Perspectives on the wild McKay correspondence

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Plan of the talk

Partly a joint work with Melanie Wood.

1. What is the McKay correspondence?
2. Motivation
3. The wild McKay correspondence conjectures
4. Relation between the weight function and the Artin/Swan conductors
5. The Hilbert scheme of points vs Bhargava's mass formula
6. Known cases
7. Possible applications
8. Future tasks

References

1. T. Yasuda, *The p-cyclic McKay correspondence via motivic integration.* `arXiv:1208.0132`, to appear in Compositio Mathematica.
2. T. Yasuda, *Toward motivic integration over wild Deligne-Mumford stacks*. `arXiv:1302.2982`, to appear in the proceedings of "Higher Dimensional Algebraic Geometry - in honour of Professor Yujiro Kawamata's sixtieth birthday".
3. M.M. Wood and T. Yasuda, *Mass formulas for local Galois representations and quotient singularities I: A comparison of counting functions.* `arXiv:1309:2879`.

What is the McKay correspondence?

A typical form of the McKay correspondence

For a finite subgroup $G \subset GL_n(\mathbb{C})$, the same invarinat arises in two totally different ways:

Singularities a resolution of singularities of $\mathbb{C}^n/G$

Algebra the representation theory of G and the given representation $G \hookrightarrow GL_n(\mathbb{C})$

For example,

Theorem (Batyrev)

Suppose

- G *is small* ($\nexists$ *pseudo-refections*), *and*
- $\exists$ *a crepant resolution* $Y \to X := \mathbb{C}^n/G$ (*i.e.* $K_{Y/X} = 0$) .

Define

$$e(Y) := \textit{the topological Euler characteristic of } Y.$$

Then

$$e(Y) = \sharp\{\textit{conj. classes in } G\} = \sharp\{\textit{irred. rep. of } G\}.$$

Motivation

Today's problem

What happens in positive characteristic? In particular, in the case where the characteristic divides $\sharp G$ (the wild case).

Motivation

- To understand singularities in positive characteristic.
- wild quotient singularities = typical "bad" singularities, and a touchstone (試金石) in the study of singularities in positive characteristic.
- [de Jong]: for any variety X over $k = \bar{k}$, $\exists$ a set-theoretic modification $Y \to X$ (i.e. an alteration with $K(Y)/K(X)$ purely insep.) with Y having only quotient singularities.

The wild McKay correspondence conjectures

For a perfect field k, let $G \subset GL_n(k)$ be a finite subgroup.
Roughly speaking, the wild McKay correspondence conjecture is an equality between:

- **Singularities** a stringy invariant of $\mathbb{A}_k^n/G$.
- **Arithmetics** a weighted count of continuous homomorphisms $G_{k((t))} \to G$ with $G_{k((t))}$ the absolute Galois group of $k((t))$. Equivalently a weighted count of etale G-extensions of $k((t))$.

One precise version:

Conjecture 1 (Wood-Y)

- $k = \mathbb{F}_q$
- $G \subset GL_n(k)$: a small finite subgroup
- $Y \xrightarrow{f} X := \mathbb{A}_k^n/G$: a crepant resolution
- $0 \in X$: the origin

Then

$$\sharp(f^{-1}(0)(k)) = \frac{1}{\sharp G} \sum_{\rho \in Hom_{cont}(G_{k((t))}, G)} q^{w(\rho)}.$$

Here w is the weight function associated to the representation $G \hookrightarrow GL_n(k)$, which measures the ramification of ρ.

More generally:

Conjecture 1' (Wood-Y)

- K is a local field with residue field $k = \mathbb{F}_q$
- $G \subset GL_n(\mathcal{O}_K)$: a small finite subgroup
- $Y \xrightarrow{f} X := \mathbb{A}^n_{\mathcal{O}_K}/G$: a crepant resolution
- $0 \in X(k)$: the origin

Then

$$\sharp(f^{-1}(0)(k)) = \frac{1}{\sharp G} \sum_{\rho \in Hom_{cont}(G_K, G)} q^{w(\rho)}.$$

The motivic and more general version:

Conjecture 2 (Y, Wood-Y)

- K is a complete discrete valuation field with perfect residue field k
- $G \subset GL_n(\mathcal{O}_K)$: a small finite subgroup
- ~~$Y \xrightarrow{f} X := \mathbb{A}^n_{\mathcal{O}_K}/G$: a crepant resolution~~
- $0 \in X(k)$: the origin

Then

$$M_{st}(X)_0 = \int_{\mathcal{M}_K} \mathbb{L}^w \qquad \text{in a modified } K_0(Var_k).$$

Here

- $\mathcal{M}_K$: the conjectural moduli space of G-covers of $Spec\, K$ defined over k.
- $M_{st}(X)_0$: the stringy motif of X at 0.

Remarks

- If $K = \mathbb{C}((t))$ and $G \subset GL_n(\mathbb{C})$, then $\mathcal{M}_K$ consists of finitely many points corresponding to conjugacy classes in G, and the RHS is of the form

$$\sum_{g \in Conj(G)} \mathbb{L}^{w(g)}.$$

  We recover Batyrev and Denef-Loeser's results.
- Both sides of the equality might be ∞ (the defining motivic integral diverges).If X has a log resolution, then this happens iff X is not log terminal. Wild quotient singularities are NOT log terminal in general.
- Conjectures would follow if the theory of motivic integration over wild DM stacks is established.

More remarks

- The assumption that the action is linear is important.
- The assumption that G is small can be removed by considering the stringy invariant of a log variety (X, Δ) with Δ a $\mathbb{Q}$-divisor.
- The ultimately general form:let (X, Δ) and (X', Δ') be two K-equivalent log DM stacks and $W \subset X$ and $W' \subset X'$ corresponding subsets. Then

$$M_{st}(X, \Delta)_W = M_{st}(X', \Delta')_{W'}.$$

(Y-: the tame case with a base k (not $\mathcal{O}_K$))

Relation between the weight function and the Artin/Swan conductors

Definition (the weight function (Y, Wood-Y))

$\rho \leftrightarrow L/K$ the corresponding etale G-extension
Two G-actions on $\mathcal{O}_L^{\oplus n}$:

- the G-action on L induces the diagonal G-action on $\mathcal{O}_L^{\oplus n}$,
- the induced representation $G \hookrightarrow GL_n(\mathcal{O}_K) \hookrightarrow GL_n(\mathcal{O}_L)$.

Put

$$\Xi := \{x \mid g \cdot_1 x = g \cdot_2 x\} \subset \mathcal{O}_L^{\oplus n}.$$

Define

$$w(\rho) := codim\,(k^n)^{\rho(I_K)} - \frac{1}{\sharp G} \cdot length\,\frac{\mathcal{O}_L^{\oplus n}}{\mathcal{O}_L \cdot \Xi}.$$

Here $I_K \subset G_K$ is the inertia subgroup.

To $G_K \xrightarrow{\rho} G \hookrightarrow GL_n(K)$, we associate the Artin conductor

$$a(\rho) = \overbrace{t(\rho)}^{\text{the tame part}} + \overbrace{s(\rho)}^{\text{the wild part (Swan cond.)}} \in \mathbb{Z}_{\geq 0}.$$

Proposition (Wood-Y)

If $G \subset GL_n(\mathcal{O}_K)$ is a permutation representation, then

$$w(\rho) = \frac{1}{2}\left(t(\rho) - s(\rho)\right).$$

The Hilbert scheme of points vs Bhargava's mass formula

- K: a local field with residue field $k = \mathbb{F}_q$
- $G = S_n$: the n-th symmetric group
- $S_n \hookrightarrow GL_{2n}(\mathcal{O}_K)$: two copies of the standard representation
- $X := \mathbb{A}^{2n}/S_n = S^n\mathbb{A}^2$: the n-th symmetric product of the affine plane (over $\mathcal{O}_K$)
- $Y := Hilb^n(\mathbb{A}^2)$: the Hilbert scheme of n points (over $\mathcal{O}_K$)
- $Y \xrightarrow{f} X$: the Hilbert-Chow morphism, known to be a crepant resolution (Beauville, Kumar-Thomsen, Brion-Kumar)

Theorem (Wood-Y)

In this setting,

$$\sharp(f^{-1}(0)(\mathbb{F}_q)) = \sum_{i=0}^{n-1} P(n, n-i) q^i = \frac{1}{n!} \sum_{\rho \in Hom_{cont}(G_K, S_n)} q^{w(\rho)}.$$

Here $P(n,j) := \sharp\{\textit{partitions of } n \textit{ into exactly } j \textit{ parts}\}$.

Bhargava's mass formula

$$\sum_{i=0}^{n-1} P(n, n-i) q^{-i} = \sum_{\substack{L/K:\text{ etale} \\ [L:K]=n}} \frac{1}{\sharp Aut(L/K)} q^{-v_K(disc(L/K))}$$

$$\overset{\text{Kedlaya}}{=} \frac{1}{n!} \sum_{\rho \in Hom_{cont}(G_K, S_n)} q^{-a(\rho)}.$$

Known cases

	1	2	3	4
	tame		wild	
	B, D-L, Y, W-Y	W-Y	Y	W-Y
group	any	$\mathbb{Z}/3\mathbb{Z} \subset GL_3$	$\mathbb{Z}/p\mathbb{Z}$	$S_n \subset GL_{2n}$
k or $\mathcal{O}_K$	k (char $\nmid \sharp G$)	$\mathcal{O}_K$ (char $k \neq 3$)	k (char p)	any
Conj 1	✓	✓	✓	✓
Conj 1'		✓		✓
Conj 2	✓		✓	

Possible applications

$$\sharp(f^{-1}(0)(k)) = \frac{1}{\sharp G} \sum_{\rho \in Hom_{cont}(G_{k((t))}, G)} q^{w(\rho)}$$

$$M_{st}(X)_0 = \int_{\mathcal{M}_K} \mathbb{L}^w$$

- Can compute the LHS's (singularities) by computing the RHS's (arithmetics), and vice versa.
- In low (resp. high) dimensions, the LHS's (resp. RHS's) seems likely easier.
- $\exists$ a log resolution of $X = \mathbb{A}^n/G$
  $\Rightarrow$ rationality and duality of the LHS's
  $\Leftrightarrow$ rationality and duality of the RHS's!!! (hidden structures on the set/space of G-extensions of a local field)

Problem

Compute the RHS's using the number theory and check whether these properties holds.

If the properties do not hold, then $\nexists$ a log resolution of X.

Future problems

- Non-linear actions on possibly singular spaces (work in progress)
- Global fields
  - $Hilb^n(\mathbb{A}^2_{\mathbb{Z}}/\mathbb{Z}) \overset{??}{\longleftrightarrow} \{L/\mathbb{Q} \mid [L:\mathbb{Q}] = n\}$
  - curve counting on wild orbifolds
- Explicit (crepant) resolution of wild quotients in low dimensions

  $\Rightarrow$ many new mass formulas
- Explicit computation of $\frac{1}{\sharp G} \sum q^{w(\rho)}$ and check the rationality or duality.
- Prove these properties in terms of the number theory.

One more problem

Problem

Does Batyrev's theorem holds also in the wild case? Namely, if

- K is a complete discrete valuation field with perfect residue field k
- $G \subset GL_n(\mathcal{O}_K)$: a small finite subgroup
- $Y \xrightarrow{f} X := \mathbb{A}^n_{\mathcal{O}_K}/G$: a crepant resolution,

then do we have the following equality?

$$e(Y \otimes_{\mathcal{O}_K} k) = \sharp\{\text{conj. classes in } G\}$$

Remark

This holds in a few cases we could compute. However, for now, I do not see any reason that this holds except the tame case.