sum_{i=1}^{\nu} \{e_i, e_i'\}_K$$

$$Q(e_i) = Q(e_i') = 0$$

$$B(e_i, e_i') = 1$$

kernel space, or hyperbolic space

$$B = \begin{pmatrix} 0 & 1_\nu \\ 1_\nu & 0 \end{pmatrix}$$

st. non-deg.

$$V = V_0 + N_\nu$$

<u>Def.</u> $V \sim V'$

$$\Leftrightarrow \quad V + {}^\exists N \cong V' + {}^\exists N'$$

$$\Leftrightarrow \quad V_0 \cong V_0'$$

eq. rel. その eq. class を type という. $[V]$ or $[Q]$

$$V \cong V' \Leftrightarrow V \sim V', \ \dim V = \dim V'$$

- Witt group W (analogy of Brauer gr.)

$$\underset{Q}{[V]} + \underset{Q'}{[V']} = \underset{Q+Q'}{[V \oplus V']}$$

$[\{0\}] = \{N\}$ zero type

$$-\underset{Q}{[V]} = \underset{-Q}{[V]} \qquad V \oplus V \supset \underset{\text{t. sing.}}{W} = \{x \oplus x \mid x \in V\}$$

(さらに $[V] \cdot [V'] = [V \otimes V']$

$$(Q \otimes Q')(x \otimes x') = Q(x)Q'(x')$$

により comm. ring.

$$V_1 \qquad Q_1(e\xi) = \xi^2$$

Ex. 1. K alg. cl.

$\mathfrak{W} \cong \mathbb{Z}_2 \qquad \xi_1^2$

Ex. 2. $K = \mathbb{R}$

$\mathfrak{W} \cong \mathbb{Z} \qquad \xi_1^2$

$Q \quad (p.q) \to p - q, \quad (p-q)(p'-q')$
$= (pp' + qq') - (pq' + qp')$

Ex. 3 $K = k_{\mathfrak{p}}$

$\mathfrak{W}$ finite !

Ex. 4. K finite field $\mathbb{F}_q$

ch. $= 2 \qquad \mathfrak{W} \cong \mathbb{Z}_2 \qquad \xi_1^2, \qquad \lambda(\xi_1^2+\xi_2^2)+\xi_1\xi_2$

ch. $\neq 2 \qquad \times (K/\wp K)$

$q \equiv 1 \ (4) \qquad \mathfrak{W} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \xi_1^2, \ \alpha\xi_1^2, \quad \xi_1^2 + \alpha\xi_2^2$

$q \equiv 3 \ (4) \qquad \mathfrak{W} \cong \mathbb{Z}_4 \qquad \pm\xi_1^2, \quad \xi_1^2 + \xi_2^2$

(∵) $\xi_1^2 + \xi_2^2 + \xi_3^2$ は 0 を表わす.
$\lambda^2 - 2\xi_1\lambda = \xi_3^2$ が K に解をもつように ξ_1, ξ_3 をとる.
$\lambda = \xi_1 + \xi_2\sqrt{-1}$ i.e. $\left(\frac{\xi_3}{2\xi_1}\right)^2 \notin \{\xi^2 - \xi \mid \xi \in K\}$

一般に

$\alpha_1\xi_1^2 + \alpha_2\xi_2^2 = c \neq 0$

の解の個数

$$\# = \begin{cases} q-1 & -\frac{\alpha_2}{\alpha_1} \sim 1 \\ q+1 & \text{〃} \not\sim 1 \end{cases}$$

Ex. 5 K ch. 2

$$\left(Q(x\lambda + y\mu) = Q(x)\lambda^2 + Q(y)\mu^2 \qquad \text{semi-linear} \right)$$

$$x = \sum e_i \xi_i$$

$$Q(x) = \sum_{i=1}^{m} (\alpha_i \xi_i^2 + \xi_i \xi_{m+i} + \beta_i \xi_{m+i}^2) + \sum_{i=2m+1}^{n} \gamma_i \xi_i^2$$

$\{\gamma_i\}$ lin. indep. $/ K^2$

$$\alpha_i = \beta_i = 0 \qquad (1 \leqslant i \leqslant \nu)$$

$$\Delta'(Q) = \sum_i \alpha_i \beta_i \quad (\mathrm{mod}\ \wp K)$$

Arf invariant

K perfect $n = 2m$ or $2m+1$

$\dim V \geqslant 3 \implies \nu > 0$

anisotropic なら ξ_1^2

$$\lambda(\xi_1^2 + \xi_2^2) + \xi_1 \xi_2 \qquad \Delta' = \lambda^2$$

Cf. $\left\{ \begin{array}{l} \text{C. Arf. Crelle 183 (1940)} \\ \text{Witt, " 193 (1954), 119–120} \\ \text{Witt – Klingenberg, " , 121–122} \end{array} \right.$

§ 3. Clifford algebra

$$\begin{cases} R & \text{comm. ring with } 1 \\ V & \text{free } R\text{-module} \qquad \text{basis } (e_1, \cdots, e_n) \\ Q & \text{q.f. on } V \end{cases}$$

- tensor algebra

$$T = T(V) = \sum_{k=0}^{\infty} T_k, \qquad T_k = \overset{k}{\otimes} V, \qquad T_0 = R$$

1) alg. w. 1 over R, gen. by $\underset{(e_1, \cdots, e_n)}{V}$

basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$

2) universal mapping property

$$\begin{tikzcd} V \arrow[r, "\varphi", "\text{linear}"'] \arrow[d, phantom, "\cap" rotate=90] & \mathfrak{A} \quad \text{alg. w. 1 over } R \\ T \arrow[ur, "\exists_1 \tilde{\varphi}"'] & \end{tikzcd}$$

hom. $(\tilde{\varphi}(1) = 1)$

T characterized by 1), 2).

- Clifford alg.

$$C = C(V, Q) = T/\mathfrak{I}$$

$$\mathfrak{I} = \mathfrak{I}_Q = \text{two-sided ideal gen. by } \{x \otimes x - Q(x)1 \mid x \in V\}$$

$\rho : V \to C$ （実は 1 : 1 ）

$$\rho(x)^2 = Q(x)$$

$$\rho(x)\rho(y) + \rho(y)\rho(x) = B(x, y)$$

1) alg. w. 1 over R, gen. by $\rho(V)$
$\rho(e_1), \cdots, \rho(e_n)$

<u>Th</u> 1 ρ は 1 : 1, C has a basis $e_{i_1} \cdots e_{i_k}$
$(i_1 < \cdots < i_k)$

$$\dim C = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

2) univ. map. prop.

$$V \xrightarrow[\text{lin.}]{\varphi} \mathfrak{A} \qquad \text{alg. w. } 1 \ / \ R$$

$$\varphi(x)^2 = Q(x)$$

$$\Longrightarrow \quad \begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\rho"'] & \mathfrak{A} \\ & C \arrow[u, "\exists_1 \psi"'] \end{tikzcd} \quad \text{hom.}$$

C : characterized by 1), 2)

<u>Ex.</u> 1 $\quad Q = 0$, $\quad C(0) = E = E(V)$ exterior alg. of V

このとき Th. 1 は既知. $\quad e_1 \wedge \cdots \wedge e_n \neq 0$

<u>Ex.</u> 2 $\quad n = 1$

$$T \cong R[X]$$

$$\mathfrak{I} \longleftrightarrow (X^2 - \alpha_1)$$

$$C \cong \begin{cases} R(\sqrt{\alpha_1}) & \alpha_1 \not\sim 1 \\ R \oplus R & \alpha_1 \sim 1 \end{cases} \qquad \text{ch.} \neq 2$$

(ch. 2 のとき radical)

$n = 2$ $\quad$ quaternion alg.

ch $\neq 2$ $\quad C = \{1, e_1, e_2, e_1 e_2\}_R$

$$e_1^2 = \alpha_1, \quad e_2^2 = \alpha_2$$

$$e_1 e_2 = - e_2 e_1$$

(α_1, α_2) とかく.

- $$\begin{array}{ccc} (V, Q) & \xrightarrow{p} & (V', Q') \\ \rho \downarrow & & \downarrow \rho' \\ C(V, Q) & \xrightarrow[\exists_1 C(p)]{} & C(V', Q') \end{array}$$

hom. of q. sp.

hom. of alg. w. 1.

$C(id) = id$

$C(p' \circ p) = C(p') \circ C(p)$ functor!

p : inj $\Rightarrow$ $C(p)$ inj

$V' \subset V$, $C(V') \subset C(V)$

- $T^+ = \sum_{k:\text{even}} T_k$, $T^- = \sum_{k:\text{odd}} T_k$

subalg.

$T = T^+ + T^-$ (dir. s.)

$C = C^+ + C^-$ (")

- $V \ni x \longrightarrow -x \in V$ $Q(-x) = Q(x)$

(ch $\neq 2$)

$C \xrightarrow{J} C$ autom. $J^2 = 1$

$J = \begin{cases} 1 & \text{on } C^+ \\ -1 & \text{on } C^- \end{cases}$ J : inner $\Leftrightarrow$ n : even

principal autom.

- $$\begin{array}{ccc} V & \xrightarrow{\rho} & C^{-1} \\ & \searrow & \nearrow \psi \\ & C & \end{array}$$

C^{-1} (= C as R-sp.)

ψ isom.

$\therefore$ $\exists$ anti-autom ι $\iota(x) = x$ for $x \in V$

$\iota^2 = 1$

principal anti-autom.

Th. 1 の証のための準備

$$T = T(V) = \sum_{k=0}^{\infty} T_k$$

$V \ni x$ に対し $\quad e_x : u \longrightarrow x \otimes u$

$$e_x(T_k) \subset T_{k+1}$$

Lemma 1. $f \in V^* \longrightarrow \exists_1\ i_f \in \mathcal{E}(T)$

(1) $\quad i_f(1) = 0$

(2) $\quad i_f \circ e_x + e_x \circ i_f = f(x)\,1 \quad$ for $x \in V$

さらに

$\quad f \longrightarrow i_f$ linear, $\quad i_f(V_k) \subset V_{k-1}$

$\quad i_f^2 = 0, \quad i_f \circ i_g + i_g \circ i_f = 0$

(*) $\quad i_f(\mathfrak{J}) = \mathfrak{J}$

∵) $\quad i_f(x \otimes u) = f(x)\,u - x \otimes i_f(u)$

$\quad i_f^2(x \otimes u) = \cancel{f(x)\,i_f(u) - f(x) \otimes} i_f(u) + x \otimes i_f^2(u)$

(*) の証

$\quad u \in \mathfrak{J}, \quad x \in V$

$\quad i_f(x \otimes u) = -x \otimes i_f(u) + f(x)\,u \in \mathfrak{J} \quad$ if $\ i_f(u) \in \mathfrak{J}$

よって $u = (x \otimes x - Q(x)1) \otimes v$ に対して $i_f(u) \in \mathfrak{J}$ をいえばよい

$$\begin{aligned} i_f((x \otimes x - Q(x)1) \otimes v) &= -x \otimes i_f(x \otimes v) + f(x)\,x \otimes v \\ &\quad - Q(x)\,i_f(v) \\ &= x \otimes x \otimes i_f(v) - \cancel{f(x)\,x \otimes v + f(x)\,x \otimes v} \\ &\quad - Q(x)\,i_f(v) \\ &= (x \otimes x - Q(x)\,1)\,i_f(v) \end{aligned}$$

A : bil. f. $V \times V \to R$

$x \in V \quad f_x^A : y \to A(x, y) \qquad f_x^A \in V^*$

$i_x^A \overset{\text{def.}}{=} i_{f_x^A}$

<u>Lemma</u> 2 $\quad A \in \mathcal{L}(V \times V, R) \longrightarrow \exists_1 \lambda_A \in \mathcal{E}(T)$

(3) $\quad \lambda_A(1) = 1$

(4) $\quad \lambda_A \circ e_x = (e_x + i_x^A) \circ \lambda_A$

(*) $\quad \lambda_A \circ i_f = i_f \circ \lambda_A$

$\because$) $\quad \lambda_A(x \otimes u) = x \otimes \lambda_A(u) + i_x^A(\lambda_A(u))$

(*) の iž induction

$$\lambda_A \circ i_f(x \otimes u) = -\lambda_A \circ e_x \circ i_f(u) + f(x)\lambda_A(u)$$

$$= -(e_x + i_x^A) \circ \lambda_A \circ i_f(u) + f(x)\lambda_A(u)$$

$$= -(e_x + i_x^A) \circ i_f \circ \lambda_A(u) + f(x)\lambda_A(u)$$

$$= i_f \circ e_x \circ \lambda_A(u) - \cancel{f(x)\lambda_A(u)}$$

$$+ i_f \circ i_x^A \circ \lambda_A(u) + \cancel{f(x)\lambda_A(u)}$$

$$= i_f \circ \lambda_A(x \otimes u)$$

<u>Lemma</u> 3 $\quad \lambda_A \circ \lambda_B = \lambda_{A+B}$

$\lambda_0 = id, \quad \lambda_{-A} = \lambda_A^{-1}$

<u>Lemma</u> 4 $\quad Q'(x) = Q(x) + A(x, x)$

$\Longrightarrow \lambda_A(\mathfrak{J}_{Q'}) = \mathfrak{J}_Q$

$(\therefore \bar{\lambda}_A : C(Q') \to C(Q))$

$\because$) $\quad \lambda_A(x \otimes u) = x \otimes \lambda_A(u) + i_x^A \cdot \lambda_A(u)$

$\in \mathfrak{J}_Q \quad$ if $\quad \lambda_A(u) \in \mathfrak{J}_Q$

$$\lambda_A\left((x\otimes x - Q'(x)1)\otimes v\right) = \lambda_A \circ e_x^2(v) - Q'(x)\lambda_A(v)$$
$$= (e_x + i_x^A)^2 \circ \lambda_A(v) - Q'(x)\lambda_A(v)$$
$$= \left(e_x^2 + A(x,x) - Q'(x)\right)\lambda_A(v)$$
$$= \left(e_x^2 - Q(x)\right)\lambda_A(v)$$
$$= (x\otimes x - Q(x)1)\otimes \lambda_A(v)$$

<u>Proof of Th. 1.</u>

$$V = \sum_{i=1}^{n} e_i R$$

$E = E(V) = C(0)$ basis $e_{i_1}\wedge\cdots\wedge e_{i_k}$

Lem. 4 において $Q' = 0$, $A(e_i, e_j) = \begin{cases} -\alpha_i & i=j \\ -\beta_{ij} & i>j \\ 0 & i<j \end{cases}$

$$\bar{\lambda}_A : E \longrightarrow C(Q)$$

$$\lambda_A(e_{i_1}\otimes\cdots\otimes e_{i_k}) = \cancel{\lambda_A} e_{i_1} \otimes \cdots \otimes \cancel{\lambda_A} e_{i_k} \quad (i_1 < \cdots < i_k)$$

$\because)$ $\lambda_A(e_{i_1}\otimes u) = e_{i_1}\otimes\lambda_A(u) + \underbrace{i^A_{e_{i_1}}(\lambda_A(u))}_{=0}$, $\lambda_A(u) = e_{i_2}\otimes\cdots\otimes e_{i_k}$

$\because$ $f_{e_{i_1}} = 0$ on $e_{i_2}, \cdots, e_{i_k}$

$$\therefore \quad \bar{\lambda}_A(e_{i_1}\wedge\cdots\wedge e_{i_k}) = \rho(e_{i_1})\cdots\cdots\rho(e_{i_k})$$

Cor. 1 $\quad (V', Q') \subset (V, Q)$ $\quad$ V'の basis が Vのそれに延長できるとき.

$$\Longrightarrow \quad C(V', Q') \subset C(V, Q)$$

Cor. 2 $\quad V = V_1 \oplus V_2$

$$C(V) \cong C(V_1) \otimes C(V_2) \quad \text{as vec. sp.}$$

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (-1)^{d(a_2)\cdot d(b_1)} (a_1 b_1) \otimes (a_2 b_2)$$

(orth. basis を fix したとき

以下, $R = K$ field, Q non-deg. st. とする.

Th. 2 $\quad V \sim 0, \quad n = 2\nu$

$$\Longrightarrow \quad C(V) \cong M_{2^\nu}(\mathbb{K})$$

$\nu > 0$ のとき

$$C^+(V) \cong M_{2^{\nu-1}}(\mathbb{K}) \oplus M_{2^{\nu-1}}(\mathbb{K})$$

∵) $\quad V = W + W'$

$$C(V) \supset C(W) = E(W) = E \qquad 2^\nu$$

$x \in E, \quad w \in W \qquad \bar{e}_w : x \longrightarrow w \cdot x$

$\qquad w' \in W' \qquad \bar{i}_{w'} : x \longrightarrow \bar{i}_f(x)$

$$f_{w'} = B(*, w')$$

$$\bar{e}_w \circ \bar{i}_{w'} + \bar{i}_{w'} \circ \bar{e}_w = B(w, w')$$

$$s(w + w') = \bar{e}_w + \bar{i}_{w'} \in \mathcal{E}(E)$$

$$s(w + w')^2 = B(w, w') = Q(w + w')$$

$$\therefore \quad s : C(Q) \longrightarrow \mathcal{E}(E).$$

(w_i) basis of W, $\quad (w_i')$ 〃 of W'

$H = (i_1, \ldots, i_h) \qquad w_H = w_{i_1} \cdots\cdots w_{i_h}$ とおく.

$j \notin H$ $\quad s(w_j')(w_H) = \bar{i}_{w_j'}(w_H) = 0$

$$s(w_j')(w_j w_H) = \bar{i}_{w_j'}\,\bar{e}_{w_j}(w_H)$$
$$= (B(w_j, w_j') - \bar{e}_{w_j} \cdot \bar{i}_{w_j'})(w_H)$$
$$= w_H$$
$$s(w_j)(w_j w_H) = 0$$

$$\therefore \quad s(w'_{H'})(w_H) = \begin{cases} 0 & \text{if } H' \not\subset H \\ \pm w_{H-H'} & \text{if } H' \subset H \end{cases}$$

よって $\quad x_{H,H'} = w_H w'_I w_{H'^c} \qquad I = (1, 2, \cdots, \nu)$

とおけば

$$s(x_{H,H'})(w_{H''}) = \pm \delta_{H',H''} w_H$$

$\therefore$ s onto $\quad \therefore$ $1:1$.

$$E = E^+ + E^-, \qquad E^\pm = E \cap C^\pm$$

$$x \in C^+ \qquad s(x) E^\pm \subset E^\pm$$

$$\therefore \quad s : C^+ \longrightarrow \mathcal{E}(E^+) \oplus \mathcal{E}(E^-)$$

$1:1$, $\therefore$ onto, q.e.d.

General case

○ <u>Def</u> of 'discriminant' $\quad (e_1, \cdots, e_n)$ basis of V/K

$$\Delta(Q) = (-1)^{\frac{n(n-1)}{2}} \det(B(e_i, e_j)) \in K^*/(K^*)^2$$
$$(R/(R^*)^2)$$

• ~~Q, Q' of the same type~~

$$\Delta(Q + Q') = (-1)^{nm} \Delta(Q)\Delta(Q')$$

• $Q \in [0] \Rightarrow \Delta(Q) = 1$

$[Q] = [Q'] \Rightarrow \Delta(Q) = \Delta(Q')$

Th. 3 K : field , B : non-deg.

1) n : even

C : simple central

C^+ : separable

$$Z^+ = \text{center of } C^+$$

If ch $\neq 2$ $= \{1, \overset{\text{orth. basis}}{e_1 \cdots e_n}\}_K \cong \begin{cases} K(\sqrt{\Delta}) & \text{if } \Delta \not\sim 1 \\ K \oplus K & \text{if } \Delta \sim 1 \end{cases}$

(If ch $= 2$ $= \{1, \overset{\text{alt. basis}}{e_1 e_2 + \cdots + e_{n-1} e_n}\} = \begin{cases} K(\wp^{-1}\Delta') & \text{if } \Delta' \not\sim 0 \\ K \oplus K & \text{if } \Delta' \sim 0 \end{cases}$)

2) n : odd (ch. $\neq 2$)

$$Z = \text{center of } C$$

$$= \{1, e_1 \cdots e_n\}_K \cong \begin{cases} K(\sqrt{2\Delta}) & \text{if } 2\Delta \not\sim 1 \\ K \oplus K & \text{if } 2\Delta \sim 1 \end{cases}$$

C : separable

C^+ : central simple ($\cong M_{2^{\frac{n-1}{2}}}(K)$ if $n = 2\nu + 1$)

∵) 1) $\bar{K}$: alg. closure of K

$$T_{\bar{K}} = T \otimes_K \bar{K} \qquad Q \sim 0 \text{ in } \bar{K}$$

$$C_{\bar{K}} = T_{\bar{K}} / \mathfrak{I}_{\bar{K}} = C \otimes_K \bar{K}$$

$$C_{\bar{K}} \cong M_{2^{\frac{n}{2}}}(\bar{K}) \Rightarrow C : \text{central simple}$$

$$C^+_{\bar{K}} \cong M_{2^{\frac{n}{2}-1}}(\bar{K}) \oplus \text{''} \Rightarrow C^+ : \text{separable}, \quad \dim_K Z^+ = 2$$

ch. $\neq 2$ $e_1, \cdots, e_n$ orth. basis

$$z = e_1 \cdots e_n \in Z^+ \qquad \therefore Z^+ = \{1, z\}_K$$

$$(2^{\frac{n}{2}} z)^2 = 2^n \cdot (-1)^{\frac{n(n-1)}{2}} Q(e_1) \cdots Q(e_n) = \Delta(Q)$$

2) $V \ni x_0$ non-isotropic

$V' = \{x_0\}_K^{\perp} \ni x'$, に対し

$Q'(x') = -Q(x_0)\,Q(x')$ とおく.

$(x_0 x')^2 = -Q(x_0)\,Q(x') = Q'(x')$

$\therefore$ $x' \longrightarrow x_0 x'$ は

$C(Q') \longrightarrow C^+(Q) \subset C(Q)$ に ext. される.

simple, 2^{n-1} 　同じ dim

$\therefore$ $C^+(Q) \cong C(Q')$ central simple

$\underset{\underset{x_0}{\parallel}}{e_1}, \cdots, e_n$ orth. basis

$z = e_1 \cdots e_n \in Z$

$$\left(2^{\frac{n+1}{2}} z\right)^2 = 2^{n+1}(-1)^{\frac{n(n-1)}{2}} Q(e_1)\cdots Q(e_n) = 2\Delta(Q)$$

Put $Z' = \{1, z\}_K$

$C^+(Q) \otimes_K Z' \longrightarrow C$

$u \otimes v \longrightarrow uv$

onto ($\because$ $C^- = z\,C^+$)

$\therefore$ 1 : 1

$\therefore$ $C \cong C^+ \otimes_K Z'$, $Z' = Z$

Z/K separable

<u>Rem.</u> B : degenerate のとき.

ch. $\neq 2$ 　radical of $C(V)$ = two-sided ideal gen. by $V^{\perp}$

$\underset{\mathcal{R}}{\parallel}$

$$C(V)/\mathcal{R} \cong C(V/V^{\perp})$$

ch. $= 2$ 　$\mathcal{R} \supset$ two-sided ideal gen. by V_*

($e_1 \in V^{\perp}$, $Q(e_1) = \alpha_1 = \beta_1^2 \Rightarrow e_1 + \beta_1 \in \mathcal{R}$

そのような e_1 が存在しなければ, $\mathcal{R} = \cdots$

∘ Expression by quaternion algebra

$$ch \neq 2. \qquad V = V_n = \{e_1, \cdots, e_n\}_K^{\text{orth. b.}}, \quad Q(e_i) = \alpha_i$$

n: even

$$C_{V_n} = C^+_{V_{n-1}} \otimes \{1,\ e_1 \cdots e_{n-1},\ e_n,\ e_1 \cdots e_n\}_K$$

$$= C_{V_{n-2}} \otimes \{1,\ e_1 \cdots e_{n-1},\ e_{n-1}e_n,\ e_1 \cdots e_{n-2}e_n\}_K$$

$$\therefore \quad C_{V_n} = C^+_{V_{n-1}} \otimes (2\Delta_{n-1},\ \alpha_n)$$

$$= C_{V_{n-2}} \otimes (2\Delta_{n-1},\ -\alpha_{n-1}\alpha_n)$$

今

$$c(V_n) = \begin{cases} \text{Brauer class of } C_{V_n} & (n : \text{even}) \\ \quad \text{〃} \quad \text{of } C^+_{V_n} & (n : \text{odd}) \end{cases}$$

とおけば

$$c(V_n) \sim \prod_{i=2}^{n} (2^{i-1}\Delta_{i-1},\ (-1)^i \alpha_i) \sim \prod_{i<j} ((-1)^{i+1}\alpha_i,\ (-1)^j \alpha_j)$$

これから

$$c(V_n \oplus W_m) = c(V_n') \cdot c(W_m) \cdot ((-1)^{n(m+1)} 2^n \Delta_n,\ (-1)^{(n+1)m} 2^m \Delta_m)$$

特に $\quad [V] = [V'] \Rightarrow c(V) = c(V')$

$$ch. = 2. \qquad V = V_n = \{e_1, \cdots, e_n\}^{\text{symp. b.}}$$

$$Q(e_i) = \alpha_i, \quad B(e_{2i-1}, e_{2i}) = 1$$

他は $= 0$

$n = 2m$

$$C_{V_n} = C_{V_{n-2}} \otimes \{1,\ e_{n-1},\ e_n,\ e_{n-1}e_n\}$$

$$\| $$

$$(\alpha_{n-1},\ \alpha_{n-1}\alpha_n]$$

$$\therefore \quad c(V_n) \sim \prod_{i=\nu+1}^{m} (\alpha_{2i-1},\ \alpha_{2i-1}\alpha_{2i}]$$

$$c(V_n \oplus W_m) = c(V_n) \cdot c(W_m)$$

§4. Structure of orthogonal group

Q : non-deg.

$$O = O(V, Q) = \{ S \in GL(V) \mid Q(Sx) = Q(x) \text{ for } \forall x \in V \}$$

($S \in \mathcal{E}(V)$ で十分. ∵ $Sx = 0 \Rightarrow x \in V^\perp,\ Q(x) = 0 \Rightarrow x = 0$)

$S \in O \qquad B(Sx, Sy) = B(x, y)$

- B : non-deg. $\Rightarrow |S| = \pm 1$

B : deg. のとき. $S = id$ on $V^\perp$

$\bar{S} = S \mid (V/V^\perp)$, $\quad \bar{B} = B \mid (\ \ ) \times (\ \ )$

$\bar{B}(\bar{S}\bar{x}, \bar{S}\bar{y}) = \bar{B}(\bar{x}, \bar{y})$

$\therefore \quad |S| = |\bar{S}| = 1$

ch $\neq 2$ のとき, SO が def. される.

ch $= 2$ 〃 , $S \underset{1:1}{\rightarrow} \bar{S}$.

$\therefore \quad O(V, Q) \subset Sp(V/V^\perp, B)$ とみることうる.

- $S \in O \qquad U_S = \{ x \in V \mid Sx = x \} (\supset V^\perp)$ とおけば

$$U_S = ((S-1)V)^\perp$$

(∵) $\subset$ は明らか. $y \in ((S-1)V)^\perp \Rightarrow y - Sy \in V^\perp$

(ch. $= 2 \quad Q(y - Sy) = B(y, Sy) = B(y, y) = 0$) ;

$\Rightarrow y = Sy$

特に $(S-1)V = \{a\}_K^{\neq 0}$. (a : non-sing. $\notin V^\perp$

$$\Longleftrightarrow \quad Sx = x - \frac{B(a, x)}{Q(a)} a,$$

symmetry w.r.t. $\{a\}_K^{(\perp)}$ という. S_a とかく. $S_a^2 = 1$

*) $S \in O_n \Rightarrow S = S_{a_1} \cdots S_{a_k}$ $(k \leq n)$ とかける.

Th. 1 (Cartan - Dieudonné) O : gen. by the symmetries *)
except $n = 4$, $K = F_2$, $\nu = 2$

∵) ch. ≠ 2 の場合. $S \in O(V)$
$V \ni a$ non-isotropic

1) $Sa = a$ $\quad S\{a\}_K^\perp = \{a\}_K^\perp$ induction

2) $Sa = -a$ $\quad S_a S a = a \longrightarrow$ 1)

3) $Sa = b \neq \pm a$ $\quad a+b$ or $a-b$ non-isotropic

$\left(\because \ Q(a+b) = Q(a-b) = 0 \Rightarrow 4Q(a) = 0 \right)$

$a \pm b = a'$ non-isot. $\quad Q(a^\mp) = Q(b) \mp B(a', b) + Q(a')$

$$S_{a'} b = b - \frac{B(a', b)}{Q(a')} a' = b \mp a' = \mp a$$

$\therefore \ S_{a'} S a = \mp a \longrightarrow$ 1)

一般の場合. (Chevalley)

1. $0 \rightarrow U_S \rightarrow V \xrightarrow{S-1} (S-1)V \rightarrow 0$

$\qquad\qquad\qquad\qquad \ni \ z = Sx - x$ non-sing. とすれば

$S_z S x = x$, $\quad S_z y = y$ for $y \in U_S$

$U_{S_z S} \supset U_S + \{x\}_K$

$\therefore$ U_S : maximal としてよい.

そのとき $(S-1)V$ t. sing.

2. G' = subgr. of O gen. by sym. normal!

$\forall$ t. sing. subsp. of dim ν は Z に conj. w.r.t. G'

言 $\underset{\text{t. sing. } \nu}{W_1, W_2} \subset V$, $\quad \dim(W_1 \cap W_2) < \nu$
$\qquad\qquad\qquad\qquad\qquad\quad \| \ \mu$

$\Rightarrow \exists S_a$ $\quad \dim(W_1 \cap S_a W_2) \geq \mu + 1$

$\because$) $W_1 + W_2 \ni {}^{\exists} a = x_1 + x_2$ non-sing

$$Q(a) = B(x_1, x_2) \neq 0 \qquad \therefore \ x_1 \notin W_2$$

$$S_a x_2 = x_2 - \frac{B(a, x_2)}{Q(a)} a = -x_1$$

$$S_a x = x \quad \text{for } x \in W_1 \cap W_2$$

$$\therefore \quad W_1 \cap S_a W_2 \supset W_1 \cap W_2 + \{x_1\}_K$$

3. $V = V_0 + W + W'$ (Witt decomp.)

$$H = \{ T \in O \mid T = id \text{ on } W^{\perp} = V_0 + W' \}$$

$$\begin{cases} T e_i = \sum \alpha_{ji} e_j + \sum \beta_{ji} e_j' + x_i \\ T e_i' = e_i' \\ T x = x \quad \text{for } x \in V_0 \end{cases}$$

$$B(e_i, e_j') = \delta_{ij} \quad \Longrightarrow \quad \alpha_{ji} = \delta_{ji}$$

$$B(e_i, \underset{V_0}{\hat{x}}) = 0 \quad \Longrightarrow \quad x_i \in V_0^{\perp}$$

$$Q(e_i) = 0 \quad \Longrightarrow \quad \beta_{ii} + Q(x_i) = 0$$

$$B(e_i, e_j) = 0 \qquad \beta_{ji} + \beta_{ij} = 0$$

$$\therefore \quad (\beta_{ij}) \text{ skew-sym.}, \quad \beta_{ii} = Q(x_i)$$

$x_0 \in V_0^{\perp}$, $Q(x_0) = \beta \neq 0$ なる

$$\begin{cases} T e_i = e_i + \beta e_i' + x_0 \\ T = id \quad \text{for other vect.} \end{cases}$$

t_s) $T = S_{\beta e_i' + x_0}$

$$H' = \{ T \in H \mid x_i = 0 \quad (1 \leq i \leq \nu) \}$$

$$\cong \{ (\beta_{ij}) \mid \text{alternating} \}$$

4. $\quad (\cancel{S}-1)V \subset W \Rightarrow U_S \supset W^{\perp} \Rightarrow S \in H$

$\therefore \quad O = H'G'$

$\quad O/G' \quad$: commutative

$\therefore \quad S, S' \in O, \qquad S, S', SS'$ conjugate

$\quad \Rightarrow \quad S \equiv S' \equiv SS' \mod G'$

$\quad \Rightarrow \quad S \in G'$

K more than 3 elem. $\qquad \exists\, \alpha \in K, \quad \alpha \neq 0, -1$

$$H' \ni S \longleftrightarrow (\beta_{ij})$$

$$S' \longleftrightarrow (\alpha\beta_{ij})$$

$$SS' \longleftrightarrow ((1+\alpha)\beta_{ij}) \qquad \text{same rank}$$

$\therefore \quad S, S', SS'$ conjugate

$\therefore \quad S \in G'$

5. $\quad K = \mathbb{F}_2 \qquad \nu = 0, 1 \qquad H' = \{1\}$

$\nu \geqq 2, \quad n > 4$

$$\begin{cases} S e_1 = e_1 + e_2' \\ S e_2 = e_2 + e_1' \\ \text{他は id} \end{cases} \qquad \left(\begin{array}{c|c|c} 1_\nu & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \\ \hline & 1_\nu & \\ \hline & & 1 \end{array}\right)$$

のときすれば十分.

$\{e_1', e_2'\}_K^{\perp} \ni f \quad$ non-sing $\quad Q(f) = 1$

$$e_1 \xrightarrow{S_{f+e_1'}} e_1 + e_1' + f \xrightarrow{S_{f+e_2'}} e_1 + e_1' + f \xrightarrow{S_{f+e_1'+e_2'}} e_1 + e_2' \xrightarrow{S_f} e_1 + e_2'$$

$$e_2 \longrightarrow e_2 \longrightarrow e_2 + e_2' + f \longrightarrow e_2 + e_1' \longrightarrow e_2 + e_1'$$

$$\therefore \quad S = S_f \cdot S_{f+e_1'+e_2'} \cdot S_{f+e_2'} \cdot S_{f+e_1'}$$

- Clifford group

$$\Gamma = \{ s \in C^* \mid s V s^{-1} = V \} \qquad \text{Clifford gr.}$$
$$\Gamma^+ = \{ s \in (C^+)^* \mid \quad \text{〃} \quad \} \qquad \text{special} \quad \text{〃}$$

- $\Gamma \ni s \Rightarrow \varphi(s) : x \to s x s^{-1}$ は $\in O$

$\because)$ $Q(s x s^{-1}) = (s x s^{-1})^2 = s x^2 s^{-1} = Q(x)$

- $a \in V$ に対し

$a \in \Gamma \iff \exists\, a^{-1} \iff Q(a) \neq 0$

そのとき $a^{-1} = Q(a)^{-1} a$

$$\therefore \quad a x a^{-1} = Q(a)^{-1} a x a = Q(a)^{-1} a \left(B(a,x) - a x \right)$$
$$= -x + \frac{B(a,x)}{Q(a)} a = - S_a x$$

以下 B non-deg. とす.

<u>Th.</u> 2

n: even のとき

$$1 \longrightarrow K^* \longrightarrow \Gamma \xrightarrow{\varphi} O \longrightarrow 1$$

一般に

$$\begin{tikzcd} 1 \arrow[r] & K^* \arrow[r] & \Gamma^+ \arrow[r, "\varphi"] & O^+ \arrow[r] & 1 \\ & & \underset{(n:\,\mathrm{odd})}{\Gamma} \arrow[ur] & \{\mathrm{sym.}\text{の偶数個の積}\} & \end{tikzcd}$$

$\because)$ n: even $\quad S \in O \to C(S)$ autom. of C

central simple

$\therefore \quad C(S)$ inner $= I_s$

$\therefore \quad S = \varphi(s) \qquad \therefore\ \varphi$ onto

$\mathrm{Ker}\ \varphi = \Gamma \cap Z = K^*$

* n : odd $\quad \varphi(\Gamma) \supset \varphi(\Gamma^+) \supseteq SO$

$(ch \neq 2) \quad \varphi(\Gamma) \not\ni -1_n$

$(\because)$ $C(-1) = J =$ principal autom. not inner $z \to -z$

$\therefore \varphi(\Gamma) = SO$

n : even $\quad \varphi(\Gamma^+) = O^+$ のは easy $\quad s e_1 \cdots e_n s^{-1}$ を考えよ

一般に $\quad O^+ \ni S = \prod_{i=1}^{h} S_{a_i} \quad (h : \text{even})$

* $\quad \therefore \quad C(S) = \underset{\substack{\| \\ 1}}{J^h} \prod I_{a_i} = I_s \qquad s = \prod_{i=1}^{h} a_i$

$\therefore \quad \varphi$ onto, $\quad \mathrm{Ker}\,\varphi = \Gamma^+ \cap Z = K^*$, $\quad$ q.e.d.

$\varphi(\Gamma^+) \supset O^+$

*

- $ch \neq 2 \qquad O^+ = SO \qquad$ trivial

一般に $\quad [O : O^+] = 2 \qquad (n \geq 1)$

<u>Lem.</u> $\quad \Gamma = Z^* \Gamma^+ \cup Z^* \Gamma^- = \begin{cases} \Gamma^+ \cup \Gamma^- & (n : \text{even}) \\ Z^* \Gamma^+ & (n : \text{odd}) \end{cases}$

特に $\quad [\Gamma : \Gamma^+] = 2 \qquad \underset{n\,:\,\text{even}}{(n \geq 1)}$

$\because)$ n : even. $\quad \Gamma \ni s = \underset{C^+}{\overset{\cap}{s'}} + \underset{C^-}{\overset{\cap}{s''}}, \qquad s', s'' \neq 0$

$s x = (\varphi(s) x) s$, $\quad \therefore \quad s' x = (\varphi(s) x) s'$

$s'' x = (\quad \text{''} \quad) s''$

$\therefore \quad s^{-1} s' x = x s^{-1} s' \qquad \therefore \quad s^{-1} s' \in Z = K$

n : odd. $\qquad \varphi(s) = \prod_{i=1}^{h} S_{a_i} \qquad s' = \prod a_i$

$\varphi(s') = (-1)^h \varphi(s) \qquad$ 共に $\in SO$

$\therefore \quad h$ even $\qquad \therefore \quad s^{-1} s' \in Z$

$ch = 2 \quad n$ even $\qquad Z^* \subset \Gamma^+$

$\therefore \quad [O : O^+] = [\Gamma : \Gamma^+] = 2$

*

- $S \in O, \quad S = \prod S_{a_i} \implies C(S) = J^r I_s, \quad s = \prod a_i$

$C(S)$: inner $\iff \begin{cases} n \text{ even} \\ n \text{ odd} \quad S \in O^+ \end{cases}$

$\Gamma^{(1)} = \{ s \in \Gamma^+ \mid N(s) = \iota(s)s = 1 \}$ ... simply conn. cov. gr. of O^+

$1 \to \{\pm 1\} \to \Gamma_K^{(1)} \to O_K^+ \xrightarrow{\theta} K^*/(K^*)^2$ [illegible]

- spinor norm

$$s \in \Gamma^+ \xrightarrow[\text{hom.}]{} \iota(s)s \in K^*$$

$$\begin{array}{ccccc} & & & & K^* \\ O^+ & & \Gamma^+ & \longrightarrow & | \\ | & & | & & | \\ O' & & \Gamma' & \longrightarrow & (K^*)^2 \\ & & | & & \\ & & K^* & & \end{array}$$

$\because)$ $\quad sx = (\varphi(s)x)s$

$x\,\iota(s) = \iota(s)(\varphi(s)x)$

$\therefore \quad (\iota(s)s)x = \iota(s)(\varphi(s)x)s = x(\iota(s)s)$

$\therefore \quad \iota(s)s \in Z \cap C^+ = K$

$S = \varphi(s)$ のとき $\qquad \theta(S) = \iota(s)s \in K^*/(K^*)^2$

spinor norm

$\theta : \quad O^+ \longrightarrow K^*/(K^*)^2$

kernel $\theta = O'$ とおく. reduced orthogonal group

$$S = \prod_{i=1}^{h} S_{a_i} \longrightarrow \theta(S) = \prod_{i=1}^{h} Q(a_i)$$

よって ~~[illegible]~~.

image $\theta = \{ Q(a)Q(b) \mid a, b \in V \text{ non-sing} \}(K^*)^2$

<u>Lemma.</u> $\quad \Gamma^+ = \{ s \in (C^+)^* \mid s^\iota s \in K^* \}$ $\qquad$ for $n \leq 5$

$\Gamma^+ = (C^+)^*$ $\qquad$ for $n \leq 3$

(See p. 40, 41, 43, 48)

- Ω_n = commutator gr. of O_n | $T \in O_n \Rightarrow T^2 \in \Omega_n$

Th. 3 (Eichler) $\nu \geqq 1$ ならば

$$1 \to \Omega_n \to O_n^+ \xrightarrow{\theta} K^*/(K^*)^2 \to 1$$

except $n = 2\nu = 4$, $K = \mathbb{F}_2$

Rem. $\nu = 0$ not true Ex. $K = \mathbb{R}$

true Ex. K alg. n. f., $n \geqq 5$ (Kneser)

Lem. 1 $\nu \geqq 1$ $\mathbb{V}$: gen. by singular vectors

∵) e_1, e_1' sing. $B(e_1, e_1') = 1$

$\forall\ x \in \{e_1, e_1'\}_K^\perp$ $B(e_1, e_1' + x) = 1$

$\exists\ y \in \{e_1, e_1' + x\}_K$ sing. $B(e_1, y) = 1$

とすると, $x \in \{e_1, e_1', y\}_K$

$T =$ id on $\{e_1, e_1'\}_K^\perp$ なる T: hyperbolic tr. という.

Lem. 2 $\nu \geqq 1$ O: gen. by hyp. tr.

(except ⋯)

∵) $T = T_a$ symmetry a: non-sing.

Lem. 1 により, $\exists\ b$: singular $B(a, b) \neq 0$

$T =$ id on $\{a, b\}_K^\perp$ ∴ hyperbolic.

<u>Lem. 3</u> $\nu \geqq 1$, $\{e_1, e_1'\}$ as above

$\forall\ T \in O_n$ can be written $T = T'T''$

T' : hyp. w.r.t. $\{e_1, e_1'\}_K$

$T'' \in \Omega_n$

(If $T \in O_n^+ \Rightarrow T' \in O_n^+$)

∵) Lem. 2 により T : hyp. としてよい. w.r.t. ~~賛~~ N

$\exists\ S \in O_n$, $N = S\{e_1, e_2'\}_K$ (Witt)

$$T = S T' S^{-1} = T'(T'^{-1} \underset{\substack{\Uparrow \\ \Omega_n}}{S\, T' S^{-1}})$$

<u>Proof of Th. 3.</u> $\theta(T) = \theta(T')$

∴ $n = 2\nu = 2$ のとき，やれば十分.

$V = \{e_1, e_1'\}_K$, $B(e_1, e_1') = 1$

$Q(e_1 \xi_1 + e_1' \xi_2) = \xi_1 \xi_2$

$$\therefore\ O_2 = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \;\middle|\; \xi \in K^* \right\}$$

$$\Gamma \ni \cancel{e_1 + e_1'\xi}\ \ e_1 \xi - e_1' \longleftrightarrow -\begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix}$$

$$1 + e_1 e_1'(\xi - 1) \longleftrightarrow \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} = T$$

$$\therefore\ O_2^+ = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \;\middle|\; \xi \in K^* \right\}$$

$$\theta(T) = (1 + e_1' e_1 (\xi - 1))(1 + e_1 e_1' (\xi - 1)) = \xi$$

∴ onto, $\theta(T) = \xi = \eta^2 \Rightarrow T = \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}^2$

$\Rightarrow T \in \Omega_2$

*) ch. $\neq 2$ のとき, $-1_n \in \Omega_n \Leftrightarrow n$: even, $(-1)^{\frac{n}{2}} \Delta \sim 1$

($\because$) $-1_n = \prod S_{e_i}$ $\therefore$ $\theta(-1_n) = \prod Q(e_i)$

$$Z_n = \begin{cases} \{1_n\} & (n \text{ odd}) \\ \{\pm 1_n\} & (n \text{ even}) \end{cases}$$

$n \geqslant 3 \Rightarrow \left(\begin{array}{ll} Z_n & : \text{center of } O_n^+ \\ \Omega_n & : \text{commutator of } O_n^+, \ \Omega_n \cap Z_n : \text{center of } \Omega_n{}^{*)} \end{array} \right.$

<u>Th.</u> 4 (Dickson - Dieudonné)

$n \geqslant 5$, $\nu \geqslant 1$ のとき

$$\Omega_n / \Omega_n \cap Z_n \quad : \quad \text{simple (non-comm.)}$$

More precisely

$\left. \begin{array}{l} N : \text{normal subgr. of } O_n^+ \\ N \not\subset Z_n \end{array} \right) \Rightarrow N \supset \Omega_n$

<u>Rem</u>. 1. $\nu = 0$ のとき ? true for $\left(\begin{array}{l} \mathbb{R}, \\ \text{alg. n.f.} \end{array} \right.$ (Kneser)

not true Ex. (Dieudonné, Sur ..., p. 34 - 39)
p. 51 - 52, p. 60.

∘ Artin, p. 179 - 186

$\left(\begin{array}{l} K \quad \text{valuation } |\ | \\ V \ni x = \sum e_i \xi_i \qquad \|x\| = \underset{1 \leq i \leq n}{\mathrm{Max}} |\xi_i| \\ \qquad |Q(x)| \leqslant 1 \Rightarrow \|x\| < c \\ \text{`elliptic space'} \end{array} \right.$

<u>Rem</u>. 2. alg. gr. / K としては

$$O_n^+ / Z_n \quad \text{simple for } n \geqslant 3, \ n \neq 4$$

$$C = C_0 + C_1 + C_2 + C_3 + C_4$$

重 ι : $+ \quad + \quad - \quad - \quad +$

$u \in C^+$ に対し

$$N(u) = \iota(u)\, u = u \cdot \iota(u) \in \begin{cases} K & n \leq 3 \\ \mathbb{Z}^+ & n = 4 \end{cases}$$

§ 5. Orthogonal group for $n \leq 6$

$n = 1.$ $\qquad O_1 = \{\pm 1\}$

$\Delta = 2\alpha_1$

$n = 2.$ $\qquad C = C_0 + C_1 + C_2 \qquad$ quaternion algebra

$\Delta = -2^2 \alpha_1 \alpha_2$

$\Delta' = \alpha_1 \alpha_2$

$, \qquad \Gamma^+ = (C^+)^*$

$$C^+ \cong \begin{cases} K(\sqrt{\Delta}) \ \text{ or } \ K(\wp^{-1}\Delta') & \Delta \neq 1, \text{ or } \Delta' \neq 0 \\ K \oplus K & \text{otherwise} \end{cases}$$

$\cong K(\sqrt{\Delta})^{(1)} \qquad (\xi \to \xi/\bar{\xi})$

$$\therefore \quad \boxed{O_2^+ \underset{alg.}{\cong} \begin{cases} K(\sqrt{\Delta})^* / K^* \ \text{ or } \ K(\wp^{-1}\Delta')^* / K^* \\ K^* \times K^* / \text{diag.} \cong K^* \end{cases}}$$

第1の場合. $\quad T \longleftrightarrow \xi \quad \Rightarrow \quad \theta(T) = N\xi$

第2 〃 $\quad \longleftrightarrow (\xi_1, \xi_2) \quad \Rightarrow \quad \theta(T) = \xi_1 \xi_2$

C^+ : field $\iff \Delta \neq 1 \ (\Delta' \neq 0) \iff \nu = 0$

$$\therefore \quad O_2' \cong \begin{cases} K(\sqrt{\Delta})^{(1)} / \{\pm 1\} \ \text{ or } \ K(\wp^{-1}\Delta')^{(1)} / \{\pm 1\} \\ K^* / \{\pm 1\} \end{cases}$$

特に $\nu = 1$

$$\Omega_2 \cong K^* / \{\pm 1\}$$

∘ Exercise : $\quad \xi \in C^+, \quad x \in V$

$$\frac{B(\varphi(\xi)x, x)}{Q(x)} = \frac{\mathrm{Tr}\, \xi^2}{N\xi}$$

$n = 3.$ $(ch \neq 2)$

$V = \{e_1, e_2, e_3\}_K$ orth. basis, $Q(e_i) = \alpha_i$

$C = C_0 + C_1 + C_2 + C_3$

$C^+ = C_0 + C_2$

$C_2 = \{f_1, f_2, f_3\}$ $f_1 = e_2 e_3$, $f_2 = e_3 e_1$, $f_3 = e_1 e_2$

$f_i^2 = \beta_i$ $(\beta_1 = -\alpha_2\alpha_3, \cdots)$, $f_i f_j = - f_j f_i$

$\therefore$ $C^+ = C(\beta_1, \beta_2)$ quaternion alg.

$C^+ \ni u = \lambda_0 + \sum_{i=1}^{3} f_i \lambda_i$ に対し

$\iota(u) = \lambda_0 - \sum f_i \lambda_i$ (can. involution)

$N(u) = u \cdot \iota(u) = \lambda_0^2 - \sum_{i=1}^{3} \beta_i \lambda_i^2$ (red. norm)

$= \lambda_0^2 + \alpha_1\alpha_2\alpha_3 \, Q\left(\frac{\lambda_1}{\alpha_1}, \frac{\lambda_2}{\alpha_2}, \frac{\lambda_3}{\alpha_3}\right)$

$\therefore$ C^+ : division $\Longleftrightarrow$ $\nu = 0$

$(\because)$ $\Longleftrightarrow$ $\forall u$: pure quat. が逆をもつ $)$

$\Gamma^+ = (C^+)^*$

$(\because)$ $\tilde{e}_I = e_1 e_2 e_3$ とおく.

$V \ni x \longrightarrow x\tilde{e}_I$ p. quat. in C^+

一方 $u \in C^+$ p. quat. $\Longleftrightarrow$ $\iota(u) = -u$ $\Longleftrightarrow$ $N(u) = -u^2$

$\therefore$ u : p. quat. $\Longrightarrow$ tut^{-1} : p. quat. for $\forall t \in C^{+*}$

$\therefore$ $\left.\begin{array}{l} x \in V \\ t \in (C^+)^* \end{array}\right) \Longrightarrow$ $txt^{-1}\tilde{e}_I = t(x\tilde{e}_I)t^{-1}$: p. quat.

$\Longrightarrow$ $txt^{-1} \in V$

又は p. 42 (3) における と 同様 に.

*) $\Delta \sim 1$ のとき, $u \in D$ に対し

$$N(u) = \lambda_0^{2*} - \sum \beta_i \lambda_i^2 = \alpha_1\alpha_2\alpha_3\, Q\left(\frac{\lambda_1}{\alpha_1}, \frac{\lambda_2}{\alpha_2}, \frac{\lambda_3}{\alpha_3}, \frac{2^2\lambda_0}{\sqrt{\Delta}}\right)$$

$$\therefore \quad \boxed{O_3^+ \underset{alg}{\cong} (C^+)^* / K^*}$$

$$O_3' \cong (C^+)^{(1)} / \{\pm 1\}$$

$\nu = 0$

$K = \mathbb{R}$ simple

$\mathbb{Q}_p$) solvable

$\mathbb{Q}$

特に $\nu = 1$

$$\Omega_3 \cong PSL_2(K)$$

simple if $K \neq \mathbb{F}_3$

$n = 4$.

ch. $\neq 2$.

$$Z^+ = \{1, e_1e_2e_3e_4\}_K \cong \begin{cases} K(\sqrt{\Delta}) & (\Delta \not\sim 1) \\ K \oplus K & (\Delta \sim 1) \end{cases}$$

$\nu = 2 \implies \Delta \sim 1$

$\nu = 1 \implies \Delta \not\sim 1$ (by the case $n = 2$)

$\nu = 0 \implies$?

D = subalg. of C^+ gen. by $\{f_1, f_2, f_3\}$

($f_1 = e_2e_3, \cdots$ as in the case $n = 3$)

Then $C^+ = D \otimes Z^+$

($\because$) $\tilde{e}_{I'} = e_1e_2e_3e_4$, $\tilde{e}_{I'}f_1 = e_1e_4(-\alpha_2\alpha_3)$, $\tilde{e}_{I'}f_2 = e_2e_4(-\alpha_3\alpha_1)$

$\tilde{e}_{I'}f_3 = e_3e_4(-\alpha_1\alpha_2)$

$\therefore$ $D \otimes Z^+ \longrightarrow C^+$ onto $\therefore$ 1 : 1.

$C^+ \ni u = \lambda_0 + \sum_{i=1}^{3} f_i\lambda_i$, $\lambda_i \in Z^+$ に対し

$$\iota(u) = \lambda_0 - \sum f_i\lambda_i$$

$$N(u) = u \cdot \iota(u) = \lambda_0^2 - \sum_{i=1}^{3} \beta_i\lambda_i^2 = N_{C^+/Z^+}(u) \quad ^{*)}$$

*) ($\Leftarrow$) $\nu = 0 \Rightarrow \nu = 0$ in $K(\sqrt{\Delta})$ ($\Rightarrow$ $D_{K(\sqrt{\Delta})}$ division)

$\because$) $Q(x + y\sqrt{\Delta}) = 0 \Rightarrow Q(x) + Q(y)\Delta = 0$, $B(x,y) = 0$

$x = e_1$, $y = e_2$ とおけば、 $\underset{\Downarrow}{}$ $Q(e_3)Q(e_4) \sim -1$

$\therefore$ $\{e_3, e_4\}_K \sim 0$ 矛盾.

- $\Gamma^+ = \{ u \in (C^+)^* \mid N(u) \in K^* \}$

$\left(\because \right)$ $\left. \begin{array}{l} N(u) \in K^* \\ x \in V \end{array} \right\} \Rightarrow$ $y = uxu^{-1} = ux\iota(u) \cdot N(u)^{-1} \in C^-$

$\iota(y) = y$

$\Rightarrow y \in V.$

逆は明らか。~~同様~~

$$\therefore \quad O_4^+ \underset{alg.}{\cong} \{ u \in (C^+)^* \mid N(u) \in K^* \} / K^*$$

$$O_4' \cong \cancel{\{ \ \text{〃} \mid N(u) = 1 \}} \; C^{+(1)}/Z^+ \; / \{\pm 1\}$$

$\Delta \nsim 1$ のとき ($\nu = 0, 1$)

$C^+ = D_{K(\sqrt{\Delta})}$ $\boxed{O_4^+ \underset{alg.}{\cong} \{ u \in D^*_{K(\sqrt{\Delta})} \mid N(u) \in K^* \} / K^*}$

特に

$\nu = 1 \overset{*)}{\Longleftrightarrow} C^+ = M_2(K(\sqrt{\Delta}))$

$$\therefore \quad \Omega_4 \cong PSL_2(K(\sqrt{\Delta}))$$

simple

(Ex. $K = \mathbb{R}$ $\Omega_4 \cong PSL_2(\mathbb{C})$ (Lorentz gr.)

$\Delta \sim 1$ のとき ($\nu = 0, 2$)

$C^+ = D \oplus D$ $\boxed{O_4^+ \underset{alg.}{\cong} \{ (u_1, u_2) \in D^* \times D^* \mid Nu_1 = Nu_2 \} / \{ (\xi, \xi) \mid \xi \in K^* \}}$

特に

$\nu = 2 \Longleftrightarrow D = M_2(K)$

$$\therefore \quad \Omega_4 \cong SL_2(K) \times SL_2(K) / \{ \pm (1_2, 1_2) \}$$

not simple

ch. $= 2$ 同様 (Δ を Δ' でおきかえる)

(例外 $\nu = 2$, $K = \mathbb{F}_2$. $O_4^+ \cong SL_2(\mathbb{F}_2) \times SL_2(\mathbb{F}_2)$

$$\begin{array}{ccccccc} O_4^+ & \supset & \Omega_4 & \supset & \text{comm. of } O_4^+ & \supset & 1 \\ \| & 2 & & 2 & & 3 \times 3 & \\ O_4' & & & & & & \end{array}$$

Rem. 1. $n=4$ と $n=3$ の関係

$u \in \Gamma^+ \cap Z^+ \iff u \in K^*$ or $\in e_{I'} K^*$

$\iff \varphi(u) = \pm 1_4$

$\therefore \quad PO_4^+ \xrightarrow[1:1]{} (C^+)^*/(Z^+)^*$

image $= \{ u \in C^+ \mid N(u) \in K^* (Z^+)^{*2} \}$

$V = V_4 \supset V_3 = \{e_1, e_2, e_3\}_K$

$C^+ = C^+(V_4) \supset D = C^+(V_3)$

$\Delta \not\sim 1$

$C^+ \cong C^+(V_{3\,K(\sqrt{\Delta})})$

$\therefore \quad PO_4^+ \xrightarrow[1:1]{} O^+(V_{3\,K(\sqrt{\Delta})})$ (alg. isom)

image $= \{ T \in O_3^+ \mid \theta(T) \in K^*(K(\sqrt{\Delta})^*)^2 \}$

$\Delta \sim 1$

$e' = \frac{1}{2}(1 + \frac{4}{\sqrt{\Delta}} e_{I'})$, $e'' = \frac{1}{2}(1 -$ 〃 $)$

$Z^+ = e'K + e''K$

$C^+ = e'D + e''D$

$\overset{\Psi}{u} = e'a + e''b \quad a, b \in D$

$Nu = e'Na + e''Nb$

$\therefore \quad Nu \in K^* \iff Na = Nb$

$PO_4^+ \longrightarrow D^*/K^* \times D^*/K^*$

$\cong O^+(V_3) \times O^+(V_3)$ (alg. isom. into)

image $= \{ (T_1, T_2) \mid \theta(T_1) = \theta(T_2) \}$

$$Q(e) = \alpha_1\alpha_2 \quad \cong \{e\}_K$$

$$n=2 \qquad \underset{Q}{V_2} \cong \underset{-N}{C_1}, \quad \{\text{p.q. in } C\} = \overset{\cong V}{C_1} \oplus C_2$$

<u>Rem. 2.</u> $n = 3$

$$V_3 \overset{\cong}{\longleftrightarrow} C_2(V_3) = \{\text{p.q. in } \overset{D}{\overset{\parallel}{C^+}}(V_3)\}$$

$$x = \sum_{i=1}^{3} e_i \frac{\lambda_i}{\alpha_i} \longleftrightarrow \hat{x} = \tilde{e}_I x = \sum f_i \lambda_i$$

$$Q(x) = \frac{1}{\alpha_1\alpha_2\alpha_3} N(\hat{x})$$

$\therefore$ metric isom.

$$T = \varphi(u) \longleftrightarrow \hat{x} \to u\hat{x}u^{-1}$$

$n = 4$, $\Delta \sim 1$ $\quad$ ~~$\alpha_4 = 1$, $\Delta = 2^4$~~ とす.

$$V_4 \cong D$$

$\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4}$

$$x = \sum_{i=1}^{3} e_i \frac{\lambda_i}{\alpha_i} + e_4\lambda_0 \longleftrightarrow \hat{x} = \lambda_0 + \sum f_i \lambda_i$$

$$Q(x) = \alpha_4 N(\hat{x})$$

$$T = \varphi(e'a + e''b) \longleftrightarrow \hat{x} \longrightarrow b\hat{x}a^{-1}$$

$$a, b \in D^*$$

$\because$)

$$\hat{x} = \frac{1}{2}(e_I x + x e_I) + \frac{1}{2}(e_4 x + x e_4)$$

$$= \underbrace{\frac{1}{2}(e_4 + e_I)}_{e'e_4 = e_4e''} x + \frac{1}{2} x (e_4 + e_I)$$

$$e_{I'} x = - x e_{I'}$$

$$y = (e'a + e''b) x (e'a^{-1} + e''b^{-1})$$

$$= e'(axb^{-1}) + e''(bxa^{-1})$$

$$\hat{y} = e_4e''(bxa^{-1}) + (bxa^{-1})e'e_4$$

$$= b(e_4e''x + xe'e_4)a^{-1}$$

$$= b\hat{x}a^{-1}$$

$$C = D \otimes \{1, e_4, \overset{(\Delta, \alpha_4)}{\overset{\parallel}{e_I}}, e_{I'}\}_K$$

$$\underset{1 \ (\text{if } \Delta \sim 1)}{\wr}$$

*) relation : $C \sim 1$ (in $K(\sqrt{\Delta})$)

n	complete sys. of invariants	condition for $\nu = 0$ ~~repr. zero~~
1	Δ	/
2	Δ, C *)	$\Delta \not\sim 1$ ($\Leftrightarrow$ $C \not\sim 1$)
3	Δ, C^+	$C^+ \not\sim 1$
4	?	($\Leftarrow$ $C \sim D \not\sim 1$ (in $K(\sqrt{\Delta})$))

◎ K has no quaternion alg. $\Longleftrightarrow$ $\nu = 0 \Rightarrow n \leq 2$

そのとき $\mathfrak{W} \underset{2}{\supset} \mathfrak{W}^+ \cong K^*/(K^*)^2$

n, Δ : compl. sys. of inv.

e.g. $K =$ F_q, alg. func. f. over alg. cl. f.

◎ $\nu = 0 \Rightarrow n \leq 4$ の場合

e.g. $K = \mathfrak{p}$-adic n. f.

n, Δ, c : complete system of invariants

∵) $n \equiv n' \pmod 2$, $\Delta = \Delta'$, $c(V) = c(V')$

$\Rightarrow [V] = [V']$ という.

$\leftarrow$

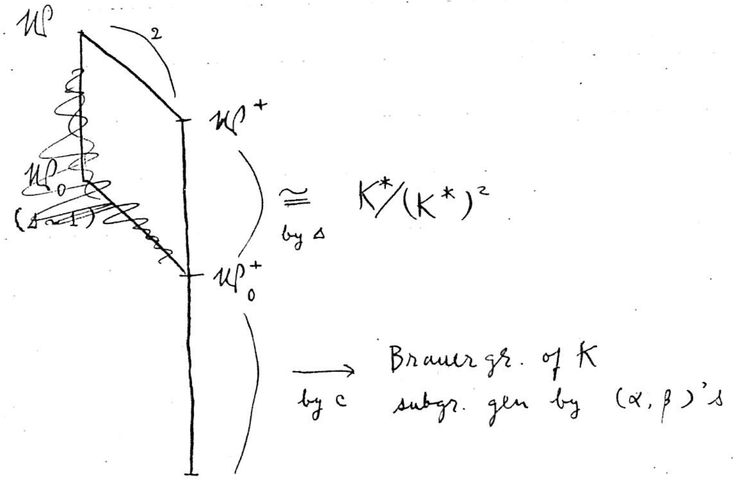
$[W] = [V] - [V'] \in \mathfrak{W}^+$, $\Delta(W) = 1$

$c(V') = c(V + W) = c(V) \cdot c(W)$

$\therefore\ c(W) = 1$

よって $\nu = 0 \Rightarrow n \leq 4$ 及び上の表から

$W \sim 0$.



$$\begin{cases} n + n' \\ \Delta(V \oplus V') = (-1)^{nn'} \Delta \Delta' \\ c(V \oplus V') = \left((-1)^{n(n'+1)} 2^n \Delta,\ (-1)^{(n+1)n'} 2^{n'} \Delta'\right) \cdot c \cdot c' \end{cases}$$

◎ quaternion alg. が group を作る.

$\Rightarrow$ Δ, c independent for $n \geqq 3$.

e.g. $K = \left(\begin{array}{l} \mathfrak{p}\text{-adic n. f., alg. n. f.,} \\ \text{alg. func. f. over finite f.} \end{array} \right.$

∵) $n = 3$. $\quad C^+ = (-\alpha_1\alpha_2, -\alpha_1\alpha_3) \sim \left(\frac{2\Delta}{\alpha_1}, \frac{2\Delta}{\alpha_2}\right)$

∴ Δ, Δ/α_1, Δ/α_2 任意にとれる.

$n > 3$. $\quad V = V_3 \oplus V'$

$\Delta = (-1)^{n'} \Delta_3 \cdot \Delta'$

$c = \left((-1)^{n'+1} 2^3 \Delta_3,\ 2^{n'} \Delta'\right) \cdot c_3 \cdot c'$

Δ, c 任意に与えられたとき

Δ_3 s.t. $(-1)^{n'+1} 2^3 \Delta_3 \sim 1$, $\quad \Delta'$ s.t. $\Delta \sim (-1)^{n'} \Delta_3 \cdot \Delta'$

により $\Delta_3 \to V' \to c_3$ の順に定めればよい.

$n = 5$.

$$\Gamma^+ = \{ u \in C^{+*} \mid N(u) \in K^* \}$$

$\because$) $\left.\begin{array}{l} u \in C^{+*}, \quad N(u) \in K^* \\ x \in V \end{array}\right\} \Rightarrow u x u^{-1} \in C^-$, ι-sym.

$$\Rightarrow u x u^{-1} = \overset{\in C_1}{y} + e_I \overset{\in C_5}{\zeta}$$

$$\therefore \quad Q(x) = Q(y) + 2 \underset{0}{\underset{\|}{y e_I \zeta}} + 2^{-5} \Delta \zeta^2$$

$\left.\begin{array}{l} \cancel{\exists x} \\ \therefore \ \forall x \end{array}\right)$ に対し $\begin{array}{l} y \neq 0 \\ \zeta = 0 \end{array}$ $\quad \because e_I \in Z$

$$\therefore \quad \boxed{\begin{array}{c} O_5^+ \cong \{ u \in (C^+)^* \mid Nu \in K^* \} / K^* \\ \text{i.e.} \quad \mathrm{Int.}(C^+, \iota) \end{array}}$$

$\dot{C}_0$	C_1	$\dot{C}_2$	C_3	$\dot{C}_4$	C_5
+	+	−	−	+	+
1	5	10	10	5	1

$C^+ \cong M_4(K)$ のとき.

$$\iota(X) = A^{-1}\, {}^t X A, \quad {}^t A = \pm A$$

$$\iota(X) = X \iff AX = \pm\, {}^t(AX)$$

$$\dim\{ X \mid \iota(X) = X \} = \begin{cases} 10 & (+) \\ 6 & (-) \end{cases}$$

$$\therefore \quad {}^t A = -A$$

$$\therefore \quad O_5' \cong PS_p(4, K)$$

$C^+ \cong M_2(\mathfrak{D})$ のとき

上と同様に $\quad \iota(X) = A^{-1}\, {}^t\bar{X} A, \quad {}^t\bar{A} = A$

$$O_5' \cong PU(2, \mathfrak{D})$$

gr. of herm. f. over $\mathfrak{D}$

$$C^+_{V_5} = C_{V_4}\,(2\Delta_5, -\alpha_5)$$

$$= C_{V_2}\,(2\Delta_3, -\alpha_3\alpha_4)\,(\quad \text{〃} \quad)$$

$$= (\alpha_1, \alpha_2)(-\alpha_1\alpha_2\alpha_3, -\alpha_3\alpha_4)(\alpha_1\cdots\alpha_4, -\alpha_5)$$

$\therefore \quad \nu = 2 \Rightarrow C^+ \sim 1$

$\nu = 1 \Rightarrow C^+ \sim (-\alpha_3\alpha_4, -\alpha_3\alpha_5) + 1$

$\nu = 0 \Rightarrow C^+ \nsim 1$ のはず

herm. f.
$\nu = 1$
$\nu = 0$

(C^+: inv. of the 1^{st} kind をもつ.
$\therefore \quad C^+ \sim 1$ or $C^+ \sim \mathfrak{D}$ (q. alg.))

$n = 6$.

$$\left.\begin{array}{ll} u \in (C^+)^*, & Nu \in K^* \\ x \in V & \end{array}\right) \text{に対し}$$

$$u x u^{-1} = y' + y'' e_I \qquad y', y'' \in V$$

$$Q(x) = Q(y') + \underbrace{(y'y'' - y''y')}_{0} e_I + 2^{-6} \Delta\, Q(y'')$$

$\therefore \quad y', y''$ lin. dep.

$$\therefore \quad u x u^{-1} = y \cdot (\lambda + e_I \mu)$$

$$u x' u^{-1} = y' (\lambda' + e_I \mu')$$

$$B(x, x') = y(\lambda + e_I \mu)\, y'(\lambda' + e_I \mu') + y'(\lambda' + e_I \mu')\, y(\lambda + e_I \mu)$$

$$= y y' (\lambda - e_I \mu)(\lambda' + e_I \mu') + y' y (\lambda + e_I \mu)(\lambda' - e_I \mu')$$

$$= B(y, y')(\lambda\lambda' + 2^{-6} \Delta\, \mu\mu') + (yy' - y'y)\, e_I\, \underbrace{(\lambda\mu' - \mu\lambda')}_{0}$$

よって

$$u x u^{-1} = T(x) \cdot \zeta^{-1} \qquad \zeta \in Z^{+*} \text{ indep. of } x$$

とおける.

$$Q(x) = Q(T(x)) \cdot N(\zeta)^{-1} \qquad T : \text{similitude}$$

$$\therefore \quad \underset{\text{multiplicator}}{\mu(T)} = N(\zeta)$$

$x, y \in V$

$$u x y u^{-1} = T(x)\, T(y) \cdot \mu(T)^{-1}$$

$$\therefore \quad u e_I u^{-1} = e_I \cdot \det(T) \cdot \mu(T)^3$$

$$\qquad\; \| \\ \qquad\; e_I$$

$$\therefore \quad \det(T) = \mu(T)^3 \qquad \text{'direct' or 'proper'}$$

逆に T, ζ as above とすれば

$x \longrightarrow Tx \cdot \zeta^{-1}$ は autom. of C, leaving e_I inv. ~~commuting with~~

に ext. できる.

$\therefore \ \exists\, u \in (C^+)^*, \quad Tx\cdot\zeta^{-1} = u x u^{-1}$ 　 2-sym.

$\therefore\ Nu \in K^*$

$$\boxed{\begin{aligned}\therefore\ \tilde{G} &= \{(T,\zeta)\mid T\in\tilde{O}_6^+,\ \zeta\in Z^{+*},\ \mu(T)=N(\zeta)\}\Big/\{(\lambda,\lambda)\mid\lambda\in K^*\}\\ &\cong \{u\in C^{+*}\mid Nu\in K^*\}/K^*\\ &\quad \text{i.e. } \widetilde{\mathrm{Int}}(C^+,\iota) \qquad C^+ = C^+_{V_5}\otimes Z^+\end{aligned}}$$

$\Delta \neq 1$ 　 C^+ simple, ι 第二種

$C^+ \cong M_4(Z^+)$ のとき

$G \cong P\tilde{U}(4, Z^+/K, H)$ 　 $\nu = 2, 1, 0$?

$C^+ \cong M_2(\mathcal{D}_{Z^+})$ のとき 　 $\mathcal{D} \sim C^+_{V_5}$

$G \cong P\tilde{U}(2, \mathcal{D}_{Z^+}/K, H)$ 　 $\nu = 1, 0$

$\Delta \sim 1$ 　 $C^+ = e' C^+_{V_5} \oplus e'' C^+_{V_5}$

$u = e'u' + e''u''$

$\iota(u) = e''\iota(u'') + e''\iota(u')$

$\therefore\ u\iota(u) \in K^* \iff \lambda = u'\iota(u'') \in K^*$ 　 red. norm of $C^+_{V_5}$ 　 $N(u') = \pm\lambda^2$

$\therefore\ \mathrm{Int}(C^+,\iota) \cong (\{(u,\lambda) \in C^{+*}_{V_5}\times K^*\ \cancel{\mid N(u')=\lambda^2}\}) \Big/ \{(\lambda,\lambda^2)\mid\lambda\in K^*\}$

又 $Z^{+*} \cong K^*\times K^*$ 　 $N(e'\lambda' + e''\lambda'') = \lambda'\lambda''$

$\therefore\ G = \tilde{O}_6^+ \cong$ ~~subgr. of~~ $(C^+_{V_5})^* \times K^* \Big/ \{(\lambda,\lambda^2)\mid\lambda\in K^*\}$

$C^+_{V_5} \cong M_4(K)$ のとき

$$SL(4,K) \underset{2}{\to} G \underset{2}{\to} PL(4,K) \qquad \nu = 3$$

$C^+_{V_5} \cong M_2(\mathfrak{D})$ のとき

$$G \cong PL(2,\mathfrak{D}) \qquad \nu = 1$$

$$\tilde{G} = \{(T,\zeta) \in \tilde{O}^+_6 \times Z^{+*} \mid \mu(T) = N(\zeta)\} / \{(\lambda,\lambda) \mid \lambda \in K^*\}$$

$$\tilde{G} \ni (T,\zeta) \ (\mathrm{mod}\ K^*) \underset{\text{rat. hom.}/K}{\longrightarrow} \zeta \ (\mathrm{mod}\ K^*) \in Z^{+*}/K^* \ \left(= R_{Z^+/K}(G_m)/G_m\right)$$

kernel $\cong O^+_6$

$$\tilde{\mathrm{Int}}(C^+,\iota) \ni u \ (\mathrm{mod}\ K^*) \longrightarrow N_{C^+/Z^+}(u) \ (\ ''\ ) \in Z^{+*}/K^*$$

kernel : $\left.\begin{array}{l} u u^\iota = a \in K^* \\ N_{C^+/Z^+}(u) = b \in K^* \end{array}\right)$ $\quad b^2 = a^4$, $\quad b = \pm a^2$

$\therefore$ component of kernel $\quad b = a^2$

$$\boxed{O^+_6 \cong \{u \in C^{+*} \mid u u^\iota \in K^*,\ N_{C^+/Z^+} u = (u u^\iota)^2\} / K^*}$$

<u>Rem.</u> 上の二つの rat. hom. を比較して

$$N_{C^+/Z^+}(u) \overset{?}{\equiv} \zeta^2 \quad (\mathrm{mod}\ K^*)$$

$\Delta \sim 1$ のとき

$$\zeta = e'\xi^2 + e'' \longleftrightarrow u = e'\xi + e''$$

よって成立.

$$\{u \in C^{+(1)} \mid uu^2 = 1\} \xrightarrow[2]{} O_6^+ \xrightarrow[2]{} \mathrm{Int}(C^+, \iota)$$

(image O_6')

$\Delta \not\sim 1$ のとき. C^+: simple, ι: 第二種

$C^+ \cong M_4(\overset{Z^+}{\text{R}})$ ($\nu = 2, \overset{?}{1}, 0$)

$$SU_4(Z^+/K) \xrightarrow[2]{} O_6^+ \xrightarrow[2]{} PU_4(Z^+/K)$$

$C^+ \cong M_2(\mathfrak{D})$ ($\nu = 1, 0$)

$$SU_2(\mathfrak{D}/Z^+/K) \xrightarrow[2]{} O_6^+ \xrightarrow[2]{} PU_2(\mathfrak{D}/Z^+/K)$$

C^+ : division ($\nu = 0$) (does not occur for alg. n. f., $\mathfrak{p}$-adic, n. f.

$$SU_1(C^+/Z^+/K) \xrightarrow[2]{} O_6^+ \xrightarrow[2]{} PU_1(C^+/Z^+/K)$$

$\Delta \sim 1$ のとき $C^+ = e' C^+_{V_5} + e'' C^+_{V_5}$, $e'^2 = e''$

$$\begin{aligned}
&\Psi \\
u &= e' u_1 + e'' u_2 \\
u^2 &= e' u_1^2 + e'' u_2^2
\end{aligned}$$

$\therefore$ $uu^2 = e'(u_1 u_2^2) + e''(u_2 u_1^2)$

$\therefore$ $uu^2 \in K^* \iff \lambda = u_1 u_2^2 \in K^*$

$$\mathrm{Int}(C^+, \iota) \cong C^{+\,*}_{V_5} \times K^* / \{(\lambda, \lambda^2) \mid \lambda \in K^*\}$$

$$u \longleftrightarrow (u_1^*, \lambda)$$

一方

$$Z^+ \ni \zeta = e'\lambda_1 + e''\lambda_2$$

$$\zeta\zeta^2 = \lambda_1 \lambda_2$$

$$\tilde{G} = \{(T, e'\mu(T) + e'') \mid T \in \tilde{O}_6^+\} \cdot \{(\lambda, \lambda) \mid \lambda \in K^*\}$$

$$\therefore \quad \tilde{O}_6^+ \cong C_{V_5}^{+\,*} \times K^* / \{(\lambda, \lambda^2) \mid \lambda \in K^*\}$$

$$\tilde{O}_6^+ \ni \xi 1_6 \longleftrightarrow (\xi 1_6,\ e'\xi^2 + e'') \in \tilde{G}$$

$$\overset{*}{\longleftrightarrow} (e'\xi + e'') \in \mathrm{Int}(C^+, \iota)$$

$$\longleftrightarrow (\xi, \xi) \in C_{V_5}^{+\,*} \times K^*$$

$* \ (\because)$

$$\xi x (e'\xi^2 + e'')^{-1}$$

$$= x(e'\xi^{-1} + e''\xi)$$

$$= (e'\xi + e'')\, x\, (e'\xi + e'')^{-1}$$

$$\{(\xi, \xi)(\lambda, \lambda^2) \mid \xi, \lambda \in K^*\} = \{(\xi, \eta) \mid \xi, \eta \in K^*\}$$

$$\therefore \quad \boxed{P\tilde{O}_6^+ \cong C_{V_5}^{+\,*} / K^*}$$

$$C_{V_5}^{+\,(1)} \xrightarrow[2]{} O_6^+ \xrightarrow[2]{} C_{V_5}^{+\,*} / K^*$$

$$C \sim C_{V_5}^+ \cong M_4(K) \qquad \nu = 3$$

$$SL_4(K) \longrightarrow O_6^+ \longrightarrow PL_4(K)$$

$$C_{V_5}^+ \cong M_2(\mathfrak{D}) \qquad \nu = 1$$

$$SL_2(\mathfrak{D}) \longrightarrow O_6^+ \longrightarrow PL_2(\mathfrak{D})$$

§ 6. Case of local fields.

K : locally compact field (char. $\neq 2$)

$d\xi =$ Haar measure of additive gr.

$d(\alpha\xi) = |\alpha|\, d\xi$

$K^* \ni \alpha \longrightarrow |\alpha|$ normalized valuation

K は ξ の valuation に関し complete.

Case (I). Archimedean val.

$K \cong \mathbb{R}$ or $\mathbb{C}$

Case (II). Non archimedean val.

$\mathfrak{o} = \{ \xi \in K \mid |\xi| \leq 1 \}$ valuation ring

$\mathfrak{p} = \{ \;\; " \;\; \mid |\xi| < 1 \}$ unique prime ideal

$\mathfrak{u} = \{ \;\; " \;\; \mid |\xi| = 1 \}$ unit

$\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$ residue class field

$$\left.\begin{array}{l} \text{discrete} \quad \therefore \quad \mathfrak{p} = (\pi) \\ \mathfrak{k} : \text{ finite field} \end{array}\right\} \text{になる.}$$

$\mathfrak{o} \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \cdots$

$\mathfrak{o} \cong \lim_{m\to\infty} \mathfrak{o}/\mathfrak{p}^m$ open, compact, totally disconnected

$N\mathfrak{p} = \# \mathfrak{k}$, $(\alpha) = \mathfrak{p}^\nu$ とすれば

$|\alpha| = N\mathfrak{p}^{-\nu}$

(II_1) $\quad \underset{\overset{\|}{p}}{\text{char. of } \bar{k}} \neq \underset{\overset{\|}{0}}{\text{char. of } K}$

K : finite ext. of $\mathbb{Q}_p$, "$\mathfrak{p}$-adic number field"

(II_2) $\quad$ char. of $\bar{k}$ = char. of K = p ($\neq 2$)

$\mathfrak{o}/\mathfrak{p}$ の代表系として $\exists_1$ 体 $k_0 \subset K$

$K \cong k_0((\pi))$. "field of formal power series"

∘ <u>Hensel's lemma</u>. $\quad f(X) \in \mathfrak{o}[X]$

$f(X) \equiv g(X) \cdot h(X) \pmod{\mathfrak{p}}$

g, h relatively prime (mod $\mathfrak{p}$)

$\Longrightarrow \quad f(X) = g'(X)\, h'(X)$,

$g \equiv g',\ h \equiv h' \pmod{\mathfrak{p}}$

$\deg g = \deg g'$.

Case (II)

Th. 1　K : Case (II), $(V, Q)/K$ について

$\nu = 0 \implies n \leq 4$

$\because$) $Q(x) = \sum_{i=1}^{5} \alpha_i \xi_i^2$ not represent zero とす.

$2^{-5}\Delta = \prod \alpha_i \sim 1$ としてよい.

$\sum_{i=1}^{4} \alpha_i \xi_i^2$ も not rep. zero.

$\therefore \quad (-\alpha_1\alpha_2, -\alpha_1\alpha_3) \neq 1$ in $K(\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4})$

$\therefore \quad \alpha_1\alpha_2\alpha_3\alpha_4 \sim 1$ in K (by Th. p. 60)

$\therefore \quad \alpha_5 \sim 1$

同様に $\alpha_i \sim 1 \quad (1 \leq i \leq 5)$

よって $Q(x) = \sum_{i=1}^{5} \xi_i^2$ としてよい.

$(-1, -1) \neq 1$ in K

$\mathfrak{p} | 2$ (i.e. K finite ext. of Q_2) でなければならぬ.

$\because$ $\mathfrak{p} \nmid 2$ ならば $X^2 + Y^2 + 1 \equiv 0 \ (\mathfrak{p})$ 解をもつ

解の個数 $= \begin{cases} N\mathfrak{p} - 1 \\ N\mathfrak{p} + 1 \end{cases}$　$N\mathfrak{p} \equiv \begin{cases} 1 & (4) \\ 3 & (4) \end{cases}$

$\xi^2 + \eta^2 + 1 \equiv 0 \ (\mathfrak{p})$

$X^2 + \eta^2 + 1 \equiv (X + \xi)(X - \xi)$

$\therefore \ \exists \xi', \quad \xi'^2 + \eta^2 + 1 = 0$ (by Hensel)

$\mathfrak{p} | 2$ のとき $\sqrt{-7} \in K$

($X^2 + 7 = 0$, $X = 1 + 2Y$, $Y^2 + Y + 2 = 0$
解あり (by Hensel))

$1^2 + 1^2 + 1^2 + 2^2 + \sqrt{-7}^2 = 0$ 矛盾!

<u>Cor.</u> (n, Δ, c) : compl. sys. of inv. になる. (by Th. p.46)

$\mathcal{B}^{(2)}(K)$ = subgr. of $\mathcal{B}(K)$ gen. by quaternion alg.

とおけば

$\mathcal{B}^{(2)}(K)$: order 2 $\{1, \mathfrak{D}\}$

$$\chi(c) = \begin{cases} 1 & c = 1 \\ -1 & c = \text{class of } \mathfrak{D} \end{cases}$$

$$\chi(V, Q) = \chi(c) = \left(\frac{c}{\mathfrak{p}}\right)$$ (Minkowski - Hasse invariant)
= Hilbert norm residue symbol $\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)$

(n, Δ, χ) : compl. sys. of inv.

Table of quad. f. over K with $\nu = 0$

n	q. f. $(\nu = 0)$	Δ	χ	
0	0	1	1	
1	$2\Delta\xi^2$	任意	1	
2	$\xi_1^2 - \Delta\xi_2^2$	$\Delta \neq 1$	1	$C = (1, -\Delta)$
	$\alpha(\xi_1^2 - \Delta\xi_2^2)$ $(\alpha \notin N(K(\sqrt{\Delta})))$		-1	$C = (\alpha, -\alpha\Delta) \sim (\alpha, \Delta)$
3	$Q \mid Q(x) - 2\Delta\xi_0^2 \sim$ (norm f. of $\mathfrak{D}$).	任意	-1	$C^+ = (-\alpha_1\alpha_2, -\alpha_1\alpha_3)$
4	norm form of $\mathfrak{D}$	1	-1	$C =$ 〃

$Q_3(x) - 2\Delta\xi_0^2 \sim (-2\Delta)(\text{n.f. of } \mathfrak{D}) \overset{*}{\sim} \text{n.f. of } \mathfrak{D}$
$Q_4(x) \sim \alpha_4 (\text{n.f. of } \mathfrak{D}) \overset{*}{\sim}$ 〃

* は $N(\mathfrak{D}^*) = K^*$ から.

∘ Local class field theory for quadratic extension

K : local field

(C) $\alpha \in K^*,\ \alpha \not\sim 1 \longrightarrow H_\alpha = N_{K(\sqrt{\alpha})/K}(K(\sqrt{\alpha})^*) \subset K^*$

$$\begin{cases} [K^* : H_\alpha] = 2 \\ \alpha \not\sim \alpha' \implies H_\alpha \neq H_{\alpha'} \end{cases}$$

$(K^*)^2 \neq K^*$ とす.

<u>Th.</u> K satisfies (C)

$\implies \exists$ unique quaternion division algebra $/K$, $\mathfrak{D}$ とおく.

∵) $(\alpha, \beta) = \{1, u, v, uv\}_K$, $u^2 = \alpha$, $v^2 = \beta$, $uv = -vu$

$(\alpha, \beta) \not\sim 1 \iff \beta \notin H_\alpha$

存在は明らか. (α, β), (α', β') 共に $\not\sim 1$ とす.

$H_\alpha = H_{\alpha'}$ のとき.

$\alpha \sim \alpha'$, $\beta'\beta^{-1} \in H_\alpha$

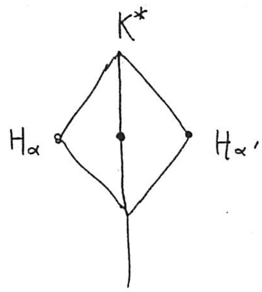
$\therefore (\alpha', \beta') \sim (\alpha, \beta') \sim (\alpha, \beta)$

$H_\alpha \neq H_{\alpha'}$ のとき

$\exists \gamma \notin H_\alpha, \notin H_{\alpha'}$

$\alpha, \alpha' \notin H_\gamma$, $\alpha'\alpha^{-1} \in H_\gamma$

$$\therefore \begin{array}{ccc} (\alpha', \beta') & \sim & (\alpha', \gamma) \\ & & \wr \\ (\alpha, \beta) & \sim & (\alpha, \gamma) \end{array}$$



<u>Cor.</u> 1 $(\mathfrak{D}^*)^2 \supsetneqq K^*$

$\left(\because\right)$ $\forall \alpha \in K^*$, $\alpha \neq 1$ に対し, $\exists \beta$ $\mathfrak{D} \cong (\alpha, \beta)$

$\therefore$ $\exists u \in \mathfrak{D}$, $u^2 = \alpha$ (実際, $u \in \mathfrak{D}^-$)

<u>Cor.</u> 2 $[K^* : (K^*)^2] > 2$ (Case (II)) のとき

$N(\mathfrak{D}^*) = K^*$

$\left(\because\right)$ $N(\mathfrak{D}^*) \supset {}^{\forall} H_\alpha$

$\alpha \neq \alpha'$ (共に $\neq 1$) $\Longrightarrow$ $\{H_\alpha, H_{\alpha'}\}$ gen. K^*

$$\mathcal{B}^{(2)}(K) = \{1, \mathfrak{D}\}$$

<u>Rem.</u> $K = \mathbb{R}$ $\Longrightarrow$ $\mathcal{B} = \mathcal{B}^{(2)}$

K : Case (II) のとき

$\mathcal{B} \ni \mathfrak{A} \sim (\pi, \chi)$ cyclic algebra

χ : unramified character

と unique にかける.

σ_0 : Frobenius automorphism とし

$\mathfrak{A} \to \left(\dfrac{\mathfrak{A}}{\mathfrak{p}}\right) = \chi(\sigma_0)$ (Hasse invariant)

この対応により

$$\mathcal{B} \cong \mathbb{Q}/\mathbb{Z}$$

$\mathfrak{A} = (\alpha, \chi) \longrightarrow \left(\dfrac{\alpha, \chi}{\mathfrak{p}}\right)$ (Chevalley's norm residue symbol)

§ 7. Case of global fields (Hasse's principle)

k : alg. n. f. or alg. func. f. over finite f.

v : (eq. class. of) valuation of k

$\begin{cases} \text{discrete, residue class f. finite} & \mathfrak{p} \\ \text{archimedean} & \lambda \quad (1 \leq \lambda \leq r_1 + r_2) \end{cases}$

$\alpha \in k^* \implies \prod_v |\alpha|_v = 1$ (Hasse's product formula)

$A_k = \prod_v' k_v$ adèle rest. product w.r.t. $\{\mathfrak{o}_\mathfrak{p}\}$

$\cup$

k discrete, A_k / k compact

$I_k = \prod_v' k_v^*$ idèle rest. product w.r.t. $\{\mathfrak{u}_\mathfrak{p}\}$

$\cup$

k^* discrete, I_k^0 / k^* compact

$I_k \ni \tilde{\alpha} = (\alpha_v)$ $|\tilde{\alpha}|_A = \prod_v |\alpha_v|_v$

<u>Approximation th.</u> $S = \{v\}$ finite set

$\forall \ \xi_v \in k_v \ (v \in S)$, $\varepsilon > 0$

$\exists \ \xi \in k$, $|\xi_v - \xi|_v < \varepsilon$

i.e. $A_S = \prod_{v \in S} k_v$, $pr_{A_S} k$ dense in A_S

∘ global class field theory for quadratic extensions

$(\tilde{C})$ 1) k'/k quad. ext., $\alpha \in k^*$ に対し

$$\alpha \in N_{k'/k}(k'^*) \iff \forall v,\ \alpha \in N_{k'_{v'}/k_v}(k'^*_{v'})$$

2) $$\alpha \in (k^*)^2 \iff \forall v,\ \alpha \in (k_v^*)^2$$

1) $$N_{k'/k}(I_{k'}) \cap k^* = N_{k'/k}(k'^*)$$

2) $$(I_k)^2 \cap k^* = (k^*)^2$$

$I_k \ni \tilde{\alpha} = (\alpha_v),\ \tilde{\beta} = (\beta_v)$

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle = \prod_v \left(\frac{\alpha_v, \beta_v}{v}\right)$$

non-degenerate, symmetric pairing $I_k/(I_k)^2 \times I_k/(I_k)^2 \to \mathbb{T}$

3) $\alpha, \beta \in k^*$

$$\langle \alpha, \beta \rangle = \prod_v \left(\frac{\alpha, \beta}{v}\right) = 1 \quad \text{(Hilbert's product formula)}$$

逆に $\chi_v = \pm 1$, $\prod_v \chi_v = 1 \Rightarrow \exists (\alpha, \beta),\ \chi_v = \left(\frac{\alpha, \beta}{v}\right)$

そのとき (using 1))

annihilator of $k^*(I_k)^2 = k^*(I_k)^2$

$$\left(I_k / k^*(I_k)^2\right)^\wedge = k^*(I_k)^2/(I_k)^2 \underset{\text{(by 2))}}{\cong} k^*/(k^*)^2$$

annihilator of $\{\alpha, (k^*)^2\} = k^* \cdot N(I_{k(\sqrt{\alpha})})$

Brauer group

$\mathcal{B}^{(2)}(k) \ni \mathfrak{a} = \langle \alpha . \beta \rangle$ とおける. (by 1), 3))

1) $\mathfrak{a} \sim 1 \iff \forall v \quad \mathfrak{a}_{k_v} \sim 1$

1), 3) $$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{B}^{(2)}(k) & \longrightarrow & \prod_v{}' \mathcal{B}^{(2)}(k_v) & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & 1 \\ & & \Psi & & \Psi \quad \Psi & & \Psi & & \\ & & \mathfrak{a} & \longrightarrow & (\mathfrak{a}_{k_v}), \ (\mathfrak{a}_{k_v}) & \longrightarrow & \prod_v \left(\frac{\mathfrak{a}_{k_v}}{v}\right) & & \end{array}$$ exact !

<u>Rem.</u> $\mathcal{B}(k) \ni \mathfrak{a} = (\alpha . \chi)$ とおける.

$$1 \rightarrow \mathcal{B}(k) \longrightarrow \prod_v{}' \mathcal{B}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 1$$ exact !

$$I_k \ni \tilde{\alpha} = (\alpha_v) \longrightarrow \langle \tilde{\alpha}, \chi \rangle = \prod_v \left(\frac{\alpha_v \,, \chi}{v} \right)$$

$\cup$

k^*

$C_k = I_k / k^*$ の order finite な char. はすべてこのようにして得られる.

◦ Hasse's principle

Lem. 1 $\quad \nu > 0 \Rightarrow \forall \mu \in K^*$, $Q(x) = \mu$ 解ける.

$\because$) $Q(x) = Q_0(x_0) + \sum_{i=1}^{\nu} \xi_i \xi_i'$

Cor. $\quad Q$ rep. $\mu \iff Q(x) - \mu\eta^2$ rep. 0

Lemma 2 $\quad \mathfrak{A} = (\alpha, \beta)$, $\mathfrak{A}_{k_v} \not\sim 1$ なる v は偶数ヶ (by 3))

Cor. $\quad Q_3(x) = 0$ in k_v が解けない v は偶数ヶ

Lemma 3 $\quad \mathfrak{p} \nmid 2$. K local f.

$$Q_3(x) = \sum_{i=1}^{3} \alpha_i \xi_i^2, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathfrak{u}$$

$\Rightarrow \nu > 0$

$\because$) $\alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 + \alpha_3 \xi_3^2 \equiv 0 \quad (\mathfrak{p})$

$\exists$ non-trivial solution $\xi_3 \neq 0$

$\therefore \quad \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 + \alpha_3 X^2 = 0$ in $\mathfrak{o}$

$\exists$ non-triv. sol. (by Hensel)

or

$\because$) $C^+ = (-\alpha_1\alpha_2, -\alpha_1\alpha_3) \sim 1$

$(V, Q) / k$

Th. 1 $\quad \nu = 0$ in $k \iff \nu = 0$ in k_v $(\exists v)$

$\because$) $(\Rightarrow)$ $n \leq 4$ p.46 の表, 1), 2), 3) より

$n \geq 5$ $\quad Q = Q'_{n-2} - Q''_2$ rep. 0 in $\forall k_v$ とする.

$S = \{v_i\} = \{\mathfrak{p} \mid \text{coeff. of } Q\} \cup \{\mathfrak{p} \mid 2\} \cup \{\lambda\}$

$$Q(x_i) = Q'(x_i) - \underset{\mu_i}{\underset{\|}{Q''(x_i)}} = 0 \quad \text{in } k_{v_i}$$

$\forall \mu_i \neq 0$ としてよい. (by Lem. 1)

e suff. large, $\xi \in k^*_{\mathfrak{p}_i}$ に対し ($\mathfrak{p}_i \in S$)

$$\xi \equiv 1 \ (\mathfrak{p}_i^e) \Longrightarrow \xi \in (k^*_{\mathfrak{p}_i})^2$$

$\exists \mu \in k^*$ s.t.

$$\begin{cases} \mu \equiv \mu_i \ (\mathfrak{p}_i^e) & \text{for } \forall \mathfrak{p}_i \in S \\ \mu \mu_i > 0 & \text{for } v_i \text{ real} \\ \exists \mathfrak{q}, \ \mu : \text{unit} & \text{for } \mathfrak{p} \notin S, \ \mathfrak{p} \neq \mathfrak{q} \end{cases}$$

∵) ~~$v_{\mathfrak{p}_i}(\mu_i) = a_i$ とおけば~~, appr. th. により $\begin{cases} \rho \equiv \mu_i \ (\mathfrak{p}_i^e) \\ \rho \mu_i > 0 \end{cases}$

$\exists \rho \in k^*$, $(\rho) = \mathfrak{a} \prod \mathfrak{p}_i^{a_i}$ $\quad \mathfrak{a}, \mathfrak{p}_i$ 素

Strahlklasse に関する算術級数の定理 により

$\exists \mathfrak{q}$ prime, $\notin S$

$$\mathfrak{q} = \mathfrak{a}(\xi), \quad \begin{cases} \xi \equiv \frac{\mu_i}{\rho} 1 \ (\mathfrak{p}_i^e) \\ \xi \cdot \frac{\mu_i}{\rho} > 0 \quad (v_i \text{ real}) \end{cases}$$

$\mu = \rho \xi$ とおけばよい.

Q', Q'' は rep. μ in $\forall k_v$

$v \in S$ に対してはよい

$\mathfrak{p} \notin S$ $\quad Q' - \mu \eta^2$ rep. 0 in $k_{\mathfrak{p}}$ $\quad\Big\}$ (by Lem. 3)

$\mathfrak{p} \notin S$, $\mathfrak{p} \neq \mathfrak{q}$, $Q'' - \mu \eta^2$ rep. 0 〃

$\mathfrak{p} = \mathfrak{q}$ $\quad$ 〃 $\quad$ in $k_{\mathfrak{q}}$ (by Lem. 1, Cor. / Lem. 2, Cor.)

∴ $Q' - \mu \eta^2$, $Q'' - \mu \eta^2$ rep. 0 in $\forall k_v$

∴ 〃 in k (by induction)

∴ Q', Q'' rep. μ in k, ∴ $Q = Q' - Q''$ rep. 0 in k.

Consequences.

Cor. 1 $\quad Q$ rep. μ in $k \iff Q$ rep. μ in $\forall k_v$

Cor. 2 $\quad [Q]=0$ in $k \iff [Q]=0$ in $\forall k_v$

(induction on n)

Cor. 3 $\quad Q \sim Q'$ in $k \iff Q \sim Q'$ in $\forall k_v$

∵) $n=n'$ のとき, $Q \sim Q' \iff [Q-Q']=0$, ∴ Cor. 2 より

Cor. 4 $\quad Q$ rep. Q' in $k \iff Q$ rep. Q' in $\forall k_v$

∵) $Q = \alpha\xi^2 + Q_1$ としてよい. $(\alpha \in k^*)$

$Q' = \alpha\xi^2 + Q_1'$

Witt's th. により

Q rep. Q' in $K \implies Q_1$ rep. Q_1' in K

∴ induction on n

∘ Witt gr.

$$0 \to \mathfrak{W} \to \prod_v{}' \mathfrak{W}_v \;\cancel{\to 0}$$

$$(Q_v) \to \prod \chi_v$$

$$0 \to \mathfrak{W}_0^+ \to \prod_v{}' \mathfrak{W}_{v0}^+ \to \{\pm 1\} \to 0$$

∘ complete system of invariants

$(n,\ \Delta,\ \chi_\mathfrak{p},\ j_\lambda)$

($n \leqq 3$ のとき, See p. 46.)

$$\chi_\lambda = \begin{cases} 1 & j_\lambda \equiv 0, 1, 2, 7 \quad (8) \\ -1 & j_\lambda \equiv 3, 4, 5, 6 \quad (8) \end{cases}$$

- Relations between invariants

$$c \sim 1 \quad \text{in } k(\sqrt{\Delta}) \qquad (n = 2)$$

$$\Delta \sim (-1)^{\frac{n(n-1)}{2} + \frac{n \mp j_\lambda}{2}} = (-1)^{\frac{n^2 \mp j_\lambda}{2}} \quad \text{in } k_\lambda$$

$$\log_{-1} \chi_\lambda \equiv \begin{cases} \frac{1}{2}\frac{n}{2}\left(\frac{n}{2}+1\right) + \frac{n+j_\lambda}{4} = \frac{n^2+4n}{8} + \frac{j_\lambda}{4} & n,\ \frac{n+j_\lambda}{2} \equiv 0 \ (2) \\ \frac{1}{2}\frac{n-1}{2}\left(\frac{n-1}{2}+1\right) + \frac{n+j_\lambda}{4} = \frac{n^2+2n-1}{8} + \text{"} & n \equiv 1,\ \frac{n+j_\lambda}{2} \equiv 0 \\ \frac{1}{2}\frac{n}{2}\left(\frac{n}{2}-1\right) + \frac{n+j_\lambda-2}{4} = \frac{n^2-4}{8} + \text{"} & n \equiv 0,\ \frac{n+j_\lambda}{2} \equiv 1 \\ \frac{1}{2}\frac{n-1}{2}\left(\frac{n-1}{2}-1\right) + \frac{n+j_\lambda-2}{4} = \frac{n^2-2n-1}{8} + \text{"} & n \equiv 1,\ \frac{n+j_\lambda}{2} \equiv 1 \end{cases}$$

$$\prod_v \chi_v = 1, \qquad j_\lambda \equiv n \quad (2), \quad |j_\lambda| \leqq n$$

これ以外に関係はない．（$n \geqq 4$）

∵）　$n = 2, 3$ のとき，上の関係により，j_λ は Δ，χ_λ から定まる．

	j_λ	Δ	χ_λ
$n = 2$	2	−1	1
	0	1	1
	−2	−1	−1
$n = 3$	3	−1	−1
	1	+1	1
	−1	−1	1
	−3	+1	−1

$n \geqq 4$

$$Q(x) = \alpha \xi^2 + Q'(x')$$

$$\begin{cases} \Delta = (-1)^{n-1} \alpha \, \Delta' \\ \chi_p = \left(\dfrac{(-1)^n 2\alpha,\ 2^{n-1} \Delta'}{p} \right) \cdot \chi_p' \\ j_\lambda = \operatorname{sign} \alpha + j_\lambda' \end{cases}$$

$(\Delta, \chi_p, j_\lambda)$ 任意に与えられたとき

$\alpha > 0$ for λ s.t. $j_\lambda = n$.

$\alpha < 0$ 〃 $j_\lambda = -n$

なる α をとり、Δ', χ_p', j_λ' を上式で def. すれば
やはり上の関係をみたす。 $\therefore$ induction on n.

- Condition for $v = 0$

  $n \geqq 5$ のとき

  $\exists \lambda \quad j_\lambda = \pm n$

  ($n \leqq 4$ のとき, See p. 46)

§ 8. $\mathcal{O}$- lattice

K : field (char. $\neq 2$)

$\mathcal{O}$: subring $\ni 1$, quot. f. of $\mathcal{O}$ $= K$.

<u>Def.</u> $V_{/K} \supset M$ $\mathcal{O}$- lattice とは

1) finitely generated $\mathcal{O}$ - module

2) $V = M \cdot K$ (i.e. M contains a basis of V/K)

<u>Rem.</u> そのとき, M has no torsion

i.e. $x \in M$, $\alpha \in \mathcal{O}$ $\alpha \neq 0$, $x\alpha = 0 \Rightarrow x = 0$

かつ $V = M \otimes_{\mathcal{O}} K$.

逆に M : fin. gen. $\mathcal{O}$ - module without torsion なら

$V = M \otimes_{\mathcal{O}} K$ とおけば, $M \cong M \otimes 1 \subset V$ $\mathcal{O}$- lattice in V.

∵) M : $\mathcal{O}$- lattice in V のとき. $M \otimes K \longrightarrow V$ の kernel は

submodule $\{ \sum (x_i \otimes \xi_i \alpha_i - x_i \alpha_i \otimes \xi_i) \mid x_i \in M, \xi_i \in K, \alpha_i \in \mathcal{O} \}$

逆に M : f. g. $\mathcal{O}$ - module w.t. のとき, $V = M \otimes_{\mathcal{O}} K$ vect. sp. $/K$

$M \ni x \longrightarrow x \otimes_{\mathcal{O}} 1 \in M \otimes_{\mathcal{O}} K$ は $1:1$

$x \otimes_{\mathcal{O}} 1 = 0 \Rightarrow x \otimes 1 = \sum (x_i \otimes \xi_i \alpha_i - x_i \alpha_i \otimes \xi_i)$

$\Rightarrow \exists \beta \in \mathcal{O}$ $x \otimes \beta = \sum (\cdots)$, $\alpha_i, \xi_i \in \mathcal{O}$

$\Rightarrow x\beta = \sum (x_i (\xi_i \alpha_i) - (x_i \alpha_i) \xi_i) = 0$

$\Rightarrow x = 0$.

$V \supset M$ $\mathcal{O}$-lattice, Q : quad. f. on V
$V' \supset M'$ 〃 Q' : 〃 V'

$$(M, Q) \cong (M', Q') \iff \begin{cases} M \xrightarrow[\cong]{\rho} M' & \text{as } \mathcal{O}\text{-module} \\ Q'(\rho(x)) = Q(x) & \text{for } x \in M \end{cases}$$

そのとき

$(V, Q) \cong (V', Q')$ by the unique ext. of ρ

よって一つの (V, Q) の中で考えれば十分.

$V \supset M, M'$

$$(M, Q) \cong (M', Q) \iff \begin{cases} \exists \rho \in O(V, Q), \\ M' = \rho(M) \end{cases}$$

<u>Prob.</u> (V, Q) given $\forall$ $\mathcal{O}$-lattice M を $\cong$ に関し分類すること

特に $\mathcal{O}$-lattice with basis に対しては.

$\mathcal{E}$ = set of all basis $(e_1, \cdots, e_n)$ of V/K

$GL(V)$ operates on $\mathcal{E}$ from left

$GL(n, K)$ 〃 from right

$\mathcal{E}/GL(n, \mathcal{O})$ = set of all $\mathcal{O}$-lattices with basis

$O(V, Q)\backslash\mathcal{E}$ = set of all symmetric matrices rep. Q

$$(e_1, \cdots, e_n) \longrightarrow A = \left(\tfrac{1}{2} B(e_i, e_j)\right)$$

$O(V, Q)\backslash\mathcal{E}/GL(n, \mathcal{O})$ = set of all classes of $\mathcal{O}$-lattices with basis

= set of all classes of sym. mat. rep. Q w.r.t. $GL(n, \mathcal{O})$

$$\left(A' \sim A \iff A' = {}^tTAT,\ T \in GL(n, \mathcal{O}) \right)$$

k : alg. n. f.

$\mathfrak{o}$: max. order in k

$k_{\mathfrak{p}}$, $\mathfrak{o}_{\mathfrak{p}}$ $\quad (\mathfrak{o}_{\mathfrak{p}_\infty} = k_{\mathfrak{p}_\infty})$

$$\begin{array}{ccc} V/k \supset M/\mathfrak{o} & & \mathfrak{o}\text{- lattice} \\ & & \downarrow \\ Vk_{\mathfrak{p}} \supset M\mathfrak{o}_{\mathfrak{p}} & & \mathfrak{o}_{\mathfrak{p}}\text{- lattice} \end{array}$$

<u>Def.</u> $M \approx M' \iff \forall \mathfrak{p} \quad M\mathfrak{o}_{\mathfrak{p}} \cong M'\mathfrak{o}_{\mathfrak{p}}$

その eq. cl. を genus という.

そのとき $\forall \mathfrak{p} \quad Vk_{\mathfrak{p}} \cong V'k_{\mathfrak{p}} \quad \therefore \quad V \cong_{/k} V'$

よって 一つの (V, Q) の中で考えれば十分.

$$\underset{\text{class}}{M \cong M'} \implies \underset{\text{genus}}{M \approx M'}$$

<u>Th.</u> A genus consists of a finite number of classes.

*) $\mathfrak{o}$ 単に order in k, i.e. subring $\ni 1$, $[\mathfrak{o} : \mathbb{Z}] = [k : \mathbb{Q}]$, で成立.

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∘ Considerations at infinite places*)

$$k_{\mathfrak{p}_{\infty,\lambda}} = k_\lambda \qquad (1 \leq \lambda \leq r) \qquad r = r_1 + r_2$$

$$k_{\mathbb{R}} = k \otimes_{\mathbb{Q}} \mathbb{R} = \sum k_\lambda$$

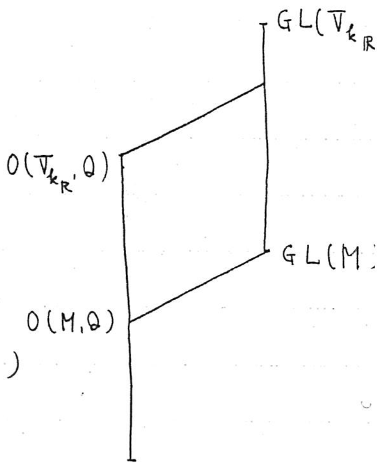
$$V_{k_{\mathbb{R}}} = V \otimes_k (k \otimes_{\mathbb{Q}} \mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R} = \sum V_{k_\lambda}$$

$$V_{k_{\mathbb{R}}} \supset V \supset M \qquad \text{dense} \quad \text{discrete}$$

$$\left(\begin{array}{lll} M = \{e_1, \cdots, e_n\}_{\mathfrak{o}} & \text{とします}. & \mathfrak{o} = \{\omega_1, \cdots, \omega_n\}_{\mathbb{Z}} \\ V = \{e_1, \cdots, e_n\}_k & & k = \{\ \ "\ \ \}_{\mathbb{Q}} \\ V_{k_{\mathbb{R}}} = \{\ \ "\ \ \}_{k_{\mathbb{R}}} & & k_{\mathbb{R}} = \{\ \ "\ \ \}_{\mathbb{R}} \end{array}\right.$$

$$GL(V_{k_{\mathbb{R}}}) = \prod_\lambda GL(V_{k_\lambda})$$

$$GL(V_{k_{\mathbb{R}}}) \supset GL(V) \supset GL(M) \qquad \text{dense} \quad \text{discrete}$$



$$O(V_{k_{\mathbb{R}}}, Q) = \prod_\lambda O(V_{k_\lambda}, Q^{(\lambda)})$$

$$\underbrace{O(V_{k_{\mathbb{R}}}, Q) \supset O(V, Q) \supset O(M, Q)}_{\text{discrete}}$$

Th. $GL(M)$, $O(M, Q)$ finitely generated

$$\left.\begin{array}{l} SL(V_{k_{\mathbb{R}}}) / SL(M) \\ O(V_{k_{\mathbb{R}}}, Q) / O(M, Q) \end{array}\right\} \text{volume finite}$$

§ 9. Minkowski's reduction theory ($k = \mathbb{Q}$)

Minkowski, Diskontinuitätsbereich für arithmetische Äquivalenz. Ges. W. II (1911), 53 - 100

Humbert, Com. Math. Helv. 12 (1939 - 40)

Siegel, Einheiten quadratischer Formen, Abh. Math. Sem. Hamb. 13 (1940), 207 - 239.

Weyl, Theory of reduction for arithmetical equivalence, I, Trans. A.M.S. 48 (1940), 126 - 164, II, " 51 (1942), 203 - 231.

——, Fundamental domains for lattice groups in division algebras Com. Math. Helv. 17 (1944-45), 283 - 306.

Weil, Discontinuous subgroups of classical groups, Chicago (1958)

Reduction des formes quadratiques d'après Minkowski et Sieg
Groupes des formes quadratiques indéfinies et des formes bilinéaires alternées, Cartan Sém. 1957-58, Exp 1, 2.

$V_{/\mathbb{R}} \supset M_{/\mathbb{Z}}$ $\qquad M = \{e_1, \cdots, e_n\}_{\mathbb{Z}}$

$G = GL(V) \cong GL(n, \mathbb{R})$

$\Gamma = GL(M) \cong GL(n, \mathbb{Z})$

G/Γ ?

$\mathcal{L}$ = space of $\forall$ lattices in V

$\mathcal{E}$ = space of $\forall$ basis of V

$$\mathcal{L} = G/\Gamma = \mathcal{E}/GL(n, \mathbb{Z}) \cong GL(n, \mathbb{R})/GL(n, \mathbb{Z})$$

$$\left(M' = \rho M = \{e_1', \cdots, e_n'\}_{\mathbb{Z}}, \quad (e_1', \cdots, e_n') = (e_1, \cdots, e_n) X \right)$$

$S = \mathcal{P}(n, \mathbb{R})$ = space of $\forall$ pos. def. sym. mat. $\qquad$ open convex cone in $\mathbb{R}^{\frac{n(n+1)}{2}}$.

$$\mathcal{P}(n, \mathbb{R}) = O(n)\backslash GL(n, \mathbb{R}) = O(V, Q_0)\backslash \mathcal{E}$$

$$(e_1', \cdots, e_n') \to A' = (Q_0(e_i', e_j'))$$

$\mathbb{F}$: fundamental dom. of $GL(n, \mathbb{Z})$ in S を求める.

$S \ni A = (\alpha_{ij}) \qquad \alpha_i = \alpha_{ii}$

$$A \prec A' \overset{def}{\Longleftrightarrow} (\alpha_1, \cdots, \alpha_n) \leq (\alpha_1', \cdots, \alpha_n')$$

in lexicographical linear order

∘ A : given $\qquad A[\Gamma] = \{ A[X] \mid X \in \Gamma = GL(n, \mathbb{Z}) \}$

の中に $\exists$ lowest elem.

$\left(\because \right) \quad X = (x^{(1)}, \cdots, x^{(n)}) \qquad A' = A[X]$

$\alpha_i' = A[x^{(i)}] \leq \alpha_i$ の解 有限

- A : lowest $\Rightarrow$ (i) $\begin{cases} A[x] \geqq \alpha_i \\ \text{for } \forall\ x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{Z}^n \neq \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} i \\ (\xi_i, \cdots, \xi_n)\ 互に素 \end{cases}$

$$\left(\begin{aligned} \because)\quad & X = \begin{pmatrix} 1 & & & \xi_1 & \\ & \ddots & & \vdots & \\ & & 1 & & \\ & & & \boxed{\begin{matrix} \xi_i & \\ \vdots & * \\ \xi_n & \end{matrix}} \end{pmatrix} \in GL(n, \mathbb{Z}) \\ & A[X] = \begin{pmatrix} \alpha_1 & & & \\ & \ddots & & \\ & & \alpha_{i-1} & \\ & & & A[x] & \\ & & & & \ddots \end{pmatrix} \end{aligned} \right.$$

<u>Def.</u> $\mathcal{P}(n, \mathbb{R}) \ni A = (\alpha_{ij})$ reduced　その全体 $\mathbb{F}$

(i) $\cdots$

(ii) $\alpha_{i,i+1} \geqq 0$

e.g. $\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$ $\alpha_1 \leqq \cdots \leqq \alpha_n$ は reduced

<u>Th.</u> 1 (Minkowski) 1) $\mathbb{F}$: closed, convex cone bounded by <u>finitely many</u> hyperplanes

2) $\mathbb{F}[\Gamma] = \mathcal{P}(n, \mathbb{R})$

3) $\mathbb{F}[X] \cap \mathbb{F} \neq \phi$ for only <u>finitely many</u> $X \in \Gamma$

$\mathbb{F}^i[X] \cap \mathbb{F} = \phi$ if $X \neq \pm 1_n$

$\mathbb{F}[X] \cap \underset{\substack{\vdots \\ \text{compact set in } \mathcal{P}(n, \mathbb{R})}}{C} = \phi$ for only <u>finite many</u> $X \in \Gamma$

4) $v(\underline{\mathbb{F}}) < \infty$

有限性以外は容易.

<u>Th.</u> 2 $\quad A = (\alpha_{ij}) \in \mathbb{F}$

1) $\alpha_1 \leqslant \cdots \leqslant \alpha_n$

2) $2|\alpha_{ij}| \leqslant \alpha_i \qquad (i<j)$

3) $\alpha_1 \cdots \alpha_n \leqslant c_n \det(A)$

1), 2) は容易. 3) 後述.

<u>Rem.</u> A not nec. pos. def. (i) $\Longrightarrow$ $A = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A' \end{array}\right)$

$A' > 0$, (i)

$$\overline{\mathbb{F}} = \bigcup_{i=0}^{n} \mathbb{F}_i$$

(closure of $\mathbb{F}$ in $\mathbb{R}^{\frac{n(n+1)}{2}}$)

<u>Ex.</u> $n=2$

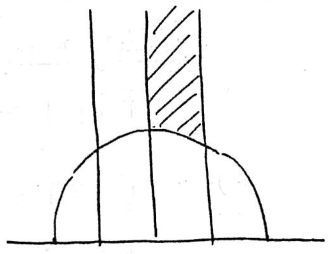
$$\alpha_1 \leqslant \alpha_2, \qquad 2|\alpha_{12}| \leqslant \alpha_1, \qquad (\alpha_{12} \geqslant 0)$$

$$\alpha_1 \xi_1^2 + 2\alpha_{12}\xi_1\xi_2 + \alpha_2\xi_2^2 = \alpha_1(\xi_1 + z\xi_2)(\xi_1 + \bar{z}\xi_2)$$

$$|z| \geqslant 1, \qquad \Re z \leqslant \frac{1}{2} \qquad (\Re z \geqslant 0)$$

$$c_2 = \frac{4}{3}$$

$n \leqslant 5$ Minkowski



∘ Jacobi transform

$$A = D[T] \quad , \quad D = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix} , \quad T = \begin{pmatrix} 1 & & \tau_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

uniquely

$$\delta_i > 0$$

一般に

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2' \end{pmatrix} \left[\begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix} \right]$$

uniquely

$$\begin{cases} A_{12} = A_1 T_{12} \\ A_2 = A_1[T_{12}] + A_2' \end{cases} \qquad \begin{cases} T_{12} = A_1^{-1} A_{12} \\ A_2' = A_2 - A_1^{-1}[A_{12}] \end{cases}$$

よって induction

$c > 0$

$$\mathfrak{S}_c = \{ A = D[T] \mid \delta_i < c\,\delta_{i+1} , \ |\tau_{ij}| < c \}$$

Siegel domain

$$c > c' \Rightarrow \mathfrak{S}_c \supset \mathfrak{S}_{c'}$$

$$\bigcup_c \mathfrak{S}_c = \mathcal{P}(n, \mathbb{R})$$

c 十分大

$$F \subset \mathfrak{S}_c$$

$\because$) $A \in F$ のとき

$$1 \leqslant \frac{\alpha_1}{\delta_1} \cdots\cdots \frac{\alpha_n}{\delta_n} \prec 1 \qquad \therefore \ \alpha_i \prec \delta_i \qquad \therefore \ \alpha_i \sim \delta_i$$

$$\alpha_{ij} = \delta_1 \tau_{1i} \tau_{1j} + \cdots + \delta_{i-1} \tau_{i-1\,i} \tau_{i-1\,j} + \delta_i \tau_{ij} \qquad (i<j)$$

$$\therefore \quad |\tau_{ij}| \prec 1 \qquad \text{by induction}$$

$$\alpha_i = \delta_i + \sum_{j=1}^{i-1} \delta_j \tau_{ij}^2 \geqslant \delta_i$$

Th. 3 (Siegel) $c > 0$, $m \geqq 1$

$S_c[X] \cap S_c \neq \phi$ for only finitely many $X \in M(n, \mathbb{Z})$, $|\det(X)| \leqq m$

後述. これから Th. 1, 1), 3) の有限性がでる.

Proof of Th. 2, 3)

Q_0 : pos. def. q. f. on V

$A = (Q_0(e_i, e_j))$ reduced

(i) $(e_1, \cdots, e_n)$ basis of $M/\mathbb{Z}$ s.t.

$e_1, \cdots, e_{i-1}$ chosen

$$\alpha_i = \mathrm{Min}\{ Q_0(x) \mid (e_1, \cdots, e_{i-1}, x, * \cdots *) \text{ basis of } M \}$$
$$= Q_0(e_i)$$

(ii) $Q(e_i, e_{i+1}) \geqq 0$

そのとき

(I) $\alpha_1 \cdots \alpha_n \leqq c_n \det(A)$

$(e_1, \cdots, e_{i-1}, e_i, * \cdots *)$ basis of M

$$\Longleftrightarrow \begin{cases} e_1, \cdots, e_i & \text{lin. indep.} \\ \{e_1, \cdots, e_i\}_{\mathbb{R}} \cap M = \{e_1, \cdots, e_i\}_{\mathbb{Z}} \end{cases}$$

このとき $e_1, \cdots, e_i$: primitive という.

$(e_1', \cdots, e_n')$ basis of V (not nec. of M)

s.t. $e_1', \cdots, e_{i-1}'$ chosen

$$\alpha_i' = \operatorname{Min} \{ Q_0(x) \mid x \in M,\ x \notin \{e_1', \cdots, e_{i-1}'\}_{\mathbb{R}} \}$$
$$= Q_0(e_i')$$

$\alpha_1' \leqslant \alpha_2' \leqslant \cdots \leqslant \alpha_n'$ successive minima

. uniquely determined by M, Q_0

($\because$) $\alpha_i' \leqslant \lambda < \alpha_{i+1}' \iff \{x \in M \mid Q_0(x) \leqslant \lambda\}$ has i lin. indep. vec.

そのとき

(II) $\alpha_1' \cdots \alpha_n' \leqslant c_n' \det(A)$ $\qquad c_n' = \dfrac{2^{2n}}{\gamma_n^2}$

(Minkowski's 2nd fund. ineq. for bounded convex body
Cf. Weyl, loc. cit or Siegel - Weyl - Mahler
Geometry of numbers, Princeton

Lem. 1. (Minkowski) $S \subset V$

$$\left. \begin{array}{l} S : \text{convex body} \\ \quad\ \text{symmetric w.r.t. } 0 \\ \quad\ v(S) > 2^n \end{array} \right\} \Rightarrow S \cap M \ni^{\exists} x \neq 0$$

$(\ v(V/M) = 1\ )$

Lem. 2 $\quad \displaystyle \operatorname*{Min}_{\substack{x \in M \\ x \neq 0}} Q_0(x) \leqslant \frac{4}{\gamma_n^{\frac{2}{n}}} \det(A)^{\frac{1}{n}}$

$$\gamma_n = \text{vol. of } n\text{-ball} = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$$

($\because$) vol. of $\{Q_0(x) \leqslant c\} = \gamma_n \det(A)^{-\frac{1}{2}} c^{\frac{n}{2}}$

Proof of (II) : $A' = (Q_0(e_i', e_j')) = D[T]$

$$(e_1', \cdots, e_n') = (e_1'', \cdots, e_n'')T$$

$$\begin{pmatrix} \xi_1'' \\ \vdots \\ \xi_n'' \end{pmatrix} = T \begin{pmatrix} \xi_1' \\ \vdots \\ \xi_n' \end{pmatrix}$$

$$Q_0(x) = \sum \delta_i \xi_i''^2$$

$$Q_1(x) = \sum \frac{\delta_i}{\alpha_i'} \xi_i''^2 \quad \text{とおく.}$$

By Lem. 2

$$\underset{\substack{x \in M \\ x \neq 0}}{\mathrm{Min}} \; Q_1(x) \leqslant \frac{4}{\gamma_n^{\frac{2}{n}}} \left(\frac{\det(A)}{\alpha_1' \cdots \alpha_n'} \right)^{\frac{1}{n}}$$

$$x \notin \{e_1', \cdots, e_{i-1}'\}_{\mathbb{R}}, \quad \in \{e_1', \cdots, e_i'\}_{\mathbb{R}}$$

$$x \longleftrightarrow \begin{pmatrix} \xi_1' \\ \vdots \\ \xi_i' \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \xi_1'' \\ \vdots \\ \xi_i'' \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$Q_1(x) = \sum_{j=1}^{i} \frac{\delta_j}{\alpha_j'} \xi_j''^2 \geqslant \frac{1}{\alpha_i'} \sum_{j=1}^{i} \delta_j \xi_j''^2 = \frac{1}{\alpha_i'} Q_0(x) \geqslant 1$$

Lem. 3 $\quad e_1, \cdots, e_{i-1} \in M$ lin. indep., primitive

$$x \in M, \quad x \notin \{e_1, \cdots, e_{i-1}\}_{\mathbb{R}}$$

$$\Longrightarrow \exists\, x' \in M \quad \{e_1, \cdots, e_{i-1}, x'\} \text{ primitive}$$

$$\{e_1, \cdots, e_{i-1}, x\}_{\mathbb{R}} = \{e_1, \cdots, e_{i-1}, x'\}_{\mathbb{R}}$$

(∵) $\left\{ \xi \,\middle|\, \sum_{j=1}^{i-1} \xi_j e_j + x\xi \in M \right\}$ is ideal in $\mathbb{Q}$ $\supset (1)$

$x' = x$ or $|\xi_j| \leqslant \frac{1}{2}$, $|\xi| \leqslant \frac{1}{2}$ とできる.

$$\therefore \quad Q_0(x')^{\frac{1}{2}} \leqslant \frac{1}{2} \left(Q_0(e_1)^{\frac{1}{2}} + \cdots + Q_0(e_{i-1})^{\frac{1}{2}} + Q_0(x)^{\frac{1}{2}} \right)$$

Lem. 4. $\alpha_i \leqslant {}^{\exists}\theta_i \, \alpha_i'$ $\qquad \theta_1 = 1$

$$\theta_i = \left(\frac{3}{2}\right)^{2(i-2)} \qquad i \geqslant 2$$

$\because$) $\alpha_1 = \alpha_1'$, $i-1$ までいえたとする.

$\exists k$, $1 \leqslant k \leqslant i$, $\quad e_k' \notin \{e_1, \cdots, e_{i-1}\}_{\mathbb{Z}}$

By Lem. 3, $\exists x \in M$

$$\{e_1, \cdots, e_{i-1}, e_k'\}_{\mathbb{R}} = \{\underbrace{e_1, \cdots, e_{i-1}, x}_{\text{primitive}}\}_{\mathbb{R}}$$

$$\therefore \quad \alpha_i \leqslant Q_0(x)^{\frac{1}{2}} \leqslant \operatorname{Max}\left\{\frac{1}{2}\left\{Q_0(e_1)^{\frac{1}{2}} + \cdots + Q_0(e_{i-1})^{\frac{1}{2}} + Q_0(e_k')^{\frac{1}{2}}\right\}, Q_0(e_k')^{\frac{1}{2}}\right\}$$

$$\leqslant \operatorname{Max}\left\{\frac{1}{2}\left(\alpha_1^{\frac{1}{2}} + \cdots + \alpha_{i-1}^{\frac{1}{2}} + \alpha_i'^{\frac{1}{2}}\right), \alpha_i'^{\frac{1}{2}}\right\}$$

$$\leqslant \operatorname{Max}\left\{\frac{1}{2}\left(1 + 1 + 1 + \cdots + \left(\frac{3}{2}\right)^{i-3}\right), 1\right\}$$

$$\frac{1}{2}\left(2 + \frac{\left(\frac{3}{2}\right)^{i-2} - 1}{\frac{3}{2} - 1}\right) = \left(\frac{3}{2}\right)^{i-2}$$

よって (I) が成立.

$$c_n = \left(\frac{3}{2}\right)^{(n-1)(n-2)} \frac{2^{2n}}{\gamma_n^2} = 2^{2n} \left(\frac{3}{2}\right)^{(n-1)(n-2)} \frac{\Gamma\left(1+\frac{n}{2}\right)^2}{\pi^n}$$

記号： $H\{X\} = {}^t\bar{X} H X$ for $X \in GL(n,*)$, $H \in \mathcal{P}(n,*)$

§10. Reduction of $GL(n, k_{\mathbb{R}}) / GL(n, \mathfrak{o})$

G : locally compact group.

K : compact subgr.

$S = K \backslash G$ $\qquad$ G operates almost effectively

i.e. $\bigcap_{g \in G} g K g^{-1} = $ finite

$G \supset \Gamma$

Γ : discrete $\iff$ Γ operates properly discontinuously on S

i.e. $\forall \underset{\text{compact}}{C} \subset S$, $\quad C\gamma \cap C \neq \phi$ for only finitely many $\gamma \in \Gamma$

$\iff$ $\forall x \in S$ $\quad \exists$ nbd. U

$U\gamma = U \qquad \gamma \in \Gamma_x$ (finite gr.)

$U\gamma \cap U = \phi \qquad \gamma \notin \Gamma_x$

$\forall x, x'$ not eq. w.r.t. Γ $\quad \exists\, U, U'$

$U\Gamma \cap U' = \phi$

$$G = GL(n, k_{\mathbb{R}}) = \prod_{\lambda=1}^{r} GL(n, k_\lambda)$$
$$K = U(n, k_{\mathbb{R}}, H_0) = \prod U(n, k_\lambda, H_0^{(\lambda)})$$
$$S = \mathcal{P}(n, k_{\mathbb{R}}) = \prod \mathcal{P}(n, k_\lambda)$$

$\Gamma = GL(n, \mathfrak{o})$ $\qquad$ ($\mathfrak{o}$: not nec. maximal)

$$M_n(k_{\mathbb{R}}) \ni X = (\underset{\substack{\cap \\ k_{\mathbb{R}}}}{\xi_{ij}}) = (X^{(\lambda)}) = (\underset{\substack{\cap \\ k_\lambda}}{\xi_{ij}^{(\lambda)}})$$

$$Tr(X) = Tr_{k_{\mathbb{R}}/\mathbb{R}}(tr(X))$$
$$N(X) = N_{k_{\mathbb{R}}/\mathbb{R}}(\det(X))$$

*) $\mathfrak{S}$: cone i.e. $A \in \mathfrak{S} \Rightarrow \alpha A \in \mathfrak{S}$ $(\alpha > 0)$

$$\mathfrak{S} \ni A = D\{T\} = {}^tT D T$$

$$D = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}, \quad T = \begin{pmatrix} 1 & & \tau_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

uniquely, $\quad \alpha_i^{(\lambda)} \geqslant \delta_i^{(\lambda)} > 0$

<u>Def.</u> $\mathcal{P}(n, k_{\mathbb{R}}) \supset \mathfrak{S} = \mathfrak{S}_c$ Siegel domain*)

$$\mathfrak{S}_c = \left\{ A = D\{T\} \;\middle|\; \begin{array}{l} \delta_i^{(\lambda)} < c\, \delta_i^{(\lambda')} \\ \delta_i^{(\lambda)} < c\, \delta_{i+1}^{(\lambda)} \\ |\tau_{ij}^{(\lambda)}| < c \end{array} \right\}$$

<u>Th.</u> 1 (Minkowski - Siegel) $\exists\, c > 0$, $m \geqslant 1$.

$\forall A \in \mathcal{P}(n, k_{\mathbb{R}})$, $\exists\, X \in M_n(\theta)$, $0 < |N(X)| \leqslant m$

s.t. $\quad A\{X\} \in \mathfrak{S}_c$

<u>Lem</u> 1. $k_{\mathbb{R}} \ni \delta = (\delta^{(\lambda)})$, $\quad \delta^{(\lambda)} > 0$

$$\frac{1}{d} \operatorname{Tr}(\delta) \geqslant N(\delta)^{\frac{1}{d}}$$

特に $\quad \operatorname{Tr}(\delta) \sim N(\delta)^{\frac{1}{d}} \iff \delta^{(\lambda)} \sim \delta^{(\lambda')} \quad (\forall \lambda, \lambda')$

そのとき

$$\operatorname{Tr}(\delta) \sim N(\delta)^{\frac{1}{d}} \sim \delta^{(\lambda)}$$

$\because$) $\quad \delta^{(\lambda_1)} = \operatorname{Max} \delta^{(\lambda)}$

$(\Rightarrow)$ $\quad \delta^{(\lambda_2)} = \operatorname{Min} \delta^{(\lambda)}$

$$\delta^{(\lambda_1)} \leqslant \operatorname{Tr}(\delta) \prec N(\delta)^{\frac{1}{d}} \leqslant \delta^{(\lambda_1)\frac{d-1}{d}} \cdot \delta^{(\lambda_2)\frac{1}{d}}$$

$$\therefore \quad \delta^{(\lambda_1)} \prec \delta^{(\lambda_2)}$$

<u>Proof of Th. 1</u> $\quad Q_0(x) = \mathrm{Tr}\, H_0(x)$ pos. def. q. f. on $V_{k_{\mathbb{R}}} / \mathbb{R}$ (dn)

$e_1', \cdots, e_n'$ as follows

$$\alpha_i' = \mathrm{Min}\,\{ Q_0(x) \mid x \in M,\ x \notin \{e_1', \cdots, e_{i-1}'\}_k \}$$
$$= Q_0(e_i')$$

$$\alpha_1' \leqq \alpha_2' \leqq \cdots \leqq \alpha_n'$$

$$(e_1', \cdots, e_n') = (e_1, \cdots, e_n) X, \qquad X \in M_n(\mathfrak{o})$$

$$A' = (H_0(e_i', e_j')) = A[X]$$
$$= D[T]$$

$$(e_1', \cdots, e_n') = (e_1'', \cdots, e_n'') T$$

e_i'': basis of $V_{k_{\mathbb{R}}} / k_{\mathbb{R}}$

$\because$) $\quad V_{k_{\mathbb{R}}} \ni x \underset{(e_i')}{\longleftrightarrow} (\xi_i) \underset{(e_i'')}{\longleftrightarrow} (\xi_i'')$

$$H_1(x) = \sum_i \frac{\delta_i}{\alpha_i'} \bar{\xi}_i'' \xi_i'', \qquad Q_1(x) = \mathrm{Tr}\, H_1(x) \text{ とおく.}$$

$$\boxed{1) \quad N(X)^2 \prod_i \left(\frac{\alpha_i'^{\,d}}{N\delta_i} \right) \prec 1}$$

$$\underset{\substack{x \in M \\ x \neq 0}}{\mathrm{Min}}\, Q_1(x) \prec \det\left(Q_1(e_i \omega_k, e_j \omega_l) \right)^{\frac{1}{nd}}$$
$$\parallel$$
$$N\left((H_1(e_i, e_j)) \right)^{\frac{1}{nd}} \det\left((\mathrm{Tr}\,(\bar{\omega}_k \omega_l)) \right)^{\frac{1}{d}}$$
$$\parallel$$
$$\left(\left(\prod \alpha_i'^{-d} \right) N(A^*) \right)^{\frac{1}{nd}} \qquad //$$

左辺 $\geqq 1$

2) $N(X) \sim 1$

$\delta_i^{(\lambda)} \sim \delta_i^{(\lambda')}$,

$\delta_i^{(\lambda)} \prec \delta_{i+1}^{(\lambda')}$

$\because$) $N(X) \geqq 1$, $\quad \alpha_i' \geqq Tr(\delta_i) \succ N(\delta_i)^{\frac{1}{d}}$

よって 1) から

$$N(X) \prec 1, \qquad \frac{\alpha_i'^{\,d}}{N(\delta_i)} \prec 1$$

$$\therefore \quad Tr(\delta_i) \sim N(\delta_i)^{\frac{1}{d}} \sim \delta_i^{(\lambda)}$$

$$\therefore \quad \delta_i^{(\lambda)} \sim \alpha_i' \leqq \alpha_{i+1}' \sim \delta_{i+1}^{(\lambda)}$$

3) $|\tau_{ij}^{(\lambda)}| \prec 1$

$\because$) $\alpha_i' \leqq Q_0 \left(e_i' + \sum_{j=1}^{h} e_j' \eta_j \right) \qquad h < i$

$$y \underset{(e_i')}{\longleftrightarrow} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_h \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \underset{(e_i'')}{\longleftrightarrow} \begin{pmatrix} \zeta_1 \\ \vdots \\ \\ \zeta_n \end{pmatrix} = T \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_h \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{cases} \zeta_1 = \eta_1 + \tau_{12}\eta_2 + \cdots + \tau_{1h}\eta_h + \tau_{1i} \\ \quad \cdots \\ \zeta_{h-1} = \eta_{h-1} + \tau_{h-1\,h}\eta_h + \tau_{h-1\,i} \\ \zeta_h = \eta_h + \tau_{h\,i} \\ \zeta_{h+1} = \tau_{h+1\,i} \\ \quad \cdots \\ \zeta_{i-1} = \tau_{i-1\,i} \\ \zeta_i = 1 \\ \zeta_{i+1} = 0 \\ \quad \cdots \end{cases}$$

$$\alpha_i' = \sum_{j=1}^{i-1} \mathrm{Tr}\,(\delta_j\,|\tau_{ji}|^2) + \mathrm{Tr}\,\delta_i \leqq \sum_{j=1}^{n} \mathrm{Tr}\,(\delta_j\,|\zeta_j|^2)$$

左辺 = $(右辺)_{\eta=0}$ より

$$\sum_{1}^{h} \mathrm{Tr}\,(\delta_j\,|\tau_{ji}|^2) \leqq \sum_{1}^{h} \mathrm{Tr}\,(\delta_j\,|\zeta_j|^2)$$

$$\therefore \quad \alpha_h' \,\mathrm{Tr}\,|\tau_{hi}|^2 \prec \alpha_h' \sum_{1}^{h} \mathrm{Tr}\,|\zeta_j|^2$$

$\exists\ \eta_1, \cdots, \eta_h$, s.t. $\zeta_1, \cdots, \zeta_h$ in a fund. dom. of $k_{\mathbb{R}}/\mathfrak{o}$

$\therefore$ bounded

$$\therefore \quad |\tau_{hi}^{(\lambda)}| \prec 1$$

<u>Th.</u> 2 (Siegel) $\mathfrak{S} = \mathfrak{S}_c$, $m > 0$

$\Rightarrow$ $\mathfrak{S}[X] \cap \mathfrak{S} \neq \phi$ only for finitely many $X \in M_n(\mathfrak{o})$,
$0 < |N(X)| \leqq m$

$\because$) $n = 1$ trivial, induction on n

1) $$X = \begin{pmatrix} X_1 & X_{12} \\ 0 & X_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$$

$A, A' \in \mathfrak{S}$, $A[X] = A'$

$$A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix} = D[T], \quad D = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}$$

$$= \begin{pmatrix} D_1[T_1] & \\ & D_2[T_2] \end{pmatrix} \left[\begin{pmatrix} 1 & T_1^{-1} T_{12} \\ 0 & 1 \end{pmatrix} \right]$$

$$\begin{cases} A_1 = D_1\{T_1\} \in \mathfrak{S}^{(n_1)} \\ A_2 - A_1^{-1}\{A_{12}\} = D_2\{T_2\} \in \mathfrak{S}^{(n_2)} \\ A_1^{-1}A_{12} = T_1^{-1}T_{12} \quad \text{bounded} \end{cases}$$

$A[X] = A'$ なら

$$\begin{cases} A_1' = A_1\{X_1\} \\ A_2' - A_1'^{-1}\{A_{12}'\} = (A_2 - A_1^{-1}\{A_{12}\})\{X_2\} \\ A_1'^{-1}A_{12}' = X_1^{-1}A_1^{-1}A_{12}X_2 + Y \end{cases}$$

induction の假定により X_1, X_2 有限, $\therefore$ Y 有限.

2) X : irreducible のとき

$X = (\xi_{ij})$ $\quad \forall\ 1 \leq i \leq n-1 \quad \exists\ h(i), k(i) \quad \xi_{h(i)k(i)} \neq 0$

$k(i) \leq i < i+1 \leq h(i)$

$A' = A\{X\}$

$$\begin{aligned} \operatorname{Tr}\delta_i' \sim \operatorname{Tr}\alpha_i' &= \operatorname{Tr} A\{x^{(i)}\} \sim \operatorname{Tr} D\{x^{(i)}\} \quad (\text{by Lem 2}) \\ &\geqq \operatorname{Tr}(\delta_j |\xi_{ji}|^2) \succ \operatorname{Tr}\delta_j \cdot \operatorname{Tr}|\xi_{ji}|^2 \\ &\succ \operatorname{Tr}\delta_j \qquad \text{if } \xi_{ji} \neq 0 \end{aligned}$$

$$\left(\underset{\neq 0}{\xi} \in \mathfrak{o} \Rightarrow \prod |\xi^{(\lambda)}| \geqq 1 \Rightarrow \exists \lambda \ |\xi^{(\lambda)}| \geqq 1 \right.$$

$$\therefore \quad \operatorname{Tr}\delta_i' \succ \operatorname{Tr}\delta_{k(i)}' \succ \operatorname{Tr}\delta_{h(i)} \succ \operatorname{Tr}\delta_{i+1} \succ \operatorname{Tr}\delta_i$$

$m_1^2 A = A'\{m_1 X^{-1}\}$ $\quad m_1 = N(X) < 1$ なら

$$\operatorname{Tr}\delta_i' \prec \operatorname{Tr}\delta_i \qquad \therefore \quad \operatorname{Tr}\delta_i \sim \operatorname{Tr}\delta_i' \sim \operatorname{Tr}\delta_{i+1}$$

$$\operatorname{Tr}\delta_i' \succ \operatorname{Tr}\delta_j \cdot \operatorname{Tr}|\xi_{ji}|^2 \qquad \therefore \quad \xi_{ji} \ \text{bounded}.$$

<u>Lem</u>. 2 $\quad A = D[T] \in \mathfrak{S} \qquad x \in k_R^n$.

$\Longrightarrow \operatorname{Tr} A[x] \sim \operatorname{Tr} D[x]$

特に $\operatorname{Tr} \alpha_i \sim \operatorname{Tr} \delta_i$

∵) $\quad y = \underset{\text{bounded}}{T} x \quad \Rightarrow \quad \operatorname{Tr} D[x] \sim \operatorname{Tr} D[y]$ といえばよい.

$$\operatorname{Tr} D[x] = \sum \operatorname{Tr}(\delta_i |\xi_i|^2) \sim \sum \operatorname{Tr}\delta_i \cdot \operatorname{Tr}|\xi_i|^2$$

$$\left(\frac{\operatorname{Tr}\delta_i \cdot \operatorname{Tr}|\eta_i|^2}{\operatorname{Tr} D[x]} \right)^{\frac{1}{2}} \prec \frac{(\operatorname{Tr}\delta_i)^{\frac{1}{2}} \left((\operatorname{Tr}|\xi_i|^2)^{\frac{1}{2}} + \sum_j (\operatorname{Tr}|\tau_{ij}\xi_j|^2)^{\frac{1}{2}} \right)}{(\operatorname{Tr} D[x])^{\frac{1}{2}}}$$

$$\leqslant 1 + \sum_{j>i} \left(\frac{\operatorname{Tr}\delta_i \ \operatorname{Tr}|\tau_{ij}\xi_j|^2}{\operatorname{Tr}\delta_j \ \operatorname{Tr}|\xi_j|^2} \right)^{\frac{1}{2}}$$

$$\prec 1$$

$\therefore \quad \operatorname{Tr} D[y] \prec \operatorname{Tr} D[x]$

$x = T^{-1} y$ から $\succ \quad \therefore \sim$.

<u>Cor</u>. $\quad \mathfrak{S}$, $m > 0$ given

$\exists$ only finitely many $X \in M_n(\mathfrak{o})$, $|N(X)| \leqslant m$

s.t.

$$^t\bar{X}\,\bar{A}^{-1} X = A \quad \text{with } A \in \mathfrak{S}$$

∵) $\quad A = D[T]$

$\bar{A}^{-1} = D^{-1}[{}^t\bar{T}^{-1}] \qquad E = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ とおく.

$\bar{A}^{-1}[E] = ED^{-1}E[E\,{}^t\bar{T}^{-1}E] \in \mathfrak{S}_{c'}$

$(\bar{A}^{-1}[E])[EX] = A$

$\therefore \quad EX$ 有限ヶ.

<u>Th</u>. 3 $\exists$ finitely many $B_i \in \tilde{\Gamma} = GL(n, k)$

s.t.

$$\Omega = \bigcup_i \mathfrak{S}\{B_i\} \qquad (\mathfrak{S} : \text{suff. large})$$

has the following properties

1) $\Omega\{\Gamma\} = \mathfrak{S} = \mathcal{P}(n, k_{\mathbb{R}})$

2) $\forall\, B \in \tilde{\Gamma}$

$\Omega\{B\} \cap \Omega\{X\} \neq \phi$ for only finitely many $X \in \Gamma$

3) $v(\underline{\Omega}) < \infty$

<u>Lem</u>. m 自然数, $\Gamma_m = \{ X \in M_n(\mathfrak{o}) \mid |N(X)| = m \}$

$\Rightarrow \Gamma_m / \Gamma$ finite

$\because$) $k = \mathbb{Q}$, $\mathfrak{o} = \mathbb{Z}$ のとき.

$\Gamma_m \ni X \Rightarrow X = YDZ$, $Y, Z \in \Gamma$, $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$

$X \equiv X' \ (\Gamma) \Leftrightarrow YD \equiv Y'D \ (\Gamma)$ $\qquad \prod d_i = m$

$\Leftrightarrow Y \equiv Y' \ (\Gamma \cap D\Gamma D^{-1})$

$\Gamma(m) = \{ X \in \Gamma \mid X \equiv 1_n \ (m) \}$ とおけば

$\Gamma \cap D\Gamma D^{-1} \supset \Gamma(m)$

$[\Gamma : \Gamma(m)] < \infty \quad \therefore \Gamma D \Gamma / \Gamma \cong \Gamma / \Gamma \cap D\Gamma D^{-1}$ finite

$k, \mathfrak{o}$ 一般のときは $(e_i \omega_k)$ に関する rep. を考えればよい.

*) 更に the group of relations も finitely generated. ($\because$ S simply connected)

<u>Proof</u> 1) S, m as in Th. 1

$$\bigcup_{1 \leq m' \leq m} \Gamma_{m'} = \bigcup_i \Gamma B_i^{-1} \qquad \text{by Lem. 1.}$$

Put
$$\Omega = \bigcup_i S\{B_i\}$$

By Th. 1
$$S = \bigcup_{\substack{X \in M_n(\mathfrak{o}) \\ 1 \leq |N(X)| \leq m}} S\{X^{-1}\} = \bigcup_{X \in \Gamma} \Omega\{X^{-1}\}.$$

2) $\Omega\{B\} \cap \Omega\{X\} \neq \emptyset \qquad X \in \Gamma$

$\Rightarrow \exists\, i, j \qquad S\{B_i B\} \cap S\{B_j X\} \neq \emptyset$

$\exists\, m_1 \qquad m_1 B_i B X^{-1} B_j^{-1} \in M_n(\mathfrak{o})$

$\therefore$ X finite in number by Th. 2

<u>Cor</u>. of 1). 2) Γ : finitely generated*)

$\because$) $\Omega\{X\} \cap \Omega \neq \emptyset$, $X \in \Gamma$ finite. $\{X_1, \cdots, X_r\}$ とす.

Γ : generated by $\{X_1, \cdots, X_r\}$

$\because$ $C = C(\Gamma)$ 1-dim simplicial complex with vertices p_X $(X \in \Gamma)$,

p_X, p_Y connected by a segment $\Longleftrightarrow$ $\Omega[X] \cap \Omega[Y] \neq \emptyset$

$\Longleftrightarrow Y = X_i X$

よって C connected をいえばよい.

C not connected とす. C_1 : connected component

$$S' = \bigcup_{X \in C_1} \Omega\{X\}, \quad S'' = \bigcup_{X \notin C_1} \Omega\{X\} \qquad \text{共に open} \neq \emptyset$$

$S = S' \cup S''$, $S' \cap S'' = \emptyset$ 矛盾.

Proof of 3) $v(\Omega) < \infty$.

Lem. 1: G : unimodular loc. compact gr.,
H : " closed subgr.

$\Rightarrow$ G/H has rel. inv. measure

$$c \int_G f(g)\, dg = \int_{G/H} d\bar{g} \int f(gh)\, dh$$

or symbolically $c\, dg = d\bar{g} \cdot dh$

(Cf. Weil, Intégration ...)

Lem. 2. $G = H \cdot N$ semi-direct product
H, N unimodular
$d(hnh^{-1}) = \delta(h)\, dn$

$$\Rightarrow \begin{cases} d_r(hn) = \delta(h)\, dh\, dn \\ d_l(hn) = dh\, dn \end{cases}$$

Lem. 3 $G \doteqdot G_1 G_2$
G, G_1 unimodular, $G_1 \cap G_2$ compact

$$\Rightarrow \quad c \int_G f(g)\, dg = \int_{G_1} \int_{G_2} f(g_1 g_2)\, dg_1\, d_r g_2$$

Lem. 4. $G = K \cdot AN$
K : compact, AN as in Lem. 2

$$\Rightarrow \quad c\, dg = dk \cdot \delta(a)\, da\, dn$$

*) $f(X) = e^{-\pi \operatorname{tr}({}^t X X)} |\det(X)|^s \quad (s = n)$ を積分せよ.

① $G = GL(n, \mathbb{R})$,

$K = O(n)$,

$A = \left\{ D = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}, \ \delta_i > 0 \right\}, \quad N = \left\{ T = \begin{pmatrix} 1 & & \tau_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$

$$dg = |\det(X)|^{-n} \, d(X)$$

$$\int_{O(n)} dk = 1, \qquad da = \det(D)^{-1} d(D) = \prod_i \frac{d\delta_i}{\delta_i}$$

$$dn = d(T) = \prod_{i<j} d\tau_{ij}$$

$$\delta(D) = \prod_{i<j} \frac{\delta_i}{\delta_j} = \prod_{i=1}^{n} \delta_i^{\,n+1-2i}$$

$$S = K \backslash G = \mathcal{P}(n, \mathbb{R})$$

$$c \int_S F(s)\, ds = \iint F(Kan)\, \delta(a)\, da\, dn$$

$$= 2^{-n} \iint F(D[T]) \prod \delta_i^{\frac{n+1}{2} - i - 1} d(D)\, d(T)$$

$$(a^2 = D, \ n = T)$$

$$= 2^{-n} \int F(X) \det(X)^{-\frac{n+1}{2}} d(X)$$

$X = D[T]$

$$\frac{\partial(x_{11}, x_{12}, \cdots, x_{22}, \cdots)}{\partial(\delta_1, \tau_{12}, \cdots, \delta_2, \cdots)} = \prod \delta_i^{\,n-i}$$

$$c = 2^{-n} \prod_{\nu=1}^{n} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\pi^{\frac{\nu}{2}}} \quad {}^{*)}$$

Lem. 4 $\quad \gamma_j = \prod_{i=1}^{n} \delta_i^{m_{ij}}, \quad \det(m_{ij}) \neq 0$

$$\Longrightarrow \quad \prod \frac{d\gamma_i}{\gamma_i} = |\det(m_{ij})| \prod \frac{d\delta_i}{\delta_i}$$

$$\left(\because \right) \quad \frac{d\gamma_j}{\gamma_j} = \sum_i m_{ij} \frac{d\delta_i}{\delta_i}$$

$$\begin{cases} \gamma_1 = \dfrac{\delta_1}{\delta_2} \\ \cdots \\ \gamma_{n-1} = \dfrac{\delta_{n-1}}{\delta_n} \\ \gamma_n = \delta_1 \cdots \delta_n \end{cases} \qquad \begin{vmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \\ 1 & 1 & \cdots & & 1 \end{vmatrix} = n$$

$$\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \\ 1 & 1 & \cdots & & 1 \end{pmatrix} \begin{pmatrix} \frac{n-1}{n} & \frac{n-2}{n} & & \frac{1}{n} \\ -\frac{1}{n} & \frac{n-2}{n} & & \frac{1}{n} \\ \vdots & -\frac{2}{n} & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ -\frac{1}{n} & -\frac{2}{n} & & \frac{1}{n} \end{pmatrix}$$

$$\therefore \quad \delta_1^{a_1} \cdots \delta_n^{a_n} = \gamma_1^{a_1 - \frac{1}{n}\sum a_i} \gamma_2^{a_1 + a_2 - \frac{2}{n}\sum a_i} \cdots \gamma_n^{\frac{1}{n}\sum a_i}$$

$$\therefore \quad \prod \delta_i^{\frac{n+1}{2} - i} = \prod \gamma_i^{\frac{i(n-i)}{2}}$$

$$\therefore \quad ds = n^{-1} 2^{-n} c^{-1} \prod_{i=1}^{n} \gamma_i^{\frac{i(n-i)}{2}} \frac{d\gamma_i}{\gamma_i} \cdot d(T)$$

*) $f(X) = e^{-2\pi \operatorname{Tr}({}^t\bar{X}X)} |\det(X)|^{2s}$ $(s=n)$ を積分せよ.

② $G = GL(n, \mathbb{C})$,

$K = U(n)$,

A, N as in ①

(Euclid measure in $\mathbb{C}$ としては

$i\, dz \wedge d\bar{z} = 2\, dx \wedge dy$

$$\delta(D) = \prod \delta_i^{2(n+1)-4i}$$

$$c'\, ds = 2^{-n} \prod \delta_i^{n-2i}\, d(D)\, d(T)$$

$$= \text{〃} \det(X)^{-n}\, d(X)$$

$$\frac{\partial(x_{11}, x'_{12}, x''_{12}, \cdots)}{\partial(\delta_1, \tau'_{12}, \tau''_{12}, \cdots)} = \prod \delta_i^{2(n-i)}$$

$$c' = 2^{-n} \prod_{\nu=1}^{n} \frac{\Gamma(\nu)}{(2\pi)^{\nu}} \quad {}^{*)}$$

$$\prod \delta_i^{n-2i+1} = \prod \gamma_i^{i(n-i)}$$

$$\therefore \quad ds = n^{-1} 2^{-n} c'^{-1} \prod_i \gamma_i^{i(n-i)-1} d\gamma_i \cdot d(T)$$

Passage to $\underline{S}$

$$G = GL(n, k_{\mathbb{R}})$$

$$\underline{G} = G/\mathbb{R}^* \supset \underline{\Gamma} = \Gamma/\{\pm 1_n\}$$

$$\underline{\mathfrak{N}} = \mathfrak{N}/\mathbb{R}^+ \subset \underline{S} = \mathcal{P}(n, k_{\mathbb{R}})/\mathbb{R}^+$$

$\underline{S}$ の inv. measure :

$$\beta_\lambda = \gamma_n^{(\lambda)} / \gamma_n^{(\lambda+1)} \quad (1 \leq \lambda \leq r-1)$$

$$\beta_r = \prod_{\lambda=1}^{r_1} \gamma_n^{(\lambda)} \prod_{\lambda=r_1+1}^{r} \gamma_n^{(\lambda)\,2}$$

$$d\underline{s} = * \prod_{\substack{1 \leq i \leq n-1 \\ 1 \leq \lambda \leq r}} \gamma_i^{(\lambda)\,\frac{i(n-i)}{2}(2)-1}\, d\gamma_i^{(\lambda)} \prod_{1 \leq \lambda \leq r-1} \beta_\lambda^{-1}\, d\beta_\lambda \cdot d(T)$$

$$\therefore \quad \int_{\underline{\mathfrak{N}}} d\underline{s} < \infty$$

∘ $\underline{\mathfrak{N}}$ は $\underline{G}$, $\underline{S}$, $\underline{\Gamma}$ に対し Th. 3, 1), 2) をみたす.

$$\therefore \quad v(\underline{S}/\underline{\Gamma}) < \infty, \quad v(\underline{G}/\underline{\Gamma}) < \infty$$

しかし, これらは non-compact for $n > 1$.

$n = 1$ のとき, unit theorem of Dirichlet.

$$G = k_{\mathbb{R}}^* = \mathbb{R}^{*\,r_1} \times \mathbb{C}^{*\,r_2}$$

$$K = \{\pm 1\}^{r_1} \times T^{r_2}, \quad A = \mathbb{R}^{+\,r}$$

$$S = \mathbb{R}^{+\,r} \qquad \underline{S} \cong \mathbb{R}^{+\,r-1}$$

$$\Gamma = \mathfrak{o}^* \qquad \underline{\Gamma} = \mathfrak{o}^*/\{\pm 1\}$$

$\underline{S}/\underline{\Gamma}$: compact !

$$G^{(1)} = \{ X \in G \mid |N(X)| = 1 \}$$

$$K,\ \Gamma \subset G^{(1)}, \qquad S^{(1)} = S \cap G^{(1)} = K \backslash G^{(1)} \approx \underline{S}$$

$\therefore$ $\Omega^{(1)} = \Omega \cap S^{(1)}$ は $S^{(1)}$, Γ について Th. 3, 1), 2).

$$v(\Omega^{(1)}) < \infty, \quad v(S^{(1)}/\Gamma) < \infty, \quad v(G^{(1)}/\Gamma) < \infty$$

Another passage to $\underline{S}$

$$\underset{\cup}{\underline{G}} = PL(n, k_{\mathbb{R}}) = G/k_{\mathbb{R}}^*$$

$\underline{\Gamma} = \Gamma/$[redacted]$\,\theta^*$

$$S \xrightarrow{\pi} \underline{S} = P\mathcal{P}(n, k_{\mathbb{R}}) = S/(\mathbb{R}^+)^r$$

$\underline{\Omega} = \pi(\Omega)$ とおけば

$$v(\underline{\Omega}) < \infty.$$

$\underline{\Omega}$ は $\underline{S}$, $\underline{\Gamma}$ について Th. 3, 1), 2) をみたす.

(1) は trivial, 2) 後述.)

$$\underset{\cup}{G^{(1)}} = SL(n, k_{\mathbb{R}})$$

$$\Gamma^{(1)} = SL(n, \theta)$$

$$S^{(1)} = S\mathcal{P}(n, k_{\mathbb{R}}), \qquad \Omega^{(1)} = S^{(1)} \cap \pi^{-1}(\underline{\Omega})$$

$\approx \underline{S}$ ~~($\Omega \cap S^{(1)}$ でも ess には同じ)~~

とおけば $\Omega^{(1)}$ は $S^{(1)}$, $\Gamma^{(1)}$ に対し Th. 3, 1), 2) をみたし

$$v(\Omega^{(1)}) < \infty.$$

○ $\underset{\sim}{\mathfrak{L}}$ が Th. 3, 2) をみたすことの証.

$$\mathfrak{S}^{(1)} = \mathfrak{S} \cap S^{(1)}$$

• $\forall A \in \mathfrak{S}$, $\exists\ \alpha = (\alpha^{(\lambda)})$, $\alpha^{(\lambda)} \sim \alpha^{(\lambda')}$

$A^{(1)} \in \mathfrak{S}'^{(1)}$ $\quad (\mathfrak{S}' = \mathfrak{S}_{c^2}$ if $\mathfrak{S} = \mathfrak{S}_c)$

$$A = \alpha A^{(1)}$$

∵) $A = D[T]$, $D = (\delta_i^{(\lambda)})$

$\alpha^{(\lambda)} = \left(\prod \delta_i^{(\lambda)}\right)^{\frac{1}{n}}$ とおく $\qquad \dfrac{\alpha^{(\lambda')}}{\alpha^{(\lambda)}} < c$

$$A^{(1)} = (\alpha^{-1} D)\{T\}$$

$$\frac{\delta_i^{(\lambda')}}{\alpha^{(\lambda')}} \Big/ \frac{\delta_i^{(\lambda)}}{\alpha^{(\lambda)}} < c^2$$

∴ $\underset{\sim}{\mathfrak{S}} \subset \underset{\sim}{\mathfrak{S}}'^{(1)}$

よって $\mathfrak{L}^{(1)} = \bigcup \mathfrak{S}^{(1)}\{B_i^{(1)}\}$ と def. しても essential には同じ.

$$(B_i^{(1)} = |\det(B_i)|^{-\frac{1}{n}} B_i)$$

$\underset{\sim}{\mathfrak{L}}\{B\} \cap \underset{\sim}{\mathfrak{L}}\{X\} \neq \phi$ とす.

$\underset{\sim}{\mathfrak{S}}\{B_i B\} \cap \underset{\sim}{\mathfrak{S}}\{B_j X\} \neq \phi$

$\underset{\sim}{\mathfrak{S}}\{Y\} \cap \underset{\sim}{\mathfrak{S}} \neq \phi \qquad Y \in M_n(\mathfrak{o}) \quad |N(Y)| \sim 1$

$A, A' \in \mathfrak{S} \qquad A\{Y\} = \kappa \cdot A' \qquad \kappa \in (\mathbb{R}^+)^r$

$$\alpha A^{(1)}\{Y\} = \kappa \alpha' A'^{(1)}$$

$Y = (Y^{(\lambda)})$, $\eta^{(\lambda)} = \det Y^{(\lambda)}$ とおけば

$$\alpha^{(\lambda)} |\eta^{(\lambda)}|^{\frac{2}{n}} = \kappa^{(\lambda)} \alpha'^{(\lambda)}$$

Y を εY, $\varepsilon \in \mathfrak{o}^*$ でおきかえてよい.

$\prod |\eta^{(\lambda)}|^{(\lambda)} \sim 1$ は Dirichlet's th. により

$\exists\ \varepsilon \in \mathfrak{o}^* \qquad |\varepsilon^{(\lambda)} \eta^{(\lambda)}| \sim 1 \qquad (\forall \lambda)$

よって Υ を $\varepsilon\Upsilon$ でおきかえれば

$$\alpha^{(\lambda)} \sim \kappa^{(\lambda)} \alpha'^{(\lambda)} \qquad (\forall \lambda)$$

$$\therefore \quad \frac{\kappa^{(\lambda')}}{\kappa^{(\lambda)}} \sim \frac{\alpha'^{(\lambda')}}{\alpha'^{(\lambda)}} \Big/ \frac{\alpha^{(\lambda')}}{\alpha^{(\lambda)}} \sim 1$$

よって $\exists\ \mathfrak{S}'$ (depending only on $\mathfrak{S}$)

$$A,\ \kappa A' \in \mathfrak{S}' \qquad \text{i.e.} \qquad \mathfrak{S}'\{\Upsilon\} \cap \mathfrak{S}' \neq \emptyset$$

よって Υ 有限ヶ.

§11. Reduction of $O(n, k_{\mathbb{R}}, Q)/O(n, \mathfrak{o}, Q)$

$$\begin{array}{ccccccc} \hat{G} & = & GL(n, k_{\mathbb{R}}) & \supset & \hat{\Gamma} & = & GL(n, \mathfrak{o}) \\ \cup & & & & \cup & & \\ G & = & G_Q = O(n, k_{\mathbb{R}}, Q) & \supset & \Gamma & = & \Gamma_Q = O(n, \mathfrak{o}, Q) \end{array}$$ とおく.

∘ Maximal compact subgroup of $O(n, *, Q)$ $\quad (* = \mathbb{R}, \mathbb{C}, k_{\mathbb{R}})$

① $G = O(n, \mathbb{R}, Q)$

$V/\mathbb{R}$

$K \subset G$ compact $\Rightarrow$ $\exists\, Q_0 > 0$, inv. under K

$$\hat{K} = O(V, Q_0)$$

$$K \in G \cap \hat{K}$$

$Q(x, y) = Q_0(Tx, y)$, $\quad T \in GL(V)$ とおけば

T : symmetric w.r.t. Q_0, Q

$\therefore \quad V = \sum V_\lambda \quad$ (orth. sum.)

$$V_\lambda = \{ x \in V \mid Tx = x\lambda \}$$

Put

$$V_+ = \sum_{\lambda > 0} V_\lambda, \quad V_- = \sum_{\lambda < 0} V_\lambda$$

$$K_\pm = O(V_\pm, Q|V_\pm) \quad \text{compact}$$

$g \in G$ について

$g \in K \Rightarrow g \in \hat{K} \Rightarrow g$ commutes with T

$\Rightarrow g$ leaves inv. V_λ 特に $V_\pm$

$$\therefore \quad K \subset G \cap \hat{K} \subset K_+ \times K_-$$

特に K : max. compact $\Rightarrow$ すべて $=$

*) $g \in SO(V, Q)$ ととることができる.

**) 実は $\exists_1$ i.e. $N(K) = K$.

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K : max. comp. のとき

$$Q_0(x) = Q(x_+) - Q(x_-) \quad \text{for } x = x_+ + x_-, \ x_\pm \in V_\pm$$

i.e. $V_{\pm} = V_{\pm 1}$, i.e. $T^2 = 1$

ととりうる. 以上により

○ K : max. compact に対し

$\exists^{**)} V_\pm$ s.t. $V = V_+ + V_-$ orth. sum

$(Q \mid V_+) > 0$, $\therefore \dim V_+ = p$

$(Q \mid V_-) < 0$

○ (V_+, V_-) as above $\underset{1:1}{\longleftrightarrow}$ $Q_0 > 0$ s.t. $Q(x, y) = Q_0(Tx, y)$

$T^2 = 1$

(matrix であれば
$Q = {}^tT Q_0 \quad \therefore (QQ_0^{-1})^2 = 1$

K は (V_+, V_-), or Q_0 から えられる.

逆に V_+' : p-dim subsp. of V とすれば

$(Q \mid V_+') > 0$

$\exists^{*)} g \in G = O(V, Q)$ $gV_+ = V_+'$ (by Witt th.)

よって V_+' から えられる max. comp. subgr. を K' とすれば

$$K' = gKg^{-1}$$

$\therefore$ $\forall$ max. comp. は 互 に conjugate.

$$S = S_Q = K \backslash G$$

$$= \{ V_+ \subset V \mid \dim p, \ (Q \mid V_+) > 0 \} \overset{\text{open}}{\subset} g_p(V)$$

$$= \{ Q_0 \in \mathcal{P}(V) \mid (QQ_0^{-1})^2 = 1 \} \overset{\text{closed}}{\subset} \hat{S} = \mathcal{P}(V)$$

$G \ni g = X$: $V_+ \to g^{-1} V_+$, $Q_0 \to Q_0[X]$

② $G = O(n, \mathbb{C}, Q)$

$K \subset G$ compact $\Longrightarrow$ $\exists\ H$ pos. def. herm. f. inv. under K

$Q(x, y) = H(\rho(x), y)$ とおく. ρ: semi-lin. trans. of $V/\mathbb{C}$

ρ: ~~symmetric~~ hermitian w.r.t. H i.e. $H(\rho(x), y) = \overline{H(x, \rho(y))}$

$\therefore$ $W \subset V$ inv. under ρ $\Longrightarrow$ $W^{\perp}$ (w.r.t. H) inv. under ρ

$\therefore$ $\exists\ (e_1', \cdots, e_n')$ n.o.b. (w.r.t. H)

$$\rho e_i' = e_i' \lambda_i$$

$\lambda_i > 0$ としてよい. $(\because)\ \rho(e_i' \alpha) = e_i' \lambda_i \bar{\alpha} = (e_i' \alpha) \lambda_i \frac{\bar{\alpha}}{\alpha}$

$\therefore$ $V_+ = \{e_1', \cdots, e_n'\}_{\mathbb{R}}$ real form of V

$$H \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad Q \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$\therefore$ $Q \mid V_+$ pos. def.

H' = herm. f. on $V/\mathbb{C}$ s.t. $H' \mid V_+ = Q \mid V_+$ とすれば

$g \in G$ に対し

$g \in K \Longrightarrow g \in U(V, H)$

$\Longrightarrow$ g commutes with ρ

$\Longrightarrow$ g leaves inv. V_+

$\Longrightarrow$ $g \in U(V, H')$

よって K: max. comp. に対しては すべて eq.

※ V_+ as above $\overset{}{\underset{1:1}{\longleftrightarrow}}$ H' i.e. H s.t. $\rho^2 = 1$

(matrix であれば $Q = {}^t\bar{T} H$

$T\bar{T} = 1$ $\therefore$ $\bar{Q} \bar{H}^{-1} Q H^{-1} = 1$

V_+' real form of V s.t. $Q | V_+' > 0$ とすれば

$\exists g \in G \qquad V_+' = g V_+$

$$\therefore \qquad K' = g K g^{-1}$$

$$S = S_Q = K \backslash G$$

$$= \{ V_+ \mid \text{real form of } V/\mathbb{C}, \ Q | V_+ > 0 \} \overset{\text{open}}{\subset} \mathfrak{g}_n(V^{2n}/\mathbb{R})$$

$$= \{ H \in \mathcal{P}(V) \mid \bar{Q} \bar{H}^{-1} Q H^{-1} = 1 \} \overset{\text{cl.}}{\subset} \mathcal{P}(V).$$

③ $\quad G = O(n, k_{\mathbb{R}}, Q) = \prod O(n, k_\lambda, Q^{(\lambda)})$

$$\hat{S} = \mathcal{P}(n, k_{\mathbb{R}}) = \prod \mathcal{P}(n, k_\lambda)$$

$\cup$

$$S_Q = \{ H \in \mathcal{P}(n, k_{\mathbb{R}}) \mid \bar{Q} \bar{H}^{-1} Q H^{-1} = 1 \}$$

$$H = (\cdots, H_\lambda, \cdots)$$
$$Q = (\cdots, Q^{(\lambda)}, \cdots)$$

- G_Q operates on S_Q transitively

isotropy gr. of $H \in S_Q$: max. compact subgr.

$G_Q \ni X$: $\qquad H \longrightarrow H\{X\}$

- 一般に

$$S_{Q[X]} = S_Q\{X\}.$$

<u>Remark</u> 1. Another interpretation of S_Q

$\mathfrak{A} = M_n(*)$ $\quad * = \mathbb{R}, \mathbb{C}, k_{\mathbb{R}}$

$\iota : \quad X \longrightarrow Q^{-1}\,{}^t X Q$ $\quad$ involution

$\underset{\underline{G}}{G \sim \operatorname{Aut}_0(\mathfrak{A}, \iota)}$ $\quad$ (connected component of the group of autom.)

° ι_0 : pos. involution とは $\operatorname{tr}(X^\iota X) \geqq 0$ $\quad$ pos. def.

$\mathfrak{A}$ の pos. inv. は

$$X \longrightarrow H^{-1}\,{}^t\bar{X} H \qquad H \in \mathcal{P}(n, *)$$

で与えられる.

° $\forall \iota \quad \exists\, \iota_0$ pos. inv. $\quad \iota_0 \iota = \iota \iota_0$

$$H^{-1}\,{}^t(\bar{Q}^{-1}\,{}^t\bar{X}\bar{Q}) H = Q^{-1}\,{}^t(H^{-1}\,{}^t\bar{X} H) Q$$

$$\therefore \quad {}^tH^{-1} Q H^{-1} \bar{Q} \in \text{center}$$

° $\underline{\hat{K}} = \operatorname{Aut}_0(\mathfrak{A}, \iota_0)$

$\underline{K} = \underline{G} \cap \underline{\hat{K}} = \operatorname{Aut}_0(\mathfrak{A}, \iota, \iota_0)$ $\quad$ max. compact

$S = \underline{K} \backslash \underline{G} = \{ \iota_0 \mid \text{pos. inv. of } \mathfrak{A}, \ \iota_0 \iota = \iota \iota_0 \}$

他の classical group のときも同様.

<u>Rem</u>. 2. $\left.\begin{array}{l} G : \text{s.s. gr. with finite center} \\ K : \text{max. compact} \end{array}\right\} \Rightarrow S = K \backslash G$ $\quad$ non-compact symmetric space

S : simply connected ($\approx$ Euclidean sp.)

depends only on the local structure of G

$$G \sim G_1 \times \cdots \times G_r \Longrightarrow S = S_1 \times \cdots \times S_r$$

∘ Reduction th.

$\hat{\Omega}$ a fund. set of $\hat{\Gamma}$ in $\hat{S}$ fix

<u>Def</u>. Q reduced $\overset{def.}{\Longleftrightarrow}$ $S_Q \cap \hat{\Omega} \neq \phi$

∘ $\forall Q$, $\exists Q' \sim Q$, reduced

(∵) $S_Q \ni H$, $\exists X \in \hat{\Gamma}$, $H\{X\} \in \hat{\Omega}$

とすると

$S_{Q[X]} \ni H\{X\}$ ∴ $Q[X]$ red.

<u>Th</u>. 1. $m > 0$ given.

There exists only a finite numb. of Q s.t.

$Q \in M_n(\mathfrak{o})$, $|N(Q)| = |N_{k/Q}(\det Q)| = m$

reduced

∵) $\hat{\Omega} = \bigcup \mathfrak{S}\{B_i\}$ $B_i^{-1} \in M_n(\mathfrak{o})$ にとる.

Q sat. above cond. とすれば

$S_Q \ni {}^{\exists}H$, $H = H_1\{B_i\}$, $H_1 \in \mathfrak{S}$

$\bar{H}^{-1}\{Q\} = H^{-1}$

∴ $\bar{H}_1^{-1}\{{}^tB_i^{-1} Q B_i^{-1}\} = H_1$

$|N({}^tB_i^{-1} Q B_i^{-1})|$ bounded

∴ Q finite in number by Cor. to Siegel's th. (p. 101)

<u>Cor</u>. 1. $0 <$ (# of red. f. in a class) $< \infty$

<u>Cor</u>. 2. (class numb. with a given norm) $< \infty$

<u>Cor</u>. 3. (class numb. in a genus) $< \infty$

<u>Th.</u> 2 $\quad \mathfrak{R}_Q = S_Q \cap \hat{\mathfrak{R}} \quad (\hat{\mathfrak{R}} : \text{suff. large}) \quad$ sat

1) $\quad \mathfrak{R}_Q\{\Gamma_Q\} = S_Q$;

2) $\quad \forall B \in O(n, k, Q)$

$\mathfrak{R}_Q\{B\} \cap \mathfrak{R}_Q\{X\} \neq \emptyset$ for only finitely many $X \in \Gamma_Q$

$\because$) $\quad Q[X_i] \quad (1 \leqslant i \leqslant \#) \quad$ set of all reduced f. in the class of Q

Put $\quad \hat{\mathfrak{R}}' = \bigcup\limits_i \hat{\mathfrak{R}}\{X_i^{-1}\}$

Proof of 1) : $\quad S_Q \ni H \qquad \exists X \in \hat{\Gamma}, \quad H\{X\} \in \hat{\mathfrak{R}}$

$H\{X\} \in S_{Q[X]} \qquad \therefore \quad Q[X]$ reduced.

$\therefore \quad \exists i \qquad Q[X] = Q[X_i]$,

$XX_i^{-1} \in \hat{\Gamma} \cap G_Q = \Gamma_Q$

$H\{XX_i^{-1}\} \in \hat{\mathfrak{R}}\{X_i^{-1}\} \subset \hat{\mathfrak{R}}'$

2) $\quad$ clear by Th. 3, 2) (p. 102).

<u>Cor.</u> $\quad \Gamma_Q$: finitely generated.

∘ Invariant meas. on S_Q

① $(e_1, \cdots, e_n)$ basis of $V/\mathbb{R}$

s.t. $\left(Q(e_i, e_j)\right)_{1 \leq i, j \leq p} > 0$ fix.

$S_Q \ni H \longleftrightarrow (V_+, V_-)$

$e_i = e_i^+ + e_i^-, \qquad e_i^\pm \in V_\pm$

$\Rightarrow e_1^+, \cdots, e_p^+$ lin. indep.

$\because)$ $\sum_1^p e_i^+ \lambda_i = 0$ とすれば

$Q\left(\sum_1^p e_i \lambda_i\right) = Q\left(\sum_1^p e_i^- \lambda_i\right) \leq 0$

$\therefore \sum_1^p e_i \lambda_i = 0$ by assumption

Put $e_j^+ = \sum_{i=1}^p e_i^+ \eta_{ij}$, $\qquad Y = \overbrace{\boxed{\eta_{ij}}}^{n-p} \Big\} p$

Then

$$Q = \begin{pmatrix} Q' & \\ & Q'' \end{pmatrix} \left[\begin{pmatrix} 1_p & Y \\ * & * \end{pmatrix} \right]$$

$$H = \begin{pmatrix} Q' & \\ & -Q'' \end{pmatrix} \left[\begin{pmatrix} 1_p & Y \\ * & * \end{pmatrix} \right]$$

$\therefore \frac{1}{2}(Q + H) = Q' \left[\left(1_p \; Y \right) \right]$

$\because)$ $(e_1, \cdots, e_n) = (e_1^+, \cdots, e_p^+, \overbrace{* \cdots *}^{V_-}) \begin{pmatrix} 1_p & Y \\ * & * \end{pmatrix}$

$$Q^{-1} = \begin{pmatrix} Q'^{-1} & \\ & Q''^{-1} \end{pmatrix} \left[\begin{pmatrix} 1_p & * \\ {}^t Y & * \end{pmatrix}^{-1} \right]$$

$$\therefore \quad Q'^{-1} = Q^{-1} \left[\begin{pmatrix} 1_p \\ {}^t Y \end{pmatrix} \right] > 0$$

よって

$$S_Q \ni H \underset{1:1}{\longleftrightarrow} Y^{(p,\, n-p)} \quad \text{s.t.} \quad Q^{-1} \left[\begin{pmatrix} 1_p \\ Y \end{pmatrix} \right] > 0 .$$

$\therefore$ S_Q : rational var. of dim $p(n-p)$.

$$\hat{G}_Q \ni X = \begin{array}{cc} \scriptstyle p & \scriptstyle n-p \\ \end{array} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$Q \longrightarrow Q[X] = Q_1$$

$$H \longrightarrow H\{X\} = H_1$$

$$Q'[1\ Y] \longrightarrow Q'\left[(1\ Y) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right]$$

$$= Q'[A + YC]\,[(1,\ (A+YC)^{-1}(B+YD))]$$

$$\therefore \quad Y \longrightarrow (A+YC)^{-1}(B+YD) = Y_1$$

$$Q' \longrightarrow Q'[A+YC] = Q'_1$$

このとき

$$\frac{d(Y_1)}{d(Y)} = \frac{|\det(D - CY_1)|^p}{|\det(A+YC)|^{n-p}}$$

$$\left(\because \right) \quad (A+YC)\,Y_1 = B + YD$$

$$dY \cdot CY_1 + (A+YC)\,dY_1 = dY \cdot D$$

$$\therefore \quad (A+YC)\,dY_1 = dY\,(D - CY_1)$$

$$\det(X) = \det(A+YC)\det(D-CY_1)$$

$\because)$ $$\begin{pmatrix}1 & Y\\ 0 & 1\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}1 & 0\\ C(A+YC)^{-1} & 1\end{pmatrix}\begin{pmatrix}A+YC & \\ & D-CY_1\end{pmatrix}\begin{pmatrix}1 & Y_1\\ 0 & 1\end{pmatrix}$$

$\therefore$ $$d(Y_1) = |\det(X)|^{p}\,|\det(A+YC)|^{-n}\,dY$$

一方

$$\det Q_1 = \det Q\cdot(\det X)^2$$
$$\det Q_1' = \det Q'\cdot(\det(A+YC))^2$$

$\therefore$ $$c\,ds = |\det Q|^{-\frac{p}{2}}\left(\det Q^{-1}\left[\begin{pmatrix}1\\ {}^tY\end{pmatrix}\right]\right)^{-\frac{n}{2}} d(Y)$$

invariant measure on S_Q

<u>Rem.</u> $Q = \begin{pmatrix}* & \\ & *\end{pmatrix}$ のとき

$$V_+ = \{e_1,\cdots,e_p\}_{\mathbb{R}} \longleftrightarrow Y = 0$$

$$g^{-1}V_+ \longleftrightarrow Y_1 = A^{-1}B$$

特に $Q = \begin{pmatrix}1_p & \\ & -1_q\end{pmatrix}$ のとき $\qquad \overset{\shortparallel}{{}^t(D^{-1}C)}$

$$X = \begin{pmatrix}X_1 & \\ & X_2\end{pmatrix}\begin{pmatrix}1 & Y\\ {}^tY & 1\end{pmatrix}$$

$\mathfrak{V}_-$ の方を考えることにより

$$c'\,ds = (\det Q)^{-\frac{n-p}{2}} \left(\det\left(-Q^{-1}\left[\begin{pmatrix} Y \\ 1 \end{pmatrix}\right]\right)\right)^{-\frac{n}{2}} d(Y)$$

$c = c'$

$\Big($ $\because$) $Q = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$ のとき，みせればよい．

$$\det(1_p - Y\,{}^tY) = \det(1_q - {}^tY\,Y)\ \Big)$$

② $V/\mathbb{C}$ $\quad (e_1, \cdots, e_n)$ basis

s.t. $(Q(e_i, e_j)) > 0$ fix

とって $\{e_1, \cdots, e_n\}_{\mathbb{R}}$ real form of V.

$$S_Q \ni H \longleftrightarrow V_+ \,, \qquad V = V_+ + V_+\sqrt{-1}$$

$$e_i = e_i^+ + e_i^-$$

$$V_+ = \{e_1^+, \cdots, e_n^+\}_{\mathbb{R}}$$ とする. (as in p. 120)

Put

$$e_j^- = \sum_{i=1}^{n} e_i^+ \sqrt{-1}\, \eta_{ij} \,, \qquad Y = (\eta_{ij})$$

or

$$(e_1, \cdots, e_n) = (e_1^+, \cdots, e_n^+)(1_n + \sqrt{-1}\, Y)$$

Then

$$Q = Q'[1_n + \sqrt{-1}\, Y]$$

$$H = Q'\{1_n + \sqrt{-1}\, Y\}$$

$$\left(\begin{array}{ll} Q = Q' - Q'[Y] \,, & Q'Y + {}^tY Q' = 0 \\ \mathrm{Re}\, H = Q' + Q'[Y] \,, & \mathrm{Im}\, H = Q'Y - {}^tYQ' = 2Q'Y \end{array} \right.$$

$\therefore \quad S_Q \ni H \longleftrightarrow Y \in M_n(\mathbb{R})$, s.t.

$$Q^{-1}[1_n + \sqrt{-1}\, {}^tY] > 0$$

i.e.

$$Q^{-1} - Q^{-1}[{}^tY] > 0 \,, \qquad QY : \text{skew sym.}$$

よって S_Q is rational var. of dim n^2

$\hat{G} = GL(n, \mathbb{C}) \ni X = A + Bi$ $\quad$ (s.t. ${}^tAQB$ skew-sym.)

$Q \longrightarrow Q[X] = Q_1$

$H \longrightarrow H\{X\} = H_1$

$$Q_1 = Q'[(1+\sqrt{-1}\,Y)(A+\sqrt{-1}\,B)]$$

$$= Q'[A - YB][\,1_n + \sqrt{-1}(A-YB)^{-1}(B+YA)]$$

$\therefore$ $Q' \longrightarrow Q'[A-YB] = Q'_1$

$Y \longrightarrow (A-YB)^{-1}(B+YA) = Y_1$

$$\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det(A+\sqrt{-1}\,B)|^2 = \det(A+BY_1)\det(A-YB)$$

$$\therefore \quad d(Y_1) = \left|\frac{\det(A+BY_1)}{\det(A-YB)}\right|^n d(Y)$$

$$= |\det X|^{2n}\,|\det(A-YB)|^{-2n}\,d(Y)$$

$$\det Q_1 = \det Q\,(\det X)^2$$

$$\det Q'_1 = \det Q'\,(\det(A-YB))^2$$

$\therefore$

$$c\;ds = (\det Q)^{-n}\,(\det \underbrace{Q^{-1}[\,1_n + \sqrt{-1}\,{}^tY]}_{\shortparallel \atop Q^{-1} - Q^{-1}[{}^tY]})^{-n}\,d(Y)$$

invariant measure on S_Q

Another expression of inv. measure ($* = \mathbb{R}, \mathbb{C}, k_{\mathbb{R}}$)

<u>Lem</u>. 1 $\quad Q = \begin{array}{c} \begin{array}{ccc} \overbrace{\ }^{m} & \overbrace{\ }^{n-2m} & \overbrace{\ }^{m} \end{array} \\ \begin{pmatrix} & & \imath \\ & Q_0 & \\ \imath & & \end{pmatrix} \end{array}, \quad \imath = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ とする

$H \in \mathfrak{S}_Q$ は

$$H = \begin{pmatrix} D_1[T_1] & & \\ & H_0 & \\ & & \imath \bar{D}_1^{-1} \imath\,[\imath\,{}^t\bar{T}_1^{-1}\imath] \end{pmatrix} \left\{ \begin{pmatrix} 1_m & T_{12}' & T_{13}' \\ & 1_{n-2m} & T_{23}' \\ & & 1_m \end{pmatrix} \right\}$$

with
$$\begin{cases} D_1[T_1] \in \hat{\mathfrak{S}}^{(m)}, \qquad H_0 \in \mathfrak{S}_{Q_0}^{(n-2m)} \\ T_{12}' = -\imath\,{}^t\bar{T}_{23}'\,Q_0 \\ \imath\,T_{13}' = -\frac{1}{2} Q_0[T_{23}'] + T_{13}'' \qquad T_{13}'' : \text{skew sym.} \end{cases}$$

と unique に書ける.

∵)
$$H = \overbrace{\begin{pmatrix} H_1 & & \\ & H_0 & \\ & & H_3 \end{pmatrix}}^{H'} \left\{ \overbrace{\begin{pmatrix} 1 & T_{12}' & T_{13}' \\ & 1 & T_{23}' \\ & & 1 \end{pmatrix}}^{T'} \right\}$$

uniquely.

$$\bar{H}^{-1}\{Q\} = H = H'\{T'\}$$
uniqueness は I より

$$\| \qquad\qquad\qquad \bar{H}'^{-1}\{Q\} = H'$$

$$\bar{H}'^{-1}\{{}^tT'^{-1}Q\} \qquad \text{i.e. } H' \in \mathfrak{S}_Q$$

$$\|$$

$$\left(\bar{H}'^{-1}\{Q\}\right)\{Q^{-1}\,{}^tT'^{-1}Q\} \qquad Q^{-1}\,{}^tT'^{-1}Q = T'$$
$$\text{i.e. } T' \in G_Q$$

$$H' \in \mathfrak{S}_Q \iff H_0 \in \mathfrak{S}_{Q_0},$$
$$H_3 = \imath\,\bar{H}_1^{-1}\,\imath \iff \begin{cases} D_3 = \imath\,\bar{D}_1^{-1}\,\imath \\ T_3 = \imath\,{}^t\bar{T}_1^{-1}\,\imath \end{cases}$$

$$\begin{pmatrix} 1 & & \\ {}^tT_{12}' & 1 & \\ {}^tT_{13}' & {}^tT_{23}' & 1 \end{pmatrix} \begin{pmatrix} & & 2 \\ & Q_0 & \\ 2 & & \end{pmatrix} \begin{pmatrix} 1 & T_{12}' & T_{13}' \\ & 1 & T_{23}' \\ & & 1 \end{pmatrix} = \begin{pmatrix} & & 2 \\ & Q_0 & \\ 2 & & \end{pmatrix}$$

$$\begin{pmatrix} & & 2 \\ & Q_0 & {}^tT_{12}'2 \\ 2 & {}^tT_{23}'Q_0 & {}^tT_{13}'2 \end{pmatrix} \quad \begin{pmatrix} & & 2 \\ & Q_0 & Q_0T_{23}' + {}^tT_{12}'2 \\ 2 & * & 2T_{13}' + Q_0[T_{23}'] + {}^tT_{13}'2 \end{pmatrix}$$

$$\therefore \quad \begin{aligned} & Q_0 T_{23}' + {}^tT_{12}' 2 = 0 \\ & 2T_{13}' + Q_0[T_{23}'] + {}^tT_{13}' 2 = 0 \end{aligned} \qquad \therefore \quad \begin{aligned} T_{12}' &= -2\,{}^tT_{23}' Q_0 \\ 2T_{13}' &= -\frac{1}{2} Q_0[T_{23}'] + \underset{\text{skew sym.}}{T_{13}''} \end{aligned}$$

<u>Cor.</u>

$$G_Q = K \cdot G_{Q_0} A' N'$$

with

$$K = K_{H^0}, \quad K \cap (G_{Q_0} A'N') = K \cap G_{Q_0} = K_{H_0^0}$$

$$H^0 = \begin{pmatrix} 1_m & & \\ & H_0^0 & \\ & & 1_m \end{pmatrix}$$

$$A' = \left\{ \begin{pmatrix} D_1 & & \\ & 1_{n-2m} & \\ & & 2D_1^{-1'}2 \end{pmatrix} \right\}$$

$$N' = \left\{ \begin{pmatrix} T_1 & & \\ & 1_{n-2m} & \\ & & 2\,{}^tT_1^{-1}2 \end{pmatrix} \begin{pmatrix} 1 & T_{12}' & T_{13}' \\ & 1 & T_{23}' \\ & & 1 \end{pmatrix} \right\}$$

<u>Rem.</u> 特に $* = \mathbb{R}, \mathbb{C}$, Q_0 : definite $\Rightarrow$ $\underset{S_{Q_0} = \{\pm Q_0\}}{G_{Q_0} = K_{H_0^0}}$

$$\therefore \quad G_Q = K \cdot A'N' \quad \text{(Iwasawa decomposition)}$$

<u>Lem.</u> 2 $\quad G_Q = K \cdot (G_{Q_0} A') N'$, inv. measure

$$c\, dg = dk \cdot \delta(g_0 a') \overset{dg_0}{\wedge} da'\, dn'$$

$$\delta(g_0 a') = \prod_{i=1}^{m} \delta_i^{\,n-2i}$$

$$da' = \prod_{i=1}^{m} \frac{d\delta_i}{\delta_i}, \qquad dn' = d(T_1)\, d(T_{12}')\, d(T_{13}'')$$

$\because$)

$$d\,(g_0 a')\, n'\, (g_0 a')^{-1} = \delta(g_0 a')\, dn'$$

$$\begin{pmatrix} D_1 & & \\ & X_0 & \\ & & 2D_1^{-1}2 \end{pmatrix} \begin{pmatrix} T_1 & T_1 T_{12}' & T_1 T_{13}' \\ & 1 & T_{23}' \\ & & 2\,{}^{t}T_1^{-1}\,2 \end{pmatrix} \begin{pmatrix} D_1^{-1} & & \\ & X_0^{-1} & \\ & & 2D_1 2 \end{pmatrix}$$

$$T_1 \longrightarrow D_1 T_1 D_1^{-1}$$

$$T_1 T_{12}' \longrightarrow D_1 T_1 T_{12}' X_0^{-1} \qquad T_{12}' \longrightarrow D_1 T_{12}' X_0^{-1}$$

$$T_1 T_{13}' \longrightarrow D_1 T_1 T_{13}'\, 2 D_1 2 \qquad T_{13}' \longrightarrow D_1 T_{13}'\, 2D_1 2$$

$$\therefore \quad 2\,T_{13}' \longrightarrow 2\,T_{13}'\,[2D_1 2]$$

$$T_{13}'' \longrightarrow T_{13}''\,[2D_1 2]$$

$$\therefore \quad \delta(g_0 a') = \prod_{i=1}^{m} \delta_i^{\,m+1-2i} \times \Big(\prod_{i=1}^{m} \delta_i\Big)^{n-2m} \times \underbrace{\Big(\prod_{i<j} \delta_i \delta_j\Big)}_{= \left(\prod_{i=1}^{m} \delta_i\right)^{m-1}}$$

$$= \prod_{i=1}^{m} \delta_i^{\,m+1-2i+n-2m+m-1}$$

$$= \prod_{i=1}^{m} \delta_i^{\,n-2i}$$

<u>Cor.</u> S_Q の inv. meas.

$$2^m c \, ds_Q = ds_{Q_0} \times \prod_{i=1}^{m} \delta_i^{\,n-2i-1} d\delta_i \times dx'$$

<u>Rem.</u>

$$Q = \begin{pmatrix} & & Q_1 \\ & Q_0 & \\ {}^tQ_1 & & \end{pmatrix} \text{ のとき}$$

$$S_Q \ni H = \begin{pmatrix} H_1 & & \\ & H_0 & \\ & & H_3 \end{pmatrix} \left\{ \begin{pmatrix} 1 & T_{12}' & T_{13}' \\ & 1 & T_{23}' \\ & & 1 \end{pmatrix} \right\}$$

with $\begin{cases} H_1 \in \hat{S}^{(m)}, \quad H_0 \in S_{Q_0}^{(n-2m)}, \qquad H_3 = \bar{H}_1^{-1}\{Q_1\} \\ T_{12}' = -{}^tQ_1^{-1}\, {}^tT_{23}'\, Q_0, \\ {}^tQ_1 T_{13}' = -\frac{1}{2} Q_0[T_{23}'] + T_{13}'' \end{cases}$

$\begin{pmatrix} D_1 & & \\ & X_0 & \\ & & Q_1^{-1} D_1^{-1} Q_1 \end{pmatrix}$ による変換により T_1, T_{12}' as above

$${}^tQ_1 T_{13}' \longrightarrow {}^tQ_1 T_{13}' [Q_1^{-1} D_1 Q_1]$$

$$\therefore \quad T_{13}'' \longrightarrow T_{13}'' [Q_1^{-1} D_1 Q_1]$$

故に ds_Q について上と同じ結果をうる.

<u>Th</u> 3. G_Q/Γ_Q : measure finite (except for $n=2, \nu=1$)

G_Q/Γ_Q : compact $\iff$ $\nu = 0$

<u>Rem</u>. G_Q/Γ_Q : meas. fin. (resp. compact)

$\iff$ S_Q/Γ_Q : 〃

$\iff$ $\mathfrak{L}_Q = \hat{\Omega} \cap S_Q$: 〃 (resp. rel. compact)

$\iff$ $\mathfrak{S} \cap S_{Q'}$: 〃 for $\forall Q' \sim Q$

($\because$) $\hat{\Omega} = \bigcup \mathfrak{S}\{B_i\}$

$\mathfrak{S}\{B_i\} \cap S_Q = (\mathfrak{S} \cap S_{Q[B_i^{-1}]})\{B_i\}$

<u>Proof of Th. 3.</u> 1. $\mathfrak{S} \cap S_Q$ not rel. compact とする.

$\mathfrak{S} \cap S_Q \supset \exists \{H\}$ sequence s.t.

$H = D\{T\}$, $D = (\delta_i^{(\lambda)})$

$\exists i_0$ $\delta_{i_0}^{(\lambda)}/\delta_{i_0+1}^{(\lambda)} \to 0$ ($\exists \lambda \therefore \forall \lambda$)

今この seq. に対し

$$\begin{cases} \delta_1^{(\lambda)} \sim \cdots \sim \delta_{n_1}^{(\lambda)} \prec \delta_{n_1+1}^{(\lambda)} \sim \cdots \sim \delta_{n_1+n_2}^{(\lambda)} \prec \cdots & \sum_{i=1}^{r} n_i = n \\ \delta_{n_1}^{(\lambda)}/\delta_{n_1+1}^{(\lambda)}, \cdots, \delta_{\sum_1^{r-1} n_i}/\delta_{\sum_1^{r-1} n_i + 1} \to 0 \\ \text{その他の } \delta_i^{(\lambda)}/\delta_{i+1}^{(\lambda)}, \ \delta_i^{(\lambda)}/\delta_i^{(\lambda')} \text{ conv. } \lim \neq 0 \\ T \text{ conv.} \end{cases}$$

としてよい.

$$\bar{Q}\cdot\bar{T}^{-1}\bar{D}^{-1}\,{}^t\bar{T}^{-1}\cdot Q\cdot T^{-1}\,D^{-1}\,{}^t\bar{T}^{-1} = 1$$

Put $\quad Q' = Q[T^{-1}] = (q'_{ij})$

$\qquad Q'^{-1} = (q''_{ij})$

$$\overline{Q'} = D\,Q'^{-1}\,D \qquad \therefore \qquad \bar{q}'_{ij} = \delta_i\delta_j\,q''_{ij}$$

今 $\lim q''^{(\lambda_0)}_{i_0 j_0} \neq 0$, $\quad n_\kappa + 1 \leqq i_0 \leqq n_{\kappa+1}$

$\qquad\qquad n_\rho + 1 \leqq j_0 \leqq n_{\rho+1}$

とすれば, $\forall k, l$ $\quad k \leqq n_\kappa$, $l \leqq n_{\rho+1}$ に対し

or $\quad k \leqq n_{\kappa+1}$, $l \leqq n_\rho$

$$\bar{q}'^{(\lambda)}_{kl}\Big/\bar{q}'^{(\lambda_0)}_{i_0 j_0} = \delta^{(\lambda)}_k\delta^{(\lambda)}_l\Big/\delta^{(\lambda_0)}_{i_0}\delta^{(\lambda_0)}_{j_0}\cdot q''^{(\lambda)}_{kl}\Big/q''^{(\lambda_0)}_{i_0 j_0} \longrightarrow 0$$

$$\therefore \quad \lim \bar{q}'^{(\lambda)}_{kl} = 0$$

$$\therefore \quad \lim Q'^{-1} = \begin{array}{ccccc} n_1 & n_2 & & & n_r \\ \end{array}\begin{pmatrix} & & & & * \\ & & & * & 0 \\ & & * & 0 & 0 \\ & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lim Q' = \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & \\ 0 & 0 & * & & \\ 0 & * & & & \\ * & & & & \end{pmatrix}$$

かつ $\quad n_1 = n_r$, $n_2 = n_{r-1}$, $\cdots$

対角線の小行列式 non-singular

$\delta_1\delta_n \sim \delta_2\delta_{n-1} \sim \cdots$

よって

$$\lim Q' = Q[\lim T^{-1}] = \begin{pmatrix} 0 & & Q_1 \\ & Q_0 & * \\ {}^tQ_1 & * & * \end{pmatrix}$$

さらに Q を $\exists T$ で変換して

$$Q[T] = \begin{pmatrix} 0 & & Q_1 \\ & Q_0 & \\ {}^tQ_1 & & 0 \end{pmatrix}$$

特に $\nu > 0$. 又 $\quad S_Q \cap S = (S_{Q[T]} \cap \underset{S'}{\underset{\cap}{S\{T\}}})\{T^{-1}\}$.

2. $\nu > 0$ のとき、1 により $Q = \begin{pmatrix} & & Q_1 \\ & Q_0 & \\ {}^tQ_1 & & \end{pmatrix}$ とし

$\mathfrak{S} \cap \mathfrak{s}_Q$ vol. finite, not compact といえばよい。
Lem. 1 から明らか。

$$\mathfrak{S} \cap \mathfrak{s}_Q \subset \mathfrak{S}^{(m)} \times (\mathfrak{S}^{(n-2m)} \cap \mathfrak{s}_{Q_0}) \times \text{compact in } N''$$

1 により

$$\delta_1^{(\lambda)} \sim \cdots \sim \delta_m^{(\lambda)} \prec \delta_{m+1}^{(\lambda)}$$

$$\prod \delta_i = \det H = |\det Q| \quad \text{故} \quad \delta_i^{(\lambda)} \ (1 \leq i \leq m) \prec 1$$

$$\delta_1/\delta_2 = \gamma_1, \ \cdots, \ \delta_{m-1}/\delta_m = \gamma_{m-1}$$

$$(d\gamma_1, \cdots, d\gamma_{m-1}, d\delta_m) = (d\delta_1, \cdots, d\delta_m) \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} n-2-1 \\ n-4-1 \\ \vdots \\ n-2m-1 \end{pmatrix} = \begin{pmatrix} n-3 \\ 2n-8 \\ \vdots \\ m(n-m-2) \end{pmatrix}$$

$m \geq 2$ のとき、 $n-m-2 \geq m-2 \geq 0$

よって

$\left. \begin{array}{l} n=2, \ m=1 \\ n-2m=2, \quad Q_0 : n_0 = 2, \ m_0 = 1 \end{array} \right\}$ の場合を除き

$$\int ds \leq \int ds_0 \times \prod_{i=1}^{m} \int_0^c \gamma_i^{i(n-i-2)} d\gamma_i \times \int dn'$$

$$< \infty \qquad \text{by induction.}$$

3. $n = 2, \quad m = 1$

$$Q = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad O = \left\{ \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix}, \begin{pmatrix} & \xi \\ \xi^{-1} & \end{pmatrix} \;\middle|\; \xi \in k^* \right\}$$

$$SO(n, Q, k_{\mathbb{R}}) = (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2}$$

$$S_Q = (\mathbb{R}^+)^r, \qquad \Gamma_Q = \{\pm 1\}$$

$n - 2m = 2$

$$\delta_1 \sim \cdots \sim \delta_m < \delta_{m+1}$$

$$\int_\delta^\varepsilon \delta_{m+1}^{-1} \, d\delta_{m+1} = (\log \varepsilon - \log \delta)$$

$$\int_0^c \delta_m^{m(n-m-2)} \log \delta_m \, d\delta_m < \infty$$

$m \geqslant 1, \quad n - m - 2 \gtreqqless m + 2 - 2 \geqslant 1$

§ 12. Idèle formulation

∘ Eichler's principle on $\mathfrak{o}$-lattice

k : alg. n. f.

$\mathfrak{o}$: max. order

$V/k \supset M/\mathfrak{o}$: $\mathfrak{o}$-lattice

$V_\mathfrak{p} = V \otimes_k k_\mathfrak{p} \supset M_\mathfrak{p} = M \otimes_\mathfrak{o} \mathfrak{o}_\mathfrak{p}$: $\mathfrak{o}_\mathfrak{p}$-lattice になる.

$M \longrightarrow \{M_\mathfrak{p}\}$

<u>Lemma</u> 1. $\begin{matrix} M \to \{M_\mathfrak{p}\} \\ M' \to \{M'_\mathfrak{p}\} \end{matrix} \Rightarrow M_\mathfrak{p} = M'_\mathfrak{p}$ for $\forall' \mathfrak{p}$

(∵) $\exists\, \alpha, \beta \in k^*$ $M\alpha \subset M' \subset M\beta$

α, β : unit for $\forall' \mathfrak{p}$

そのような $\mathfrak{p}$ に対し, $M_\mathfrak{p} = M'_\mathfrak{p}$.)

特に $M^* = \sum_i e_i \mathfrak{o}$ とすれば $M'_\mathfrak{p} = \sum_i e_i \mathfrak{o}_\mathfrak{p}$ for $\forall' \mathfrak{p}$

<u>Lemma</u> 2. M : $\mathfrak{o}$-lattice

$M'_\mathfrak{p}$: $\mathfrak{o}_\mathfrak{p}$-lattice , $M'_\mathfrak{p} = M_\mathfrak{p}$ for $\forall' \mathfrak{p}$

Put $M' = \bigcap_\mathfrak{p} (M'_\mathfrak{p} \cap V)$

$\Rightarrow$ M' : $\mathfrak{o}$-lattice, $M'_\mathfrak{p} = M' \otimes_\mathfrak{o} \mathfrak{o}_\mathfrak{p}$.

∵) $M = \sum e_i \mathfrak{o}$ としてよい.

1. $k = \mathbb{Q}$ のとき.

$\forall p$ $M'_p \supset M_p$ としてよい.

(∵) $M'_{p_i} \not\supset M_{p_i}$ for finite p_i, $M'_{p_i} p_i^{-m_i} \supset M_{p_i}$

$a = \prod p_i^{m_i}$, $M'_p a^{-1} \supset M_p$ for $\forall p$

∴ $\{M'_p\}$ を $\{M'_p a^{-1}\}$ でおきかえる.)

同様に $\exists\, a, b \in \mathbb{Q}^*$, $\quad M_p^* a \subset M'_p \subset M_p b$

$M = \bigcap_p (M_p \cap V)$ は明らか $\quad \therefore \quad M a \subset M' \subset M b$

$\therefore \quad M' : \quad \mathbb{Z}$ - lattice

$$M'_p \ni x = \sum e_i \xi_i \qquad \xi_i \in \mathbb{Q}_p$$

$$\xi_i \equiv a p^{-m} = \eta_i \pmod{\mathbb{Z}_p}$$

$$\eta_i \in \mathbb{Q}, \quad \eta_i \in \mathbb{Z}_q \quad \text{for } q \neq p$$

$$\therefore \quad x = \underbrace{\sum e_i (\xi_i - \eta_i)}_{\cap\ M_p} + \underbrace{\sum e_i \eta_i}_{\cap\ M'_p \cap \bigcap_{q \neq p} M_q \subset M'} \in M_p + M' \subset M' \otimes \mathbb{Z}_p$$

$$\therefore \quad M'_p = M' \otimes \mathbb{Z}_p$$

2. 一般の場合. $\quad [k : \mathbb{Q}] = d \qquad V/k/\mathbb{Q} \quad nd$ - dim $/\mathbb{Q}$

$$V_p = \sum_{\mathfrak{p}|p} V_{\mathfrak{p}}$$

$$M'_p \overset{\text{def}}{=} \sum_{\mathfrak{p}|p} M'_{\mathfrak{p}} = M_p = \sum_{\mathfrak{p}|p} M_{\mathfrak{p}} \qquad \text{for } \forall' p$$

よって 1. により

$$M' = \bigcap_p M'_p : \ \mathbb{Z}\text{-lattice}, \qquad M'_p = M' \otimes \mathbb{Z}_p$$

M' : $\mathfrak{o}$ - module $\therefore$ $\mathfrak{o}$ - lattice

$\left(\because\right) \quad \alpha \in \mathfrak{o} \qquad M'_p \alpha = \sum M'_{\mathfrak{p}} \alpha \subset M'_p$

$\qquad \therefore \quad M' \alpha \subset M'$

$$M'_p = M' \otimes \mathbb{Z}_p = (M' \otimes_{\mathfrak{o}} \mathfrak{o}) \otimes \mathbb{Z}_p = M' \otimes_{\mathfrak{o}} (\mathfrak{o} \otimes \mathbb{Z}_p)$$

$$= M' \otimes_{\mathfrak{o}} \Big(\sum_{\mathfrak{p}|p} \mathfrak{o}_{\mathfrak{p}}\Big) = \sum_{\mathfrak{p}|p} M' \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$$

$$\therefore \quad M'_{\mathfrak{p}} = M' \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$$

- Idèle group of $\hat{G} = GL(V)$

$$\hat{G}_k = GL(V_{/k})$$

$$\hat{G}_{k_\mathfrak{p}} = GL(V_\mathfrak{p}) \quad : \quad \text{loc. compact}$$

$$\cup$$

$$\hat{U}_{M_\mathfrak{p}} = GL(M_\mathfrak{p}) \quad : \quad \text{open compact for finite } \mathfrak{p}$$

実は max. compact

$$\hat{\mathfrak{g}} = \hat{G}_A = \prod{}' \hat{G}_{k_\mathfrak{p}} \qquad \text{restricted direct product w.r.t. } \{\hat{U}_{M_\mathfrak{p}}\}$$

$$\cup \qquad \text{indep. of } M \text{ (by Lem. 1)}$$

$$\hat{\mathfrak{U}}_M = \prod \hat{U}_{M_\mathfrak{p}}$$

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \times \hat{\mathfrak{g}}_\infty, \qquad \hat{\mathfrak{U}}_M = \hat{\mathfrak{U}}_{M0} \times \hat{\mathfrak{g}}_\infty$$

open compact in $\hat{\mathfrak{g}}_0$

- $\hat{\mathfrak{g}}$ operates transitively on $\{M\}$

$\hat{\mathfrak{g}} \ni \tilde{\rho} = (\rho_\mathfrak{p})$ に対し

$M \longleftrightarrow \{M_\mathfrak{p}\}$

Put $\qquad \tilde{\rho} M \longleftrightarrow \{\rho_\mathfrak{p} M_\mathfrak{p}\} \qquad$ ($\tilde{\rho}_0$ のみに dep)

Lem. 1, 2 により $\quad \tilde{\rho} M$: $\mathfrak{o}$-lattice 定まる

$\forall M'$: $\mathfrak{o}$-lattice $\quad \exists\, \tilde{\rho} \in \hat{\mathfrak{g}}, \quad M' = \tilde{\rho} M$

$$\therefore \quad \hat{\mathfrak{g}} / \hat{\mathfrak{U}}_M = \text{space of } \forall\ \mathfrak{o}\text{-lattices}$$

(i.e. There exists only one genus w.r.t $\hat{G}$.)

- M, M' : $\mathfrak{o}$-lattices

$M \cong M'$ as $\mathfrak{o}$-lattice $\iff \exists\, \rho \in \hat{G}_k \quad M' = \rho M$

$\iff$ corr. to the same double coset*

$$\hat{G}_k \backslash \hat{\mathfrak{g}} / \hat{\mathfrak{U}}_M.$$

Th. 1) $\forall$ $\mathfrak{o}$-lattice M

$$M = \sum_{i=1}^{n} e_i \mathfrak{a}_i \qquad \mathfrak{a}_i : \text{ideal in } k$$

2) $M' = \sum e_i' \mathfrak{a}_i'$

$$M \cong M' \text{ as } \mathfrak{o}\text{-lattice} \iff \prod \mathfrak{a}_i \sim \prod \mathfrak{a}_i'$$

(Cf. Steinitz, Math. Ann. 71 (1912), 72
Chevalley, L'arithmétique dans les algèbres de matrices, Actualités 323

$\therefore \quad \# \; \hat{G}_k \backslash \hat{\mathfrak{g}} / \hat{\mathfrak{U}}_M = h =$ class number of k

$M_i = \tilde{p}_i M$ representation of classes

$$\hat{\mathfrak{g}} = \bigcup_{i=1}^{h} \hat{G}_k \tilde{p}_i \hat{\mathfrak{U}}_M = \bigcup_{i=1}^{h} \hat{\mathfrak{U}}_M \tilde{p}_i^{-1} \hat{G}_k$$

$$\therefore \quad \hat{\mathfrak{g}} / \hat{G}_k = \bigcup_i \hat{\mathfrak{U}}_M \tilde{p}_i^{-1} \hat{G}_k / \hat{G}_k$$

$$\approx \bigcup_i \underbrace{(\tilde{p}_i \hat{\mathfrak{U}}_M \tilde{p}_i^{-1})}_{= \hat{\mathfrak{U}}_{M_i}} \hat{G}_k / \hat{G}_k$$

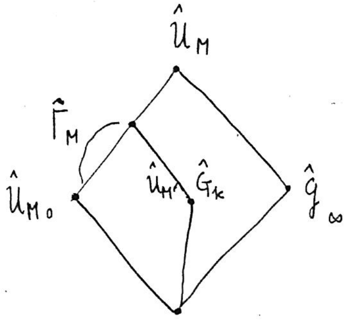
$$\approx \bigcup_i \hat{\mathfrak{U}}_{M_i} / \hat{\mathfrak{U}}_{M_i} \cap \hat{G}_k$$

Put $\hat{\Gamma}_{M_i} = GL(M_i/\mathfrak{o})$ considered as $\subset \hat{\mathfrak{g}}_\infty = GL(V_{k_\mathbb{R}})$

$$= pr_\infty (\hat{\mathfrak{U}}_{M_i} \cap \hat{G}_k)$$

$\hat{\Gamma}_{M_i}$: discrete subgr. of $\hat{\mathfrak{g}}_\infty$

$\therefore \quad \hat{G}_k$: " of $\hat{\mathfrak{g}}$



$\hat{\mathfrak{U}}_M / \hat{\mathfrak{U}}_M \cap \hat{G}_k$ (fibre : $\hat{\mathfrak{U}}_{M_0}$
base sp : $\hat{\mathfrak{g}}_\infty / \hat{\Gamma}_M$

- $\hat{\Gamma}_M$'s are commensurable

$\because)$ $\exists\, \alpha, \beta \in k^*$ $\quad M\alpha \subset M' \subset M\beta$

Put $\gamma = \alpha/\beta$, $\quad \hat{\Gamma}_M(\gamma) = \{ X \in \hat{\Gamma}_M \mid X \equiv 1 \pmod{\gamma} \}$

$\left(\begin{array}{l} \text{i.e. } X = 1 + \gamma X_1 \\ X_1 M \subset M \end{array} \right.$

$[\hat{\Gamma}_M : \hat{\Gamma}_M(\gamma)] < \infty$

$\hat{\Gamma}_M(\gamma) \subset \hat{\Gamma}_{M'}$

$\left(\begin{array}{l} \because)\ x \in M', \quad X \in \hat{\Gamma}_M(\gamma), \quad X = 1 + \gamma X_1 \\ X x = \underbrace{x + X_1 x \cdot \frac{\alpha}{\beta}}_{\in M'} \quad \left| \begin{array}{l} x\beta^{-1} \in M \Rightarrow X_1(x\beta^{-1}) \in M \\ \Rightarrow X_1(x\beta^{-1})\alpha \in M' \end{array} \right. \end{array} \right.$

- $[\hat{\Gamma}_{M'} : \hat{\Gamma}_M] = [\hat{\Gamma}_{M'} : \hat{\Gamma}_M \cap \hat{\Gamma}_{M'}] / [\hat{\Gamma}_M : \hat{\Gamma}_M \cap \hat{\Gamma}_{M'}]$
  $= 1$

$\because)$ $\quad [\hat{\Gamma}_{\rho M} : \hat{\Gamma}_M] = 1 \quad$ for $\rho \in \hat{G}_k$ $\quad$ (cf [E], p.106)

$\left(\because\ v(g_\infty^{1\text{階}} / \hat{\Gamma}_M) < \infty, \quad \hat{\Gamma}_{\rho M} = \rho \hat{\Gamma}_M \rho^{-1} \right.$

よって Th. 1 より

$$M = \sum e_i \mathfrak{o}$$

$$M' = \sum_{i=1}^{n-1} e_i \mathfrak{o} + e_n \mathfrak{a} \qquad \text{としてよい}$$

$$\hat{\Gamma}_{M'} = \left\{ X \in \begin{pmatrix} \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{a}^{-1} \\ & \cdots & & \vdots \\ \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{a}^{-1} \\ \mathfrak{a} & \cdots & \mathfrak{a} & \mathfrak{o} \end{pmatrix}, \ \det(X) \text{ unit} \right\}$$

$$\therefore \quad \hat{\Gamma}_M \cap \hat{\Gamma}_{M'} = \left\{ X \in \begin{pmatrix} \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{a} & \cdots\ \mathfrak{a} & \mathfrak{o} \end{pmatrix}, \ '' \right\}$$

$$[\hat{\Gamma}_M : \hat{\Gamma}_M \cap \hat{\Gamma}_{M'}] = \#\, (\mathfrak{o}/\mathfrak{a})^{n-1} / (\mathfrak{o}/\mathfrak{a})^*$$

$$\Vert$$

$$[\hat{\Gamma}_{M'} : \quad '' \quad] = \#\, (\mathfrak{a}^{-1}/\mathfrak{o})^{n-1} / (\mathfrak{o}/\mathfrak{a})^*$$

○ $G^{(1)} = SL(V)$ の場合.

$$M \approx M' \iff M \cong M'$$

$\because$) $(\Rightarrow)$ $M' = \tilde{\rho} M$ i.e. $M'_{\mathfrak{p}} = \rho_{\mathfrak{p}} M$ $\rho_{\mathfrak{p}} \in G^{(1)}{}_{k_{\mathfrak{p}}}$

$$M = \sum e_i \mathfrak{o} + e_n \mathfrak{a}, \qquad \mathfrak{a} = \prod \mathfrak{p}^m$$
$$M' = \sum e'_i \mathfrak{o} + e'_n \mathfrak{a}', \qquad \mathfrak{a}' = \prod \mathfrak{p}^{m'}$$

とすれば

$$M_{\mathfrak{p}} = (e_1, \cdots, e_{n-1}, e_n \pi^m)$$
$$M'_{\mathfrak{p}} = (e'_1, \cdots, e'_{n-1}, e'_n \pi^{m'})$$
$$(e'_1, \cdots, e'_n) = (e_1, \cdots, e_n) A \qquad A \in \hat{G}_k$$

$$\therefore \quad \det \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \pi^{-m} \end{pmatrix} A \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \pi^{m'} \end{pmatrix} = \text{unit}$$

$\therefore$ $(\det A) = \mathfrak{a}/\mathfrak{a}'$ $\therefore$ $\mathfrak{a} = \mathfrak{a}'$ としてよい.

$$\therefore \quad M \cong M'$$

∘ $G = O(V, Q)$ の場合.

$$\begin{aligned} \mathfrak{g} &= G_A = {\prod}' G_{k_\mathfrak{p}} \\ &\cup \\ \mathfrak{U}_M &= \prod U_{M_\mathfrak{p}} \end{aligned}$$

$$M \approx M' \iff \exists\, \tilde{\rho} \in \mathfrak{g}, \quad M' = \tilde{\rho} M$$

$$M \cong M' \iff \exists\, \rho \in G_k, \quad M' = \rho M$$

$$\therefore \quad \mathfrak{g}/\mathfrak{U}_M = \text{genus of } M$$

$$\# \; G_k \backslash \mathfrak{g} / \mathfrak{U}_M = \# \text{ of classes in the genus of } M < \infty$$

∵) $M \approx M'$ under G, $M \cong M'$ under $\hat{G}$

なる M' の class numb. $< \infty$ をいえばよい.

$$\left.\begin{aligned} M &= \sum e_i \mathfrak{o} + e_n \mathfrak{a} \\ M' &= \sum e_i' \mathfrak{o} + e_n' \mathfrak{a} \end{aligned}\right) \rho \in G$$

$$M \cong M' \text{ under } G \iff \underset{\hat{G}}{\underset{\cap}{\rho}}(e_1' \cdots e_n') = (e_1 \cdots e_n) \underset{\hat{\Gamma}_M}{\underset{\cap}{T}}$$

$$\iff (Q(e_i', e_j')) = {}^tT\,(Q(e_i, e_j))\,T$$

$$M \approx M' \text{ under } G \Rightarrow |N(Q(e_i, e_j))| = |N(Q(e_i', e_j'))|$$

∴ p. 118, Th. 1 により class numb. $< \infty$

$$\because \quad \hat{\Gamma}_M = \left\{ X \in \begin{pmatrix} \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{a}^{-1} \\ \vdots & \ddots & & \vdots \\ \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{a}^{-1} \\ \mathfrak{a} & \cdots & \mathfrak{a} & \mathfrak{o} \end{pmatrix}, \ \det X = \text{unit} \right\}$$

$$\sim \hat{\Gamma}$$

$M_i = \tilde{p}_i M$ representatives of classes in the genus of M

$$\mathcal{G} = \bigcup_i \mathcal{U}_M \tilde{p}_i^{-1} G_k$$

$$\mathcal{G}/G_k \approx \bigcup_i \mathcal{U}_{M_i} / \mathcal{U}_{M_i} \cap G_k$$

$\mathcal{G}/G_k$ volume finite (except $n = 2 \ldots$)

" compact $\Longleftrightarrow$ $\nu = 0$

§ 13. Tamagawa measures, Siegel's th.

G : alg. gr. / k alg. n. f.

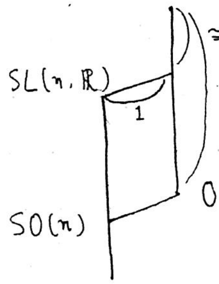
$\omega = F(t)\, dt_1 \wedge \cdots \wedge dt_N$ inv. n-form

$|\omega|_{\mathfrak{p}} = |F(t)|_{\mathfrak{p}}\, |dt_1|_{\mathfrak{p}} \cdots |dt_N|_{\mathfrak{p}}$ inv. measure on $G_{k_{\mathfrak{p}}}$

$$|dt|_{\mathfrak{p}} = \begin{cases} dt & (\mathbb{R}) \\ \sqrt{-1}\, dt\, d\bar{t} = 2\, d\tau'\, d\tau'' & t = \tau' + \sqrt{-1}\,\tau'' \quad (\mathbb{C}) \\ \int_{\mathfrak{o}} = 1 & (k_{\mathfrak{p}}) \end{cases}$$

⓪ $\hat{G} = GL(n)$ $\omega = \det(X)^{-n}\, d(X)$

① $GL(n, \mathbb{R})$



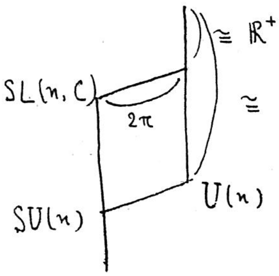
$\cong \mathbb{R}^+ \quad (\xi \to \xi^2)$

$\cong \mathcal{P}(n, \mathbb{R})$: $\det(X)^{-\frac{n+1}{2}}\, d(X)$

$$\int_{O(n)} = \left(\prod_{\nu=1}^{n} \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{\nu}{2}}} \right)^{-1} = \kappa_n^{-1}$$

特に $\kappa_1 = 1$ (cf. p. 105)

② $GL(n, \mathbb{C})$



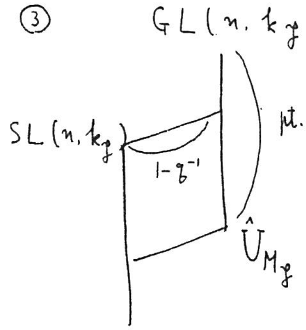
$\cong \mathbb{R}^+ \quad (\xi \to |\xi|^2)$

$\cong \mathcal{P}(n, \mathbb{C})$: $\det(X)^{-n}\, d(X)$

$$\int_{U(n)} = \left(\prod_{\nu=1}^{n} \frac{\Gamma(\nu)}{(2\pi)^{\nu}} \right)^{-1} = {\kappa'_n}^{-1}$$

特に $\kappa'_1 = (2\pi)^{-1}$ (cf. p. 107)

③



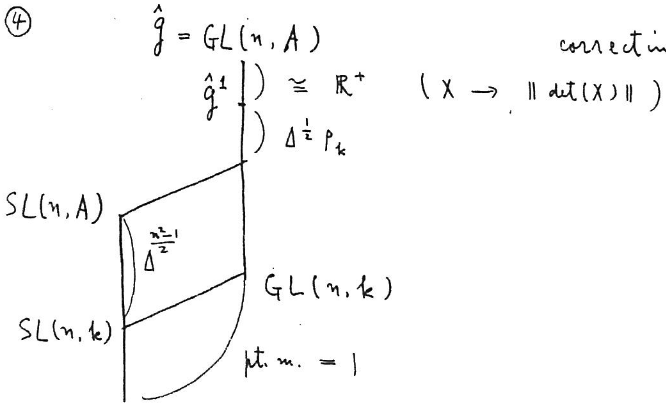
$$\int_{\hat{U}_{M_\mathfrak{p}}} = \prod_{\nu=1}^{n} (1 - q^{-\nu})$$

indep. of $M_\mathfrak{p}$

(一般に $U_{M_\mathfrak{p}} \to G_{\mathfrak{o}/\mathfrak{p}}$ onto のとき

$$\int_{U_{M_\mathfrak{p}}} |\omega|_\mathfrak{p} = \frac{\# G_{\mathfrak{o}/\mathfrak{p}}}{q^{\dim G}}$$)

④



correcting factor $1 - q^{-1}$

。 $G^{(1)} = SL(n)$ の場合

$$g^{(1)} = \mathcal{U}_M^{(1)} \cdot G_k^{(1)}$$

$$\therefore \quad g^{(1)} / G_k^{(1)} = \mathcal{U}_M^{(1)} / \mathcal{U}_M^{(1)} \cap G_k^{(1)}$$

$$\therefore \quad v(g^{(1)} / G_k^{(1)}) = v(\mathcal{U}_{M_0}^{(1)}) \cdot v(g_\infty^{(1)} / \Gamma_M^{(1)})$$

$$\Delta^{\frac{n^2-1}{2}} \tau(G^{(1)}) \qquad \prod_{\mathfrak{p}} \prod_{\nu=2}^{n} (1 - N\mathfrak{p}^{-\nu}) \qquad \frac{1}{(w, n)} v(K^{(1)}) \cdot v(S^{(1)} / \Gamma_M^{(1)})$$

$$(\zeta_k(2) \cdots \zeta_k(n))^{-1}$$

$w = \#$ roots of 1 in k

*) 特に $n=2$, $r_1 = d = [k:\mathbb{Q}]$ (Hilbert modular gr. のとき)

$$v(S^{(1)}/\Gamma_M^{(1)}) = 2\Delta^{\frac{3}{2}}\pi^{-d}\zeta_k(2)$$

• $\tau(G^{(1)}) = 1$

$$\therefore \quad v(S^{(1)}/\Gamma_M^{(1)}) = (w,n)\,\Delta^{\frac{n^2-1}{2}}\prod_{\nu=2}^{n}\left\{\zeta_k(\nu)\left(\frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{\nu}{2}}}\right)^{r_1}\left(\frac{\Gamma(\nu)}{(2\pi)^{\nu}}\right)^{r_2}\right\}$$

$$= (w,n)\,\Delta^{\frac{n(n-1)}{4}}\prod_{\nu=2}^{n}\xi_k(\nu) \quad {}^{*)}$$

$$\left(\begin{aligned} \xi_k(s) &= \left(\frac{\Delta^{\frac{1}{2}}}{2^{r_2}\pi^{\frac{n}{2}}}\right)^{s}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_k(s) \\ &= \Delta^{\frac{s}{2}}\left(\frac{\Gamma(\frac{s}{2})}{\pi^{\frac{s}{2}}}\right)^{r_1}\left(\frac{\Gamma(s)}{(2\pi)^{s}}\right)^{r_2}\zeta_k(s) \\ \xi_k(s) &= \xi_k(1-s) \end{aligned}\right.$$

∘ $\hat{G} = GL(n)$ の場合

$$\hat{g}^1 = \bigcup_{i=1}^{h}\hat{\mathfrak{U}}_M\tilde{p}_i^{-1}\hat{G}_k$$

$$\therefore \quad \hat{g}^1/\hat{G}_k = \bigcup_{i=1}^{h}\hat{\mathfrak{U}}_{M_i}/\hat{\mathfrak{U}}_{M_i}\cap\hat{G}_k$$

$$\therefore \quad v(\hat{g}^1/\hat{G}_k) = \sum v(\hat{\mathfrak{U}}_{M_i0})\cdot v(\hat{g}^1_\infty/\hat{\Gamma}_{M_i})$$

$$= h\,\underbrace{v(\hat{\mathfrak{U}}_{M0})}_{=\prod_{\nu=2}^{n}\zeta_k(\nu)^{-1}}\cdot\underbrace{v(\hat{g}^1_\infty/\hat{\Gamma}_M)}_{=\frac{2}{w}v(\hat{K})\cdot v(\hat{S}^1/\hat{\Gamma}_M)}$$

$$= \frac{2h}{w}(2\pi)^{r_2}\prod_{2}^{n}\{\cdots\}^{-1}\,v(\hat{S}^1/\hat{\Gamma}_M)$$

($w = \#(\gamma\in\hat{\Gamma}_M$ which is id. on $\hat{S}^1$)
$2 = (\hat{g}_\infty/\hat{g}^1_\infty$ と $\hat{S}/\hat{S}^1$ の measure の比)

特に $n=1$ のとき

$$v(\hat{S}^1/\hat{\Gamma}_M) = v((\mathbb{R}^+)^{n-1}/\theta^*) \overset{\text{def}}{=} 2^{r_1-1} R$$

$$\left(\begin{array}{l} k_{\mathbb{R}}^* = \mathbb{R}^{*\,r_1} \times \mathbb{C}^{*\,r_2} \\ \theta^* \subset \quad = (\{\pm 1\}^{r_1} \times T^{r_2}) \times \mathbb{R}^{+\,r_1+r_2} \\ \qquad\qquad\quad 1 \qquad\quad 2\pi \qquad\quad (\xi_1, \cdots, \xi_{r_1+r_2}) \\ \qquad\qquad\qquad\qquad\qquad\quad 2^{r_1+r_2} \prod \frac{d\xi_i}{\xi_i} \end{array} \right.$$

$$v(I_k/k^*) = \frac{2h}{w}(2\pi)^{r_2} \cdot 2^{r_1-1} R = \frac{hR}{w} \cdot 2^{r_1}(2\pi)^{r_2}$$

$$= \Delta^{\frac{1}{2}} \rho_k$$

$$\rho_k = \mathrm{Res}_{s=1} \zeta_k(s)$$

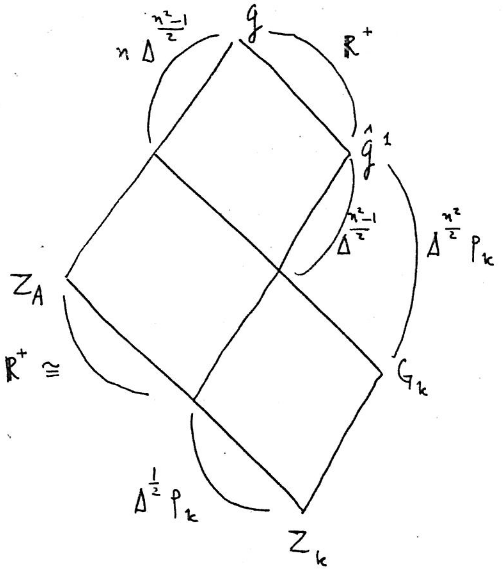
$$\therefore \quad v(\hat{g}^1/\hat{G}_k) = \Delta^{\frac{n^2}{2}} \rho_k$$

$$v(\hat{S}^1/\hat{\Gamma}_M) = \Delta^{\frac{n^2-1}{2}} 2^{r_1-1} R \prod_{\nu=2}^{n} \left\{ \zeta_k(\nu) \left(\frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{\nu}{2}}} \right)^{r_1} \left(\frac{\Gamma(\nu)}{(2\pi)^{\nu}} \right)^{r_2} \right\}$$

$$= \frac{w}{2h} \cdot \Delta^{\frac{n(n-1)}{4}} \cdot \mathrm{Res}_{s=1} \xi_k(s) \cdot \prod_{\nu=2}^{n} \xi_k(\nu)$$

$$\left(\begin{array}{l} \mathrm{Res}_{s=1} \xi_k(s) = \Delta^{\frac{1}{2}} (2\pi)^{-r_2} \cdot \rho_k \\ \qquad\qquad\qquad = 2^{r_1} \cdot \frac{hR}{w} \end{array} \right.$$

∘ $PL(n) = \hat{G}/Z$ の場合



$\therefore \ \tau(\hat{G}/Z) = n$

◎ $G = G_Q$: $\omega_Q(X)$

$\hat{G}$: $\det(X)^{-n} d(X)$

$G_Q \backslash \hat{G}$: $\det(Y)^{-\frac{n+1}{2}} d(Y)$

$$\int_{\hat{G}} f(X) \det(X)^{-n} d(X) = \int_{G_Q \backslash \hat{G}} \left(\int_{G_Q} f(XX_1)\, \omega_Q(X) \right) \det(\underset{\underset{Q[X_1]}{\shortparallel}}{Y})^{-\frac{n+1}{2}} d(Y)$$

$$\omega_Q(X) = \omega_{Q[T]}(T^{-1}XT)$$

① $G_Q = O(n, \mathbb{R}, Q)$

$K_{Q,Q_0} = O(Q) \cap O(Q_0)$

$S_Q = K_{Q,Q_0} \backslash G_Q$: $|\det(Q)|^{-\frac{p}{2}} \left(\det Q^{-1} \begin{bmatrix} 1_p \\ {}^tZ \end{bmatrix} \right)^{-\frac{n}{2}} d(Z)$

$$\int_{K_{Q,Q_0}} = (\kappa_p \kappa_{n-p})^{-1}$$

($Q = \begin{pmatrix} 1_p & \\ & -1_{n-p} \end{pmatrix}$, $Q_0 = 1_n$ のとき 計算すればよい.)

② $G_Q = O(n, \mathbb{C}, Q)$

$K_{Q,H} = O(Q) \cap U(H)$

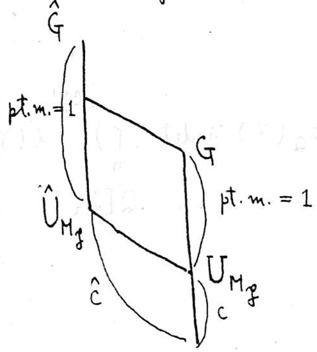
$S_Q = K_{Q,H} \backslash G_Q$: $\det(Q)^{-n} \det(Q^{-1} - Q^{-1}[{}^tZ])^{-n} d(Z)$

$$\int_{K_{Q,H}} = \kappa_n^{-2}$$

($Q = H = 1_n$ のとき 計算すればよい.)

③ $G_Q = O(n, k_\mathfrak{p}, Q)$

$M_\mathfrak{p} = \sum e_{i,\mathfrak{p}}\, \mathfrak{o}$



char. func. of $\hat{U}_{M_\mathfrak{p}}$ を積分.

$$\hat{c} = c\,|\det(Q)|_\mathfrak{p}^{-\frac{n+1}{2}} \int_{\{Y = Q[X_1] \mid X_1 \in \hat{U}_{M_\mathfrak{p}}\}} |d(Y)|_\mathfrak{p}$$

$$= \lim_{m\to\infty} \frac{\#\hat{G}(\mathfrak{o}/\mathfrak{p}^m)}{q^{m\frac{n(n+1)}{2}}\,\#G_Q(\mathfrak{o}/\mathfrak{p}^m)}$$

$$\#\hat{G}(\mathfrak{o}/\mathfrak{p}^m) = q^{(m-1)n^2}\,\#\hat{G}(\mathfrak{o}/\mathfrak{p}) = q^{mn^2}\underbrace{(1-q^{-1})\cdots(1-q^{-n})}_{\hat{c}}$$

$$\therefore\quad c = |\det(Q)|_\mathfrak{p}^{\frac{n+1}{2}} \underbrace{\lim_{m\to\infty} \frac{\#G_Q(\mathfrak{o}/\mathfrak{p}^m)}{q^{m\frac{n(n-1)}{2}}}}$$

$2\alpha_\mathfrak{p}$ とおく. dep. only on the genus of $M_\mathfrak{p}$

($m \gg 0$ const.)

④ $G^+ = G_Q^+ = O^+(Q)$

$\mathfrak{g}^+ = G_A^+$

$$v(\mathfrak{U}^+_{M,0}) = \prod v(U^+_{M_\mathfrak{p}})$$

$$= N(\det(Q))^{-\frac{n+1}{2}} \prod_{\mathfrak{p}} \alpha_\mathfrak{p}$$

conv. dep. only on genus of M.

$\left(\begin{array}{ll} n=2,\ v=0 & 条件収束 \\ \ \ \text{"} \ ,\ v=1 & 発散 \end{array} \right)$

$$\mathfrak{g}^+ / G_k^+ \approx \bigcup \mathfrak{U}^+_{M_i} / \mathfrak{U}^+_{M_i} \cap G_k^+$$

$$\therefore \ \Delta^{\frac{n(n-1)}{4}} \tau(G_Q^+) = \sum \underset{N(\det Q)^{-\frac{n+1}{2}} \prod \alpha_\mathfrak{p}}{\underset{\parallel}{v(\mathfrak{U}^+_{M_i,0})}} \ \underset{\frac{1}{(2,n)} v(K)\cdot v(S_Q/\Gamma^+_{M_i})}{\underset{\parallel}{v(\mathfrak{g}^+_\infty / \Gamma^+_{M_i})}}$$

$$= \frac{\prod_\mathfrak{p} \alpha_\mathfrak{p}}{2^r N(\det Q)^{\frac{n+1}{2}} \cdot \prod_{i=1}^{r_1} (\kappa_{p_i} \kappa_{n-p_i}) \cdot \kappa_n^{2r_2}} \cdot \frac{1}{(2,n)} \sum_i v(S_Q / \Gamma^+_{M_i})$$

(ただし $r_1 = d$, Q : tot. def. のとき, Γ_{M_i} : finite
最後の factor は $\sum_i \frac{1}{\# \Gamma^+_{M_i}}$ でおきかえる.

- $\tau(G_Q^+) = 2$

$$\therefore \ \left. \begin{array}{l} \dfrac{1}{(2,n)} \displaystyle\sum_i v(S_Q / \Gamma^+_{M_i}) \\ \text{or} \\ \displaystyle\sum_i \dfrac{1}{\# \Gamma^+_{M_i}} \end{array} \right\} = 2^{r+1} \Delta^{\frac{n(n-1)}{4}} N(\det Q)^{\frac{n+1}{2}} \left\{ \begin{array}{l} \dfrac{\prod_{i=1}^{r_1} (\kappa_{p_i} \kappa_{n-p_i}) \cdot \kappa_n^{2r_2}}{\prod_\mathfrak{p} \alpha_\mathfrak{p}} \\ \kappa_n^{r_1} \Big/ \prod_\mathfrak{p} \alpha_\mathfrak{p} \end{array} \right.$$

(Minkowski - Siegel)

<u>Rem.</u> 1. representation number についても同様の公式

$$\tau = \begin{cases} 2 & n - n' = 1 \\ 1 & n - n' > 1 \end{cases}$$

2. $n = 2$, $k = \mathbb{Q}$ のとき Dirichlet の公式

$n = 3, 4$ quaternion

3. hermitian form の場合.

Hel Braun, Ramanathan, Weil

4 Witt th.

2 Clifford alg.

2 Orth. gr.

2. 2 – 6. isomorphism

2. alg. n.f. $\mathfrak{p}$-adic n.f.

2 Hasse's th.

2 finiteness th.

4 q.f. / $\mathfrak{p}$-adic n.f. Minkowski's reduction th.

Siegel's th.

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Klingenberg und Witt, 193 (1954), 121 – 122
Witt — 193 (1954), 119 – 120

K ch. 2, Q non-deg. ($\gamma=0$), $V^{\perp}$ ($n-2m$ dim)

$$Q(x\lambda + y\mu) = Q(x)\lambda^2 + Q(y)\mu^2$$

$$x = \sum e_i \xi_i$$

$$Q(x) = \sum_{i=1}^{m} \left(\alpha_i \xi_i^2 + \xi_i \xi_{m+i} + \beta_i \xi_{m+i}^2\right) + \sum_{i=2m+1}^{n} \gamma_i \xi_i^2$$

$\{\gamma_i\}$ lin. indep. / K^2.

$\alpha_i = \beta_i = 0 \quad (1 \leq i \leq \nu)$

K perfect $n = 2m$ or $2m+1$

$\dim V \geq 3 \Rightarrow V \ \exists$ isotropic vect.

anisotropic まで ? ξ_1^2

$\lambda(\xi_1^2 + \xi_2^2) + \xi_1\xi_2 \quad | \quad \xi_1^2 + \xi_1\xi_2 + \mu\xi_2^2$.

$K(\theta) \quad (\lambda\theta^2 + \theta + \lambda = 0)$ が inv. $\quad \mu \pmod{\wp K}$

$$A = \begin{pmatrix} 2a_1 & & a_{ij} \\ & \ddots & \\ a_{ji} & & 2a_n \end{pmatrix} \qquad \alpha = \text{Pfaffian of } \begin{pmatrix} 0 & & a_{ij} \\ & \ddots & \\ -a_{ji} & & 0 \end{pmatrix}$$

$$(-1)^m |A| \alpha^{-2} = 1 + 4\Delta(A) \quad \Big| \quad \Delta(A)\alpha^2 \in \mathbb{Z}[a_i, a_{ij}]$$

$\Delta(A) \pmod{\gamma^2+\gamma}$ inv. of $\sum a_i \xi_i^2 + \sum_{i<j} a_{ij}\xi_i\xi_j$

$$A = \left(\begin{array}{c|c} A_1 & 1_m \\ \hline 0 & A_2 \end{array}\right) \qquad \Delta(A) = \mathrm{tr}(A_1 A_2) \pmod{\gamma^2+\gamma}$$