The *wild* McKay correspondence

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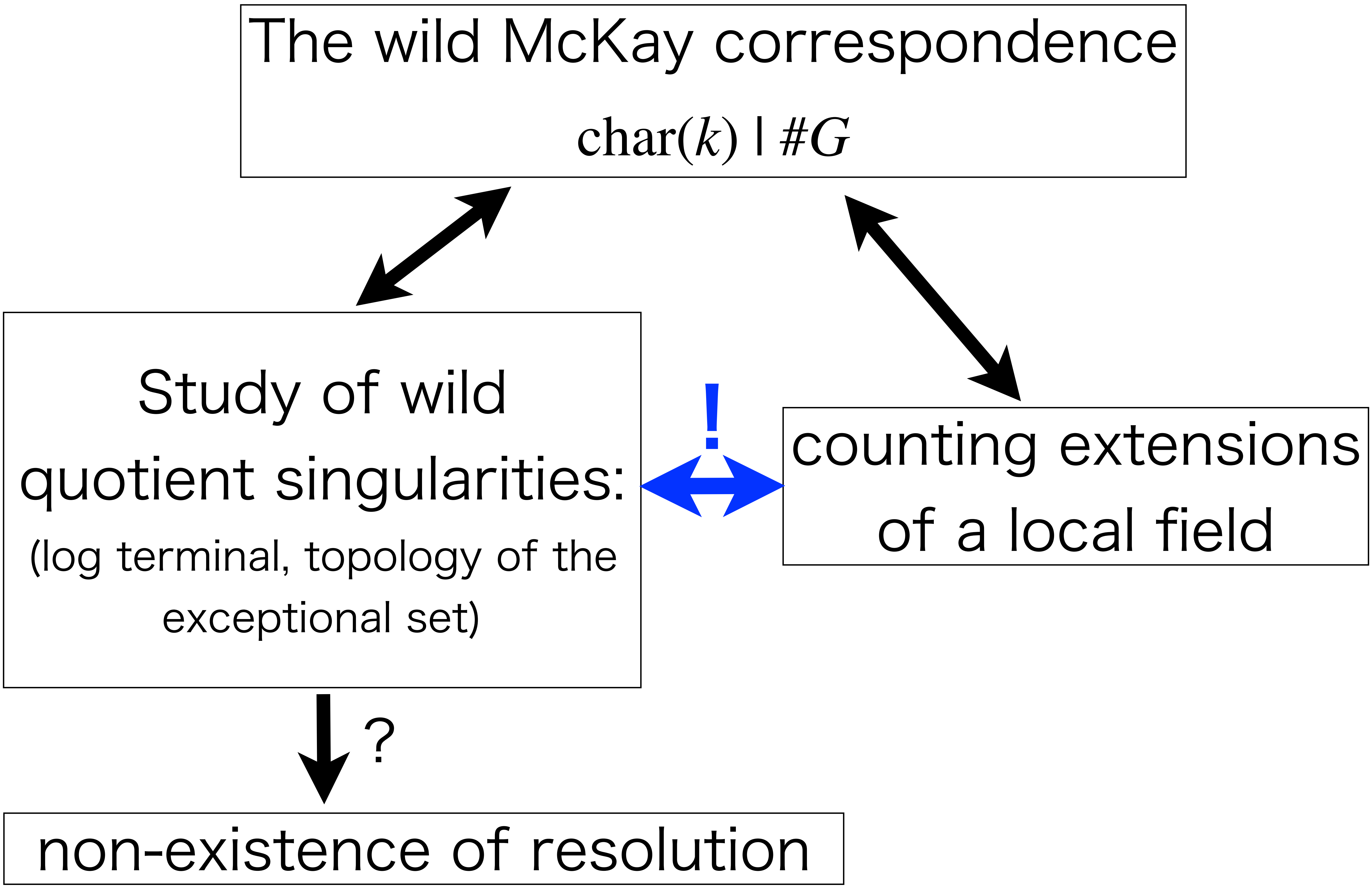
The wild McKay correspondence
char(k) | $\#G$
Study of wild
quotient singularities:
(log terminal, topology of the
exceptional set)
!
counting extensions
of a local field
?
non-existence of resolution

Plan of the talk

1. Stringy motifs and the McKay correspondence
2. Motivic integration over wild DM stacks
3. Counting extensions of local fields
4. Future tasks and summary

1. Stringy motifs and the McKay correspondence

- Wild group actions are not too much wild. -

Stringy motifs

X: $\mathbf{Q}$-Gorenstein

$W \subset X$: a closed subset

the stringy motif of X along W

$$\rightsquigarrow \quad M_{\mathrm{st}}(X)_W := \int_{(J_\infty X)_W} \mathbf{L}^{F_X} \, d\mu_X$$

arcs meeting W

coming from singularities

The M_{st} contains a lot of information on resolutions:

1. minimal discrepancy
2. topology of the exceptional set of a resolution
3. non-existence of resolution

$f\colon Y \to X$ a resolution

$\Downarrow$ change of variables

$$M_{\mathrm{st}}(X)_W = M_{\mathrm{st}}(Y, -K_f)_{f^{-1}W} := \int_{(J_\infty Y)_{f^{-1}W}} \mathbf{L}^{-F_{K_f}} d\mu_Y$$

if f is crepant $\|$

$[f^{-1}W]$

$\Uparrow$

explicit if K_f is a SNC divisor

$$M_{\mathrm{st}}(X)_W = M_{\mathrm{st}}(Y, -K_f)_{f^{-1}W}$$

<u>Conjecture A</u>

- This holds even if Y is a **DM stack,**
- if the RHS is suitably defined as a motivic integral over the space of **twisted arcs**.
- This will follow from the **change of variables formula**.

<u>Theorem</u> [Y]

Conjecture holds if Y is a **tame** DM stack

The McKay correspondence

Settieng:

- $G \subset GL_d(k)$ a finite group without reflection
- $X := k^d/G$ the quotient variety
- $\mathcal{X} := [k^d/G]$ the quotient stack
- $\mathcal{X} \to X$ a stacky crepant resolution
- $f : Y \to X$ a crepant resolution as a variety

$$\Downarrow \text{ Conjecture A}$$

$$[f^{-1}W] = M_{\mathrm{st}}(Y)_{f^{-1}W} = M_{\mathrm{st}}(X)_W = M_{\mathrm{st}}(\mathcal{X})_{\mathcal{W}}$$

The (wild) McKay correspondence

$$[f^{-1}W] = M_{\mathrm{st}}(\mathcal{X})_{\mathcal{W}}$$

$$\Downarrow \quad \text{if } \mathrm{char}(k) \nmid \#G$$

$$e_{\mathrm{top}}(f^{-1}(0)) = \#\mathrm{Conj}(G) \quad (0 \in X)$$

the tame McKay correspondence

(conjectured by Reid,
proved by Batyrev,
our approach essentially
due to Denef-Loeser.)

$$[f^{-1}W] = M_{\mathrm{st}}(\mathcal{X})_{\mathcal{W}}$$

[Y] $\Downarrow$ $\mathrm{char}(k) = \#G = p > 0$

$$e_{\mathrm{top}}(f^{-1}(0)) = p \quad (0 \in X)$$

the p-cyclic McKay correspondence

Other consequences in the p-cyclic case

$G \subset GL_d(k) \rightsquigarrow$ a numerical invariant

$D_G \in \mathbf{Z}$

$M_{\mathrm{st}}(X)_0 \neq \infty \quad \Leftrightarrow D_G \geq p$

$\Downarrow \quad \Uparrow$ ∃ log res.

log terminal

∃ crepant res. $\Rightarrow D_G = p$

crepant

Non-existence of resolution

$\mathcal{X}$: not necessarily linear

$$M_{\mathrm{st}}(Y, -K_f)_{f^{-1}W} = M_{\mathrm{st}}(X)_W = M_{\mathrm{st}}(\mathcal{X})_{\mathcal{W}}$$

⬆ explicit formula

⬆ compute

crepant

$\exists f \Rightarrow$ Properties of the invariant:

* rationality polynomial

* Poincaré duality

(* topology of the dual complex of the exceptional set [Kerz-Saito])

If contradict

* $D_G = p$ (the p-cyclic case)

$\Rightarrow$

$\nexists f$ crepant

2. Motivic integration over wild DM stacks

- Twist arcs wildly! -

Expect that stringy motifs of stacks will be defined as motivic integrals over spaces of **twisted arcs**.

Spaces of (twisted) arcs	
X: a variety	$J_\infty X = \mathrm{Hom}(\mathrm{Spec}\, k[[t]], X)$
$\mathcal{X}$: a tame DM stack	$\mathcal{J}_\infty \mathcal{X} = \bigsqcup_{l \geq 1} \mathrm{Hom}_{\mathrm{rep}}([\mathrm{Spec}\, k[[t^{1/l}]]/\mu_l], \mathcal{X})$
$\mathfrak{X}$: a wild DM stack	$\mathfrak{J}_\infty \mathfrak{X}$ ∃ infinitely many E's $= \bigsqcup_{E \to \mathrm{Spec}\, k[[t]]:\ \text{Gal. cov.}} \mathrm{Hom}_{\mathrm{rep}}([E/\mathrm{Gal}], \mathfrak{X})$

Change of variables for varieties

$f : Y \to X$ a proper birational morphism of varieties

$f_\infty : J_\infty Y \to J_\infty X$ the induced map of arc spaces

Theorem [Kontsevich, Denef-Loeser, Sebag]

measurable fn on C (↓ F)   Jac. order fn of f (↓ $\mathrm{ord}\,\mathrm{Jac}_f$)

$$\int_C \mathbf{L}^F d\mu_X = \int_{f_\infty^{-1}(C)} \mathbf{L}^{F \circ f_\infty - \mathrm{ord}\,\mathrm{Jac}_f} d\mu_Y$$

$C \subset J_\infty X$

Change of variables for stacks

$f:\mathcal{Y}\to\mathcal{X}$ a proper birational morphism of (tame or wild) DM stacks

$f_\infty: J_\infty\mathcal{Y}\to J_\infty\mathcal{X}$ the induced map of twisted arc spaces

Conjecture B

weight fns coming from local group actions

$$\int_C \mathbf{L}^{F+w_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int_{f_\infty^{-1}(C)} \mathbf{L}^{F\circ f_\infty - \mathrm{ord}\,\mathrm{Jac}_f + w_{\mathcal{Y}}} d\mu_{\mathcal{Y}}$$

Known for the tame and p-cyclic cases [Y] group actions

singularities

$$M_{\mathrm{st}}(\mathcal{X})_{\mathcal{W}} := \int_{(\mathcal{J}_\infty \mathcal{X})_{\mathcal{W}}} \mathbf{L}^{F_{\mathcal{X}} + w_{\mathcal{X}}} d\mu_{\mathcal{X}}$$

the change of variables

$$\Downarrow$$

Conjecture A

$f : \mathcal{Y} \to X$ a proper birational morphism with $\mathcal{Y}$ a DM stack

$$M_{\mathrm{st}}(X)_W = M_{\mathrm{st}}(\mathcal{Y}, -K_f)_{f^{-1}W}$$

3. Counting extensions of local fields

- Singularities encode the number theory. -

Setting

a finite group G

$\rightsquigarrow$ $G\text{-Cov}(D)$ the (hypothetical) space of

G-covers of $D = \operatorname{Spec} k[[t]]$

- G acts on $G\text{-Cov}(D)$ by conjugation

Go back to the situation:

- $G \subset GL_d(k)$ a finite group without reflection
- $X := k^d/G$ the quotient variety
- $\mathcal{X} := [k^d/G]$ the quotient stack

Conjecture C

$w_{\mathcal{X}}$ factors as $w_{\mathcal{X}}: J_\infty \mathcal{X} \to G\text{-Cov}(D)/G \xrightarrow{w_G} \mathbf{Q}$

&

$$\begin{aligned} M_{\mathrm{st}}(X)_0 &= M_{\mathrm{st}}(\mathcal{X})_0 \\ &= \int_{(\mathcal{J}_\infty \mathcal{X})_0} \mathbf{L}^{w_{\mathcal{X}}} d\mu_{\mathcal{X}} \\ &= \sum_{w \in \mathbf{Q}} [(G\text{-Cov}(D)/G)_{w_G = w}] \mathbf{L}^w \end{aligned}$$

a weighted "count" of extensions of the local field $k((t))$

$$\sum_{w \in \mathbf{Q}} [(G\text{-Cov}(D)/G)_{w_G = w}] \mathbf{L}^w$$

$\Updownarrow$ analogous

K a local field with residue field having q elements

[Serre] $$\sum_{\substack{[L:K]=n \\ \text{totally ramified}}} \frac{1}{\sharp \mathrm{Aut}(L/K)} \cdot q^{-v_K(\mathrm{Disc}(L))} = q^{-n+1}$$

Generalized by Bhargava (étale extensions),

Kedlaya and Wood (Galois representations)

<u>Note</u>: We can use $\mathbf{Z}_p$ instead of $k((t))$.

$f: Y \to X$ a crepant resolution, $E = f^{-1}(0)$

$\Downarrow$

$$[E] = \sum_{w \in \mathbf{Q}} [(G\text{-Cov}(D)/G)_{w_G = w}] \mathbf{L}^w$$

The tame case

$$e_{\mathrm{top}}(E) = \#\mathrm{Conj}(G) = \#(G\text{-Cov}(D)/\mathrm{G})$$

The p-cyclic case

$$\sharp E(\mathbf{F}_q) = 1 + \frac{p-1}{p} \sum_{\substack{[L:\mathbf{F}_q((t))]=p \\ \text{Galois, totally ramified}}} q^{w_G(L)}$$

4. Future tasks and summary

Future tasks

1. The motivic integration over DM stacks:
   i. Construct the space of twisted arcs.
   ii. Prove the change of variables formula.
2. Compute stringy invariants by computing
   i. G-$\mathrm{Cov}(D)/\mathrm{G}$ and w_G, or
   ii. a resolution of k^d/G.
3. What quotient singularity admits a crepant resolution?
4. The wild **categorical** McKay correspondence?

Summary

1. The motivic integration over wild DM stacks would prove the wild McKay correspondence.
2. A quotient singularity would encode a weighted count of a local field.
3. Computing stringy invariants, one might be able to disprove resolution.