Motivic integration, the McKay correspondence and wild ramification (tentative title)

Takehiko Yasuda

Department of Mathematics, Graduate School of Science, Osaka University Toyonaka, Osaka 560-0043, JAPAN

Email address: takehikoyasuda@math.sci.osaka-u.ac.jp.ac.jp

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CHAPTER 1

Introduction

1.1. Notation, terminology and convention

Throughout the book, we fix a base field k. We denote by L a field extension of k unless otherwise noted.

We assume that all schemes are separated.

By a k-variety, we mean an integral scheme of finite type over k.

By a ring, we mean a unital commutative ring unless otherwise noted.

For a ring R, we denote by $R[\![t]\!]$ (resp. $R(\!(t)\!)$) the ring of formal power series (resp. Laurent power series) with coefficients in R. Note that for a domain R, $R(\!(t)\!)$ is not generally the same as the fraction field of $R[\![t]\!]$; this is different from some authors' notation.

For a morphism $f\colon Y \to X$ of schemes and an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we denote by $f^{-1}\mathcal{I}$ the pullback of $\mathcal{I}$ as an ideal sheaf, which was denoted by $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ or $\mathcal{I} \cdot \mathcal{O}_Y$ in [**Har77**, p. 163]. When X is an affine scheme $\operatorname{Spec} R$, we often identify $f^{-1}\mathcal{I}$ with the corresponding ideal of R.

For a k-scheme X, a k-algebra R and a subset $C \subset X$, we denote by $C(R)$ the subset of $X(R)$ consisting of R-points $\operatorname{Spec} R \to X$ with image contained in C.

CHAPTER 2

The Grothendieck ring of varieties and realization maps

2.1. The Grothendieck ring of varieties

Motivic integration take values in the complete Grothendieck ring of varieties, denoted by $\widehat{\mathcal{M}}_k$, or a variant of it. Elements of this ring is considered as a toy version of motives that Grothendieck devised in order to unify various cohomology theories.

DEFINITION 2.1.1. We define the *Grothendieck ring of k-varieties*, denoted by $\mathrm{K}_0(\mathbf{Var}_k)$, to be the quotient group of the free abelian group $\bigoplus \mathbb{Z}\{X\}$ generated by isomorphism classes $\{X\}$ of k-schemes of finite type modulo the *scissor relation*: if Y is a closed subscheme of X, then

$$\{X\} = \{Y\} + \{X \setminus Y\}.$$

Namely we take the quotient of $\bigoplus \mathbb{Z}\{X\}$ by the submodule generated by the elements of the form

$$\{X\} - \{Y\} - \{X \setminus Y\}.$$

The multiplication on the additive group $\mathrm{K}_0(\mathbf{Var}_k)$ is given by

$$\{X\} \cdot \{Y\} := \{X \times_k Y\}.$$

By abuse of terminology, we call the class $\{X\}$ of a k-scheme X in $\mathrm{K}_0(\mathbf{Var}_k)$ the *motive* of X and denote it sometimes by $\mathrm{M}(X)$ (but more often keep using the notation $\{X\}$).

REMARK 2.1.2. It is more common to denote the class of a scheme X in $\mathrm{K}_0(\mathbf{Var}_k)$ by $[X]$ rather than $\{X\}$. We reserve the brackets $[\cdot]$ to denote quotient stacks.

It is easy to see that $\mathrm{K}_0(\mathbf{Var}_k)$ becomes a commutative ring. The identities for addition and multiplication are respectively $0 = \{\emptyset\}$ and $1 = \{\mathrm{Spec}\, k\}$.

The scissor relation in particular shows that the class $\{X\}$ of a scheme X in $\mathrm{K}_0(\mathbf{Var}_k)$ is independent of the scheme structure of X. Namely we have $\{X\} = \{X_{\mathrm{red}}\}$ with X_{red} the associated reduced scheme of X. Therefore, for a locally closed subset C of a scheme X of finite type, the class $\{C\} \in \mathrm{K}_0(\mathbf{Var}_k)$ is well-defined.

LEMMA 2.1.3. *If a scheme X of finite type is the disjoint union of locally closed subsets $C_i \subset X$, $1 \le i \le n$, then $\{X\} = \sum_{i=1}^{n} \{C_i\}$.*

PROOF. The proof is by Noetherian induction on the pair

$$p(X) := (\dim X, \text{the number of irreducible components of } X) \in \mathbb{N}^2.$$

Here $\mathbb{N}^2$ is given the lexicographic order. Thus we need to show the lemma for a given X, assuming that the lemma holds for every scheme X' with $p(X') < p(X)$.

First consider the case where X is reducible. Let $X_1 \subset X$ be an irreducible component and $X_2 \subset X$ its complement. From the scissor relation, $\{X\} = \{X_1\} + \{X_2\}$. Let $C_{i,j} := C_i \cap X_j$ for $1 \leq i \leq n$ and $j = 1, 2$, which are locally closed subsets of X_j. Since $p(X_j) < p(X)$, we have $\{X_j\} = \sum_i \{C_{i,j}\}$. We also have $\{C_i\} = \{C_{i,1}\} + \{C_{i,2}\}$ from the scissor relation. Thus

$$\{X\} = \{X_1\} + \{X_2\} = \sum_{i,j} \{C_{i,j}\} = \sum_i \{C_i\}.$$

Next consider the case where X is irreducible. Each C_i is written as the intersection $Y \cap U$ of a closed subset Y and an open subset U. If C_i contains the generic point, then the closed subset Y is the whole variety X and $C_i = U$. Thus exactly one of C_i's, say C_1, is an open dense subset of X. From the scissor relation, $\{X\} = \{C_1\} + \{X \setminus C_1\}$. Since $p(X \setminus C_1) < p(X)$, we have $\{X \setminus C_1\} = \sum_{i=2}^n \{C_i\}$. Thus

$$\{X\} = \{C_1\} + \{X \setminus C_1\} = \{C_1\} + \sum_{i=2}^n \{C_i\} = \sum_{i=1}^n \{C_i\}.$$

□

REMARK 2.1.4. Every k-scheme X of finite type is the disjoint union of finitely many integral subschemes $X_i \subset X$. It follows that $\mathrm{K}_0(\mathbf{Var}_k)$ is in fact generated by the classes of k-varieties. Similarly we get the same ring if we use k-varieties in Definition 2.1.1 instead of k-schemes of finite type.

DEFINITION 2.1.5. Let X be a k-scheme of finite type. For a locally closed subset $C \subset X$, we can define $\{C\} \in \mathrm{K}_0(\mathbf{Var}_k)$ by giving any scheme structure to C. A subset $C \subset X$ is said to be *constructible* if there exists a decomposition $C = \bigsqcup_i C_i$ into finitely many locally closed subsets C_i. For such a C, we define an element $\{C\} \in \mathrm{K}_0(\mathbf{Var}_k)$ to be $\sum_i \{C_i\}$.

LEMMA 2.1.6. *The class $\{C\} \in \mathrm{K}_0(\mathbf{Var}_k)$ of a constructible subset C is well-defined.*

PROOF. Let $C = \bigsqcup_i C_i$ and $C = \bigsqcup_j C_j$ be decompositions by finitely many locally closed subsets. If we put $C_{i,j} := C_i \cap C_j$, then $C_i = \bigsqcup_j C_{i,j}$ and $C_j = \bigsqcup_i C_{i,j}$. By Lemma 2.1.3, $\{C_i\} = \sum_j \{C_{i,j}\}$ and $\{C_j\} = \bigsqcup_i \{C_{i,j}\}$. Thus

$$\sum_i \{C_i\} = \sum_{i,j} \{C_{i,j}\} = \sum_j \{C_j\}.$$

□

DEFINITION 2.1.7. We denote the element $\{\mathbb{A}_k^1\} \in \mathrm{K}_0(\mathbf{Var}_k)$ associated to an affine line $\mathbb{A}_k^1$ by $\mathbb{L}$.

EXAMPLE 2.1.8. In $\mathrm{K}_0(\mathbf{Var}_k)$, we have

$$\begin{aligned} \{\mathbb{G}_{m,k}\} &= \mathbb{L} - 1, \\ \{\mathbb{A}_k^n\} &= \mathbb{L}^n, \\ \{\mathbb{P}_k^n\} &= \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1. \end{aligned}$$

The first equality follows from the scissor relation. The second one follows from the definition of multiplication. The third one follows from the scissor relation and the decomposition $\mathbb{P}_k^n = \mathbb{A}_k^n \sqcup \mathbb{A}_k^{n-1} \sqcup \cdots \sqcup \operatorname{Spec} k$.

EXAMPLE 2.1.9. For a vector bundle $V \to X$ of rank r, we have $\{V\} = \{X\}\mathbb{L}^r$. Indeed, there exists a stratification $X = \bigsqcup X_i$ into locally closed subsets such that for each i, $V|_{X_i} \cong X_i \times \mathbb{A}^r_k$. Thus

$$\{V\} = \sum_i \{V|_{X_i}\} = \sum_i \{X_i \times \mathbb{A}^r_k\} = \sum_i \{X_i\}\mathbb{L}^r = \{X\}\mathbb{L}^r.$$

The following compactness property of constructible subsets is useful and will be used without explicit mention.

LEMMA 2.1.10. *Let X be a k-scheme of finite type and let C and C_i, $i \in I$ be constructible subsets of X. If $C \subset \bigcup_{i\in I} C_i$, then there exists a finite subset $I' \subset I$ such that $C \subset \bigcup_{i\in I'} C_i$. In particular, if $C = \bigsqcup_{i\in I} C_i$ is a stratification of C, then I is finite.*

PROOF. The proof is by induction. If $\dim C \leq 0$, then C has at most finitely many points and the lemma holds. When $\dim C > 0$, then we take a finite set $I_0 \subset I$ such that every point of C of maximum dimension is contained in some C_i, $i \in I_0$ (there are only finitely many points of maximum dimension). Then $C \setminus \bigcup_{i\in I_0} C_i$ has less dimension than C. By induction hypothesis, there exists a finite subset $I_1 \subset I$ such that $C \setminus \bigcup_{i\in I_0} C_i \subset \bigcup_{i\in I_1} C_i$. We can take the desired finite set I' to be $I_0 \cup I_1$. □

2.2. Realization maps

The ring $\mathrm{K}_0(\mathbf{Var}_k)$ is huge and it is convenient to have maps from it to simpler rings such as polynomial rings or the ring of integers. We call such maps *realization maps* or simply *realizations.* We can associate a realization map to each generalized Euler characteristic.

DEFINITION 2.2.1. Let R be a ring and let $\mathbf{Var}_k$ be the category of k-varieties. A map $\chi\colon \mathbf{Var}_k \to R$ is called a *generalized Euler characteristic* if the following conditions hold:

(1) If Y is a closed subvariety of X, then $\chi(X) = \chi(Y) + \chi(X \setminus Y)$.
(2) For two varieties X and Y, we have $\chi(X \times_k Y) = \chi(X)\chi(Y)$.

For a generalized Euler characteristic χ, from the first condition and the definition of $\mathrm{K}_0(\mathbf{Var}_k)$, we have a unique group homomorphism $\chi'\colon \mathrm{K}_0(\mathbf{Var}_k) \to R$ making the following diagram commutative.

$$\begin{tikzcd}
\mathbf{Var}_k \arrow[r, "\mathrm{M}"] \arrow[dr, "\chi"'] & \mathrm{K}_0(\mathbf{Var}_k) \arrow[d, "\chi'"] \\
 & R
\end{tikzcd}$$

The second condition then shows that χ' is a ring homomorphism. Namely the natural map

$$\mathrm{M}\colon \mathbf{Var}_k \to \mathrm{K}_0(\mathbf{Var}_k),\ X \mapsto \mathrm{M}(X) = \{X\}$$

is the universal generalized Euler characteristic. We will usually denote the induced realization χ' by the same symbol as the original map χ.

REMARK 2.2.2. We can define $\chi(C)$ for a constructible subset C of a k-scheme of finite type in the same way as defining $\{C\}$.

EXAMPLE 2.2.3. If k is a finite field, then the *point counting map*

$$\sharp\colon \mathbf{Var}_k \to \mathbb{Z},\ X \mapsto \sharp X(k)$$

is a generalized Euler characteristic. Therefore we get the corresponding realization map $\sharp\colon \mathrm{K}_0(\mathbf{Var}_k) \to \mathbb{Z}$. For each $r \in \mathbb{Z}_{>0}$, we also have the generalized Euler characteristic

$$\sharp_r\colon \mathbf{Var}_k \to \mathbb{Z},\ X \mapsto \sharp X(k_r)$$

with k_r/k the extension of degree r and get a realization $\sharp_r\colon \mathrm{K}_0(\mathbf{Var}_k) \to \mathbb{Z}$.

EXAMPLE 2.2.4. For a $\mathbb{C}$-variety X, the *topological Euler characteristic* $\mathrm{e}_{\mathrm{top}}(X) \in \mathbb{Z}$ is given by

$$\mathrm{e}_{\mathrm{top}}(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} \mathrm{H}^i_{\mathrm{c}}(X(\mathbb{C}), \mathbb{Q}),$$

where $\mathrm{H}^*_{\mathrm{c}}(X(\mathbb{C}), \mathbb{Q})$ are singular cohomology groups of $X(\mathbb{C})$ (given with the analytic topology) with compact support. For an arbitrary field k, we can similarly define $\mathrm{e}_{\mathrm{top}}(X)$, using l-adic cohomology groups $\mathrm{H}^i_{\mathrm{c}}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)$ instead. Here l is the prime number different from the characteristic of k. It is known that the definition using l-adic cohomology is independent of the choice of l [**Nic11**, pp. 198–199] and gives the same value as the one defined in terms of singular cohomology. If X is smooth, then from the Poincaré duality, we may use the usual cohomology H^i rather than the one with compact support. The map $e_{\mathrm{top}}\colon \mathbf{Var}_k \to \mathbb{Z}$ is a generalized Euler characteristic.

EXAMPLE 2.2.5. The *Poincaré polynomial* of a smooth proper k-variety X is defined to be

$$\mathrm{P}(X) = \mathrm{P}(X;t) := \sum_i (-1)^i b_i(X) t^i \in \mathbb{Z}[t],$$

where $b_i(X)$ denotes the i-th Betti number $\dim_{\mathbb{Q}_l} \mathrm{H}^i(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)$ for l-adic cohomology, which is known to be independent of l. There exists a (necessarily unique) generalized Euler characteristic $\mathrm{P}\colon \mathbf{Var}_k \to \mathbb{Z}[t]$ such that for smooth proper X, $\mathrm{P}(\{X\}) = \mathrm{P}(X)$ (see [**Nic11**, Appendix]). When k is a finitely generated field, we can express $\mathrm{P}(X)$ for a general (not necessarily smooth or proper) variety X in terms of weight filtration on $\mathrm{H}^i_{\mathrm{c}}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)$.

EXAMPLE 2.2.6. If k is a subfield of $\mathbb{C}$, we can also define the *E-polynomial* (also called the *Hodge-Deligne polynomial*), denoted by $\mathrm{E}(X) = \mathrm{E}(X;u,v) \in \mathbb{Z}[u,v]$ such that if X is smooth and proper, then

$$\mathrm{E}(X;u,v) = \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} h^{p,q}(X(\mathbb{C})) u^p v^q,$$

where $h^{p,q}$ are Hodge numbers. In the general case, $\mathrm{E}(X)$ can be expressed in terms of the mixed Hodge structure on $\mathrm{H}^i_{\mathrm{c}}(X(\mathbb{C}), \mathbb{Q})$. The map $\mathrm{E}\colon \mathbf{Var}_k \to \mathbb{Z}[u,v]$ is a generalized Euler characteristic.

LEMMA 2.2.7 (Properties of $\mathrm{P}(X)$ and $\mathrm{E}(X)$). *Let X be a k-scheme of finite type. In the following assertions, those equalities and statements involving E-polynomials are restricted to the case $k \subset \mathbb{C}$.*

(1) *We have* $\mathrm{P}(X;1) = \mathrm{e}_{\mathrm{top}}(X)$, $\mathrm{E}(X;t,t) = \mathrm{P}(X;t)$ *and* $\mathrm{E}(X;1,1) = \mathrm{e}_{\mathrm{top}}(X)$.
(2) *We have* $\dim X = (\deg \mathrm{P}(X))/2$ *and* $\dim X = (\deg \mathrm{E}(X))/2$, *with the convention* $\dim \emptyset = \deg 0 = -\infty$.

(3) *If $X \neq \emptyset$, then the coefficient of $T^{2\dim X}$ in* $\mathrm{P}(X)$ *is equal to the number of irreducible components of $X \otimes_k k^{\mathrm{sep}}$ of maximal dimension* $\dim X$.
(4) *If $X \neq \emptyset$, then the coefficient of $(uv)^{\dim X}$ in* $\mathrm{E}(X)$ *is equal to the number of irreducible components of $X \otimes_k k^{\mathrm{sep}}$ of dimension* $\dim X$ *and this term is the only term of degree* $2\dim X$.
(5) *If X is smooth and proper, then* $\mathrm{P}(X)$ *and* $\mathrm{E}(X)$ *satisfy the Poincaré duality, that is, the following functional equations hold;*

$$\mathrm{P}(X; t^{-1})t^{2\dim X} = \mathrm{P}(X; t),$$
$$\mathrm{E}(X; u^{-1}, v^{-1})(uv)^{2\dim X} = \mathrm{E}(X; u, v).$$

Sketch of proof. (1) The first equality follows from the definitions of $\mathrm{e}_{\mathrm{top}}$ and P. The second one follows from the Hodge decomposition of cohomology groups. The first two equalities imply the last.

(2), (3) and (4) We sketch the proof only when k has characteristic zero. See [**Nic11**, Prop. 8.7] for the general case. If X is smooth and proper, then the assertions are clear. From the additivity of $\mathrm{P}(X)$, by compactification and resolution, we can write

$$\mathrm{P}(X) = \mathrm{P}(X') - \mathrm{P}(Y)$$

where X' is smooth and proper and Y is of dimension $< \dim X$. The assertions for $\mathrm{P}(X)$ follows by induction. Similarly for $\mathrm{E}(X)$.

(5) From the usual Poincaré duality, we have the following equalities of Betti numbers and Hodge numbers

$$b_i = b_{2d-i} \text{ and } h^{p,q} = h^{d-p,d-q} \quad (d := \dim X),$$

which show the assertion. □

Example 2.2.8. We can construct a realization map from "nice" cohomology functors with compact support

$$\mathrm{H}^i_{\mathrm{c}} \colon \mathbf{Var}_k \to \mathbf{A},$$

where $\mathbf{A}$ is some abelian category having tensor products. The Grothendieck ring of $\mathbf{A}$, denoted $\mathrm{K}_0(\mathbf{A})$, is the quotient of the free $\mathbb{Z}$-module $\bigoplus \mathbb{Z}\{M\}$ generated by isomorphism classes $\{M\}$ of objects modulo the submodule generated by elements $\{M_1\} - \{M_2\} + \{M_3\}$ for all short exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

The product on $\mathrm{K}_0(\mathbf{A})$ is given by $\{M\}\{N\} = \{M \otimes N\}$. Let us define

$$\chi \colon \mathbf{Var}_k \to \mathrm{K}_0(\mathbf{A})$$

by $\chi(X) := \sum (-1)^i \{\mathrm{H}^i_{\mathrm{c}}(X)\}$. For a k-variety X and a closed subvariety $Y \subset X$, we have the long exact sequence,

$$\cdots \to \mathrm{H}^i_{\mathrm{c}}(X \setminus Y) \to \mathrm{H}^i_{\mathrm{c}}(X) \to \mathrm{H}^i_{\mathrm{c}}(Y) \to \mathrm{H}^{i+1}_{\mathrm{c}}(X \setminus Y) \to \cdots.$$

This and the Künneth formula show that χ is a generalized Euler characteristic.

If $k = \mathbb{C}$, we can put $\mathbf{A} = \mathbf{MHS}$, the category of mixed Hodge structures say over $\mathbb{Q}$ and $\mathrm{H}^i_{\mathrm{c}}(X) = \mathrm{H}^i_{\mathrm{c}}(X(\mathbb{C}), \mathbb{Q})$ and get a generalize Euler characteristic

$$\chi_{\mathrm{Hodge}} \colon \mathbf{Var}_{\mathbb{C}} \to \mathrm{K}_0(\mathbf{MHS}).$$

For an arbitrary k, we denote by $\mathfrak{G}_k$ the absolute Galois group of k. We may take $\mathbf{A} = \mathbf{Rep}_l(\mathfrak{G}_k)$, the category of continuous representations over $\mathbb{Q}_l$ with l a prime number different from the characteristic of k. Using the étale cohomology groups $\mathrm{H}^i_{\mathrm{c}}(X) = \mathrm{H}^i_{\mathrm{c}}(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)$, we get a generalized Euler characteristic

$$\chi_l \colon \mathbf{Var}_k \to \mathrm{K}_0(\mathbf{Rep}_l(\mathfrak{G}_k)).$$

If k is finite, we have the well-define map

$$\phi_r \colon \mathrm{K}_0(\mathbf{Rep}_l(\mathfrak{G}_k)) \to \mathbb{Q}_l,\ \{V\} \mapsto \mathrm{Tr}(F^r|V),$$

where F is the geometric Frobenius action. From the Grothendieck-Lefschetz trace formula [**Mil80**, Th. 13.4], we have the commutative diagram.

$$\begin{tikzcd}
\mathrm{K}_0(\mathbf{Var}_k) \arrow[r, "\sharp_r"] \arrow[d, "\chi_l"'] & \mathbb{Z} \arrow[d, hook] \\
\mathrm{K}_0(\mathbf{Rep}_l(\mathfrak{G}_k)) \arrow[r, "\phi_r"'] & \mathbb{Q}_l
\end{tikzcd}$$

Lemma 2.2.9. *If k is a finite field, then for every positive integer r, the map $\sharp_r \colon \mathbf{Var}_k \to \mathbb{Z}$ factors through χ_l.*

Example 2.2.10. When k has characteristic zero, there exists a realization map $\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{CHM}_k)$ to the Grothendieck ring of Chow motives, which follows from results of [**GS96**, **GNA02**].

2.3. The localization $\mathcal{M}_k$

Definition 2.3.1. We define $\mathcal{M}_k$ to be the localization $\mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ of $\mathrm{K}_0(\mathbf{Var}_k)$ by $\mathbb{L}$. An *effective element* of $\mathcal{M}_k$ is an element of the form $\{X\}\mathbb{L}^n$ with X a k-scheme of finite type and n a (possibly negative) integer.

We will need effective elements with negative exponent n when defining the motivic measure on an arc space. As a group, $\mathcal{M}_k$ is generated by effective elements.

Remark 2.3.2. If a realization map $\chi \colon \mathrm{K}_0(\mathbf{Var}_k) \to R$ sends $\mathbb{L}$ to an invertible element, then it uniquely extends to a map $\mathcal{M}_k \to R$, which we keep denoting by the same symbol, say χ in this case.

Lemma 2.3.3. *We have:*

$$\mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l(-d) & (i = 2d) \\ 0 & (i \neq 2d) \end{cases}$$

Similarly, when $k = \mathbb{C}$, we have:

$$\mathrm{H}^i_{\mathrm{c}}(\mathbb{C}^d, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-d) & (i = 2d) \\ 0 & (i \neq 2d) \end{cases}$$

Proof. From [**MR073**, XV, Cor. 2.2], we have:

$$\mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l) \cong \mathrm{H}^i_{\mathrm{c}}(\mathrm{Spec}\, k^{\mathrm{sep}}, \mathbb{Q}_l) \cong \begin{cases} \mathbb{Q}_l & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$

When $k = \mathbb{C}$, similar isomorphisms for singular cohomology follow from the fact that $\mathbb{C}^d$ is contractible. The lemma follows from the Poincaré duality [**Mil80**, Cor. 11.2], [**PS08**, Th. 6.23 and Cor. B.25]. $\square$

EXAMPLE 2.3.4. The above lemma shows

$$\mathrm{e}_{\mathrm{top}}(\mathbb{A}^1_k) = 1,\ \mathrm{P}(\mathbb{A}^1_k) = t^2,\ \mathrm{E}(\mathbb{A}^1_{\mathbb{C}}) = uv,$$
$$\chi_l(\mathbb{A}^1_k) = \{\mathbb{Q}_l(-1)\},\ \chi_{\mathrm{Hodge}} = \{\mathbb{Q}(-1)\}.$$

From Remark 2.3.2, the topological Euler characteristic realization $\mathrm{e}_{\mathrm{top}}\colon \mathrm{K}_0(\mathbf{Var}_k) \to \mathbb{Z}$ (Example 2.2.4) uniquely extends to a map

$$\mathrm{e}_{\mathrm{top}}\colon \mathcal{M}_k \to \mathbb{Z}.$$

As for the Poincaré polynomial realization, we may extend the target ring $\mathbb{Z}[t]$ to $\mathbb{Z}[t^{\pm}]$ to get

$$\mathrm{P}\colon \mathcal{M}_k \to \mathbb{Z}[t^{\pm}].$$

Similarly, from the E-polynomial realization, we induce a map

$$\mathrm{E}\colon \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[u^{\pm}, v^{\pm}].$$

Realizations χ_{Hodge} and $\chi_{\text{ét}}$ send $\mathbb{L}$ to an invertible element and induce

$$\chi_{\mathrm{Hodge}}\colon \mathcal{M}_{\mathbb{C}} \to \mathrm{K}_0(\mathbf{MHS}),$$
$$\chi_l\colon \mathcal{M}_k \to \mathrm{K}_0(\mathbf{Rep}_l(\mathfrak{G}_k))$$

respectively. When k is a finite field, then the maps $\sharp_r$ extends to

$$\sharp_r\colon \mathcal{M}_k \to \mathbb{Q}.$$

2.4. The completion $\widehat{\mathcal{M}}_k$

We will need to consider infinite sums of effective elements and discuss their convergence. For this purpose, we define a completion of $\mathcal{M}_k$ and get a topological ring.

DEFINITION 2.4.1. The *dimension* of an effective element $\{X\}\mathbb{L}^n$ is defined by

$$\dim\{X\}\mathbb{L}^n := \dim X + n.$$

LEMMA 2.4.2. *The dimension of an effective element is independent of the choice of its expression as $\{X\}\mathbb{L}^n$.*

PROOF. From Lemma 2.2.7, for an effective element $\alpha = \{X\}\mathbb{L}^n$,

$$\dim\alpha = \dim X + n = \frac{1}{2}\deg \mathrm{P}(X) + n = \frac{1}{2}\deg \mathrm{P}(\alpha).$$

The last term is clearly independent of the choice of the expression. □

DEFINITION 2.4.3. For $m \in \mathbb{Z}$, we define $F_m \subset \mathcal{M}_k$ to be the subgroup generated by effective elements of dimension $\le -m$. We get a descending filtration $\{F_m\}_{m\in\mathbb{Z}}$ such that $F_mF_n \subset F_{m+n}$ and a projective system of abelian groups $\{\mathcal{M}_k/F_m\}_{m\in\mathbb{Z}}$. We define the *complete Grothendieck ring of k-varieties* to be the projective limit

$$\widehat{\mathcal{M}}_k := \varprojlim \mathcal{M}_k/F_m.$$

This becomes a complete topological ring as follows. We give the discrete topologies to $\mathcal{M}_k/F_m$ and the limit topology to $\widehat{\mathcal{M}}_k$, which makes $\widehat{\mathcal{M}}_k$ a complete topological group. If we define $\widehat{F}_m$ to be the projective limit of $\{F_m/F_n\}_n$, then $\{\widehat{F}_m\}_m$ is a fundamental system of open neighborhoods of 0. Since $F_mF_n \subset F_{m+n}$, for two elements $(\alpha_m)_{m\in\mathbb{Z}}$ and $(\beta_m)_{m\in\mathbb{Z}}$ of $\widehat{\mathcal{M}}_k$ with $\alpha_m, \beta_m \in \mathcal{M}_k/F_m$, products

$\alpha_m\beta_m$ are well-defined as elements of $\mathcal{M}_k/F_{m^2}$. Thus the sequence $(\alpha_m\beta_m)_{m\in\mathbb{Z}}$ defines an element of $\widehat{\mathcal{M}}_k$. We define the product of $(\alpha_m)_{m\in\mathbb{Z}}$ and $(\beta_m)_{m\in\mathbb{Z}}$ to be this element.

Definition 2.4.4. The image of an effective element $\{X\}\mathbb{L}^n \in \mathcal{M}_k$ is again called an *effective element* and denoted by $\{X\}\mathbb{L}^n$.

In $\widehat{\mathcal{M}}_k$, an effective element of dimension n belongs to $\widehat{F}_{-n}$. Therefore a series $(\alpha_n)_{n\geq 0}$ of effective elements with $\dim \alpha_n \to -\infty$ converges to 0 and the infinite sum $\sum_n \alpha_n$ converges to some element in $\widehat{\mathcal{M}}_k$. More generally:

Definition 2.4.5. Let $\alpha_i \in \widehat{\mathcal{M}}_k$, $i \in I$ be effective elements indexed by a countable set. We say that the *sum* $\sum_{i\in I}\alpha_i$ *converges* if for every integer m, there are at most finitely many α_i of dimension $\geq m$. We call the element given by a convergent sum $\sum_{i\in I}\alpha_i$ a *pseudo-effective element*. We define its *dimension* to be $\max\{\dim \alpha_i\}$. When the sum $\sum_{i\in I}\alpha_i$ does not converge, we say that it *diverges*. If it is the case, we formally put $\sum_{i\in I}\alpha_i := \infty$.

Thus a countable sum $\sum_{i\in I}\alpha_n$ of effective elements always give an element of $\widehat{\mathcal{M}}_k \cup \{\infty\}$. Then we can generalize all these to countable sums of pseudo-effective elements as follows:

Definition 2.4.6. Let $\alpha_i \in \widehat{\mathcal{M}}_k$, $i \in I$ be pseudo-effective elements indexed by a countable set. We say that the sum $\sum_{i\in I}\alpha_i$ *converges* if for every integer m, there are at most finitely many α_i's of dimension $\geq m$. If it is the case, we define the element $\sum_{i\in I}\alpha_i \in \widehat{\mathcal{M}}_k$ in the obvious way. Otherwise we say that the sum *diverges* and we put $\sum_{i\in I}\alpha_i := \infty$.

Divergent sums don't give an element of $\widehat{\mathcal{M}}_k$. When an infinite sum of pseudo-effective elements diverges, it is sometimes useful to classify them depending on how "large" infinities they are.

Definition 2.4.7. For a (not necessarily convergent) countable sum $\sum_{i\in I}\alpha_i$ of pseudo-effective elements, we define its *dimension* to be $\sup\{\dim \alpha_i\} \in \mathbb{Z}\cup\{\infty\}$. If the sum diverges and has dimension d, then we write

$$\sum_{i\in I}\alpha_i := \infty_d,$$

We say that the sum is *dimensionally bounded* if it has finite dimension.

If we write $\{\infty_*\} := \{\infty_d \mid d \in \mathbb{Z}\cup\{\infty\}\}$, a countable sum $\sum_{i\in I}\alpha_i$ of pseudo-effective elements thus defines an element of

$$\widehat{\mathcal{M}}_k \cup \{\infty_*\},$$

which refines the one defined in $\widehat{\mathcal{M}}_k \cup \{\infty\}$.

The following lemma will be used to show that motivic integrals are well-defined in $\widehat{\mathcal{M}}_k \cup \{\infty_*\}$.

Lemma 2.4.8. *Let I be an at most countable set. For each $i \in I$, let β_{ij}, $j \in J_i$ be at most countably many pseudo-effective elements such that $\alpha_i := \sum_{j\in J_i}\beta_{ij}$ converges (note that this is automatic if J_i is finite). Then*

$$\sum_{i\in I}\alpha_i = \sum_{i\in I, j\in J_i}\beta_{ij} \quad \text{in } \widehat{\mathcal{M}}_k \cup \{\infty_*\}.$$

In particular, one side converges (resp. dimensionally bounded) if and only if so does the other side.

Proof. Since $\dim \alpha_i = \max\{\dim \beta_{ij} \mid j \in J_i\}$, we have that

$$\sup_i \dim \alpha_i = \sup_{i,j} \dim \beta_{ij}.$$

Therefore $\sum \alpha_i$ is dimensionally bounded if and only if so does $\sum \beta_{ij}$.

Let $m \in \mathbb{Z}$. If there are at most finitely many β_{ij}'s of dimension $\geq m$, then there are at most finitely many α_i's of dimension $\geq m$. Conversely suppose that $I_m := \{i \in I \mid \dim \alpha_i \geq m\}$ is a finite set. For $i \in I \setminus I_m$, we have $\dim \beta_{ij} < m$. For $i \in I_m$, since $\sum_{j \in J_i} \beta_{ij}$ converges, $J_{i,m} := \{j \in J_i \mid \dim \beta_{ij} \geq m\}$ is a finite set. Therefore

$$\{(i,j) \mid \dim \beta_{ij} \geq m\} = \bigcup_{i \in I_m} J_{i,m},$$

which is a finite set. In conclusion, there are at most finitely many β_{ij}'s of dimension $\geq m$ if and only if there are at most finitely many α_i's of dimension $\geq m$. We have proved that $\sum \alpha_i$ converges if and only if so does $\sum \beta_{ij}$.

The equality $\sup_i \dim \alpha_i = \sup_{i,j} \dim \beta_{ij}$ now shows the equality $\sum_i \alpha_i = \sum_{i,j} \beta_{ij}$ when these sums diverge. It remains to show that when they converge, their limits are the same. This holds because the two sums reduce to the same finite sum modulo $\widehat{F}_m$ for every $m \in \mathbb{Z}$. □

2.5. Realization maps from $\widehat{\mathcal{M}}_k$

We can extend some of realization maps discussed above further from $\mathcal{M}_k$ to $\widehat{\mathcal{M}}_k$ by completing the target ring with respect to a filtration compatible the one of $\mathcal{M}_k$.

Example 2.5.1. Consider the Poincaré polynomial realization $\mathrm{P}\colon \mathcal{M}_k \to \mathbb{Z}[t^{\pm}]$ (see Examples 2.2.5 and 2.3.4). The ring of Laurent power series

$$\mathbb{Z}(\!(t^{-1})\!) = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \mid a_i \in \mathbb{Z},\, a_i = 0\,(i \gg 0) \right\}$$

is the completion of $\mathbb{Z}[t^{\pm}]$ with respect to the descending filtration $\{F_m\}_{m \in \mathbb{Z}}$, where $F_m := t^{-m}\mathbb{Z}[t^{-}]$, the subgroup of Laurent polynomials of degree $\leq -m$. Since $\mathrm{P}(F_m) \subset F_{-2m}$, the above realization map induces the map between the completions,

$$\mathrm{P}\colon \widehat{\mathcal{M}}_k \to \mathbb{Z}(\!(t^{-1})\!).$$

Similarly, if $k = \mathbb{C}$, we can extend the E-polynomial realization (see Examples 2.2.6 and 2.3.4) to get

$$\mathrm{E}\colon \widehat{\mathcal{M}}_k \to \mathbb{Z}(\!(u^{-1}, v^{-1})\!).$$

Example 2.5.2. We can extend $\chi_{\mathrm{Hodge}}\colon \mathcal{M}_{\mathbb{C}} \to \mathrm{K}_0(\mathbf{MHS})$ as follows. For $m \in \mathbb{Z}$, we define $F_m \subset \mathrm{K}_0(\mathbf{MHS})$ to be the subgroup generated by mixed Hodge structures of weight $\leq -m$. We then define the completion

$$\widehat{\mathrm{K}_0}(\mathbf{MHS}) := \varprojlim \mathrm{K}_0(\mathbf{MHS})/F_m.$$

This has a natural ring structure of ring like $\widehat{\mathcal{M}}_{\mathbb{C}}$. Since $\chi_{\mathrm{Hodge}}(F_m) \subset F_{2m}$, the above map χ_{Hodge} induces

$$\chi_{\mathrm{Hodge}} \colon \widehat{\mathcal{M}}_{\mathbb{C}} \to \widehat{\mathrm{K}_0}(\mathbf{MHS}).$$

For a finitely generated field k, there exists a full abelian subcategory $\mathbf{mRep}_l(\mathfrak{G}_k) \subset \mathbf{Rep}_l(\mathfrak{G}_k)$ of *mixed* representations, which are equipped with weight filtration. As in the case of mixed Hodge structures, we can define the completion $\widehat{\mathrm{K}_0}(\mathbf{mRep}_l(\mathfrak{G}_k))$ of the Grothendieck ring $\mathrm{K}_0(\mathbf{mRep}_l(\mathfrak{G}_k))$ and extend the realization map χ_l to

$$\chi_l \colon \widehat{\mathcal{M}}_k \to \widehat{\mathrm{K}_0}(\mathbf{mRep}_l(\mathfrak{G}_k)).$$

As was already noted, motivic integration take values in $\widehat{\mathcal{M}}_k$ or some variant of it. As a special and important case, if we integral the constant function 1 on the whole arc space of a smooth variety X, we get the value $\mathrm{M}(X) = \{X\}$ in $\widehat{\mathcal{M}}_k$. It is essential to know what information on X can be extracted from this value. Firstly we can use realization maps P and E to extract some numerical data.

PROPOSITION 2.5.3. *Let X and Y be k-varieties such that $\{X\} = \{Y\}$ in $\widehat{\mathcal{M}}_k$.*

1. *We have $\mathrm{P}(X) = \mathrm{P}(X)$ and $\mathrm{e}_{\mathrm{top}}(X) = \mathrm{e}_{\mathrm{top}}(Y)$.*
2. *If $k = \mathbb{C}$, we also have $\mathrm{E}(X) = \mathrm{E}(Y)$.*
3. *If k is a finite field and if k_r/k is the degree r extension, then $\sharp X(k_r) = \sharp Y(k_r)$.*
4. *If X and Y are smooth and proper, then they have the same Betti numbers (either for the l-adic cohomology or the singular cohomology in the case $k \subset \mathbb{C}$): for every $i \in \mathbb{Z}$, $b_i(X) = b_i(Y)$.*
5. *If X and Y are smooth and proper and if $k = \mathbb{C}$, then they have the same Hodge numbers: for every $p, q \in \mathbb{Z}$, $h^{p,q}(X) = h^{p,q}(Y)$.*

PROOF. (1) The equality $\mathrm{P}(X) = \mathrm{P}(Y)$ is obtained by sending $\{X\} = \{Y\}$ by the map $\mathrm{P} \colon \widehat{\mathcal{M}}_k \to \mathbb{Z}(\!(t^{-1})\!)$. Note that since the completion map $\mathbb{Z}[t^{\pm}] \to \mathbb{Z}(\!(t^{-1})\!)$ is injective, having $\mathrm{P}(X) = \mathrm{P}(Y)$ in $\mathbb{Z}(\!(t^{-1})\!)$ is equivalent to having the same equality in $\mathbb{Z}[t^{\pm}]$. The other equality follows by substituting 1 for t;

$$\mathrm{e}_{\mathrm{top}}(X) = \mathrm{P}(X; 1) = \mathrm{P}(Y; 1) = \mathrm{e}_{\mathrm{top}}(Y).$$

(2) We can use the realization map $\mathrm{E} \colon \widehat{\mathcal{M}}_k \to \mathbb{Z}(\!(u^{-1}, v^{-1})\!)$ to deduce this assertion.

(3) Since $\mathrm{K}_0(\mathbf{mRep}_l(\mathfrak{G}_k)) \to \widehat{\mathrm{K}_0}(\mathbf{mRep}_l(\mathfrak{G}_k))$ is injective [**Yas06**, p. 728], we get equalities $\chi_{\text{ét}}(X) = \chi_{\text{ét}}(Y)$ in $\mathrm{K}_0(\mathbf{mRep}_l(\mathfrak{G}_k))$. From the commutative diagram at the end of Example 2.2.8,

$$\sharp X(k_r) = \sharp_r(\{X\}) = \sharp_r(\{Y\}) = \sharp Y(k_r).$$

.

(4) When X is smooth and proper, the Poincaré polynomial is just the generating function of Betti numbers. The assertion follows from $\mathrm{P}(X) = \mathrm{P}(Y)$.

(5) Similarly, the E-polynomial is the generating function of Hodge numbers in the smooth and proper case. The equality $\mathrm{E}(X) = \mathrm{E}(Y)$ implies the assertion. □

PROPOSITION 2.5.4. *Let X and Y be smooth and proper k-varieties such that $\{X\} = \{Y\}$ in $\widehat{\mathcal{M}}_k$.*

(1) *If k is finitely generated, then we have isomorphisms of $\mathfrak{G}_k$-representations*
$$\mathrm{H}^i(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)^{\mathrm{ss}} \cong \mathrm{H}^i(Y \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)^{\mathrm{ss}}.$$
Here the superscript ss *means the semi-simplification.*

(2) *If $k = \mathbb{C}$, then we have isomorphisms of Hodge structures*
$$\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \cong \mathrm{H}^i(Y(\mathbb{C}), \mathbb{Q}) \quad (i \in \mathbb{Z}).$$

Sketch of proof. Since the maps $\mathrm{K}_0(\mathbf{mRep}_l(\mathfrak{G}_k)) \to \widehat{\mathrm{K}_0}(\mathbf{mRep}_l(\mathfrak{G}_k))$ and $\mathrm{K}_0(\mathbf{MHS}) \to \widehat{\mathrm{K}_0}(\mathbf{MHS})$ are injective [**Yas06**, p. 728], we get equalities $\chi_l(X) = \chi_l(Y)$ and $\chi_{\mathrm{Hodge}}(X) = \chi_{\mathrm{Hodge}}(Y)$ respectively in the non-complete Grothendieck ring. Since cohomology groups H^i have pure weight $2i$, we can deduce degree-wise equalities $\{\mathrm{H}^i(X)\} = \{\mathrm{H}^i(Y)\}$. In general, the equality $\{M\} = \{N\}$ in the Grothendieck group of an abelian category implies an isomorphism $M^{\mathrm{ss}} \cong N^{\mathrm{ss}}$ of semi-simplifications. The first assertion now follows. For the second assertion, the Hodge structure on $\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$ is polarizable and hence semi-simple [**PS08**, Cor. 2.12]. (The polarizability is well-known when X is projective. In the general case, we can apply Chow's lemma to embed cohomology groups into ones of a smooth projective variety.) Thus the semi-simplification does not change Hodge structures on cohomology groups. $\square$

2.6. Piecewise trivial $\mathbb{A}^n$-bundles

Definition 2.6.1. Let $f \colon W \to V$ be a morphism of k-schemes of finite type. Let $D \subset W$ and $C \subset V$ be constructible subsets with $f(D) \subset C$. We say that the induced map $f|_D \colon D \to C$ is a *piecewise trivial $\mathbb{A}^n$-bundle* if there exists a stratification $C = \bigsqcup_{i=1}^{l} C_i$ by locally closed subsets $C_i \subset V$ such that for each i, $(f|_D)^{-1}(C_i) \subset W$ is a locally closed subset and C_i-isomorphic to $\mathbb{A}^n_{C_i} = \mathbb{A}^n_k \times_k C_i$ with the reduced structures on C_i and $(f|_D)^{-1}(C_i)$.

Lemma 2.6.2. *With the above notation, if $f|_D \colon D \to C$ is a piecewise trivial $\mathbb{A}^n$-bundle, then $\{D\} = \{C\}\mathbb{L}^n$.*

Proof. Let $C = \bigsqcup_i C_i$ be as above. Then
$$\{D\} = \bigsqcup_i \{(f|_D)^{-1}(C_i)\} = \bigsqcup_i (\{C_i\}\mathbb{L}^n) = \left(\bigsqcup_i C_i\right)\mathbb{L}^n = \{C\}\mathbb{L}^n.$$
$\square$

The following lemma is useful to show that some map of constructible subsets is a piecewise trivial $\mathbb{A}^n$-bundle.

Lemma 2.6.3. *We keep the above notation. Suppose that for every point $c \in C$, $(f|_D)^{-1}(c)$ is a closed subset of the scheme-theoretic fiber $f^{-1}(c)$. Then there exists a (necessarily finite) stratification $C = \bigsqcup C_i$ into locally closed subsets $C_i \subset V$ such that $(f|_D)^{-1}(C_i)$ are locally closed subsets of W .*

Proof. Let $\eta \in C$ be a point of maximal dimension and let E be the Zariski closure of $(f|_D)^{-1}(\eta)$ in W. The sets D and E coincide when restricted to $f^{-1}(\eta)$. Therefore there exists a locally closed subset $C' \subset V$ such that $\eta \in C' \subset C$ and such that D and E coincide when restricted to $(f|_D)^{-1}(C')$. In particular, $f^{-1}(C')$ is a locally closed subset of W. It is now easy to show the lemma by induction on the dimension and the number of points of maximal dimension. $\square$

CHAPTER 3

Jet schemes and arc schemes

The type of motivic integration that we discuss mainly in this book is integration over arc spaces of varieties. Arc spaces are constructed as limits of jet schemes. To define measures on arc spaces, we need properties of truncation maps between jet schemes. Throughout the chapter, X and Y denote k-schemes of finite type.

3.1. Notation of power series rings, formal disks, *etc.*

For a k-algebra R, we denote by $R[\![t]\!]$ the ring of formal power series with coefficients in R and by $R(\!(t)\!)$ the localization $R[\![t]\!]_t$ by t, that is, the ring of Laurent power series with coefficients in R. In particular, if L is a field, $L(\!(t)\!)$ is also a field. We define $\mathrm{D}_R := \operatorname{Spec} R[\![t]\!]$ and call it the *formal disk over R.* If R is an algebraically closed field, we call it also a *geometric formal disk.* For $n \in \mathbb{Z}_{\geq 0}$, we put $\mathrm{D}_{R,n} := \operatorname{Spec} R[\![t]\!]/(t^{n+1}) = \operatorname{Spec} R[t]/(t^{n+1})$. Sometimes, following the convention $t^{\infty} = 0$, we write $\mathrm{D}_R = \mathrm{D}_{R,\infty}$.

3.2. Jets

Definition 3.2.1. Let $n \in \mathbb{Z}_{\geq 0}$ and let R be a k-algebra. An *n-jet of X over R* is a k-morphism $\mathrm{D}_{R,n} \to X$. A *geometric n-jet of* X is an n-jet of X over an algebraically closed field.

A geometric 0-jet is just a geometric point and a geometric 1-jet is a Zariski tangent vector (over some algebraically closed field).

Remark 3.2.2. In the context of complex analytic spaces, if $X = \mathbb{C}^d$, then a morphism $(\mathbb{C}, 0) \to X$ from the germ of $\mathbb{C}$ at the origin is given by a tuple $(h_1, \ldots, h_d)$ of convergent power series h_i. Considering an n-jet of X amounts to looking only at the terms of h_i of degree $\leq n$ by ignoring the terms of degree $> n$.

Definition 3.2.3. An *n-th jet scheme* of X, denoted by $\mathrm{J}_n X$, is (a k-scheme representing) the functor:

$$(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$$
$$\operatorname{Spec} R \mapsto \operatorname{Hom}_{\mathbf{Sch}_k}(\mathrm{D}_{R,n}, X)$$

For a morphism $f\colon Y \to X$ of k-schemes, we denote the induced morphism $\mathrm{J}_n Y \to \mathrm{J}_n X$ by f_n.

Here we follow the usual convention of identifying a scheme with the associated functor $(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$ (for instance, [**EH00**, VI]). As we will see shortly in Proposition 3.2.8, the functor $\mathrm{J}_n X$ is indeed representable by a scheme.

Lemma 3.2.4. *We have an isomorphism*

$$\mathrm{J}_n(\mathbb{A}^d_k) \cong \mathbb{A}^{d(n+1)}_k = \operatorname{Spec} k[x_i^{(j)} \mid 1 \leq i \leq d,\, 0 \leq j \leq n]$$

such that an R-point $(r_i^{(j)}) \in \mathbb{A}_k^{d(n+1)}(R)$ corresponds to the n-jet:

$$\mathrm{D}_{R,n} = \operatorname{Spec} R[t]/(t^{n+1}) \to \mathbb{A}_k^d$$
$$r_i^{(0)} + r_i^{(1)}t + \cdots + r_i^{(n)}t^n \leftarrow\!\shortmid x_i$$

PROOF. We have the following one-to-one correspondences, which are functorial in R:

$$\begin{aligned}(\mathrm{J}_n\,\mathbb{A}_k^d)(\operatorname{Spec} R) &\leftrightarrow \operatorname{Hom}_{\mathbf{Alg}_k}(k[x_1,\dots,x_d], R[t]/(t^{n+1}))\\ &\leftrightarrow \left(R[t]/(t^{n+1})\right)^{\oplus d}\\ &\leftrightarrow R^{\oplus d(n+1)}\\ &\leftrightarrow \mathbb{A}_k^{d(n+1)}(\operatorname{Spec} R).\end{aligned}$$

Therefore there is an isomorphism $\mathrm{J}_n(\mathbb{A}_k^d) \cong \mathbb{A}_k^{d(n+1)}$ which induces the correspondence of the lemma between points of $\mathbb{A}_k^{d(n+1)}$ and n-jets. □

Let us write $k[x_*^{(*)}] = k[x_i^{(j)} \mid 1 \le i \le d,\, 0 \le j \le n]$. The lemma in particular shows that the universal n-jet

$$u\colon (\mathrm{J}_n\,\mathbb{A}_k^d) \times_k \operatorname{Spec} k[t]/(t^{n+1}) \to \mathbb{A}_k^d$$

is given by:

$$\operatorname{Spec} k[x_*^{(*)}][t]/(t^{n+1}) \to \mathbb{A}_k^d$$
$$x_i^{(0)} + x_i^{(1)}t + \cdots + x_i^{(n)}t^n \leftarrow\!\shortmid x_i$$

LEMMA 3.2.5. *For a closed subscheme $X = V(f_1,\dots,f_l) \subset \mathbb{A}_k^d$, we define $F_\lambda^{(j)} \in k[x_*^{(*)}]$ ($1 \le \lambda \le l$, $0 \le j \le n$) by*

$$u^*(f_\lambda) = f_\lambda(u^*(x_1),\dots,u^*(x_d)) = F_\lambda^{(0)} + F_\lambda^{(1)}t + \cdots + F_\lambda^{(n)}t^n,$$

where u is the universal n-jet. Then the isomorphism $\mathrm{J}_n(\mathbb{A}_k^d) \cong \mathbb{A}_k^{d(n+1)}$ of Lemma 3.2.4 induces the isomorphism of subschemes,

$$\mathrm{J}_n\,X \cong V(F_\lambda^{(j)} \mid 1 \le \lambda \le l,\, 0 \le j \le n).$$

PROOF. Let γ be an n-jet of $\mathbb{A}_k^d$ over R and let $(r_i^{(j)})_{i,j} \in \mathbb{A}_k^{d(n+1)}(R)$ be the corresponding point. We have

$$\begin{aligned}&\gamma^*(f_\lambda)\\ &= f_\lambda(\gamma^*(x_1),\dots,\gamma^*(x_d))\\ &= f_\lambda(r_1^{(0)} + r_1^{(1)}t + \cdots + r_1^{(n)}t^n, \dots, r_d^{(0)} + r_d^{(1)}t + \cdots + r_d^{(n)}t^n)\\ &= F_\lambda^{(0)}(r_*^{(*)}) + F_\lambda^{(1)}(r_*^{(*)})t + \cdots + F_\lambda^{(n)}(r_*^{(*)})t^n.\end{aligned}$$

Therefore the following conditions are equivalent:

(1) The n-jet γ gives an n-jet of X.
(2) For every λ, $\gamma^*(f_\lambda) = 0$.
(3) For every λ and j, $F_\lambda^{(j)}(r_*^{(*)}) = 0$.
(4) The point $(r_*^{(*)})$ lies in the subscheme defined by $F_\lambda^{(j)}$'s.

This shows that the isomorphism $\mathrm{J}_n(\mathbb{A}_k^d) \cong \mathbb{A}_k^{d(n+1)}$ restricts to the isomorphism of the lemma of subfunctors. □

EXAMPLE 3.2.6. Consider the plane curve $X = V(y^2 + x^3) \subset \mathbb{A}^2_k$. Then

$$\begin{aligned} u^*(f) =& (y^{(0)} + y^{(1)}t + \cdots + y^{(n)}t^n)^2 + (x^{(0)} + x^{(1)}t + \cdots + x^{(n)}t^n)^3 \\ =& (y^{(0)})^2 + (x^{(0)})^3 + \left(2y^{(0)}y^{(1)} + 3(x^{(0)})^2 x^{(1)}\right) t \\ &+ \left(2y^{(0)}y^{(2)} + (y^{(1)})^2 + 3(x^{(0)})^2 x^{(2)} + 3x^{(0)}(x^{(1)})^2\right) t^2 + \cdots . \end{aligned}$$

Thus $\mathrm{J}_1\, X$ is the closed subscheme of $\mathbb{A}^4_k$ with coordinates $x^{(0)}, x^{(1)}, y^{(0)}, y^{(1)}$ defined by

$$(y^{(0)})^2 + (x^{(0)})^3 = 2y^{(0)}y^{(1)} + 3(x^{(0)})^2 x^{(1)} = 0$$

and $\mathrm{J}_2\, X$ is the closed subscheme of $\mathbb{A}^6_k$ with coordinates $x^{(0)}, x^{(1)}, x^{(2)}, y^{(0)}, y^{(1)}, y^{(2)}$ defined by

$$\begin{aligned} &(y^{(0)})^2 + (x^{(0)})^3 = 2y^{(0)}y^{(1)} + 3(x^{(0)})^2 x^{(1)} \\ &= 2y^{(0)}y^{(2)} + (y^{(1)})^2 + 3(x^{(0)})^2 x^{(2)} + 3x^{(0)}(x^{(1)})^2 = 0. \end{aligned}$$

We have the morphism $\mathrm{J}_n\, X \to X$ such that for each k-algebra R, the map $(\mathrm{J}_n\, X)(R) \to X(R)$ sends an n-jet $\mathrm{D}_{R,n} \to X$ to the R-point

$$\operatorname{Spec} R \hookrightarrow \mathrm{D}_{R,n} \to X.$$

LEMMA 3.2.7. *Let $f\colon Y \to X$ be an étale morphism (e.g. an open immersion). Then we have a natural isomorphism of functors $(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$,*

$$\mathrm{J}_n\, Y \cong (\mathrm{J}_n\, X) \times_X Y.$$

PROOF. Giving an R-point of the right side is equivalent to giving a commutative diagram of solid arrows with the vertical arrows given:

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{D}_{R,n} \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

From the formal étaleness (for definition, see [**Gro67**, Def. 17.1.1]) of f, this is in turn equivalent to giving the dashed arrow, that is, an R-point of the left side of the isomorphism. We have got the desired correspondence of points. □

PROPOSITION 3.2.8. *The functor $\mathrm{J}_n\, X$ is a scheme of finite type which is affine over X.*

PROOF. The functor is the Weil restriction

$$\mathrm{R}_{(k[t]/(t^{n+1}))/k}\left(X \otimes_k k[t]/(t^{n+1})\right)$$

and hence the proposition follows from a general result [**BLR90**, p. 195]. But we give a more ad-hoc proof.

Let $X = \bigcup U_i$ be an affine open covering and let $U_{ij} := U_i \cap U_j$, which are again affine (we assume that all schemes are separated). From Lemma 3.2.5, $\mathrm{J}_n\, U_i$ and $\mathrm{J}_n\, U_{ij}$ are affine finite type schemes. From Lemma 3.2.7, morphisms $\mathrm{J}_n\, U_{ij} \to \mathrm{J}_n\, U_i$ and $\mathrm{J}_n\, U_{ij} \to \mathrm{J}_n\, U_j$ are open immersions. We can glue $\mathrm{J}_n\, U_i$ along $\mathrm{J}_n\, U_{ij}$ to get the scheme $\mathrm{J}_n\, X$. Since $\mathrm{J}_n\, X \times_X U_i \cong \mathrm{J}_n\, U_i$ are affine and of finite type, the morphism $\mathrm{J}_n\, X \to X$ is affine (in particular, separated) and of finite type. This shows the proposition. □

LEMMA 3.2.9. *If $\iota\colon Z \to X$ is an immersion (resp. an open immersion, a closed immersion), then so is the induced morphism $\iota_n\colon \mathrm{J}_n Z \to \mathrm{J}_n X$.*

PROOF. The assertion for open immersions follows from Lemma 3.2.7. The one for closed immersions follows from the local description of jet schemes in Lemma 3.2.5. The one for general immersions follows from these two cases. □

3.3. Truncation morphisms

For integers $n' \geq n$, we have the natural surjection $R[t]/(t^{n'+1}) \to R[t]/(t^{n+1})$, which maps

$$r_0 + r_1 t + \cdots + r_{n'} t^{n'} \mapsto r_0 + r_1 t + \cdots + r_n t^n.$$

Namely this truncates polynomials by cutting off the terms of degree $> n$. The map corresponds to the closed immersion $\mathrm{D}_{R,n} \hookrightarrow \mathrm{D}_{R,n'}$.

DEFINITION 3.3.1. Let X be a k-scheme. We define the *truncation morphism*

$$\pi^{n'}_n\colon \mathrm{J}_{n'} X \to \mathrm{J}_n X$$

by mapping an n'-jet

$$\mathrm{D}_{R,n'} \xrightarrow{\gamma} X$$

to the n-jet

$$\mathrm{D}_{R,n} \hookrightarrow \mathrm{D}_{R,n'} \xrightarrow{\gamma} X.$$

EXAMPLE 3.3.2. When $X = \mathbb{A}^d_k$, through the isomorphism of Lemma 3.2.4, the truncation map $\pi^{n'}_n\colon \mathrm{J}_{n'} \mathbb{A}^d_k \to \mathrm{J}_n \mathbb{A}^d_k$ corresponds to the morphism $\mathbb{A}^{d(n'+1)}_k \to \mathbb{A}^{d(n+1)}_k$ mapping

$$(r^{(j)}_i)_{1\leq i\leq d,\, 0\leq j\leq n'} \mapsto (r^{(j)}_i)_{1\leq i\leq d,\, 0\leq j\leq n}.$$

The corresponding k-algebra homomorphism is the inclusion

$$k[x^{(j)}_i \mid 1 \leq i \leq d,\, 0 \leq j \leq n] \hookrightarrow k[x^{(j)}_i \mid 1 \leq i \leq d,\, 0 \leq j \leq n'].$$

LEMMA 3.3.3. *Truncation morphisms $\pi^{n'}_n\colon \mathrm{J}_{n'} X \to \mathrm{J}_n X$ are affine.*

PROOF. Take an affine open covering $X = \bigcup_i U_i$. This induces affine open coverings $\mathrm{J}_n X = \bigcup_i \mathrm{J}_n U_i$ and $\mathrm{J}_{n'} X = \bigcup_i \mathrm{J}_{n'} U_i$. The lemma holds, since $\mathrm{J}_{n'} U_i$ is the preimage of $\mathrm{J}_n U_i$ by the map $\pi^{n'}_n\colon \mathrm{J}_{n'} X \to \mathrm{J}_n X$. □

The following proposition is essential when defining a measure on an arc space.

PROPOSITION 3.3.4. *Let X be a smooth k-variety of dimension d. Then the morphism $\pi^{n'}_n\colon \mathrm{J}_{n'} X \to \mathrm{J}_n X$ is a Zariski locally trivial fibration with fiber $\mathbb{A}^{d(n'-n)}_k$. Namely, there exists an open covering $\mathrm{J}_n X = \bigcup U_i$ such that $(\pi^{n'}_n)^{-1}(U_i)$ is U_i-isomorphic to $U_i \times_k \mathbb{A}^{d(n'-n)}_k$. In particular, truncation morphisms are surjective.*

PROOF. There exists an open covering $X = \bigcup V_j$ such that each V_j has an étale morphism to $\mathbb{A}^d_k$ (given by a local coordinate system). From Lemma 3.2.7, we have the Cartesian diagram:

$$\begin{tikzcd}
\mathrm{J}_{n'} V_j \arrow[r] \arrow[d] & \mathrm{J}_{n'} \mathbb{A}^d_k \arrow[d] \\
\mathrm{J}_n V_j \arrow[r] & \mathrm{J}_n \mathbb{A}^d_k
\end{tikzcd}$$

Since $\mathrm{J}_n V_j$ form a Zariski covering of $\mathrm{J}_n X$, it suffices to show the proposition in the case $X = \mathbb{A}^d_k$. This case follows from the explicit description in Example 3.3.2. (Note that the surjectivity of truncation morphisms follows also from the definition of formal smoothness.) $\square$

3.4. Arcs

DEFINITION 3.4.1. Let R be a k-algebra. An *arc of* X *over* R is a k-morphism $\mathrm{D}_R \to X$ from the formal disk over R. When R is an algebraically closed field, we call it a *geometric arc of* X.

DEFINITION 3.4.2. We define the *arc scheme of* X, denoted by $\mathrm{J}_\infty X$, to be the projective limit of jet schemes,

$$\mathrm{J}_\infty X := \varprojlim \mathrm{J}_n X.$$

For a morphism $f\colon Y \to X$ of k-schemes, we denote the induced morphism $\mathrm{J}_\infty Y \to \mathrm{J}_\infty X$ by f_∞.

Note that since truncation morphisms $\mathrm{J}_{n+1} X \to \mathrm{J}_n X$ are affine, the projective limit exists [**Gro66**, Prop. 8.2.3].

REMARK 3.4.3. If X has positive dimension, then $\mathrm{J}_\infty X$ is neither of finite type nor Noetherian.

LEMMA 3.4.4 ([**Bha16**]). *The scheme* $\mathrm{J}_\infty X$ *represents the following functor:*

$$\begin{aligned} \mathrm{J}_\infty X\colon (\mathbf{Aff}_k)^{\mathrm{op}} &\to \mathbf{Set} \\ \operatorname{Spec} R &\mapsto \operatorname{Hom}_{\mathbf{Sch}_k}(\mathrm{D}_R, X) \end{aligned}$$

PROOF. When X is an affine scheme $\operatorname{Spec} S$, then

$$\begin{aligned} (\mathrm{J}_\infty X)(R) &= \varprojlim \operatorname{Hom}_{\mathbf{Alg}_k}(S, R[t]/(t^{n+1})) \\ &= \operatorname{Hom}_{\mathbf{Alg}_k}(S, R[\![t]\!]) \\ &= \operatorname{Hom}_{\mathbf{Sch}_k}(\mathrm{D}_R, X). \end{aligned}$$

We refer the reader to [**Bha16**] for the general case. $\square$

REMARK 3.4.5. In general, the underlying set of a k-scheme X is identified with the set of equivalence classes of geometric points $\operatorname{Spec} L \to X$; two geometric points $\operatorname{Spec} L_1 \to X$ and $\operatorname{Spec} L_2 \to X$ are *equivalent* if there exist an algebraically closed field L_3 and morphisms $\operatorname{Spec} L_3 \to \operatorname{Spec} L_i$, $i = 1, 2$, such that the following diagram is commutative.

$$\begin{tikzcd} \operatorname{Spec} L_3 \arrow[r] \arrow[d] & \operatorname{Spec} L_1 \arrow[d] \\ \operatorname{Spec} L_2 \arrow[r] & X \end{tikzcd}$$

Applying this to $\mathrm{J}_\infty X$ (resp. $\mathrm{J}_n X$), we can identify the underlying set of $\mathrm{J}_\infty X$ with the set of equivalence classes of geometric arcs $\mathrm{D}_L \to X$. Two arcs $\mathrm{D}_{L_1} \to X$ and $\mathrm{D}_{L_2} \to X$ are *equivalent* if there exist an algebraically closed field L_3 and morphisms

$\operatorname{Spec} L_3 \to \operatorname{Spec} L_i$, $i = 1, 2$, such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{D}_{L_3} \arrow[r] \arrow[d] & \mathrm{D}_{L_1} \arrow[d] \\ \mathrm{D}_{L_2} \arrow[r] & X \end{tikzcd}$$

DEFINITION 3.4.6. For each $n \in \mathbb{Z}_{\geq 0}$, the natural map $\mathrm{J}_\infty X \to \mathrm{J}_n X$ is denoted by π_n and again called a *truncation morphism.*

By the identification $R[t]/(t^{n+1}) = R[\![t]\!]/(t^{n+1})$, we can regard $\mathrm{D}_{R,n}$ as a closed subscheme of D_R. The morphism π_n sends an arc $\gamma\colon \mathrm{D}_R \to X$ to the induced n-jet $\mathrm{D}_{R,n} \hookrightarrow \mathrm{D}_R \xrightarrow{\gamma} X$.

LEMMA 3.4.7. *Let $f\colon Y \to X$ be an étale morphism (e.g. an open immersion). Then we have a natural isomorphism of functors $(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$,*

$$\mathrm{J}_\infty Y \cong (\mathrm{J}_\infty X) \times_X Y.$$

PROOF. From Lemma 3.2.7,

$$\mathrm{J}_\infty Y = \varprojlim \mathrm{J}_n Y \cong \varprojlim \left((\mathrm{J}_n X) \times_X Y \right).$$

Recall that a projective limit and a fiber product interchange [**Gro66**, Lem. 8.2.6], or more generally, two limits in a category interchange [**Rie17**, Th. 3.8.1]. Thus the last limit is isomorphic to

$$\left(\varprojlim \mathrm{J}_n X \right) \times_X Y = (\mathrm{J}_\infty X) \times_X Y.$$

□

LEMMA 3.4.8. *If $\iota\colon Z \to X$ is an immersion (resp. an open immersion, a closed immersion), then so is the induced morphism $\iota_\infty\colon \mathrm{J}_\infty Z \to \mathrm{J}_\infty X$.*

PROOF. The assertion for open immersions follows from Lemma 3.4.7. As for closed immersions, we may assume that X is affine. Then $\mathrm{J}_n X$ are affine, say $\operatorname{Spec} A_n$. From Lemma 3.2.9, we can write $\mathrm{J}_n Z = \operatorname{Spec} A_n/I_n$ for some ideal $I_n \subset A_n$. Then $\mathrm{J}_\infty X = \operatorname{Spec} \varinjlim A_n$ and $\mathrm{J}_\infty Z = \operatorname{Spec} \varinjlim (A_n/I_n)$. Since the direct limit is an exact functor, the natural map $\varinjlim A_n \to \varinjlim (A_n/I_n)$ is surjective. The assertion for closed immersions follows and so does the assertion for general immersions. □

REMARK 3.4.9. Since every geometric arc $\operatorname{Spec} L[\![t]\!] \to X$ factors through the associated reduced scheme X_{red}, the morphism $\mathrm{J}_\infty X_{\mathrm{red}} \to \mathrm{J}_\infty X$ is bijective and we may identify these spaces set-theoretically.

3.5. Order functions

DEFINITION 3.5.1. Let $Z \subsetneq X$ be a proper closed subscheme defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. We define a function

$$\mathrm{ord}_Z = \mathrm{ord}_{\mathcal{I}}\colon \mathrm{J}_\infty X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

as follows: if a point $\gamma \in \mathrm{J}_\infty X$ is represented by a geometric arc $\gamma\colon \mathrm{D}_L \to X$ (see Remark 3.4.5) and if $\gamma^{-1}\mathcal{I} = (t^m)$, then $\mathrm{ord}_{\mathcal{I}}(\gamma) := m$. Here we follow the convention that $(t^0) = (1)$ and $(t^\infty) = (0)$.

LEMMA 3.5.2.

(1) *We have* $\operatorname{ord}_Z(\gamma) = \infty$ *if and only if* $\gamma \in \mathrm{J}_\infty Z$. *Namely* $\mathrm{J}_\infty Z = \operatorname{ord}_Z^{-1}(\infty)$.
(2) *For* $n \in \mathbb{Z}_{\geq 0}$, $\operatorname{ord}_Z(\gamma) \geq n+1$ *if and only if* $\pi_n(\gamma) \in \mathrm{J}_n Z$.

Proof. For the first assertion, having $\operatorname{ord}_Z(\gamma) = \infty$ is by definition equivalent to $\gamma^{-1}\mathcal{I} = (0)$. In turn, the last equality is equivalent to that $\gamma\colon \mathrm{D}_L \to X$ factors through Z, that is, $\gamma \in \mathrm{J}_\infty Z$. For the second assertion, we can show in turn that the following conditions are equivalent:

- $\operatorname{ord}_Z(\gamma) \geq n+1$.
- $\gamma^{-1}\mathcal{I} \subset (t^{n+1})$.
- $\gamma_n^{-1}\mathcal{I} = 0$ with $\gamma_n := \pi_n(\gamma)$.
- $\gamma_n\colon \mathrm{D}_{L,n} \to X$ factors through Z.
- $\gamma_n \in \mathrm{J}_n Z$.

□

We can also define order functions on jet schemes as follows:

Definition 3.5.3. Keeping the notation, we define

$$\operatorname{ord}_Z = \operatorname{ord}_{\mathcal{I}}\colon\ \mathrm{J}_n X \to \{0, 1, \dots, n, \infty\}$$

as follows: if $\gamma^{-1}\mathcal{I} = (t^m)$, then $\operatorname{ord}_{\mathcal{I}}(\gamma) := m$ again with the convention $(t^0) = (1)$ and $(t^\infty) = (0)$.

3.6. Jacobian ideals of varieties

The Jacobian criterion characterizes smooth points of an affine variety $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_l)$ in terms of the Jacobian matrix $(\partial f_i/\partial x_j)_{i,j}$. The Jacobian ideal $\mathrm{Jac}_X \subset \Gamma(X, \mathcal{O}_X)$ is generated by c-minors of the Jacobian matrix with c the codimension of X in $\mathbb{A}_k^n$. This gives a natural scheme structure to the singular locus X_{sing}. The Jacobian ideal plays an important role in the study of jet schemes as well as motivic integration.

For later use, it is convenient to define the Jacobian ideal in a more intrinsic way using Fitting ideals. Let M be a finitely generated module over a Noetherian ring R and let

$$F_1 \to F_0 \to M \to 0$$

be a free presentation of it. For each $j \in \mathbb{Z}_{\geq 0}$, the map $\bigwedge^j F_1 \to \bigwedge^j F_0$ induces the map $\bigwedge^j F_1 \otimes \bigwedge^j F_0^* \to R$, where F_0^* denotes the dual R-module of F_0. In fact, the image of the last map is independent of the choice of the free presentation.

Definition 3.6.1. For $i \in \mathbb{Z}_{\geq 0}$, the *$i$-th Fitting ideal* of M, denoted by $\mathrm{Fitt}_i(M)$, is defined to be the image of $\bigwedge^{\operatorname{rank} F_0 - i} F_1 \otimes \bigwedge^{\operatorname{rank} F_0 - i} F_0^* \to R$.

If we fix bases of F_1 and F_0, then the map $F_1 \to F_0$ corresponds to a matrix A with entries in R. The ideal $\mathrm{Fitt}_i(M)$ is then generated by the minors of A of size $\operatorname{rank} F_0 - i$. If $\mathcal{M}$ is the coherent sheaf on $\operatorname{Spec} R$ corresponding to M, then *i-th Fitting ideal sheaf* $\mathrm{Fitt}_i(\mathcal{M}) \subset \mathcal{O}_X$ is the one corresponding to $\mathrm{Fitt}_i(M)$. For a coherent sheaf $\mathcal{M}$ on a general Noetherian scheme X, i-th Fitting ideal sheaves defined on affine open subschemes glue together to give the global *i-th Fitting ideal sheaf* $\mathrm{Fitt}_i(\mathcal{M}) \subset \mathcal{O}_X$. If there exists a presentation $\mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{M}$ such that $\mathcal{F}_0$ and $\mathcal{F}_1$ are locally free of constant rank, then $\mathrm{Fitt}_i(\mathcal{M})$ coincides with the image of $\bigwedge^{\operatorname{rank} \mathcal{F}_0 - i} \mathcal{F}_1 \otimes \bigwedge^{\operatorname{rank} \mathcal{F}_0 - i} \mathcal{F}_0^* \to \mathcal{O}_X$. Basic properties of Fitting ideals are as follows:

LEMMA 3.6.2 (see [**Eis95**, Section 20.2]). *Let X be a Noetherian scheme and let $\mathcal{M}$ be a coherent sheaf on X.*

(1) *For $x \in X$, we have $x \in V(\mathrm{Fitt}_r(\mathcal{M}))$ if and only if the stalk $\mathcal{M}_x$ cannot be generated by r elements as an $\mathcal{O}_{X,x}$-module.*
(2) *For a morphism $f\colon Y \to X$ of Noetherian schemes, we have*

$$f^{-1}\,\mathrm{Fitt}_r(\mathcal{M}) = \mathrm{Fitt}_r(f^*\mathcal{M}).$$

DEFINITION 3.6.3. Let X be a k-scheme of finite type of pure dimension d. We define the *Jacobian ideal sheaf* $\mathrm{Jac}_X \subset \mathcal{O}_X$ to be $\mathrm{Fitt}_d(\Omega_{X/k})$. We denote the associated order function $\mathrm{ord}_{\mathrm{Jac}_X}$ by j_X.

For an affine variety $X = \operatorname{Spec} R$ with $R = k[x_1, \dots, x_n]/(f_1, \dots, f_l)$, we have the free presentation

$$R^l \xrightarrow{(\partial f_j/\partial x_i)_{i,j}} R^n \to \Omega_{R/k} \to 0.$$

See [**Mat89**, p. 195] or [**Eis95**, p. 390]. Using this, we see that the last definition of the Jacobian ideal coincides with the one given by the Jacobian matrix.

A version of the Jacobian criterion says that X is k-smooth at a point x if and only if the stalk $\Omega_{X/k,x}$ is a free $\mathcal{O}_{X,x}$-module of rank d [**Gro67**, Prop. 17.15.5]. From a property of Fitting ideals, the closed subset defined by Jac_X is exactly the non-smooth locus of X.

3.7. Hensel's lemma and lifting of jets

Hensel's lemma is intimately related to the geometry of jet schemes and arc schemes. We first recall the following simple version:

PROPOSITION 3.7.1 (cf. [**Eis95**, Theorem 7.3]). *Let $f(x) \in k[\![t]\!][\![x]\!]$ and let $a \in (t) \subset k[\![t]\!]$. Let $e := \mathrm{ord}(df/dx)(a)$. If*

$$f(a) \equiv 0 \pmod{t^{2e+1}},$$

then there exists $b \in k[\![t]\!]$ such that

$$f(b) = 0 \text{ and } b \equiv a \pmod{t^{e+1}}.$$

This is generalized to a system of multivariate power series, which is a slightly weaker form of [**CLNS18**, Lemma 1.3.3] (cf. [**Bou85**, chap. III, §4, p. 271, Cor. 3]):

PROPOSITION 3.7.2 (Hensel's lemma). *Let $f_1, \dots, f_l \in k[\![t]\!][\![x_1, \dots, x_d]\!]$ with $l \le d$. Let $\underline{a} = (a_1, \dots, a_d) \in k[\![t]\!]^d$ with $\mathrm{ord}\, a_i > 0$ and let e be the order of*

$$\det\left(\frac{\partial f_j}{\partial x_i}(\underline{a})\right)_{1 \le i,j \le l} \in k[\![t]\!].$$

Let n be an integer $\ge e$. If we have

$$f_1(\underline{a}) \equiv \cdots \equiv f_l(\underline{a}) \equiv 0 \pmod{t^{n+e+1}},$$

then there exists $\underline{b} = (b_1, \dots, b_d) \in k[\![t]\!]^d$ such that

$$f_1(\underline{b}) = \cdots = f_l(\underline{b}) = 0 \text{ and } \underline{b} \equiv \underline{a} \pmod{t^{n+1}}.$$

To recover Proposition 3.7.1, we put $l = d = 1$ and $n = e$. We can interpret Hensel's lemma in terms of jets and arcs from a more geometric viewpoint:

COROLLARY 3.7.3 (a geometric version of Hensel's lemma). *Let X be a k-scheme of finite type which is a local complete intersection. Let $\alpha\colon \mathrm{D}_{L,n} \to X$ be an n-jet with $e := \mathrm{j}_X(\alpha) \le n$. If α lifts to an $(n+e)$-jet $\mathrm{D}_{L,n+e} \to X$, then α lifts to an arc $\mathrm{D}_L \to X$. In particular, if we define a closed subscheme $Z := V(\mathrm{Jac}_X) \subset X$, then*

$$\pi_n^{2n}(\mathrm{J}_{2n} X) \setminus \mathrm{J}_n Z \subset \pi_n(\mathrm{J}_\infty X).$$

PROOF. By base change, we may suppose that $L = k$. We may also assume that X is a complete intersection $V(f_1, \dots, f_l)$ in an affine space $\mathbb{A}_k^d$ with coordinates $x_1, \dots, x_d$ and that the image of α is the origin of $\mathbb{A}_k^d$. For a suitable order of coordinates, $\alpha^* \det(\partial f_i/\partial x_j)_{1 \le i,j \le l}$ has order e. Then any tuple $\underline{a} \in k[\![t]\!]^d$ giving α satisfies the condition in Proposition 3.7.2. Then there exists $\underline{b} = (b_1, \dots, b_d) \in k[\![t]\!]^d$ such that

$$f_1(\underline{b}) = \cdots = f_l(\underline{b}) = 0 \text{ and } \underline{b} \equiv \underline{a} \pmod{t^{n+1}}.$$

The arc corresponding to $\underline{b}$ is a lift of α. □

As an application of Hensel's lemma, we obtain the following result of Greenberg [**Gre66**]. It looks similar to Hensel's lemma, but has no condition on Jacobian order.

PROPOSITION 3.7.4 (Greenberg lifting theorem). *Let X be a finite type k-scheme. There exist integers $a \ge 1$ and $b \ge 0$ such that for every $n \in \mathbb{Z}_{\ge 0}$, if an n-jet $\alpha\colon \mathrm{D}_{L,n} \to X$ lifts to an $(an+b)$-jet $\mathrm{D}_{L,an+b} \to X$, then α lifts to an arc $\mathrm{D}_L \to X$. In particular, $\pi_n(\mathrm{J}_\infty X) = \pi_n^{an+b}(\mathrm{J}_{an+b} X)$.*

SKETCH OF PROOF. We prove this only when k is a perfect infinite field and X has pure dimension. For the general case, we refer the reader to the original paper of Greenberg cited above. The proof uses reduction to the case where X is reduced, reduction to the case where X is a complete intersection and induction on dimension.

Reduction to the reduced case: If $\mathcal{I} \subset \mathcal{O}_X$ is the nilradical, then for some $m > 0$, $\mathcal{I}^m = 0$. If α is a geometric $(mn+m-1)$-jet of X, then $(\alpha^{-1}\mathcal{I})^m = 0$ and hence $\alpha^{-1}\mathcal{I} \subset (t^{n+1})$. Therefore the n-jet $\pi_n^{nm+m-1}(\alpha)$ is a jet of X_{red}. If $a'n+b'$ is a function as in the proposition for X_{red}, then we can take a function for X as $m(a'n+b')+m-1$. Indeed, if an n-jet β of X lifts to an $(m(a'n+b')+m-1)$-jet $\widetilde{\beta}$, then it induces an $(a'n+b')$-jet $\widetilde{\beta}_{a'n+b'}$ of X_{red} which is a lift of β. Thus the assertion for X follows from the one for X_{red}. This also shows the proposition in the case $\dim X = 0$.

Reduction to complete intersections: We can locally embed X into a reduced complete intersection Y of the same dimension and write $Y = X \cup W$ with W the union of extra irreducible components. Suppose that the proposition holds for Y. Let Z denote the scheme-theoretic intersection $X \cap W$. Let $an+b$ and $a'n+b'$ be functions for Y and Z as in the proposition respectively. Suppose that a geometric n-jet β of X lifts an $(a'n+b')$-jet β'. If β' is a jet of Z, then β lifts to an arc of Z, which is also an arc of X. Otherwise, if β' lifts to a $a(a'n+b')+b$-jet, then it lifts to an arc $\widetilde{\beta}$ of Y. Since β' is a jet of X but not one of Z, it is not a jet of W. Therefore $\widetilde{\beta}$ is an arc of X.

Complete intersections: If X is a reduced scheme of pure dimension $d > 0$ which is a complete intersection, then we can take a closed subscheme $Y \subset X$ of pure dimension $d-1$ which contains $V(\mathrm{Jac}_X)$. Let $an+b$ be a function for Y as in the proposition, which exists by the assumption of induction. Moreover we can choose one such that $a \geq 2$. Suppose that a geometric n-jet β of X lifts an $(an+b)$-jet β'. If β' is a jet of Y, then β lifts to an arc of Y, which is also an arc of X. Otherwise, if β' lifts to a $2(an+b)$-jet, then from the geometric Hensel lemma, β' lifts to an arc of X. $\square$

COROLLARY 3.7.5. *Let X be a finite type k-scheme. For every n, $\pi_n(\mathrm{J}_\infty X)$ is a constructible subset of $\mathrm{J}_n X$.*

PROOF. From the Greenberg lifting theorem, $\pi_n(\mathrm{J}_\infty X) = \pi_n^{n'}(\mathrm{J}_{n'} X)$ for some $n' \geq n$. The right side is a constructible subset from Chevalley's theorem below. $\square$

PROPOSITION 3.7.6 (Chevalley's theorem (see [**Har77**, Ch. II, Exercise 3.19])). *Let W, V be k-schemes of finite type, let $f\colon W \to V$ be a morphism and let $C \subset W$ be a constructible subset. Then $f(C)$ is a constructible subset of V.*

3.8. Cylinders

DEFINITION 3.8.1. Let X be a k-scheme of finite type and let $n \in \mathbb{Z}_{\geq 0}$. A subset $C \subset \mathrm{J}_\infty X$ is called a *cylinder of level* n if there exists a constructible subset $C' \subset \mathrm{J}_n X$ such that $\pi_n^{-1}(C') = C$. A subset $C \subset \mathrm{J}_\infty X$ is called a *cylinder* if it is a cylinder of some level.

Below are basic properties of cylinders.

LEMMA 3.8.2. (1) *If C is a cylinder of level n, then it is a cylinder of level n' for every $n' \geq n$.*
(2) *If C is a cylinder of level n, then $C = \pi_n^{-1}(\pi_n(C))$.*
(3) *If C is a cylinder, then $\pi_n(C)$ is constructible for every n.*
(4) *Cylinders in $\mathrm{J}_\infty X$ form a finitely additive class, that is, they are closed under taking finite unions and complements.*

PROOF. (1), (2) Obvious.
(3) Let $C' \subset \mathrm{J}_m X$ be a constructible subset such that $m \geq n$ and $C = \pi_m^{-1}(C')$. Then $\pi_n(C) = \pi_n^m(C') \cap \pi_n(\mathrm{J}_\infty X)$. From Corollary 3.7.5 and Chevalley's theorem, this is constructible.
(4) This easily follows from the fact that constructible subsets of a finite type scheme form a finitely additive class. $\square$

LEMMA 3.8.3. *Let A and C_i, $i \in I$ be cylinders in $\mathrm{J}_\infty X$. If $A \subset \bigcup_{i\in I} C_i$, then $A \subset \bigcup_{i \in J} C_i$ for some finite subset $J \subset I$.*

PROOF. We first claim that the same statement holds for constructible subsets A and C_i, $i \in I$ of a finite type scheme. Indeed we can find a finite subset I' such that $A \setminus \bigcup_{i \in I'} C_i$ has less dimension than A does. Induction on dimension shows the claim.

Now we switch to the situation of the lemma. To prove the lemma by contradiction, we suppose that there is no such finite subset J. Let m be a level of A and let n_i, $i \in I$ be levels of C_i respectively. For $n \in \mathbb{Z}_{\geq 0}$, we set $I_n := \{i \in I \mid n_i \leq n\}$ so that for $i \in I_n$, C_i is a cylinder of level n. For $n \geq m$, let

$$B_n := \pi_n(A) \setminus \bigcup_{i \in I_n} \pi_n(C_i) = \pi_n\left(A \setminus \bigcup_{i \in I_n} C_i\right).$$

Every B_n is nonempty. (Indeed, if it was not the case, then the above claim would show that $\pi_n(A)$ is covered by a finite subcollection of $\pi_n(C_i)$, $i \in I_n$ and that A is covered by a finite subcollection of C_i, $i \in I_n$, a contradiction.) For $n' \geq n \geq m$, since $I_n \subset I_{n'}$, we have

$$\pi_n^{n'}(B_{n'}) = \pi_n \left(A \setminus \bigcup_{i \in I_{n'}} C_i \right) \subset \pi_n \left(A \setminus \bigcup_{i \in I_n} C_i \right) = B_n.$$

From the lemma below, $\bigcap_{n' \geq n} \pi_n^{n'}(B_{n'}) \neq \emptyset$. Therefore there exists a sequence of $\bar{k}$-points $b_n \in B_n(\bar{k})$, $n \geq m$ such that for $n' \geq n \geq m$, $\pi_n^{n'}(b_{n'}) = b_n$. This defines the limit point $b_\infty \in A(\bar{k})$. This point is contained some of C_i's, say C_{i_0}. For $n \geq \max\{m, n_{i_0}\}$, the point $b_n = \pi_n(b_\infty)$ lies in $\pi_n(C_{i_0})$. At the same time, since b_n lies in B_n, it does not lie in C_{i_0}. This is a contradiction and we have proved the lemma. □

Lemma 3.8.4. *Let X be a scheme of finite type over k and let $C_1 \supset C_2 \supset \cdots$ be a descending chain of nonempty constructible subsets of X. Then $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.*

Proof. On the contrary, suppose that $\bigcap_{i=1}^{\infty} C_i = \emptyset$. Then, for $j \gg i$, every generic point of C_i is not contained in C_j and hence $\dim C_j < \dim C_i$. This shows that for $i \gg 0$, C_i is empty, which contradicts the assumption. □

CHAPTER 4

Motivic integration over smooth varieties

In this chapter, we discuss motivic integration over smooth varieties. Restriction to the smooth case allows us to grasp the essence of the theory quickly. Note however that most results in these sections will be eventually generalized in later chapters to the singular case with suitable modification.

Throughout the chapter, X and Y denote smooth k-varieties of dimension d.

4.1. Motivic measures

DEFINITION 4.1.1. We define the *motivic measure* on $\mathrm{J}_\infty X$ to be the map

$$\mu_X \colon \{\text{cylinders in } \mathrm{J}_\infty X\} \to \widehat{\mathcal{M}}_k$$

defined as follows: for a cylinder $C \subset \mathrm{J}_\infty X$ of level n,

$$\mu_X(C) := \{\pi_n(C)\}\mathbb{L}^{-dn}.$$

We call $\mu_X(C)$ the *measure* or *volume* of C.

Note that $\mu_X(C)$ is always an effective element (see Definition 2.3.1).

LEMMA 4.1.2. *The measure $\mu_X(C)$ is independent of the choice of the level n.*

PROOF. Let $n' \geq n$ be two levels of C. From Lemma 3.3.4, there exists a stratification $\pi_n(C) = \bigsqcup_i B_i$ into locally closed subsets B_i such that for every i,

$$(\pi_n^{n'})^{-1}(B_i) \cong B_i \times_k \mathbb{A}_k^{d(n'-n)},$$

in particular, $\{(\pi_n^{n'})^{-1}(B_i)\} = \{B_i\}\mathbb{L}^{d(n'-n)}$ in $\widehat{\mathcal{M}}_k$. Since $\pi_{n'}(C) = \bigsqcup_i (\pi_n^{n'})^{-1}(B_i)$, we have

$$\begin{aligned}\{\pi_{n'}(C)\}\mathbb{L}^{-dn'} &= \sum_i \{(\pi_n^{n'})^{-1}(B_i)\}\mathbb{L}^{-dn'}\\ &= \sum_i \{B_i\}\mathbb{L}^{d(n'-n)}\mathbb{L}^{-dn'}\\ &= \sum_i \{B_i\}\mathbb{L}^{-dn}\\ &= \{\pi_n(C)\}\mathbb{L}^{-dn}.\end{aligned}$$

$\square$

DEFINITION 4.1.3. Let $A \subset \mathrm{J}_\infty X$ be a subset. A function $h \colon A \to \mathbb{Z}$ is said to be *cylindrical* if there exists countably many cylinders A_i, $i \in I$ such that $A = \bigsqcup_{i \in I} A_i$ and h is constant on each A_i.

EXAMPLE 4.1.4. Let $W \subsetneq X$ be a proper closed subscheme. The function

$$\mathrm{ord}_W|_{\mathrm{J}_\infty X \setminus \mathrm{J}_\infty W} \colon \mathrm{J}_\infty X \setminus \mathrm{J}_\infty W \to \mathbb{Z}$$

is cylindrical. To see this, we observe that for $\gamma \in \mathrm{J}_\infty X$ and for $n \in \mathbb{Z}_{>0}$, $\mathrm{ord}_W(\gamma) \geq n$ if and only if $\pi_{n-1}(\gamma) \in \mathrm{J}_{n-1} W$. Namely, for $n > 0$, we have

$$\mathrm{ord}_W^{-1}(\geq n) := \mathrm{ord}_W^{-1}(\mathbb{Z}_{\geq n} \cup \{\infty\}) = \pi_{n-1}^{-1}(\mathrm{J}_{n-1} W).$$

These as well as $\mathrm{ord}_W^{-1}(\geq 0) = \mathrm{J}_\infty X$ are cylinders. Therefore $\mathrm{ord}_W^{-1}(n) = \mathrm{ord}_W^{-1}(\geq n) \setminus \mathrm{ord}_W^{-1}(\geq n+1)$ are cylinders. The stratification $\mathrm{J}_\infty X \setminus \mathrm{J}_\infty W = \bigsqcup_{n \in \mathbb{N}} \mathrm{ord}_W^{-1}(n)$ is a desired one.

DEFINITION 4.1.5. For a cylindrical function $h \colon A \to \mathbb{Z}$, we define the *integral of* $\mathbb{L}^h$ to be

$$\int_A \mathbb{L}^h \, d\mu_f := \sum_{i \in I} \mu_X(A_i) \mathbb{L}^{h(A_i)} \in \widehat{\mathcal{M}}_k \cup \{\infty_*\}.$$

Note that each term in the above sum is an effective element of $\widehat{\mathcal{M}}_k$. From Definition 2.4.6, the sum converges and defines an element of $\widehat{\mathcal{M}}_k$ if for every $m \in \mathbb{Z}$, there are only finitely many terms with $\dim \mu_X(h^{-1}(n))\mathbb{L}^n < m$. Otherwise it diverge and defined to be ∞.

LEMMA 4.1.6. *The definition of the integral above is independent of the choice of the stratification* $A = \bigsqcup_{i \in I} A_i$.

PROOF. Let $A = \bigsqcup_{j \in J} B_j$ be another such stratification. We have

$$A_i = \bigsqcup_{j \in J} A_i \cap B_j \text{ and } B_j = \bigsqcup_{i \in I} A_i \cap B_j.$$

From 3.8.3, these are finite stratifications. Therefore

$$\begin{aligned} \sum_{i \in I} \mu_X(A_i) \mathbb{L}^{h(A_i)} &= \sum_{i \in I} \sum_{j \in J} \mu_X(A_i \cap B_j) \mathbb{L}^{h(A_i \cap B_j)} \\ &= \sum_{j \in J} \sum_{i \in I} \mu_X(A_i \cap B_j) \mathbb{L}^{h(A_i \cap B_j)} \\ &= \sum_{i \in J} \mu_X(B_j) \mathbb{L}^{h(B_j)}. \end{aligned}$$

□

EXAMPLE 4.1.7. The whole arc scheme $\mathrm{J}_\infty X$ is a cylinder of level 0. Therefore

$$\mu_X(\mathrm{J}_\infty X) = \{X\}.$$

By taking the constant function $h \equiv 0$ on $\mathrm{J}_\infty X$, we get the following integral expression of the motive $\mathrm{M}(X) = \{X\}$:

$$\int_{\mathrm{J}_\infty X} 1 \, d\mu_X := \int_{\mathrm{J}_\infty X} \mathbb{L}^0 \, d\mu_X = \mu_X(\mathrm{J}_\infty X) = \{X\}.$$

Note that these are not true for singular varieties. But, when X has only mild singularities (in precise, $\mathbb{Q}$-Gorenstein singularities), we can define the *stringy motive* $\mathrm{M}_{\mathrm{st}}(X)$ (Definitions **??** and 6.3.6) as an integral over $\mathrm{J}_\infty X$ which coincides with $\mathrm{M}(X)$ if X is smooth.

For a proper closed subscheme $Z \subsetneq X$, $\mathrm{J}_\infty Z$ is not generally a cylinder as a subset of $\mathrm{J}_\infty X$, but a *negligible subset* (see Definition 5.6.1). Namely it has measure zero in a sense which will be justified later. We can ignore such a subset, as far as measures and integrals are concerned. In particular, we have the following lemma:

LEMMA 4.1.8. *Let $Z \subsetneq X$ be a proper closed subscheme and let $h\colon A \to \mathbb{Z}$ be a cylindrical function with $A \subset \mathrm{J}_\infty X$. Then the restriction of h to $A \setminus \mathrm{J}_\infty Z$ is also cylindrical and*

$$\int_A \mathbb{L}^h \, d\mu_X = \int_{A \setminus \mathrm{J}_\infty Z} \mathbb{L}^h \, d\mu_X.$$

We don't prove this lemma now, but will prove a more general statement later (Lemma 5.7.2). This enables us to slightly generalize motivic integrals as follows.

4.2. Almost bijectivity

The most important theorem in motivic integration is the change of variables formula (Theorem 4.4.6). It describes how a motivic integral is transformed under a birational transform of the given variety. In the first place, this formula is based on the following result.

PROPOSITION 4.2.1 (Almost bijectivity). *Let $f\colon Y \to X$ be a birational (resp. proper birational) morphism. Let $W \subset Y$ and $V \subset X$ be proper closed subvarieties such that f induces the isomorphism $Y \setminus W \to X \setminus V$. For every extension L/k, the map*

$$(\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W)(L) \to (\mathrm{J}_\infty X \setminus \mathrm{J}_\infty V)(L)$$

is injective (resp. bijective).

PROOF. Let $\gamma \in (\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W)(L)$. Since Y is separated over k, from the valuative criterion for separatedness, the γ is determined by the induced morphism $\operatorname{Spec} L(\!(t)\!) \to X \setminus V = Y \setminus W$. This shows the injectivity of $(\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W)(L) \to (\mathrm{J}_\infty X \setminus \mathrm{J}_\infty V)(L)$. Let $\beta \in (\mathrm{J}_\infty X \setminus \mathrm{J}_\infty V)(L)$. If f is proper, then from the valuative criterion for properness, we have a unique lift $\gamma \in (\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W)(L)$ of β as indicated in the following diagram.

$$\begin{tikzcd} \operatorname{Spec} L(\!(t)\!) \arrow[r] \arrow[d, hook] & Y \arrow[d, "f"] \\ \operatorname{Spec} L[\![t]\!] \arrow[r] \arrow[ur, dotted, "\exists !"] & X \end{tikzcd}$$

This shows the bijectivity of $(\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W)(L) \to (\mathrm{J}_\infty X \setminus \mathrm{J}_\infty V)(L)$ in the proper case. $\square$

REMARK 4.2.2. In the situation of Proposition 4.2.1, also the map $\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty W \to \mathrm{J}_\infty X \setminus \mathrm{J}_\infty V$ (of underlying topological spaces) is injective (resp. bijective). This follows from the proposition and the description of the point set of a scheme in terms of its geometric points in Remark 3.4.5.

4.3. Jacobian ideals of morphisms

DEFINITION 4.3.1. Let Y, X be smooth k-varieties of the same dimension d and let $f\colon Y \to X$ be a generically étale morphism (that is, étale on an open dense subset of Y). Let $\omega_Y := \Omega^d_{Y/k}$ be the canonical sheaf of Y and similarly for ω_X. By the natural injective map, we regard $f^*\omega_X$ as a subsheaf of ω_Y. We define the *Jacobian ideal sheaf* $\mathrm{Jac}_f \subset \mathcal{O}_Y$ by the equality

$$f^*\omega_X = \mathrm{Jac}_f \cdot \omega_Y.$$

We denote the associate order function $\mathrm{ord}_{\mathrm{Jac}_f}$ by $\mathfrak{j}_f$.

Since $f^*\omega_X$ and ω_Y are invertible sheaves, such an ideal uniquely exists and is locally principal. In terms of local coordinates, if f is locally given by a tuple

$$(f_1(y_1, \dots, y_d), \dots, f_d(y_1, \dots, y_d))$$

of d functions, then Jac_f is locally generated by the Jacobian determinant,

$$\det\left(\partial f_i/\partial y_j\right)_{i,j}.$$

The closed subset defined by Jac_f is the locus where f is not étale.

DEFINITION 4.3.2. We call the effective divisor on Y defined by Jac_f the *relative canonical divisor* for $f\colon Y \to X$ and denote it by $K_{Y/X}$.

REMARK 4.3.3. It is customary to write $K_{Y/X} = K_Y - f^*K_X$ somehow ambiguously (sometimes symbols "$\equiv$" or "$\sim_{\mathbb{Q}}$" are used instead of "$=$"), as canonical divisors are determined only up to linear equivalence. But we can justify it using the canonical map between canonical sheaves as in the above definition (see [**Kol13**, Notation 2.6]).

4.4. The change of variables formula

In this subsection, we prove the change of variables formula for a proper birational morphism $f\colon Y \to X$ of smooth varieties. As in Proposition 4.2.1, the map $f_\infty\colon \mathrm{J}_\infty Y \to \mathrm{J}_\infty X$ is almost bijective (bijective outside negligible subsets). Thanks to this, we may expect to be able to transform motivic integrals on $\mathrm{J}_\infty Y$ to ones on $\mathrm{J}_\infty X$ and vice versa. To realize it, we need to understand how measures of subsets change under the almost bijection. It turns out that the change is controlled by the Jacobian of f, as in the case of the change of variables formula for integrals in multivariate calculus.

We begin with proving auxiliary results. The following lemma roughly says that for two arcs γ, γ' of Y, if $f_\infty(\gamma)$ and $f_\infty(\gamma')$ are close to each other (t-adically), then so are γ and γ'.

LEMMA 4.4.1 (Fiber inclusion lemma). *Let $f\colon Y \to X$ be a proper birational morphism. Let $\gamma, \gamma' \in (\mathrm{J}_\infty Y)(L)$ with L/k an extension. Let $n \in \mathbb{Z}_{\geq 0}$ and suppose that $f_n\pi_n(\gamma) = f_n\pi_n(\gamma')$. Suppose that $\mathfrak{j}_f(\gamma) =: e \leq n/2$ for the function $\mathfrak{j}_f$ given in Definition 4.3.1. Then $\pi_{n-e}(\gamma) = \pi_{n-e}(\gamma')$. Namely, the fiber of $f_n\colon \mathrm{J}_n Y \to \mathrm{J}_n X$ over $f_n\pi_n(\gamma)$ is contained in the fiber of $\pi^n_{n-e}\colon \mathrm{J}_n Y \to \mathrm{J}_{n-e} Y$ over $\pi_{n-e}(\gamma)$.*

PROOF. We claim that there exists $\delta \in (\mathrm{J}_\infty Y)(L)$ such that $f_\infty(\delta) = f_\infty(\gamma)$ and $\pi_{n-e}(\delta) = \pi_{n-e}(\gamma')$. If $Z = V(\mathrm{Jac}_f) \subset Y$ is the exceptional locus of f, then assumption $\mathfrak{j}_f(\gamma) < \infty$ in particular shows that $\gamma \notin \mathrm{J}_\infty Z$. From the almost bijectivity (Proposition 4.2.1), we have $\delta = \gamma$. Thus the lemma follows from the above claim.

To show the claim, we may suppose that $L = k$ by base change. Let $y := \pi_0(\gamma) \in Y$ and $x := f(y)$. Choosing local coordinates, we express f by a tuple

$$\underline{f} = (f_1, \dots, f_d) \quad (f_i \in \widehat{\mathcal{O}_{Y,y}} = k[\![y_1, \dots, y_d]\!])$$

and γ by a tuple

$$\underline{\gamma} = (\gamma_1, \dots, \gamma_d) \in k[\![t]\!]^d.$$

Namely $f^*\colon \widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{Y,y}}$ is given by $x_i \mapsto f_i$ and $\gamma^*\colon \widehat{\mathcal{O}_{Y,y}} \to k[\![t]\!]$ is given by $y_i \mapsto \gamma_i$. Similarly for γ'. The system of equations

$$\underline{f}(\underline{y}) = \underline{f}(\underline{\gamma'})$$

has the approximate solution $\underline{y} = \underline{\gamma}$ modulo t^{n+1}. From the Hensel lemma (Proposition 3.7.2), there exists a genuine solution $\underline{y} = \underline{\delta}$ such that $\underline{\delta} \equiv \underline{\gamma} \pmod{t^{n-e+1}}$. The corresponding arc $\delta \in (\mathrm{J}_\infty Y)(k)$ satisfies the claim. $\square$

Lemma 4.4.2 (The $\mathbb{A}^e$-fibration lemma). *With the notation as in Lemma 4.4.1, the fiber of $\mathrm{J}_n Y \to \mathrm{J}_n X$ over $f_n\pi_n(\gamma)$ is isomorphic to $\mathbb{A}^e_L$ over L. Moreover, if $H_{n,e}$ denotes the locus in $\mathrm{J}_n Y$ with $\mathfrak{j}_f = e$, then*

$$f_n|_{H_{n,e}}\colon H_{n,e} \to f_n(H_{n,e})$$

is a piecewise trivial $\mathbb{A}^e$-bundle.

Proof. Let F denote this fiber. This is included in the fiber of $\pi^e_{n-e}\colon \mathrm{J}_n Y \to \mathrm{J}_{n-e} Y$ over $\pi_{n-e}(\gamma)$, which is isomorphic to $\mathbb{A}^{ed}_L$, where $d = \dim Y$. The L-points of F correspond to solutions $\underline{\delta} \in (L[\![t]\!]/(t^e))^d$ of the system of equations expressed with the multi-index notation,

$$\underline{f}(\underline{\gamma} + \underline{\delta} t^{n-e+1}) \equiv \underline{f}(\underline{\gamma}) \pmod{t^{n+1}}.$$

Let $J := (\partial f_j/\partial t_i(\underline{\gamma})) \in M_n(L[\![t]\!])$ be the Jacobian matrix of $\underline{f}$ evaluated at $\underline{\gamma}$. Since $2(n-e+1) \geq n+1$, by Taylor expansion, this is equivalent to

$$\underline{f}(\underline{\gamma}) + t^{n-e+1} J\underline{\delta} \equiv \underline{f}(\underline{\gamma}) \pmod{t^{n+1}}$$

and hence to the $L[\![t]\!]$-linear equation

$$\left(t^{n-e+1}J\right)\underline{\delta} \equiv 0 \pmod{t^{n+1}}.$$

For some invertible matrices $P, Q \in GL_n(L[\![t]\!])$, we have

$$P^{-1}JQ = \operatorname{diag}(t^{e_1}, \dots, t^{e_d})$$

with $\sum e_i = e$ (see [**Bou81**, VII. 21]). Thus, if we put $\underline{\epsilon} := Q^{-1}\underline{\delta} \in (L[\![t]\!]/(t^e))^d$, then we are reduced to solving the equation

$$t^{n-e+e_i+1}\epsilon_i \equiv 0 \pmod{t^{n+1}} \quad (1 \leq i \leq d).$$

Thus the solution space is

$$\{\underline{\epsilon} \mid \operatorname{ord} \epsilon_i \geq e - e_i\} \cong L^e,$$

which is an L-linear subspace of $(L[\![t]\!]/(t^e))^d \cong L^{de}$. Thus we have identification $F(L) = \mathbb{A}^e_L(L)$. To get an isomorphism $F \cong \mathbb{A}^e_L$ of L-schemes, consider an arbitrary L-algebra R. By the same argument, the R-points of F also correspond to the solutions in $(R[\![t]\!]/(t^e))^d$ of any of the above systems of equations. Therefore the solution set is identified with R^e, functorially in R. As a consequence, we get an isomorphism $F \cong \mathbb{A}^e_L$ of functors $(\mathbf{Aff}_L)^{\mathrm{op}} \to \mathbf{Set}$. The first assertion of the lemma follows.

For the second assertion, we first note that from the fiber inclusion lemma, $f_n^{-1} f_n(H_{n,e}) = H_{n,e}$. It suffices to show that for any irreducible locally closed subvariety $V \subset \mathrm{J}_n X$, the morphism $W := f_n^{-1}(V) \to V$ is a piecewise trivial $\mathbb{A}^e$-fibration. For a generic point $\eta \in V$, we have an isomorphism $\mathbb{A}^e_{\kappa(\eta)} \cong W_\eta$, where $\kappa(\eta)$ denotes the residue field of η and W_η denotes the fiber of $W \to V$ over η. By spreading out (see Lemma 4.4.3 below), we get an isomorphism morphism $\phi \colon \mathbb{A}^e_U \to W_U$ over some open neighborhood $\eta \in U \subset W$. The second assertion follows by induction. $\square$

Lemma 4.4.3 (Spreading out an affine space). *Let V be a k-variety with the generic point η and let $f \colon W \to V$ be a morphism of finite type such that the generic fiber $f^{-1}(\eta)$ is isomorphic to $\mathbb{A}^n_\eta$ over η (with identifying η with* $\operatorname{Spec} \kappa(\eta)$*). Then there exists an open dense subvariety $U \subset V$ such that $f^{-1}(U)$ is isomorphic to $\mathbb{A}^n_U$ over U.*

Proof. The isomorphism $f^{-1}(\eta) \to \mathbb{A}^n_{\kappa(\eta)}$ extends to a morphism $f^{-1}(U) \to \mathbb{A}^n_U$ for some open subvariety $U \subset V$. Its non-étale locus is a closed subset which does not dominate U. Shrinking U, we may suppose that $f^{-1}(U) \to \mathbb{A}^n_U$ is étale. Its image is an open subset of $\mathbb{A}^n_U$ containing $\mathbb{A}^n_{\kappa(\eta)}$. Therefore, shrinking U further, we may suppose that $f^{-1}(U) \to \mathbb{A}^n_U$ is also surjective. Now it is étale surjective morphism of degree one, hence an isomorphism. (Indeed, since the morphism is étale of degree one, from the lower semi-continuity of the cardinality of a fiber [**Gro66**, Prop. 15.5.1], every geometric fiber has at most one point, that is, the morphism is universally injective (also called radiciel) [**Gro60**, Chap. I, (3.5.4) and (3.5.5)]. From [**Gro67**, Th. 17.9.1], a universally injective and étale morphism is an open immersion. A surjective open immersion is an isomorphism.) $\square$

Corollary 4.4.4. *Let $f \colon Y \to X$ be a proper birational morphism and let $C \subset \mathrm{J}_\infty X$ be a cylinder such that j_f takes a constant value $e < \infty$ on $f_\infty^{-1}(C)$. Then*

$$\mu_Y(f_\infty^{-1}(C)) = \mathbb{L}^e \mu_X(C).$$

Proof. Suppose that C is a cylinder of level n with $n \ge 2e$ and let $D := \pi_n(C) \subset \mathrm{J}_n X$. Then the map

$$\pi_n(f_\infty^{-1}(C)) = f_n^{-1}(D) \to D$$

is a piecewise trivial $\mathbb{A}^e$-bundle from Lemma 4.4.2. Thus

$$\begin{aligned} \mu_Y(f_\infty^{-1}(C)) &= \{\pi_n(f_\infty^{-1}(C))\}\mathbb{L}^{-nd} \\ &= \{D\}\mathbb{L}^e\mathbb{L}^{-nd} \\ &= \mathbb{L}^e \mu_X(C). \end{aligned}$$

$\square$

Corollary 4.4.5. *Let $f \colon Y \to X$ be a proper birational morphism and let $C \subset \mathrm{J}_\infty Y$ be a cylinder such that $C \cap \mathrm{j}_f^{-1}(\infty) = \emptyset$. Then $f_\infty(C)$ is a cylinder.*

Proof. Note that for every $e \in \mathbb{Z}_{\ge 0}$, $\mathrm{j}_f^{-1}(\le e)$ is a cylinder. From 3.8.3, there exists e_0 such that $\mathrm{j}_f|_C \le e_0$. Similarly we may assume that $\mathrm{ord}_{f^{-1}\mathcal{I}}|_C \le e_0$ for the defining ideal sheaf $\mathcal{I}$ of the closed subset $f(V(\mathrm{Jac}_f))$. Suppose that C is a cylinder of level $n - e_0$ for an integer $n \geq e_0$. We claim that $f_\infty(C)$ is a cylinder of level n. Indeed, $\pi_n(f_\infty(C)) = f_n(\pi_n(C))$ is a constructible subset from 3.7.5 and Chevalley's theorem. Let $\beta \in (\mathrm{J}_\infty Y)(L)$ be a geometric arc of Y lying on C and let $\alpha \in (\mathrm{J}_\infty X)(L)$ be a geometric arc with $\pi_n(\alpha) = f_n\pi_n(\beta)$. It suffices to show that α lies on $f_\infty(C)$. We have $\mathrm{ord}_{\mathcal{I}}(\alpha) \leq e_0 < \infty$, in particular, $\alpha \notin \mathrm{J}_\infty(V(\mathcal{I}))$. From the almost bijectivity (Proposition 4.2.1), we have the corresponding geometric arc $\widetilde{\alpha}$ of Y. From the fiber inclusion lemma, we have $\pi_{n-e_0}(\widetilde{\alpha}) = \pi_{n-e_0}(\beta)$. Since C is a cylinder of level $n - e_0$, $\widetilde{\alpha}$ lies on C and α lies on $f_\infty(C)$. □

Theorem 4.4.6 (The change of variables formula). *Let $f\colon Y \to X$ be a proper birational morphism of smooth k-varieties. Let $B \subset \mathrm{J}_\infty Y$ be a subset and let $h\colon f_\infty(B) \to \mathbb{Z}$ be a cylindrical function. Then*

$$\int_{f_\infty(B)} \mathbb{L}^h \, d\mu_X = \int_B \mathbb{L}^{h\circ f_\infty - \mathrm{j}_f} \, d\mu_Y \quad \text{in } \widehat{\mathcal{M}}_k \cup \{\infty_*\}.$$

Proof. Replacing B with $B \setminus \mathrm{j}_f^{-1}(\infty)$, we may assume that $\mathrm{j}_f|_B$ takes only finite values. Let $f_\infty(B) = \bigsqcup A_i$ be a stratification into countably many cylinders such that $h|_{A_i}$ are constant. Let $B_i := f_\infty^{-1}(A_i)$ and let $B_{i,e} := B_i \cap \mathrm{j}_f^{-1}(e)$ for $e \in \mathbb{Z}_{\geq 0}$. From Corollary 4.4.5, $A_{i,e} := f_\infty(B_{i,e})$ are also cylinders. From Corollary 4.4.4,

$$\mu_Y(B_{i,e}) = \mu_X(A_{i,e})\mathbb{L}^e.$$

We have

$$\begin{aligned}\int_{f_\infty(B)} \mathbb{L}^h \, d\mu_X &= \sum_{i,e} \mu_X(A_{i,e})\mathbb{L}^{h(A_{i,e})} \\ &= \sum_{i,e} \mu_Y(B_{i,e})\mathbb{L}^{h(A_{i,e})-e} \\ &= \int_B \mathbb{L}^{h\circ f_\infty - \mathrm{j}_f} \, d\mu_Y.\end{aligned}$$

□

4.5. Strong K-equivalence

The first application of the change of variables formula is a theorem of Batyrev and Kontsevich, the invariance of Hodge numbers of smooth and proper complex varieties by K-equivalence. First, Batyrev proved the invariance of Betti numbers in the case of Calabi-Yau varieties by using mod p reduction, the Weil conjecture and p-adic integration. Then Kontsevich invented motivic integration as an analog of p-adic integration and used it to prove the invariance of Hodge numbers in a more natural way so that one can avoid the use of mod p reduction and the Weil conjecture.

Later it was found another proof in terms of the weak factorization theorem. However the proof by motivic integration is still meaningful, because it can be generalized to positive and mixed characteristics and to Deligne-Mumford stacks.

Definition 4.5.1. Smooth k-varieties X and Y are said to be *strongly K-equivalent* if there exist a smooth k-variety Z and proper birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ such that $K_{Z/X} = K_{Z/Y}$.

REMARK 4.5.2. We will introduce the more general notion of K-equivalence by loosening the assumption that Z is smooth. The two notions coincide in the situation where resolution of singularities is available, in particular, in characteristic zero.

EXAMPLE 4.5.3. Let k be a field of characteristic zero. We say that a smooth proper k-variety X is a *Calabi-Yau variety* if $\omega_X \cong \mathcal{O}_X$. A more general notion is that of minimal models; a smooth proper k-variety X is a *minimal model* if K_X is nef, that is, for any curve $C \subset X$, we have $K_X \cdot C \geq 0$. If X and Y are two minimal models birational to each other, then they are strongly K-equivalent (see [**KM98**, Prop. 3.51]).

THEOREM 4.5.4. *Let X and Y be smooth k-varieties which are strongly K-equivalent. Then $\{X\} = \{Y\}$ in $\widehat{\mathcal{M}}_k$.*

PROOF. Let $f\colon Z \to X$ and $g\colon Z \to Y$ be proper birational morphisms as in Definition 4.5.1. The equality $K_{Z/X} = K_{Z/Y}$ shows that $\mathrm{Jac}_f = \mathrm{Jac}_g$ and $\mathrm{j}_f = \mathrm{j}_g$. Let $V \subset Z$ be the common exceptional locus of f and g and let $W \subset X$ be its image. Then

$$\begin{aligned}
\{X\} &= \int_{\mathrm{J}_\infty X} \mathbb{L}^0 \, d\mu_X && \text{(Example 4.1.7)} \\
&= \int_{\mathrm{J}_\infty X \setminus \mathrm{J}_\infty W} \mathbb{L}^0 \, d\mu_X && \text{(Lemma 4.1.8)} \\
&= \int_{\mathrm{J}_\infty Y \setminus \mathrm{J}_\infty V} \mathbb{L}^{-\mathrm{j}_f} \, d\mu_X. && \text{(Theorem 4.4.6)}
\end{aligned}$$

Similarly $\{Y\}$ is also equal to the last integral. □

COROLLARY 4.5.5. *Let X and Y be smooth k-varieties which are strongly K-equivalent.*

(1) *We have* $\mathrm{P}(X) = \mathrm{P}(Y)$ *and* $\mathrm{e}_{\mathrm{top}}(X) = \mathrm{e}_{\mathrm{top}}(Y)$.
(2) *If $k \subset \mathbb{C}$, then* $\mathrm{E}(X) = \mathrm{E}(Y)$.
(3) *If k is a finite field, then for every finite extension k'/k, $\sharp X(k') = \sharp Y(k')$.*

Moreover, if X and Y are proper, then:

(1) *We have $b_i(X) = b_i(Y)$, $i \in \mathbb{Z}$.*
(2) *If k is a finitely generated field, then we have isomorphisms of $\mathfrak{G}_k$-representations $\mathrm{H}^i(X \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)^{\mathrm{ss}} \cong \mathrm{H}^i(Y \otimes_k k^{\mathrm{sep}}, \mathbb{Q}_l)^{\mathrm{ss}}$, $i \in \mathbb{Z}$. Here the superscript* ss *means semisimplification.*
(3) *If $k \subset \mathbb{C}$, we have $h^{p,q}(X) = h^{p,q}(Y)$, $p, q \in \mathbb{Z}$.*
(4) *If $k \subset \mathbb{C}$, we have isomorphisms of Hodge structures $\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \cong \mathrm{H}^i(Y(\mathbb{C}), \mathbb{Q})$, $i \in \mathbb{Z}$.*

PROOF. These follow from Theorem 4.5.4 and Proposition 2.5.3. □

REMARK 4.5.6. The assumption of strong K-equivalence in Theorem 4.5.4 and Corollary 4.5.5 can be weakened to the one of K-equivalence (Corollary 6.4.8).

4.6. Fractional powers of $\mathbb{L}$

We will consider $\mathbb{Q}$-divisors (that is, divisors with rational coefficients). The order functions associated to them take values in $\mathbb{Q}$. This gives rise to functions of the form $\mathbb{L}^h$ for a $\mathbb{Q}$-valued function h. Thus we first need to add fractional powers of $\mathbb{L}$ to the Grothendieck ring of varieties. For the sake of simplicity, we just add the power $\mathbb{L}^{1/r}$ for some sufficiently factorial r rather than adding all the powers $\mathbb{L}^a$, $a \in \mathbb{Q}$; we choose r so that all the relevant rational numbers lie in $\frac{1}{r}\mathbb{Z}$. We then modify the ring further by localization and completion as before. The notions of effective elements and their dimensions generalize to this setting in a natural way.

DEFINITION 4.6.1. For $r \in \mathbb{Z}_{>0}$, we define

$$\mathrm{K}_0(\mathbf{Var}_k)_r := \mathrm{K}_0(\mathbf{Var}_k)[x]/(x^r - \mathbb{L})$$

and denote the class of x in this ring by $\mathbb{L}^{1/r}$. We define $\mathcal{M}_{k,r}$ to be the localization of $\mathrm{K}_0(\mathbf{Var}_k)_r$ by $\mathbb{L}$ (or equivalently by $\mathbb{L}^{1/r}$). Elements of the form $\{X\}\mathbb{L}^a$, $a \in \frac{1}{a}\mathbb{Z}$ are called *effective*. The *dimension* of $\{X\}\mathbb{L}^a$ is defined to be $\dim X + a$. For $m \in \frac{1}{r}\mathbb{Z}$, we define F_m to be the subgroup of $\mathcal{M}$ generated by effective elements of dimension $\leq -m$. We define the completion

$$\widehat{\mathcal{M}}_{k,r} := \varprojlim \mathcal{M}_{k,r}/F_m.$$

For every r, there exists a natural map $\widehat{\mathcal{M}}_k \to \widehat{\mathcal{M}}_{k,r}$. Using this, we may redefine the motivic measure μ_X for a smooth variety X to take values in $\widehat{\mathcal{M}}_{k,r}$. We denote it again by μ_X. We can define $\frac{1}{r}\mathbb{Z}$-valued cylindrical functions in the same way as $\mathbb{Z}$-valued ones as in Definition 4.1.3. For a smooth variety X, a subset $A \subset \mathrm{J}_\infty X$ and a cylindrical function $h\colon A \to \frac{1}{r}\mathbb{Z}$, we similarly define the integral

$$\int_A \mathbb{L}^h \, d\mu_X \in \widehat{\mathcal{M}}_{k,r} \cup \{\infty_*\}.$$

Here

$$\{\infty_*\} := \left\{ \infty_d \mid d \in \frac{1}{r}\mathbb{Z} \cup \{\infty\} \right\}.$$

The change of variables formula (Theorem 4.4.6) holds also for a $\frac{1}{r}\mathbb{Z}$-valued cylindrical function h.

The Poincaré polynomial realization $\mathrm{P}\colon \widehat{\mathcal{M}}_k \to \mathbb{Z}(\!(t^{-1})\!)$ in Example 2.5.1 uniquely extends to

$$\mathrm{P}\colon \widehat{\mathcal{M}}_{k,r} \to \mathbb{Z}(\!(t^{-1/r})\!)$$

by sending $\mathbb{L}^{1/r}$ to $t^{2/r}$. Similarly, if $k = \mathbb{C}$, we have the E-polynomial realization

$$\mathrm{E}\colon \widehat{\mathcal{M}}_{\mathbb{C},r} \to \mathbb{Z}(\!(u^{-1/r}, v^{-1/r})\!).$$

We can also extend the realizastion χ_{Hodge} to

$$\chi_{\mathrm{Hodge}}\colon \widehat{\mathcal{M}}_{\mathbb{C},r} \to \widehat{\mathrm{K}_0}(\mathbf{MHS}^{1/r}),$$

where $\mathbf{MHS}^{1/r}$ denotes the category of $\frac{1}{r}\mathbb{Z}$-indexed Hodge structures [**Yas06**, Section 3.8]. As for the l-adic realization χ_l, if we replace k with a finite extension of it, then we can construct fractional Tate twists $\mathbb{Q}_l(a)$, $a \in \frac{1}{r}\mathbb{Z}$ as Galois representations with fractional weight filtration [**Ito04**, Section 5.3]. This enables us to extend χ_l to $\widehat{\mathcal{M}}_{k,r}$ after a finite extension of k.

4.7. Explicit formula

In this subsection, we show an explicit formula for the motivic integral associated to a $\mathbb{Q}$-divisor with simple normal crossing support.

Let X be a smooth k-variety and let $D = \sum_{i \in I} a_i D_i$ ($a_i \in \mathbb{Q}$) be a $\mathbb{Q}$-divisor, where I is a finite set and D_i, $i \in I$ are prime divisors such that $\bigcup_{i \in I} D_i$ is simple normal crossing.

Definition 4.7.1. We define the *order function*

$$\operatorname{ord}_D \colon \mathrm{J}_\infty X \setminus \bigcup_i \mathrm{J}_\infty D_i \to \mathbb{Q}$$

by

$$\operatorname{ord}_D(\gamma) := \sum_{i \in I} a_i \operatorname{ord}_{D_i}(\gamma).$$

This is a cylindrical function.

Definition 4.7.2. For a subset $J \subset I$, we define

$$D_J^\circ := \bigcap_{j \in J} D_j \setminus \bigcup_{i \in I \setminus J} D_i.$$

Namely D_J° consists of points that are contained in D_j, $j \in J$ but not in D_i, $i \notin J$. By convention, we put $D_\emptyset^\circ = X \setminus \bigcup_{i \in I} D_i$.

We have the stratification

$$X = \bigsqcup_{J \subset I} D_J^\circ.$$

Lemma 4.7.3. *For $\underline{m} = (m_i) \in (\mathbb{Z}_{\geq 0})^I$, we have*

$$\mu_X \left(\bigcap_{i \in I} \operatorname{ord}_{D_i}^{-1}(m_i) \right) = \{D_{\operatorname{Supp}(\underline{m})}^\circ\} (\mathbb{L} - 1)^{\sharp \operatorname{Supp}(\underline{m})} \mathbb{L}^{-\sum_{i \in I} m_i}.$$

Here $\operatorname{Supp}(\underline{m}) := \{i \in I \mid m_i > 0\}$. More generally, for a constructible subset $C \subset X$, we have

$$\mu_X \left(\bigcap_{i \in I} \operatorname{ord}_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(C) \right) = \{D_{\operatorname{Supp}(\underline{m})}^\circ \cap C\} (\mathbb{L} - 1)^{\sharp \operatorname{Supp}(\underline{m})} \mathbb{L}^{-\sum_{i \in I} m_i}.$$

Proof. We first consider the case where $X = \mathbb{A}_k^d$ with coordinates $x_1, \ldots, x_d$, $I = \{1, \ldots, c\} \subset \{1, \ldots, d\}$ and $D_i = V(x_i)$. Then the k-points of $\bigcap_{i \in I} \operatorname{ord}_{D_i}^{-1}(m_i)$ is identified with

$$\{\underline{\gamma} \in k[\![t]\!] \mid \forall i \in \{1, \ldots, c\}, \operatorname{ord}(\gamma_i) = m_i\}.$$

For $n \geq \max\{m_i\}$, the image of this set in $\mathrm{J}_n X$ is

$$\begin{aligned} &\{\underline{\gamma} \in k[t]/(t^{n+1}) \mid \forall i \in \{1, \ldots, c\}, \operatorname{ord}(\gamma_i) = m_i\} \\ &\cong (k^*)^c \times k^{\sum_{i \leq c}(n - m_i)} \times k^{\sum_{i > c}(n+1)}. \end{aligned}$$

Similarly for R-points for any k-algebra R. Thus we see

$$\begin{aligned}\pi_n\left(\bigcap_{i\in I}\operatorname{ord}_{D_i}^{-1}(m_i)\right) &\cong \mathbb{G}_{m,k}^{c}\times\mathbb{A}_k^{d(n+1)-c-\sum_{i\le c}m_i}\\ &\cong \left(\mathbb{G}_{m,k}^{c-\sharp\operatorname{Supp}(\underline{m})}\times\mathbb{A}_k^{d-c}\right)\times\mathbb{G}_{m,k}^{\sharp\operatorname{Supp}(\underline{m})}\times\mathbb{A}_k^{dn-\sum m_i}\\ &\cong D_{\operatorname{Supp}(\underline{m})}^{\circ}\times\mathbb{G}_{m,k}^{\sharp\operatorname{Supp}(\underline{m})}\times\mathbb{A}_k^{dn-\sum m_i}.\end{aligned}$$

This shows the lemma in this case.

For the general case, from the additivity of both sides, we may assume that X admits a system of coordinates $x_1,\dots,x_d\in\Gamma(X,\mathcal{O}_X)$, that I is a subset of $\{1,\dots,d\}$, say $\{1,\dots,c\}$, and that $D_i=V(x_i)$. In other words, we have an étale morphism $X\to\mathbb{A}_k^d$ and each D_i is the pullback of some coordinate hyperplane. Since jet schemes are compatible with étale morphisms (Lemma 3.2.7), from the above case of affine space, we have

$$\pi_n\left(\bigcap_{i\in I}\operatorname{ord}_{D_i}^{-1}(m_i)\right)\cong D_J^{\circ}\times\mathbb{G}_{m,k}^{\sharp\operatorname{Supp}(\underline{m})}\times\mathbb{A}_k^{dn-\sum m_i}.$$

For a constructible subset $C\subset X$, we also have

$$\pi_n\left(\bigcap_{i\in I}\operatorname{ord}_{D_i}^{-1}(m_i)\cap\pi_0^{-1}(C)\right)\cong (D_J^{\circ}\cap C)\times\mathbb{G}_{m,k}^{\sharp\operatorname{Supp}(\underline{m})}\times\mathbb{A}_k^{dn-\sum m_i}.$$

The lemma follows. $\square$

The following proposition allows us to compute stringy motive explicitly. Note that for $b\in\frac{1}{r}\mathbb{Z}_{>0}$, we have

$$\frac{1}{\mathbb{L}^b-1}=\frac{\mathbb{L}^{-b}}{1-\mathbb{L}^{-b}}=\mathbb{L}^{-b}+\mathbb{L}^{-2b}+\cdots\quad\text{in }\widehat{\mathcal{M}}_{k,r}.$$

Proposition 4.7.4 (Explicit formula). *Let $C\subset X$ be a constructible subset. Consider the following integral*

$$S:=\int_{\pi_0^{-1}(C)\setminus\bigcup_{i\in I}\mathrm{J}_\infty D_i}\mathbb{L}^{\operatorname{ord}_D}\,d\mu_X.$$

This integral converges (resp. is dimensionally bounded) if and only if for every $i\in I$ with $C\cap D_i\neq\emptyset$, we have $a_i<1$ (resp. ≤ 1). When it converges, we have

$$S=\sum_{J\subset I}\{D_I^{\circ}\cap C\}\prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{1-a_i}-1}\in\widehat{\mathcal{M}}_{k,r}.$$

PROOF. We have

$$
\begin{aligned}
S &= \sum_{\underline{m}\in(\mathbb{Z}_{\geq 0})^I} \mu_X\left(\bigcap_{i\in I} \mathrm{ord}_{D_i}^{-1}(m_i)\cap \pi_0^{-1}(C)\right)\mathbb{L}^{\sum a_i m_i} \\
&= \sum_{J\subset I}\sum_{\mathrm{Supp}(\underline{m})=J} \mu_X\left(\bigcap_{i\in I} \mathrm{ord}_{D_i}^{-1}(m_i)\cap \pi_0^{-1}(C)\right)\mathbb{L}^{\sum a_i m_i} \\
&= \sum_{J\subset I}\sum_{\mathrm{Supp}(\underline{m})=J} \{D_J^\circ\cap C\}(\mathbb{L}-1)^{\sharp J}\mathbb{L}^{\sum_{i\in I}(a_i-1)m_i} \\
&= \sum_{J\subset I}\{D_J^\circ\cap C\}(\mathbb{L}-1)^{\sharp J}\prod_{i\in J}\sum_{m>0}\mathbb{L}^{(a_i-1)m}. \qquad (4.7.1)
\end{aligned}
$$

Therefore the integral converges if and only if the geometric series $\sum_{m>0}\mathbb{L}^{(a_i-1)m}$ converges for every $J\subset I$ with $D_J^\circ\cap C\neq\emptyset$ and every $i\in J$. Using $D_i=\bigsqcup_{i\in J}D_J^\circ$, the last condition is equivalent to that the geometric series $\sum_{m>0}\mathbb{L}^{(a_i-1)m}$ converges for every $i\in J$ with $D_i\cap C\neq\emptyset$. Similarly for the dimensional boundedness. If the integral converges, then the formula of the proposition follows from

$$
(\mathbb{L}-1)^{\sharp J}\prod_{i\in J}\sum_{m>0}\mathbb{L}^{(a_i-1)m}=\prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{1-a_i}-1}.
$$

□

For future application to singularities, we also give a formula for the dimension of the integral S which applies also to the dimensionally bounded case.

PROPOSITION 4.7.5. *We keep the notation of Proposition 4.7.4. We assume that S is dimensionally bounded. Then*

$$
\dim S=\max\left\{\dim(D_J^\circ\cap C)+\sum_{i\in J}a_i \;\middle|\; J\subset I\right\}.
$$

Moreover, if $C=\bigcup_{i\in K}D_i$ for some subset $K\subset I$, then

$$
\dim S=\max\{d-1+a_i\mid i\in K\}.
$$

PROOF. From Proposition 4.7.4, if $D_J^\circ\cap C\neq\emptyset$, then for every $i\in J$, we have $a_i-1\leq 0$. Therefore, for every $J\subset I$, we have

$$
\begin{aligned}
\dim\{D_J^\circ\cap C\}(\mathbb{L}-1)^{\sharp J}\prod_{i\in J}\sum_{m>0}\mathbb{L}^{(a_i-1)m} &= \dim\{D_J^\circ\cap C\}+\sharp J+\sum_{i\in J}(a_i-1) \\
&= \dim\{D_J^\circ\cap C\}+\sum_{i\in J}a_i.
\end{aligned}
$$

Note that if $D_J^\circ\cap C=\emptyset$, then the both sides are $-\infty$. The first assertion now follows from formula (4.7.1).

Suppose that $C=\bigcup_{i\in K}D_i$ for some $K\subset I$. We need to show

$$
\max\left\{\dim(D_J^\circ\cap C)+\sum_{i\in J}a_i \;\middle|\; J\subset I\right\}=\max\{d-1+a_i\mid i\in K\}. \qquad (4.7.2)
$$

We have

$$
D_J^\circ\cap C=\begin{cases} D_J^\circ\neq\emptyset & (D_J\neq\emptyset \text{ and } J\cap K\neq\emptyset)\\ \emptyset & (\text{otherwise}).\end{cases}
$$

In particular, for $i \in K$,

$$\dim(D^\circ_{\{i\}} \cap C) + a_i = d - 1 + a_i.$$

This shows the inequality $\geq$ in (4.7.2).

To show the opposite inequality, we need to show the inequality

$$\dim(D^\circ_J \cap C) + \sum_{i \in J} a_i \leq \max\{d - 1 + a_i \mid i \in K\}$$

for every $J \subset I$ be such that $D_J \neq \emptyset$ and $J \cap K \neq \emptyset$. If J is a singleton, then this is obvious. Suppose that J has at least two elements and let $j \in J \cap K$. For any $i \in J \setminus \{i\}$, we have

$$\emptyset \neq D_J \subset D_i \cap D_j \subset D_i \cap C.$$

From the assumption of dimensional boundedness and Proposition (4.7.4), we have $a_i - 1 \leq 0$. Therefore

$$\begin{aligned}
\dim(D^\circ_J \cap C) + \sum_{i \in J} a_i &= d - \sharp J + \sum_{i \in J} a_i \\
&= d + \sum_{i \in J}(a_i - 1) \\
&\leq d + a_j - 1.
\end{aligned}$$

Thus the inequality $\leq$ in (4.7.2) holds. $\square$

CHAPTER 5

Motivic integration over singular varieties

Throughout this chapter, X denotes a geometrically reduced k-schemes of finite type which have pure dimension d.

5.1. Jacobian orders in terms of modules of differentials

To reduce notational complexity, we introduce the following notation.

DEFINITION 5.1.1. For a ring R, we denote the ideal (t) of $R[\![t]\!]$ or $R[t]/(t^{n+1})$ by $\mathfrak{t}$. We also write $\mathfrak{t}^m_{m'} := \mathfrak{t}^m/\mathfrak{t}^{m'}$ for $m' \geq m$. When we need to specify the ring R, we also write them as $\mathfrak{t}_R$ and $\mathfrak{t}^m_{m',R}$ respectively.

LEMMA 5.1.2. *Let $\alpha \in (\mathrm{J}_\infty X)(L)$ be an arc over a field L. We have that $\mathfrak{j}_X(\alpha) < \infty$ if and only if $\mathrm{rank}_{L[\![t]\!]}\, \alpha^*\Omega_{X/k} = d$. Moreover, if it is the case, then*

$$\mathfrak{j}_X(\alpha) = \dim_L(\alpha^*\Omega_{X/k})_{\mathrm{tors}}.$$

Here $(-)_{\mathrm{tors}}$ means the torsion part.

PROOF. We first note that if η denotes the generic point of D_L and if $x = \alpha(\eta) \in X$, then

$$\mathrm{rank}_{L[\![t]\!]}\, \alpha^*\Omega_{X/k} = \dim_{L(\!(t)\!)}(\alpha|_\eta)^*\Omega_{X/k} = \dim_{\kappa(x)} \Omega_{X/k} \otimes_{\mathcal{O}_X} \kappa(x) \geq d.$$

Here the last inequality follows from the upper semicontinuity [**Har77**, Ch. 2, Ex. 5.8] and the fact that $\Omega_{X/k}$ has rank d. Recall that $\mathfrak{j}_X$ is the order of the ideal $\alpha^{-1}\mathrm{Jac}_X$ and that Jac_X is the d-th Fitting ideal of $\Omega_{X/k}$. By a property of Fitting ideals (Proposition 3.6.2), we have $\alpha^{-1}\mathrm{Jac}_X = \mathrm{Fitt}_d(\alpha^*\Omega_{X/k})$. Let us write $\alpha^*\Omega_{X/k} \cong \bigoplus_{j=1}^l L[\![t]\!]/\mathfrak{t}^{e_j} \oplus L[\![t]\!]^{\oplus r}$ with $e_j \in \mathbb{Z}_{>0}$. Then we have the free presentation

$$L[\![t]\!]^{\oplus l} \to L[\![t]\!]^{\oplus l+r} \to \alpha^*\Omega_{X/k} \to 0$$

given by the matrix

$$M = (t^{e_i}\delta_{ij})_{1\leq i\leq l+r,\, 1\leq j\leq l}.$$

Here δ_{ij} denotes the Kronecker delta. The ideal $\mathrm{Fitt}_d(\alpha^*\Omega_{X/k})$ is generated by $(l+r-d)$-minors of M. Thus

$$\mathrm{Fitt}_d(\alpha^*\Omega_{X/k}) = \begin{cases} \mathfrak{t}^{\sum_i e_i} & (r = d) \\ 0 & (r > d) \end{cases}$$

and

$$\mathfrak{j}_X(\alpha) = \begin{cases} \sum_i e_i & (r = d) \\ \infty & (r > d). \end{cases}$$

This shows the lemma. □

5.2. The derivation induced by two jets

In defining motivic measure or in proving the change of variables formula, it is the key to show that some maps of spaces of the form $\pi_n(\mathrm{J}_\infty X)$, spaces of jets liftable to arcs, have fibers isomorphic to affine spaces of expected dimension. We will show such a result by realizing those fibers as a linear subspace of a larger subset of $\mathrm{J}_n X$ also isomorphic to an affine space. To do so, we relate jets with derivation and with the module of differentials.

Suppose that X is an affine scheme $\operatorname{Spec} S$. Let $n \in \mathbb{Z}_{\geq 0}$, let R be a k-algebra and let $\alpha, \beta \in (\mathrm{J}_{2n+1} X)(R)$ be two $(2n+1)$-jets of X over R such that $\pi_n^{2n+1}(\alpha) = \pi_n^{2n+1}(\beta) \in (\mathrm{J}_n X)(R)$. Then the map

$$\beta^* - \alpha^* \colon S \to R[\![t]\!]/\mathfrak{t}^{2n+2}$$

has image included in $\mathfrak{t}^{n+1}$.

Proposition 5.2.1. *Let us regard $R[t]/\mathfrak{t}^{2n+2}$ as an S-module by the map $\alpha^* \colon S \to R[t]/\mathfrak{t}^{2n+2}$. Then the map*

$$\beta^* - \alpha^* \colon S \to \mathfrak{t}^{n+1}_{2n+2}$$

is a k-derivation. Similarly for the induced map $S \to \mathfrak{t}^{n+1}_{m+1}$ for $n \leq m \leq 2n+1$.

Proof. For $u, v \in S$, since $(\beta^* - \alpha^*)(v) \in \mathfrak{t}^{n+1}$ and $\alpha^*(u) \equiv \beta^*(u) \pmod{\mathfrak{t}^{n+1}}$, we have, modulo $\mathfrak{t}^{2(n+1)}$,

$$\begin{aligned}
&\alpha^*(u)(\beta^* - \alpha^*)(v) + \alpha^*(v)(\beta^* - \alpha^*)(u) \\
&= \beta^*(u)(\beta^* - \alpha^*)(v) + \alpha^*(v)(\beta^* - \alpha^*)(u) \\
&= \beta^*(u)\beta^*(v) - \alpha^*(u)\alpha^*(v) \\
&= (\beta^* - \alpha^*)(uv).
\end{aligned}$$

Thus $\beta^* - \alpha^*$ is a derivation, which is clearly k-linear. It is easy to see that the induced maps $S \to \mathfrak{t}^{n+1}_{m+1}$, $n \leq m \leq 2n+1$ are also k-derivations. □

The converse is also true:

Proposition 5.2.2. *Let $m \in \mathbb{Z}$ be such that $n \leq m \leq 2n+1$ and let $\delta \colon S \to \mathfrak{t}^{n+1}_{m+1}$ be a k-derivation, where we again regard $\mathfrak{t}^{n+1}_{m+1}$ as an S-module by α^*. Then there exists a unique m-jet β such that $\delta = \beta^* - \alpha^*$ and $\pi_n^m(\beta) = \pi_n^m(\alpha)$.*

Proof. We claim that the map

$$\beta^* := \delta + \alpha \colon S \to R[t]/\mathfrak{t}^{m+1}$$

is a k-algebra homomorphism. Indeed this is clearly k-linear. For $u, v \in S$, since δ is a derivation,

$$\begin{aligned}
\beta^*(uv) &= \delta(uv) + \alpha^*(uv) \\
&= \alpha^*(u)\delta(v) + \alpha^*(v)\delta(u) + \alpha^*(u)\alpha^*(v).
\end{aligned}$$

Since $\delta(u)\delta(v) = 0$ modulo $\mathfrak{t}^{2n+2}$ as well as modulo $\mathfrak{t}^{m+1}$, we can continue the above equalities as

$$\begin{aligned}
&= (\delta(u) + \alpha^*(u))(\delta(v) + \alpha^*(v)) \\
&= \beta^*(u)\beta^*(v).
\end{aligned}$$

Thus β^* is a k-algebra homomorphism and corresponds to an m-jet β. The uniqueness is obvious. $\square$

From this proposition, the map $\beta^* - \alpha^*$ corresponds to an S-linear map

$$\Omega_{X/k} \to \mathfrak{t}_{2n+2}^{n+1}.$$

In turn, it corresponds to an $R[\![t]\!]$-module homomorphism

$$\alpha^*\Omega_{X/k} \to \mathfrak{t}_{2n+2}^{n+1}.$$

When α lifts to an arc $\tilde{\alpha}$ over R, then we also get the $R[\![t]\!]$-module homomorphism

$$\tilde{\alpha}^*\Omega_{X/k} \twoheadrightarrow \alpha^*\Omega_{X/k} \to \mathfrak{t}_{2n+2}^{n+1}.$$

Definition 5.2.3. For $m \in \mathbb{Z}$ with $n \le m \le 2n+1$, we denote the induced maps $\alpha^*\Omega_{X/k} \to \mathfrak{t}_{m+1}^{n+1}$ and $\tilde{\alpha}^*\Omega_{X/k} \to \mathfrak{t}_{m+1}^{n+1}$ by $\delta_\alpha(\beta)_m$ and $\delta_{\tilde{\alpha}}(\beta)_m$ respectively. Fixing α or $\tilde{\alpha}$, we often abbreviate them as $\delta(\beta)_m$.

Propositions 5.2.1 and 5.2.2 show:

Corollary 5.2.4. *Let $\alpha \in (\mathrm{J}_{2n+1} X)(R)$, let $m \in \mathbb{Z}$ be such that $n \le m \le 2n+1$ and let*

$$F := (\pi_n^m)^{-1}(\pi_n(\alpha)) = (\mathrm{J}_m X) \times_{\mathrm{J}_n X} \operatorname{Spec} R,$$

where the fiber product is taken for the truncation $\pi_n^m \colon \mathrm{J}_m X \to \mathrm{J}_n X$ and the morphism $\pi_n^{2n+1}(\alpha)\colon \operatorname{Spec} R \to \mathrm{J}_n X$. Then, for R-algebras Q, we have functorial bijections:

$$\begin{aligned} F(Q) &\to \operatorname{Hom}_{R[\![t]\!]}(\alpha^*\Omega_{X/k}, \mathfrak{t}_{m+1,Q}^{n+1}) \\ \beta &\mapsto \delta_\alpha(\beta)_m \end{aligned}$$

Proposition 5.2.5. *We keep the assumption of Corollary 5.2.4. In addition, we suppose that $\alpha^*\Omega_{X/k}$ is of the form $\bigoplus_{i=1}^l R[\![t]\!]/\mathrm{t}^{e_i}$, $0 \le e_i \le 2n+2$ (this is automatic if R is a field).*

1. *Let $\omega_1, \dots, \omega_l$ be the corresponding generators of $\alpha^*\Omega_{X/k}$. Then we have functorial bijections:*

$$\begin{aligned} \operatorname{Hom}_{R[\![t]\!]}(\alpha^*\Omega_{X/k}, \mathfrak{t}_{m+1,Q}^{n+1}) &\to \bigoplus_{i=1}^{l} Q^{\oplus s_i} = Q^{\oplus s} \\ \delta &\mapsto (\delta(\omega_1), \dots, \delta(\omega_l)) \end{aligned}$$

   Here $s_i := \min\{e_i, m-n\}$ and $s = \sum s_i$.
2. *We have an isomorphism $F = (\pi_n^m)^{-1}(\pi_n(\alpha)) \cong \mathbb{A}_R^s$ over R.*

Proof. (1) The identifications

$$\begin{aligned} \operatorname{Hom}_{R[\![t]\!]}\left(\bigoplus_{i=1}^{l} R[\![t]\!]/\mathrm{t}^{e_i}, \mathfrak{t}_{m+1,Q}^{n+1}\right) &= \bigoplus_{i=1}^{l} \operatorname{Hom}_{R[\![t]\!]}\left(R[\![t]\!]/\mathrm{t}^{e_i}, \mathfrak{t}_{m+1,Q}^{n+1}\right) \\ &= \bigoplus_{i=1}^{l} \mathfrak{t}_{m+1,Q}^{m-s_i+1} = \bigoplus_{i=1}^{l} Q^{\oplus s_i} \end{aligned}$$

show this assertion.

(2) This follows from (1) and Corollary 5.2.4. $\square$

REMARK 5.2.6. In the situation of the last proposition, if α lifts to $\tilde{\alpha} \in (\mathrm{J}_\infty X)(R)$, then we have

$$\mathrm{Hom}_{R[\![t]\!]}(\alpha^*\Omega_{X/k}, \mathfrak{t}^{n+1}_{m+1,Q}) = \mathrm{Hom}_{R[\![t]\!]}(\tilde{\alpha}^*\Omega_{X/k}, \mathfrak{t}^{n+1}_{m+1,Q}).$$

5.3. Bundle structure of truncation maps

DEFINITION 5.3.1. For $0 \le e \le n \le \infty$, we define a subset $\mathrm{J}_n^{(e)} X \subset \mathrm{J}_n X$ to be $\mathfrak{j}_X^{-1}(e)$ and a subset $\mathrm{J}_n^{(\le e)} X$ to be $\mathfrak{j}_X^{-1}(\mathbb{Z}_{\le e})$.

LEMMA 5.3.2. *Let $e, n \in \mathbb{Z}_{\ge 0}$ with $e \le n$.*

(1) *$\mathrm{J}_n^{(\le e)} X$ is a closed subset of $\mathrm{J}_n X$.*
(2) *$\mathrm{J}_n^{(e)} X$ is a locally closed subset of $\mathrm{J}_n X$.*
(3) *$\pi_n(\mathrm{J}_\infty^{(\le e)} X)$ and $\pi_n(\mathrm{J}_\infty^{(e)} X)$ are constructible subsets of $\mathrm{J}_n X$.*

PROOF. (1) Consider the universal n-jet

$$u\colon (\mathrm{J}_n X) \times_k \mathrm{D}_{n,k} \to X.$$

Since the source of this morphism shares the underlying topological space with $\mathrm{J}_n X$, we may think of its structure sheaf as the coherent sheaf of $\mathcal{O}_{\mathrm{J}_n X}$-algebras and denote it by $\mathcal{A}$. We denote by $\mathcal{T}$ the ideal sheaf generated by t in $\mathcal{A}$ and let $\mathcal{I} := u^{-1}\mathrm{Jac}_X$. The subset $\mathrm{J}_n^{(\le e)} X$ is the locus where $\mathcal{I}$ properly contains $\mathcal{T}^{e+1}$, which is expressed as the support of the coherent sheaf $(\mathcal{I}+\mathcal{T}^{e+1})/\mathcal{T}^{e+1}$. Therefore it is closed.

(2) This is because $\mathrm{J}_n^{(e)} X = \mathrm{J}_n^{(\le e)} X \setminus \mathrm{J}_n^{(\le e-1)} X$.

(3) This follows from Corollary 3.7.5 and the first two assertions. □

DEFINITION 5.3.3. Let $\alpha\colon \mathrm{D}_L \to X$ be an arc over a field L. We define the *flat pullback* $\alpha^\flat\Omega_{X/k}$ to be the free $L[\![t]\!]$-module $\alpha^*\Omega_{X/k}/(\alpha^*\Omega_{X/k})_{\mathrm{tors}}$.

PROPOSITION 5.3.4. *Let $m \ge n$ be non-negative integers such that $m \le 2n + 1 - e$ (e.g. $m = n+1$ such that $n \ge e$).*

(1) Let L be a field and let $\alpha \in (\mathrm{J}_\infty^{(\le e)} X)(L)$. Let F and $F^\flat$ be the fibers of

$$\mathrm{J}_m X \to \mathrm{J}_n X \text{ and}$$
$$\pi_m(\mathrm{J}_\infty X) \to \pi_n(\mathrm{J}_\infty X)$$

over $\pi_n(\alpha)$ respectively. Then we have

$$F^\flat(L) = \mathrm{Hom}_{L[\![t]\!]}(\alpha^\flat\Omega_{X/k}, \mathfrak{t}^{n+1}_{m+1,L})$$

under the identification given in Proposition 5.2.5 and Remark 5.2.6. Moreover, $F^\flat$ is a $d(m-n)$-dimensional L-linear subspace (in particular, a closed subscheme) of $F = \mathbb{A}^s_L$.

(2) The map

$$\pi_m(\mathrm{J}_\infty^{(\le e)} X) \to \pi_n(\mathrm{J}_\infty^{(\le e)} X)$$

is a piecewise trivial $\mathbb{A}^{d(m-n)}$-bundle.

PROOF. The problem is local in X. We may assume that X is an affine scheme so that we can apply results of Section 5.2.

(1) We first consider the case where X is a complete intersection $V(f_1, \dots, f_l) \subset \mathbb{A}_k^n$ with $l = n - d$. If $\beta \in F(L)$ lifts to an arc, in particular, lifts to a $(2n+1)$-jet, then the corresponding map

$$\delta \colon \alpha^* \Omega_{X/k} \to \mathfrak{t}_{m+1,L}^{n+1}$$

is induced from a map

$$\widetilde{\delta} \colon \alpha^* \Omega_{X/k} \to \mathfrak{t}_{2n+2,L}^{n+1}.$$

Since the torsion part of $\alpha^* \Omega_{X/k}$ is killed by t^e and since $(2n+2) - (m+1) \geq e$, the map δ kills the torsion and factors through $\alpha^\flat \Omega_{X/k}$.

Conversely, if a map $\delta \colon \alpha^* \Omega_{X/k} \to \mathfrak{t}_{m+1,L}^{n+1}$ kills the torsion, then it lifts to a map $\widetilde{\delta} \colon \alpha^* \Omega_{X/k} \to \mathfrak{t}_{2n+2,L}^{n+1}$, which means that the m-jet $\beta \in F(L)$ corresponding to δ lifts to a $(2n+1)$-jet $\widetilde{\beta}$. Since $n \geq e$, from the geometric Hensel lemma (Corollary 3.7.3), β lifts to an arc. As a consequence, the subset

$$\mathrm{Hom}_{L[\![t]\!]}(\alpha^\flat \Omega_{X/k}, \mathfrak{t}_{m+1,L}^{n+1})$$

corresponds to $F^\flat(L)$, the set of jets liftable to arcs. Since $\alpha^\flat \Omega_{X/k}$ is a free $L[\![t]\!]$-module of rank d, $F^\flat(L)$ is a $d(m-n)$-dimensional L-linear subspace of $F(L)$.

For an algebraic closure $\overline{L}$ of L, the same argument shows that $F^\flat(\overline{L}) \subset F(\overline{L})$ is a $d(m-n)$-dimensional $\overline{L}$-linear subspace. It follows that $F^\flat \subset F$ is a $d(m-n)$-dimensional L-linear subspace, in particular, isomorphic to $\mathbb{A}_L^{d(m-n)}$. We have proved assertion (1) when X is a complete intersection.

Next we consider the case where X is a general affine variety $V(f_1, \dots, f_m) \subset \mathbb{A}_k^n$. Since the Jacobian ideal Jac_X is generated by minors of the Jacobian matrix $(\partial f_i / \partial x_j)$ of size $l = n - d$, reordering $f_1, \dots, f_m$ and $x_1, \dots, x_n$ if necessary, we may suppose that

$$(\alpha^* \det(\partial f_i / \partial x_j)_{1 \leq i,j \leq l}) = \mathfrak{t}_L^{e'} \quad (e' := \mathrm{j}_X(\alpha) \leq e).$$

Let $Y = V(f_1, \dots, f_l)$. This contains X and is a local complete intersection of dimension d around the image of α such that $\mathrm{j}_Y(\alpha) = \mathrm{j}_X(\alpha) = e'$. Moreover the natural map $\alpha^* \Omega_{Y/k} \to \alpha^* \Omega_{X/k}$ is an isomorphism, because it is surjective and the two modules have the same rank and the torsion parts of the same length. Thus we get an isomorphism $\alpha^\flat \Omega_{Y/k} \to \alpha^\flat \Omega_{X/k}$ and can identify $\mathrm{Hom}_{L[\![t]\!]}(\alpha^\flat \Omega_{X/k}, \mathfrak{t}_{m+1,L}^{n+1})$ with $\mathrm{Hom}_{L[\![t]\!]}(\alpha^\flat \Omega_{Y/k}, \mathfrak{t}_{m+1,L}^{n+1})$. It suffices to show the claim that the fibers of $\pi_m(\mathrm{J}_\infty Y) \to \pi_n(\mathrm{J}_\infty Y)$ and $\pi_m(\mathrm{J}_\infty X) \to \pi_n(\mathrm{J}_\infty X)$ over $\pi_n(\alpha)$ are identical. In turn, it suffices to show that the fiber H of $\mathrm{J}_\infty Y \to \mathrm{J}_n Y$ over $\pi_n(\alpha)$ is contained in $\mathrm{J}_\infty X$. From the Hensel lemma and assertion (1) for complete intersections, for $n' \geq n$, every fiber of $\pi_{n'+1}(\mathrm{J}_n^{(\leq e)} Y) \to \pi_{n'}(\mathrm{J}_n^{(\leq e)} Y)$ is isomorphic to an affine space of dimension d. It follows that the fiber H of $\mathrm{J}_\infty Y \to \mathrm{J}_n Y$ over $\pi_n(\alpha)$ is irreducible. Let us write $Y = X \cup W$ with W a k-scheme of the same pure dimension. Then $\mathrm{J}_\infty Y = \mathrm{J}_\infty X \cup \mathrm{J}_\infty W$ and H is contained in either $\mathrm{J}_\infty X$ or $\mathrm{J}_\infty W$. Since $\mathrm{j}_Y(\alpha) < \infty$, α does not factor through the singular locus of Y. In particular, α does not factor through $X \cap W$. Therefore α is not an arc of W. Thus W cannot be contained in $\mathrm{J}_\infty W$. We conclude that H is contained in $\mathrm{J}_\infty X$. We have completed the proof of assertion (1).

(2) From assertion (1) and Lemma 2.6.3, there exists a stratification $\pi_n(\mathrm{J}_\infty^{(\leq e)} X) = \bigsqcup C_i$ into finitely many locally closed subsets such that

$$D_i := (\pi_n^m)^{-1}(C_i) \cap \pi_m(\mathrm{J}_\infty X)$$

are locally closed in $\mathrm{J}_m X$. From Lemma 4.4.3 (spreading out), it suffices to show that for every point $c \in C_i(L)$ with L a field, the fiber $F^\flat$ of $D_i \to C_i$ over c is isomorphic to $\mathbb{A}_L^{d(m-n)}$ over L. To show this, we first claim that there exists a finite Galois extension M/L such that the induced M-point $c_M \in C_i(M)$ lifts to an arc α. Let us take a finite Galois extension M/L such that c_M lifts to an $an+b$-jet for a and b as in Proposition 3.7.4, Greenberg lifting theorem. Then c_M also lifts to an arc, thus the above claim holds. Then, from assertion (1), the fiber $F_M^\flat$ over c_M is isomorphic to $\mathbb{A}_M^{d(m-n)}$. We see that $F_M^\flat$ has a natural structure of a $\mathrm{Gal}(M/L)$-equivariant vector bundle over $\operatorname{Spec} M$ because the Galois action on $F_M^\flat$ is induced from the one on the $M[\![t]\!]$-module $\alpha^\flat \Omega_{X/k}$. By the Galois descent (see Corollary A.2.5), we see that the fiber $F^\flat$ is a vector bundle over $\operatorname{Spec} L$, that is, $\mathbb{A}_L^{d(m-n)}$. □

5.4. Boundedness of fiber dimensions

LEMMA 5.4.1. *Every fiber of $\pi_{n+1}(\mathrm{J}_\infty X) \to \pi_n(\mathrm{J}_\infty X)$ has dimension $\le d$.*

PROOF. By the standard base change argument, we may assume that k is algebraically closed and it is enough to show that the fiber over every k-point of $\pi_n(\mathrm{J}_\infty X)$ has dimension $\le d$. We may also assume X is affine, say $V(f_1, \dots, f_l) \subset \mathbb{A}_k^n$. Let $\alpha \in (\mathrm{J}_\infty X)(k)$ be an arc given by a tuple $\underline{\alpha} \in k[\![t]\!]^n$. The fiber of $\pi_{n+1}(\mathrm{J}_\infty X)(k) \to \pi_n(\mathrm{J}_\infty X)(k)$ over $\pi_n(\alpha)$ is identified with

$$F := \{(\underline{\beta} \bmod t) \in k^n \mid \underline{\beta} \in k[\![t]\!]^n \text{ and } \underline{f}(\underline{\alpha} + \underline{\beta} t^{n+1}) = \underline{0}\}.$$

Let

$$\mathcal{X} := \operatorname{Spec} k[\![t]\!][\underline{x}]/(f_1(\underline{\alpha} + \underline{x} t^{n+1}), \dots, f_l(\underline{\alpha} + \underline{x} t^{n+1}))$$

and let $0, \eta \in \operatorname{Spec} k[\![t]\!]$ be the closed and generic points respectively. The generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ is isomorphic to $X \otimes_k k(\!(t)\!)$, in particular, has dimension d. Indeed, $\mathcal{X}_\eta \subset \mathbb{A}^n_{k(\!(t)\!)}$ is sent to $X \otimes_k k(\!(t)\!)$ by the coordinate change $\underline{x} \mapsto \underline{\alpha} + \underline{x} t^{n+1}$. Let $\mathcal{X}'$ be the Zariski closure of $\mathcal{X}_\eta$ in $\mathcal{X}$, which is a flat $k[\![t]\!]$-scheme of relative dimension d. The above set F consists of the points $s(0)$ with s ranging over all sections $\operatorname{Spec} k[\![t]\!] \to \mathcal{X}$ and thus is a subset of $\mathcal{X}'_0(k)$ with $\mathcal{X}'_0$ the special fiber of $\mathcal{X}'$. Since $\mathcal{X}'_0$ has dimension $\le d$, so does F. □

The following is a direct consequence of this lemma.

COROLLARY 5.4.2. *We have*

$$\dim \pi_n(\mathrm{J}_\infty X) \le (n+1)d.$$

5.5. Ordinary cylinders

DEFINITION 5.5.1. We say that a geometric arc $\alpha\colon \mathrm{D}_L \to X$ is *ordinary* if the generic point of D_L maps into the smooth locus X_{sm}, equivalently, if the corresponding L-point of $\mathrm{J}_\infty X$ lies on $\mathrm{J}_\infty^{(<\infty)} X = \mathrm{J}_\infty X \setminus \mathrm{J}_\infty(X_{\mathrm{sing}})$. We say that a cylinder $C \subset \mathrm{J}_\infty X$ is *ordinary* if every point of C corresponds to an ordinary arc, that is, C is included in $\mathrm{J}_\infty^{(<\infty)} X$.

LEMMA 5.5.2. *Every ordinary cylinder $C \subset \mathrm{J}_\infty X$ is included in $\mathrm{J}_\infty^{(\le e)} X$ for $e \gg 0$.*

PROOF. This follows from Lemma 3.8.3. □

LEMMA 5.5.3. *Let $C \subset \mathrm{J}_\infty X$ be an ordinary cylinder. Then, for $n \gg 0$, the map $\pi_{n+1}(C) \to \pi_n(C)$ is a piecewise trivial $\mathbb{A}^d$-bundle.*

PROOF. This follows from the last lemma and Proposition 5.3.4. $\square$

DEFINITION 5.5.4. Let $C \subset \mathrm{J}_\infty X$ be an ordinary cylinder. We define the *measure* of C to be

$$\mu_X(C) := \{\pi_n(C)\}\mathbb{L}^{-nd} \quad (n \gg 0).$$

This is well-defined thanks to Lemma 5.5.3.

5.6. Negligible subsets

DEFINITION 5.6.1. A subset $N \subset \mathrm{J}_\infty X$ is said to be *negligible* if it is included in the arc space $\mathrm{J}_\infty Z$ of some closed subscheme $Z \subset X$ of positive codimension.

LEMMA 5.6.2. *Let $e \in \mathbb{Z}_{\geq 0}$ and $N \subset \mathrm{J}_\infty^{(\leq e)} X$ be a negligible subset. For every $m \in \mathbb{Z}$, there exists an ordinary cylinder C such that* $\dim \mu_X(C) \leq m$ *and $N \subset C$.*

PROOF. For $n \geq e$, let

$$C_n := \pi_n^{-1}(\pi_n(N)),$$

which are ordinary cylinders with $N \subset C_n$. From 5.4.2, we have

$$\dim\{\pi_n(N)\} \leq (n+1)(d-1)$$

and

$$\begin{aligned} \dim \mu_X(C_n) &= \dim\{\pi_n(N)\} - nd \\ &\leq (n+1)(d-1) - nd \\ &= -n + d - 1. \end{aligned}$$

For $n \gg 0$, C_n becomes a desired ordinary cylinder. $\square$

LEMMA 5.6.3. *Let I be a set which is at most countable. Let S and T_i, $i \in I$ be ordinary cylinders of $\mathrm{J}_\infty X$ and let N be a negligible subset of $\mathrm{J}_\infty X$. Suppose that $S = \bigsqcup_{i\in I} T_i \sqcup N$. Then the sum $\sum_{i\in I} \mu_X(T_i)$ converges to $\mu_X(S)$.*

PROOF. Let $e \in \mathbb{Z}_{\geq 0}$ be such that $S \subset \mathrm{J}_\infty^{(\leq e)} X$. Let m be any integer. From Lemma 5.6.2, there exists an ordinary cylinder C such that $\dim \mu_X(C) \leq m$ and $N \subset C$. Then

$$S \setminus C = \bigsqcup_{i\in I} (T_i \setminus C).$$

From Lemma 3.8.3, this is a finite disjoint union. It shows that $T_i \subset C$ and $\dim \mu_X(T_i) \leq m$ for all but finitely many i. We conclude that the sum $\sum_{i\in I} \mu_X(T_i)$ converges and

$$\mu_X(S) \equiv \sum_{i\in I} \mu_X(T_i) \pmod{\widehat{F}_{-m}}.$$

Since the equality holds for any m, we have $\mu_X(S) = \sum_{i\in I} \mu_X(T_i)$. $\square$

5.7. Admissible functions and motivic integrals

Definition 5.7.1. Let $A \subset \mathrm{J}_\infty X$ be a subset, let $r \in \mathbb{Z}_{>0}$ and let $f\colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$. We say that f is *admissible* if there exists a stratification $A = \bigsqcup_{i \in I} A_i \sqcup N$ into countably many ordinary cylinders A_i, $i \in I$ and a negligible subset N such that for every i, the restriction $f|_{A_i}$ is constant with value different from ∞. (Thus the value ∞ is taken only on N.) For such f, we define the *integral of* $\mathbb{L}^f$ to be

$$\int_A \mathbb{L}^f \, d\mu_X := \sum_{i \in I} \mu_X(A_i)\mathbb{L}^{f(A_i)} \in \widehat{\mathcal{M}}_{k,r} \cup \{\infty\}.$$

It is clear that a $\mathbb{Q}$-linear combination of finitely many admissible functions is again admissible.

Lemma 5.7.2. *The integral of $\mathbb{L}^f$ is independent of the stratification $A = \bigsqcup_{i \in I} A_i \sqcup N$.*

Proof. Let $A = \bigsqcup_{j \in J} B_j \sqcup O$ be another such stratification. For each $i \in I$, we have

$$A_i = \bigsqcup_{j \in J} (A_i \cap B_j) \sqcup (A_i \cap O).$$

From Lemma 5.6.3, we have

$$\mu_X(A_i) = \sum_{j \in J} \mu_X(A_i \cap B_j).$$

From Lemma 2.4.8 (or its obvious generalization to $\widehat{\mathcal{M}}_{k,r}$), we have

$$\sum_{i \in I} \mu_X(A_i)\mathbb{L}^{f(A_i)} = \sum_{(i,j) \in I \times J} \mu_X(A_i \cap B_j)\mathbb{L}^{f(A_i \cap B_j)}.$$

Similarly

$$\sum_{j \in J} \mu_X(B_j)\mathbb{L}^{f(B_j)} = \sum_{(i,j) \in I \times J} \mu_X(A_i \cap B_j)\mathbb{L}^{f(A_i \cap B_j)}.$$

□

Lemma 5.7.3. *Let $Z \subset X$ be a closed subscheme of positive codimension with the defining ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The order function $\mathrm{ord}_{\mathcal{I}} = \mathrm{ord}_Z$ as well as its restriction to any cylinder is admissible.*

Proof. Let $C \subset \mathrm{J}_\infty X$ be a cylinder. For $e, n \in \mathbb{Z}_{\geq 0}$, we define ordinary cylinders $C_{e,n} := C \cap (\mathrm{ord}_Z)^{-1}(n) \cap \mathfrak{j}_X^{-1}(e)$ and a negligible subset $N := C \cap ((\mathrm{ord}_Z)^{-1}(\infty) \cup \mathfrak{j}_X^{-1}(\infty))$. We have

$$C = \bigsqcup_{e,n} C_{e,n} \sqcup N.$$

Since ord_Z take the constant value n on each $C_{e,n}$, the function $\mathrm{ord}_Z|_C$ is admissible. □

5.8. Jacobian orders for morphisms

Let Y be another geometrically reduced k-schemes of finite type which have pure dimension d and let $f\colon Y \to X$ be a generically étale morphism. Throughout the rest of the present chapter, we will keep this setting.

DEFINITION 5.8.1. We say that an arc $\alpha\colon \mathrm{D}_L \to Y$ is *f-ordinary* if Y is smooth at $\alpha(\eta)$ and f is étale at $\alpha(\eta)$, where η is the generic point of D_L. A subset $C \subset \mathrm{J}_\infty Y$ is said to be *f-ordinary* if every point of C is f-ordinary.

Clearly, being f-ordinary implies being ordinary. Note also that for an f-ordinary arc α, X is smooth at $f \circ \alpha(\eta)$. If $Z \subset Y$ is the union of the non-smooth locus of Y and the non-étale locus of f, then α is f-ordinary if and only if $\alpha \notin \mathrm{J}_\infty Z$.

For an f-ordinary arc α, the flat pullbacks $\alpha^\flat\Omega^d_{Y/k}$ and $(f\circ\alpha)^\flat\Omega^d_{X/k}$ (see Definition 5.3.3) are free $L[\![t]\!]$-modules of rank 1 and the map

$$(f\circ\alpha)^\flat\Omega^d_{X/k} \to \alpha^\flat\Omega^d_{Y/k}$$

is injective. We regard $(f\circ\alpha)^\flat\Omega^d_{X/k}$ as a submodule of $\alpha^\flat\Omega^d_{Y/k}$ by this injection.

DEFINITION 5.8.2. We define the *Jacobian order of f at an f-ordinary arc α* to be

$$\mathrm{j}_f(\alpha) := \dim_L \frac{\alpha^\flat\Omega^d_{Y/k}}{(f\circ\alpha)^\flat\Omega^d_{X/k}}.$$

Putting $\mathrm{j}_f(\alpha) := \infty$ for ones not f-ordinary, we get the *Jacobian order function of f*,

$$\mathrm{j}_f\colon\ \mathrm{J}_\infty Y \to \mathbb{Z}_{\ge 0} \cup \{\infty\}.$$

DEFINITION 5.8.3. Suppose that Y is smooth. Then we define the *Jacobian ideal sheaf* $\mathrm{Jac}_f \subset \mathcal{O}_X$ *of f* to be the 0-th Fitting ideal sheaf of $\Omega_{Y/X}$.

LEMMA 5.8.4. *If Y is smooth, then $\mathrm{j}_f = \mathrm{ord}_{\mathrm{Jac}_f}$.*

PROOF. If α is not f-ordinary, then $\mathrm{j}_f(\alpha) = \mathrm{ord}_{\mathrm{Jac}_f}(\alpha) = \infty$. Suppose that α is f-ordinary. We have the exact sequence of $L[\![t]\!]$-modules

$$(f\circ\alpha)^*\Omega_{X/k} \to \alpha^*\Omega_{Y/k} \to \alpha^*\Omega_{Y/X} \to 0.$$

Since $\alpha^*\Omega_{Y/k}$ is free, the left map kills the torsion part of $(f\circ\alpha)^*\Omega_{X/k}$ and we get the exact sequence

$$(f\circ\alpha)^\flat\Omega_{X/k} \to \alpha^*\Omega_{Y/k} \to \alpha^*\Omega_{Y/X} \to 0.$$

Choosing bases of $(f\circ\alpha)^\flat\Omega_{X/k}$ and $\alpha^*\Omega_{Y/k}$, the map $(f\circ\alpha)^\flat\Omega_{X/k} \to \alpha^*\Omega_{Y/k}$ is given by a square matrix $A \in M_d(L[\![t]\!])$. We see that both $\mathrm{j}_f(\alpha)$ and $\mathrm{ord}_{\mathrm{Jac}_f}(\alpha)$ are the order of $\det(A) \in L[\![t]\!]$. $\square$

5.9. Fiber inclusion lemma

LEMMA 5.9.1. *Let $n \in \mathbb{Z}_{\ge 0}$ and let $\beta \in (\mathrm{J}_\infty Y)(L)$ and $\alpha \in (\mathrm{J}_\infty X)(L)$ be such that $f_n\pi_n(\beta) = \pi_n(\alpha)$. Let $e := \mathrm{j}_f(\beta)$, $e_X := \mathrm{j}_X(f_\infty(\beta))$ and $e_Y := \mathrm{j}_Y(\beta)$. Suppose that $n \ge \max\{2e + e_Y, e_X\}$. Then there exists $\gamma \in (\mathrm{J}_\infty Y)(L)$ such that $\pi_{n-e}(\gamma) = \pi_{n-e}(\beta)$ and $f_\infty(\gamma) = \alpha$.*

PROOF. From $n \geq e_X$ and Proposition 5.3.4, we can identify the fiber $F^\flat(L)$ of the map

$$(\pi_{n+1}(\mathrm{J}_\infty X))(L) \to (\pi_n(\mathrm{J}_\infty X))(L)$$

over $\pi_n f_\infty(\alpha)$ with

$$\mathrm{Hom}_{L[\![t]\!]}((f \circ \beta)^\flat \Omega_{X/k}, \mathfrak{t}_{n+2,L}^{n+1}).$$

By assumption, the $(n+1)$-jet $\pi_{n+1}(\alpha)$ belongs to $F^\flat(L)$ and corresponds to the map

$$\delta_{f_\infty(\beta)}(\alpha)_{n+1} \colon (f \circ \beta)^\flat \Omega_{X/k} \to \mathfrak{t}_{n+2,L}^{n+1},$$

according to the notation of Definition 5.2.3. For some bases, the map

$$(f \circ \beta)^\flat \Omega_{X/k} \to \beta^\flat \Omega_{Y/k}$$

is represented by a diagonal matrix $\mathrm{diag}(t^{e_1}, \dots, t^{e_d})$ with $\sum e_i = e$, in particular, $e_i \leq e$ for every i. Therefore the map $\delta_{f_\infty(\beta)}(\alpha)_{n+1}$ is induced from a map

$$\widetilde{\delta} \colon \beta^\flat \Omega_{Y/k} \to \mathfrak{t}_{n+2,L}^{n-e+1}.$$

The assumption $n \geq 2e + e_Y$ is equivalent to

$$n + 1 \leq 2(n-e) + 1 - e_Y.$$

Again from Proposition 5.3.4, the map $\widetilde{\delta}$ is of the form $\delta_\beta(\gamma^{(1)})_{n+1}$ for some $\gamma^{(1)} \in (\mathrm{J}_\infty Y)(L)$. By construction, we have

$$\pi_{n-e}(\gamma^{(1)}) = \pi_{n-e}(\beta) \text{ and } \pi_{n+1} f_\infty(\gamma^{(1)}) = \pi_{n+1}(\alpha).$$

Note that $\mathrm{j}_f(\gamma^{(1)}) = \mathrm{j}_f(\beta)$ and $\mathrm{j}_X(f_\infty(\gamma^{(1)})) = \mathrm{j}_X(f_\infty(\beta))$. Applying the same argument to $\gamma^{(1)}, \alpha, n+1$ in place of β, α, n, we see that there exists some $\gamma^{(2)} \in (\mathrm{J}_\infty Y)(L)$, which satisfies

$$\pi_{n+1-e}(\gamma^{(2)}) = \pi_{n+1-e}(\gamma^{(1)}) \text{ and } \pi_{n+2} f_\infty(\gamma^{(2)}) = \pi_{n+2}(\alpha).$$

Repeating this, we get a sequence $\gamma^{(i)} \in (\mathrm{J}_\infty Y)(L)$, $i \in \mathbb{Z}_{>0}$ such that

$$\pi_{n+i-1-e}(\gamma^{(i)}) = \pi_{n+i-1-e}(\gamma^{(i-1)}) \text{ and } \pi_{n+i} f_\infty(\gamma^{(i)}) = \pi_{n+i}(\alpha).$$

The limit $\gamma \in (\mathrm{J}_\infty Y)(L)$ of $\gamma^{(i)}$'s has the desired property. □

DEFINITION 5.9.2. Let $C \subset \mathrm{J}_\infty Y$ be a subset. We say that $f_\infty|_C$ is *geometrically injective* if for every field L, the induced map $C(L) \to f_\infty(C(L))$ is injective, equivalently, if for every *algebraically closed* field L, the induced map $C(L) \to f_\infty(C(L))$ is injective.

LEMMA 5.9.3 (Fiber inclusion lemma; the singular case). *Let L be a field, let $\beta, \beta' \in (\mathrm{J}_\infty Y)(L)$. Let $n \in \mathbb{Z}_{\geq 0}$ and suppose that $f_n\pi_n(\beta) = f_n\pi_n(\beta')$. Let $e := \mathrm{j}_f(\beta)$, $e_X := \mathrm{j}_X(f_\infty(\beta))$ and $e_Y := \mathrm{j}_Y(\beta)$. Suppose that $n \geq \max\{2e + e_Y, e_X\}$. Suppose also that $\beta, \beta' \in C(L)$ for a cylinder C of level $n - e$ and that $f_\infty|_C$ is geometrically injective. Then $\pi_{n-e}(\beta) = \pi_{n-e}(\beta')$.*

PROOF. For $\alpha := f_\infty(\beta')$, let $\gamma \in (\mathrm{J}_\infty Y)(L)$ as in Lemma 5.9.1. Since C is a cylinder of level $n - e$ and $\pi_{n-e}(\beta) = \pi_{n-e}(\gamma)$, we see that $\gamma \in C(L)$. From the geometric injectivity, the equality $f_\infty(\gamma) = f_\infty(\beta')$ implies $\gamma = \beta'$, which shows the lemma. □

5.10. Preservation of cylinders under f_∞

LEMMA 5.10.1. *Let $C \subset \mathrm{J}_\infty Y$ be a cylinder of level $n-e$. Suppose that the inequalities $\mathfrak{j}_f \le e$, $\mathfrak{j}_X \circ f_\infty \le e_X$ and $\mathfrak{j}_Y \le e_Y$ hold on C and that $n \ge \max\{2e+e_Y, e_X\}$. Then $f_\infty(C)$ is a cylinder of level n.*

PROOF. We first note that $\pi_n(f_\infty(C)) = f_n(\pi_n(C))$ is a constructible subset of $\mathrm{J}_n X$. Let $\beta \in C(L)$ be a geometric point and let $\alpha \in (\mathrm{J}_\infty X)(L)$ be such that $f_n\pi_n(\beta) = \pi_n(\alpha)$. From Lemma 5.9.1, there exists $\gamma \in (\mathrm{J}_\infty Y)(L)$ such that $\pi_{n-e}(\gamma) = \pi_{n-e}(\beta)$ and $f_\infty(\gamma) = \alpha$. Since C is a cylinder of $n-e$, we have $\gamma \in C(L)$. Therefore $f_\infty(\gamma) = \alpha \in (f_\infty(C))(L)$. This shows that $f_\infty(C)$ is a cylinder of level n. $\square$

COROLLARY 5.10.2. *Let $C \subset \mathrm{J}_\infty Y$ be an f-ordinary cylinder. Then $f_\infty(C)$ is an ordinary cylinder.*

PROOF. From Lemma 3.8.3, the functions $\mathfrak{j}_f$ and $\mathfrak{j}_X \circ f_\infty$ are bounded on C. The corollary follows from Lemma 5.10.1. $\square$

5.11. The $\mathbb{A}^e$-fibration lemma

LEMMA 5.11.1 (The $\mathbb{A}^e$-fibration lemma; the singular case). *Let $n, e, e_X, e_Y \in \mathbb{Z}_{\ge 0}$ be such that $n \ge \max\{2e+e_Y, e_X\}$. Let $C \subset \mathrm{J}_\infty Y$ be a cylinder of level $n-e$ such that $\mathfrak{j}_f|_C \le e$, $(\mathfrak{j}_X \circ f_\infty)|_C \le e_X$ and $\mathfrak{j}_Y|_C \le e_Y$. Suppose that $f_\infty|_C$ is geometrically injective. Then*

$$f_n|_{\pi_n(C)} \colon \pi_n(C) \to f_n(\pi_n(C))$$

is a piecewise trivial $\mathbb{A}^e$-bundle. In particular,

$$\mu_Y(C) = \mu_X(f_\infty(C))\mathbb{L}^e.$$

PROOF. From Lemma 5.10.1, $f_\infty(C)$ is a cylinder of level n. Consider an L-point

$$\alpha_n \in (f_n(\pi_n(C)))(L).$$

For an extension M/L, let $\alpha_{n,M}$ denotes the induced M-point. For a suitable finite Galois extension M/L, there exists $\beta \in C(M)$ such that $f_n(\pi_n(\beta)) = \alpha_{n,M}$. Indeed we can lift α_n to a point of $\pi_n(C)$ after a finite extension of L. In turn, from Proposition 5.3.4, this point lifts to C without further extension of the field. By enlarging M if necessary, we may take M/L to be Galois.

Let H_M be the fiber of $\pi_n(C) \to f_n(\pi_n(C))$ over $\alpha_{n,M} = f_n(\pi_n(\beta))$ and let $F_M^\flat$ be the fiber of $\pi_n(C) \to \pi_{n-e}(C)$ over $\pi_{n-e}(\beta)$. From the fiber inclusion lemma, we have $H_M \subset F_M^\flat$. Since $n \le 2(n-e)+1-e_Y$, we can apply Proposition 5.3.4 so that for any extension N/M, we may identify

$$F_M^\flat(N) = \mathrm{Hom}_{M[\![t]\!]}(\beta^\flat\Omega_{Y/k}, \mathfrak{t}_{n+1,N}^{n-e+1}).$$

We see that $H_M(N)$ is then identified with the kernel of

$$\mathrm{Hom}_{M[\![t]\!]}(\beta^\flat\Omega_{Y/k}, \mathfrak{t}_{n+1,N}^{n-e+1}) \to \mathrm{Hom}_{M[\![t]\!]}((f\circ\beta)^\flat\Omega_{X/k}, \mathfrak{t}_{n+1,N}^{n-e+1}).$$

For some bases, the map

$$M[\![t]\!]^{\oplus d} = (f\circ\beta)^\flat\Omega_{X/k} \to \beta^\flat\Omega_{Y/k} = M[\![t]\!]^{\oplus d}$$

is represented by a diagonal matrix $\mathrm{diag}(t^{e_1}, \dots, t^{e_d})$ with $\sum e_i = e$. Therefore the above map of Hom modules is identified with the map

$$\left(\mathfrak{t}_{n+1,N}^{n-e+1}\right)^{\oplus d} \to \left(\mathfrak{t}_{n+1,N}^{n-e+1}\right)^{\oplus d}$$

represented by the same matrix. It has the kernel

$$\bigoplus_i \mathfrak{t}_{n+1,N}^{n-e_i+1} \cong N^{\oplus e}.$$

This shows that H_M is an e-dimensional M-linear subspace of $F_M^\flat$ and a closed subset of $(\mathrm{J}_n Y) \otimes_k M$. Moreover, H_M is a $\mathrm{Gal}(M/L)$-equivariant vector subbundle of $F_M^\flat$ over $\operatorname{Spec} M$. By the Galois descent (Corollary A.2.5), the fiber H_L of $\pi_n(C) \to f_n(\pi_n(C))$ over α_n is isomorphic to $\mathbb{A}_L^e$. From Lemma 2.6.3 and spreading out (Lemma 4.4.3), the map $\pi_n(C) \to f_n(\pi_n(C))$ is a piecewise trivial $\mathbb{A}^e$-bundle. This shows

$$\begin{aligned} \mu_Y(C) &= \{\pi_n(C)\}\mathbb{L}^{-nd} \\ &= \{f_n(\pi_n(C))\}\mathbb{L}^{-nd+e} \\ &= \mu_X(f_\infty(C))\mathbb{L}^e. \end{aligned}$$

□

5.12. Preservation of admissible functions

PROPOSITION 5.12.1. (1) *Let $A \subset \mathrm{J}_\infty X$ be a subset and let $h\colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be an admissible function. Then $h \circ f_\infty\colon f_\infty^{-1}(A) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is admissible.*

(2) *Let $B \subset \mathrm{J}_\infty Y$ be a subset such that $f_\infty|_B$ is geometrically injective. Let $h\colon B \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be an admissible function. Then the function induced from h,*

$$h'\colon f_\infty(B) \xrightarrow{(f_\infty)^{-1}} B \xrightarrow{h} \frac{1}{r}\mathbb{Z} \cup \{\infty\},$$

is admissible.

PROOF. (1) Let $Z \subset Y$ be the non-étale locus of f and let W be the Zariski closure of $f(Z) \cup X_{\mathrm{sing}}$. There exists a stratification $A = \bigsqcup_i A_i \sqcup N$ into countably many ordinary cylinders A_i and a negligible subset N such that each restriction $h|_{A_i}$ is constant. Replacing N with $N \cup (f_\infty(B) \cap \mathrm{ord}_W^{-1}(\infty))$ and A_i with countably many cylinders $A_i \cap \mathrm{ord}_W^{-1}(j)$, we may take A_i to be disjoint from $\mathrm{ord}_W^{-1}(\infty)$. Then $B_i := f_\infty^{-1}(A_i)$ are ordinary cylinders and $f_\infty^{-1}(N)$ is negligible. Therefore $h \circ f_\infty$ is admissible.

(2) Let $B = \bigsqcup B_i \sqcup N$ be a stratification as in the definition of admissible functions. By a similar argument as above, we may suppose that B_i are f-ordinary. From Corollary 5.10.2, $f_\infty(B_i)$ are ordinary cylinders. It is easy to see that $f_\infty(N)$ is negligible. We conclude that $h'\colon f_\infty(B) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ is admissible. □

PROPOSITION 5.12.2. *The Jacobian order function of f, $\mathfrak{j}_f\colon \mathrm{J}_\infty Y \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, is admissible. The same is true for the restriction $\mathfrak{j}_f|_C$ to any cylinder C.*

PROOF. There exists a proper birational morphism $g\colon Z \to Y$ such that the torsion-free pullbacks $g^\flat \Omega_{Y/k}^d := g^* \Omega_{Y/k}^d / (g^* \Omega_{Y/k}^d)_{\mathrm{tors}}$ and $(f \circ g)^\flat \Omega_{X/k}^d := (f \circ$

$g)^*\Omega^d_{X/k}/((f\circ g)^*\Omega^d_{X/k})_{\mathrm{tors}}$ are locally free (for instances, see [**OZ91**, **VU06**]). Let $\mathcal{I}\subset\mathcal{O}_Z$ be the ideal sheaf given by

$$(f\circ g)^\flat\Omega^d_{X/k}=\mathcal{I}\cdot g^\flat\Omega^d_{Y/k}.$$

Then the equality $\mathfrak{j}_f\circ g_\infty=\mathrm{ord}_{\mathcal{I}}$ holds for g-ordinary arcs γ such that $g_\infty(\gamma)$ is f-ordinary. In particular, the equality holds outside a negligible subset. Since $\mathrm{ord}_{\mathcal{I}}$ is admissible by Lemma 5.7.3, so is $\mathfrak{j}_f\circ g_\infty$. It follows that $\mathfrak{j}_f\circ g_\infty|_{\mathrm{J}_\infty Y\setminus\mathfrak{j}_g^{-1}(\infty)}$ is also admissible. From Proposition 5.12.1, $\mathfrak{j}_f$ is admissible outside $g_\infty(\mathfrak{j}_g^{-1}(\infty))$. Since $g_\infty(\mathfrak{j}_g^{-1}(\infty))$ is negligible, the whole function $\mathfrak{j}_f$ is also admissible. The second assertion is now obvious. □

5.13. The change of variables formula

Definition 5.13.1. Let $C\subset\mathrm{J}_\infty Y$ be a subset. We say that $f_\infty|_C$ is *almost geometrically injective* if there exists a negligible subset N such that $f_\infty|_{C\setminus N}$ is geometrically injective (Definition 5.9.2).

Theorem 5.13.2 (The change of variables formula for singular varieties). *Let $B\subset\mathrm{J}_\infty Y$ be a subset such that $f_\infty|_B$ is almost geometrically injective. Let $h\colon f_\infty(B)\to\frac{1}{r}\mathbb{Z}\cup\{\infty\}$ be an admissible function. Then*

$$\int_{f_\infty(B)}\mathbb{L}^h\,d\mu_X=\int_B\mathbb{L}^{h\circ f_\infty-\mathfrak{j}_f}\,d\mu_Y.$$

Proof. Removing a negligible subset from B, we may assume that B is f-ordinary, $f_\infty|_B$ is geometrically injective and there exists a stratification $f_\infty(B)=\bigsqcup_i A_i$ into countably many ordinary cylinders such that $h|_{A_i}$ are constant. Then $B_i:=f_\infty^{-1}(A_i)$ are f-ordinary cylinders. If we put $B_{i,e}:=B_i\cap\mathfrak{j}_f^{-1}(e)$, $e\in\mathbb{Z}_{\ge0}$, we have $B_i=\bigsqcup_{e\ge0}B_{i,e}$ and $A_i=\bigsqcup_{e\ge0}f_\infty(B_{i,e})$. From Lemma 5.11.1, we have

$$\mu_X(f_\infty(B_{i,e}))=\mu_Y(B_{i,e})\mathbb{L}^{-e}.$$

Therefore

$$\begin{aligned}\int_{f_\infty(B)}\mathbb{L}^h\,d\mu_X&=\sum_i\mu_X(A_i)\mathbb{L}^{h(A_i)}\\&=\sum_{i,e}\mu_X(f_\infty(B_{i,e}))^{h(f_\infty(B_{i,j}))}\\&=\sum_{i,e}\mu_Y(B_{i,e})^{h(f_\infty(B_{i,e}))-\mathfrak{j}_f(B_{i,e})}\\&=\int_B\mathbb{L}^{h\circ f_\infty-\mathfrak{j}_f}\,d\mu_Y.\end{aligned}$$

□

5.14. Group actions

We now consider the situation where a finite group G acts on our k-variety X. For simplicity, we suppose that every G-orbit in X is contained in an affine open subset. This implies that the quotient X/G is a scheme. Then every G-orbit in $\mathrm{J}_n X$, $n\in\mathbb{Z}_{\ge0}\cup\{\infty\}$, is also contained in an affine open subset and the quotients $(\mathrm{J}_n X)/G$ exist as schemes. We define the motivic measure on the set $|\,\mathrm{J}_\infty X|/G=|(J_\infty X)/G|$.

CHAPTER 6

Stringy motives

In section 4.5, we saw that the motive $\mathrm{M}(X) = \{X\}$ of a smooth variety X is invariant under strong K-equivalences. To generalize this fact to varieties having mild singularities, we introduce in this chapter the notion of stringy motives, a variant of motives incorporating information of singularities.

We work over a perfect field k in this chapter except the last section 6.7. This assumption in particular implies that a k-variety is geometrically reduced and a normal k-variety is smooth in codimension one.

6.1. Singularities in the minimal model program

To develop the minimal model program in dimension ≥ 3, it was essential to allow varieties to have mild singularities. Among others, four important classes of singularities are terminal singularities, canonical singularities, klt singularities and log canonical singularities in the order of mildness. We briefly recall these notions. We refer the reader to [**Kol13**, Section 2.1] for details.

Let X be a normal k-variety of dimension d. We begin with fixing our terminology on divisors as follows:

Definition 6.1.1. A *divisor* (resp. $\mathbb{Q}$-divisor) on X means a Weil divisor, that is, a $\mathbb{Z}$-linear combination (resp. $\mathbb{Q}$-linear combination) of prime divisors. We identify a Cartier divisor on X with the (Weil) divisor associated to it in the obvious way. Thus Cartier divisors form a subclass of divisors. For a positive integer r, a divisor D on X is said to be r-*Cartier* if rD is Cartier. A divisor is said to be $\mathbb{Q}$-*Cartier* if it is r-Cartier for some $r > 0$. A $\mathbb{Q}$-divisor D is said to be $\mathbb{Q}$-*Cartier* if for some positive integer r, rD has coefficients in $\mathbb{Z}$ and is Cartier.

For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D and for a dominant morphism of normal varieties $f\colon Y \to X$, we can define the *pull-back f^*D to be the $\mathbb{Q}$-divisor $\frac{1}{r}f^*(rD)$ for a positive integer as above.*

Definition 6.1.2. The *canonical sheaf* of X, denoted by ω_X, is defined to be the double dual $(\Omega^d_{X/k})^{\vee\vee}$ of $\Omega^d_{X/k}$, that is, the unique reflexive sheaf $\mathcal{F}$ such that $\mathcal{F}|_{X_{\mathrm{sm}}} \cong \Omega^d_{X_{\mathrm{sm}}/k}$ (see Remark 6.1.3 for reflexive sheaves). A *canonical divisor* of X, denoted by K_X, is a Weil divisor D on X such that $\mathcal{O}_X(D) \cong \omega_X$. This is unique modulo linear equivalence and we may call this divisor *the* canonical divisor.

Remark 6.1.3 (Reflexive sheaves [**Har80**]). Let X be a normal variety. A coherent sheaf $\mathcal{F}$ on X is said to be *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ to the double dual is an isomorphism. This is equivalent to that there exists an open subset $U \subset X$ with the inclusion map $\iota\colon U \hookrightarrow X$ such that the complement U^c has codimension ≥ 2, $\mathcal{F}|_U$ is locally free and the natural map $\mathcal{F} \to \iota_*\mathcal{F}|_U$ is an isomorphism. The dual $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ of a coherent sheaf $\mathcal{F}$ is always reflexive and so is the double dual $\mathcal{F}^{\vee\vee} = (\mathcal{F}^{\vee})^{\vee}$. The double dual $\mathcal{F}^{\vee\vee}$ is also call the *reflexive hull* of $\mathcal{F}$. If $\mathcal{F}$ is locally free in codimension one, then $\mathcal{F}^{\vee\vee}$ is the unique reflexive sheaf which coincide with $\mathcal{F}$ in codimension one.

The coherent sheaf $\mathcal{O}_X(D)$ associated to a divisor D is defined as a subsheaf of the constant sheaf $K(X)$ in terms of orders of poles and zeroes along prime divisors appearing in D, just like in the case of a smooth variety. This sheaf $\mathcal{O}_X(D)$ is reflexive and sending D to $\mathcal{O}_X(D)$ gives a one-to-one correspondence of divisors and reflexive subsheaves of $K(X)$. A divisor D is Cartier if and only if $\mathcal{O}_X(D)$ is invertible. Moreover, two divisors are linearly equivalent if and only if the corresponding sheaves are isomorphic.

Definition 6.1.4. Let $r \in \mathbb{Z}_{>0}$. We say that X is *r-Gorenstein* if rK_X is a Cartier divisor, equivalently if the r-th reflexive power $\omega_X^{[r]} := (\omega_X^{\otimes r})^{\vee\vee}$ is invertible. We say that X is *$\mathbb{Q}$-Gorenstein* if it is r-Gorenstein for some r, that is, K_X is $\mathbb{Q}$-Cartier.

Let $f\colon Y \to X$ be a generically étale morphism of normal varieties. Suppose that X is r-Gorenstein. By the natural map, we may regard $f^*\omega_X^{[r]}$ as an invertible subsheaf of $\omega_Y^{[r]} \otimes K(Y)$.

Definition 6.1.5. With the above notation, we define the *relative canonical divisor* $K_{Y/X}$ to be the $\mathbb{Q}$-divisor on Y given by

$$K_{Y/X} = K_Y - f^*K_X.$$

In precise, this means that for $r > 0$ as above, we have $f^*\omega_X^{[r]} \otimes \mathcal{O}_Y(rK_{Y/X}) = \omega_Y^{[r]}$ as subsheaves of $\omega_Y^{[r]} \otimes K(Y)$.

Definition 6.1.6. For a prime divisor E on Y, the *discrepancy* $a(E;X) \in \mathbb{Q}$ of E over X is defined to be the multiplicity of E in $K_{Y/X}$ so that we can write $K_{Y/X} = \sum_E a(E;X)E$ with E running over prime divisors on Y. The *log discrepancy* $a_{\log}(E;X)$ is then defined to be $a(E;X) + 1$.

Definition 6.1.7. Let X be a k-variety. A *modification* of X is a proper birational morphism $Y \to X$ or a variety Y given with such a morphism. We say that a modification $Y \to X$ is *normal* if Y is normal. A *divisor over* a variety X means a prime divisor on some normal modification Y of X. The *center* of a divisor E over X, denoted by $\mathrm{center}_X(E)$, is the image of E on X, which is an irreducible closed subset of X. We say that a divisor E over X is *exceptional* if $\mathrm{center}_X(E)$ has codimension ≥ 2. Two divisors E, E' over X, say lying on normal modifications Y, Y' respectively, are said to be *equivalent* if they correspond to each other by the natural birational map between Y and Y'.

A divisor over a variety X gives a discrete valuation $K(X)^* \to \mathbb{Z}$ on the function field $K(X)$, associating to a rational function its vanishing order or minus the order of pole along the divisor. Two divisors over X are equivalent if and only if they give the same valuation. We usually identify equivalent divisors over the given variety. Two equivalent divisors over X share the discrepancy as well as the center.

Definition 6.1.8. Let X be a normal $\mathbb{Q}$-Gorenstein variety with the singular locus X_{sing} and let $W \subset X$ be a closed subset. We define the *minimal log discrepancy of X along W* to be

$$\mathrm{mld}(W;X) := \inf_{\mathrm{center}_X(E) \subset W} a_{\log}(E;X),$$

where E runs over divisors over X with $\mathrm{center}_X(E) \subset W$.

It is known that we have either $\mathrm{mld}(W; X) \geq 0$ or $\mathrm{mld}(W; X) = -\infty$ [**KM98**, Cor. 2.31].

DEFINITION 6.1.9. Let X be a normal $\mathbb{Q}$-Gorenstein variety. We say that X is

$$\begin{cases} \textit{terminal} & (\mathrm{mld}(X_{\mathrm{sing}}; X) > 1) \\ \textit{canonical} & (\mathrm{mld}(X_{\mathrm{sing}}; X) \geq 1) \\ \textit{klt} & (\mathrm{mld}(X_{\mathrm{sing}}; X) > 0) \\ \textit{log canonical} & (\mathrm{mld}(X_{\mathrm{sing}}; X) \geq 0). \end{cases}$$

We also say that X *has only terminal (resp. canonical, klt, log canonical) singularities.*

6.2. Log pairs

It is also customary in birational geometry to consider not only singular varieties but also pairs of singular varieties and $\mathbb{Q}$-divisors.

DEFINITION 6.2.1. A *log pair* X means the pair (X, D) of a normal k-variety and a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. The divisor D is called the *boundary* or *boundary divisor* of the log pair.

REMARK 6.2.2. Although boundary divisors are often supposed to be effective in the literature, we don't impose this restriction in this book.

REMARK 6.2.3. We think of log pairs as generalization of $\mathbb{Q}$-Gorenstein varieties, identifying a $\mathbb{Q}$-Gorenstein variety X with the pair $(X, 0)$. The divisor $K_X + D$ is then regarded as the canonical divisor of the pair (X, D) and sometimes called a *log canonical divisor.*

DEFINITION 6.2.4. Let (X, D) be a log pair and let $f \colon Y \to X$ be a normal modification. We define a $\mathbb{Q}$-divisor $K_{Y/(X,D)}$ on Y by

$$K_Y = f^*(K_X + D) + K_{Y/(X,D)}.$$

The multiplicity of a prime divisor E in $K_{Y/(X,D)}$ is denoted by $a(E; X, D)$ and called the *discrepancy* of E with respect to (X, D). The *log discrepancy* $a_{\log}(E; X, D)$ is then defined to be $a(E; X, D) + 1$. For a nonempty closed subset $W \subset X$, we define the *minimal log discrepancy* of (X, D) along W to be

$$\mathrm{mld}(W; X, D) := \inf_{\mathrm{center}_X(E) \subset W} a_{\log}(E; X, D)$$

with E running over divisors over X with $\mathrm{center}_X(E) \subset W$.

REMARK 6.2.5. There seem to be two slightly different definitions of the minimal log discrepancy in the literature. Our definition above coincides, for example, with the one in [**EM06**], but not with the one in [**Kol13**]. In the latter reference, the infimum is taken over E with $\mathrm{center}_X(E) = W$.

DEFINITION 6.2.6. We say that a log pair (X, D) is

$$\begin{cases} \textit{klt} & (\mathrm{mld}(X; X, D) > 0) \\ \textit{log canonical} & (\mathrm{mld}(X; X, D) \geq 0). \end{cases}$$

If we define the singular locus $(X,D)_{\mathrm{sing}}$ of (X,D) to be $X_{\mathrm{sing}} \cup \mathrm{Supp}(D)$, then the above inequalities are equivalent to

$$\mathrm{mld}\left((X,D)_{\mathrm{sing}}; X, D\right) > 0 \quad (\text{resp. } \geq 0).$$

Therefore this definition is compatible with the one of a normal $\mathbb{Q}$-Gorenstein variety; X is klt (resp. log canonical) if and only if the pair $(X,0)$ is so.

REMARK 6.2.7. For a prime divisor E on X, if E has multiplicity e in D, then

$$a_{\log}(E; X, D) = -e + 1.$$

Therefore, for a log pair being klt (resp. log canonical), all the coefficients of the boundary must be < 1 (resp. ≤ 1).

A priori, it would be necessary to look at all the modifications of the given variety in order to compute the minimal log discrepancy. The following propositions enable us to compute it with a single resolution.

PROPOSITION 6.2.8 ([**Kol13**, Cor. 2.12]). *Let X be a $\mathbb{Q}$-Gorenstein variety and let $f\colon Y \to X$ be a resolution. Then X is terminal (resp. canonical) if and only if for every exceptional prime divisor E on Y, $a_{\log}(E;X) > 1$ (resp. ≥ 1).*

DEFINITION 6.2.9. Let (X,D) be a log pair, let $W \subset X$ be a closed subset and let $f\colon Y \to X$ be a resolution. We say that f is a *log resolution* of (X,D) if $f^{-1}D$, $\mathrm{Exc}(f)$ and $f^{-1}D \cup \mathrm{Exc}(f)$ are all simple normal crossing divisors. We say that f is a *log resolution* of the triple (X,D,W) if $W = X$ and f is a log resolution of (X,D) or if $f^{-1}D$, $f^{-1}W$, $\mathrm{Exc}(f)$ and $f^{-1}D \cup f^{-1}W \cup \mathrm{Exc}(f)$ are all simple normal crossing divisors.

PROPOSITION 6.2.10 (cf. [**Kol13**, Cor. 2.13]). *Let (X,D) be a log pair, let $W \subset X$ be a closed subset and let $f\colon Y \to X$ be a log resolution of (X,D).*

[**Kol13**, Cor. 2.13] *Then (X,D) is klt (resp. lc) if and only if $a_{\log}(E;X,D) > 0$ (resp. ≥ 0).*

(1) *Let $W \subset X$ be a closed subset. Suppose that f is also a log resolution of the triple (X,D,W) and that (X,D) is log canonical in a open neighborhood of W. Then*

$$\mathrm{mld}(W; X, D) = \min_{E \subset f^{-1}(W)} a_{\log}(E; X, D),$$

where E runs over prime divisors on Y contained in $f^{-1}(W)$.

The second assertion follows from Propositions 4.7.5, 6.4.6 and 6.6.1.

6.3. Stringy motives

DEFINITION 6.3.1. Suppose that X is r-Gorenstein. Let $\mathcal{I}_{X,r} \subset \mathcal{O}_X$ be the ideal sheaf given by

$$\mathrm{Im}\left((\Omega^d_{X/k})^{\otimes r} \to \omega_X^{[r]}\right) = \mathcal{I}_{X,r} \cdot \omega_X^{[r]}.$$

For a constructible subset $C \subset X$, we then define the *stringy motive of X along C* to be the integral

$$\mathrm{M}_{\mathrm{st}}(X)_C := \int_{\pi_0^{-1}(C)} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,r}}} d\mu_X \in \widehat{\mathcal{M}}_{k,r} \cup \{\infty_*\}.$$

When $C = X$, we just call it the *stringy motive of X* and often omit the subscript C.

REMARK 6.3.2. Note that for $n \geq 0$, we have $\mathcal{I}_{X,rn} = (\mathcal{I}_{X,r})^n$. Thus the function $\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,r}}$ is independent of the choice of r. It follows that $\mathrm{M}_{\mathrm{st}}(X)_C$ is essentially independent of r. More precisely, if $M_r \in \widehat{\mathcal{M}}_{k,r} \cup \{\infty_*\}$ and $M_{r'} \in \widehat{\mathcal{M}}_{k,r'} \cup \{\infty_*\}$ are the ones defined for r and r', then they have the same image in $\widehat{\mathcal{M}}_{k,rr'} \cup \{\infty_*\}$ via the canonical maps. In practice, we will fix a sufficiently factorial r so that all $\mathbb{Q}$-Gorenstein varieties under consideration be r-Gorenstein and all $\mathbb{Q}$-divisors have coefficients in $\frac{1}{r}\mathbb{Z}$.

Roughly speaking, the function $\frac{1}{r}\operatorname{ord}_{\mathcal{I}}$ measures the difference between $\Omega^d_{X/k}$ and the canonical divisor K_X; the former is directly related to the change of variables formula, while the latter is more important in the birational geometry.

To generalize stringy motives to log pairs, we need to consider fractional ideal sheaves.

DEFINITION 6.3.3. A *coherent fractional ideal sheaf* on X is a nonzero coherent $\mathcal{O}_X$-submodule $\mathcal{I}$ of the constant function field sheaf $K(X)$.

For a coherent fractional ideal sheaf $\mathcal{I}$ on X, there exists an open dense subset $U \subset X$ such that $\mathcal{I}|_U = \mathcal{O}_U$ as subsheaves of $K(X)$. Indeed, if $\mathcal{I}$ is generated by rational functions $f_1, \dots, f_l \in K(X)$ on an open subset $V \subset X$, then we only need to remove their poles to get such an open subset U. Moreover there exists the largest open subset with this property; for the sake of uniqueness, we choose it as U. Let $\alpha\colon \mathrm{D}_L \to X$ be an arc such that $\alpha(\eta) \in U$. The equality $\mathcal{I}|_U = \mathcal{O}_U$ induces the injection

$$\alpha^\flat \mathcal{I} \hookrightarrow \alpha^*\mathcal{I} \otimes_{L[\![t]\!]} L(\!(t)\!) = (\alpha|_\eta)^*\mathcal{O}_U = L(\!(t)\!).$$

Thus the flat pullback $\alpha^\flat\mathcal{I}$ is regarded as a fractional ideal of $L[\![t]\!]$; we denote this fractional ideal as $\alpha^{-1}\mathcal{I}$.

DEFINITION 6.3.4. For an arc $\alpha\colon \mathrm{D}_L \to X$ with $\alpha(\eta) \in U$, we define $\operatorname{ord}_{\mathcal{I}}(\alpha)$ to be an integer n such that $\alpha^{-1}\mathcal{I} = t^n \cdot L[\![t]\!]$. If $\alpha(\eta) \notin U$, we define $\operatorname{ord}_{\mathcal{I}}(\alpha) := \infty$. We get a function

$$\operatorname{ord}_{\mathcal{I}}\colon \mathrm{J}_\infty X \to \mathbb{Z} \cup \{\infty\}.$$

LEMMA 6.3.5. *For a coherent fractional ideal $\mathcal{I}$, the function* $\operatorname{ord}_{\mathcal{I}}$ *is admissible.*

PROOF. By means of an affine open covering, the problem is reduced to the case where X is affine. Then $\mathcal{I}$ is generated by rational functions $f_1/g_1, \dots, f_l/g_l$, where f_i and g_i are regular functions on X. Let D be the effective Cartier divisor defined by $g_1 g_2 \cdots g_l = 0$. Then $\mathcal{I} \subset \mathcal{O}_X(-D)$. Thus there exists an ideal sheaf $\mathcal{J}$ with $\mathcal{I} = \mathcal{J}\mathcal{O}_X(-D)$. We see that

$$\operatorname{ord}_{\mathcal{I}} = \operatorname{ord}_{\mathcal{J}} - \operatorname{ord}_D$$

outside the negligible subset $\mathrm{J}_\infty Z$ with $Z = D \cup V(\mathcal{J})$. Since $\operatorname{ord}_{\mathcal{J}}$ and ord_D are admissible, $\operatorname{ord}_{\mathcal{I}}$ is also admissible. $\square$

DEFINITION 6.3.6. Let (X, D) be a log pair such that $r(K_X + D)$ is Cartier. We regard $\mathcal{O}_X(rK_X)$ and $\mathcal{O}_X(r(K_X + D))$ as submodules of the constant sheaf

$$\mathcal{O}_X(rK_X) \otimes K(X) = (\Omega^d_{X/k})^{\otimes r} \otimes K(X).$$

We define a coherent fractional ideal sheaf $\mathcal{I}_{X,D,r} \subset K(X)$ by

$$\operatorname{Im}\Big((\Omega^d_{X/k})^{\otimes r} \to \mathcal{O}_X(rK_X)\Big) = \mathcal{I}_{X,D,r} \cdot \mathcal{O}_X(r(K_X + D)).$$

For a constructible subset $C \subset X$, we then define the *stringy motive of* (X, D) *along* C to be the integral

$$\mathrm{M}_{\mathrm{st}}(X, D)_C := \int_{\pi_0^{-1}(C)} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,D,r}}} \, d\mu_X \in \widehat{\mathcal{M}}_{k,r} \cup \{\infty\}.$$

REMARK 6.3.7. If X is smooth, then $\mathcal{I}_{X,D,r}$ is the defining ideal sheaf of the divisor rD. In particular,

$$\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,D,r}} = \operatorname{ord}_D .$$

When D has simple normal crossing support, we have explicit formulas for $\mathrm{M}_{\mathrm{st}}(X, D)_C$ and its dimension (Propositions 4.7.4 and 4.7.5). In particular, we have a criterion for whether or not $\mathrm{M}_{\mathrm{st}}(X, D)_C$ converges and one for whether or not it is dimensionally bounded.

6.4. Basic properties of stringy motives

PROPOSITION 6.4.1. *If X is smooth, then* $\mathrm{M}_{\mathrm{st}}(X)_C = \mathrm{M}_{\mathrm{st}}(X, 0)_C = \{C\}$.

PROOF. In this case, the function $\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,r}} = \frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,0,r}}$ is constant zero. Thus

$$\mathrm{M}_{\mathrm{st}}(X)_C = \mathrm{M}_{\mathrm{st}}(X, 0)_C = \int_{\pi_0^{-1}(C)} d\mu_X = \mu_X(\pi_0^{-1}(C)) = \{C\}.$$

$\square$

DEFINITION 6.4.2. A *morphism* $f\colon (Y, E) \to (X, D)$ of log pairs means just a morphism $f\colon Y \to X$ of underlying varieties. We say that a morphism $f\colon (Y, E) \to (X, D)$ is *generically étale* (resp. *proper*, *birational*, *being a modification*) if it is so as a morphism of varieties.

DEFINITION 6.4.3. We say that a generically étale morphism $f\colon (Y, E) \to (X, D)$ of log pairs is *crepant* if $K_Y + E = f^*(K_X + D)$. (Note that this notion depends on not only underlying varieties but also boundary divisors.) We say that two log pairs (Y, E) and (X, D) are *K-equivalent* if there exist crepant modifications $(Z, F) \to (Y, E)$ and $(Z, F) \to (X, D)$ from a third log pair (Z, F). (In other words, (X, D) and (Y, E) are K-equivalent if there are normal modifications $Z \to X$ and $Z \to Y$ such that $K_{Z/(X,D)} = K_{Z/(Y,E)}$.)

Unlike the case of strong K-equivalence (Definition 4.5.1), we don't assume the above Z to be smooth. Thus K-equivalence is a weaker condition than strong K-equivalence.

EXAMPLE 6.4.4. Two crepant resolutions of the same normal $\mathbb{Q}$-Gorenstein varieties are clearly K-equivalent. Two birational Calabi-Yau varieties are K-equivalent. Here a Calabi-Yau variety means a smooth proper variety X with $\omega_X \cong \mathcal{O}_X$. More generally, two minimal models of the same log pair are K-equivalent [**Kol13**, Prop. 1.21].

LEMMA 6.4.5. *Let $(Y, E) \to (X, D)$ be a crepant morphism of log pairs. Then*

$$\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,D,r}} \circ f_\infty - \mathfrak{j}_f = \frac{1}{r}\operatorname{ord}_{\mathcal{I}_{Y,E,r}} .$$

PROOF. Let α be a geometric arc of Y and let

$$a := \frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,D,r}} \circ f_\infty(\alpha),\ b := \frac{1}{r}\operatorname{ord}_{\mathcal{I}_{Y,E,r}}(\alpha) \text{ and } e := \mathfrak{j}_f(\alpha).$$

Suppose that all these values are finite. It holds for α's outside a negligible subset. We have

$$\begin{aligned} t^{re}\cdot\alpha^\flat(\Omega^d_{Y/k})^{\otimes r} &= (f\circ\alpha)^\flat(\Omega^d_{X/k})^{\otimes r} \\ &= t^{ra}\cdot(f\circ\alpha)^*\mathcal{O}_X(r(K_X+D)) \\ &= t^{ra}\cdot\alpha^*\mathcal{O}_Y(r(K_Y+E)) \\ &= t^{ra-rb}\cdot\alpha^\flat(\Omega^d_{Y/k})^{\otimes r}. \end{aligned}$$

Thus $e = a - b$ and the claim holds. $\square$

PROPOSITION 6.4.6. *Let $(Y,E)\to(X,D)$ be a crepant modification of log pairs and let $C\subset X$ be a constructible subset. Then*

$$\mathrm{M}_{\mathrm{st}}(Y,E)_{f^{-1}(C)} = \mathrm{M}_{\mathrm{st}}(X,D)_C.$$

PROOF. This follows from Lemma 6.4.5 and the change of variables formula. $\square$

The following corollary is a direct consequence of this proposition:

COROLLARY 6.4.7. *Let (Y,E) and (X,D) be K-equivalent log pairs. Then*

$$\mathrm{M}_{\mathrm{st}}(Y,E) = \mathrm{M}_{\mathrm{st}}(X,D).$$

More generally, if $g\colon Z\to Y$ and $f\colon X\to Y$ are normal modifications with $K_{Z/(X,D)} = K_{Z/(Y,E)}$ and if $C\subset Y$ and $B\subset X$ are constructible subsets with $g^{-1}(C) = f^{-1}(B)$, then

$$\mathrm{M}_{\mathrm{st}}(Y,E)_C = \mathrm{M}_{\mathrm{st}}(X,D)_B.$$

The following corollary strengthens Theorem 4.5.4 by weakening the strong K-equivalence condition to K-equivalence condition.

COROLLARY 6.4.8. *Let X and Y be K-equivalent smooth varieties. Then $\{X\} = \{Y\}$ in $\widehat{\mathcal{M}}_k$.*

PROOF. From Corollary 6.4.7, we have

$$\{X\} = \mathrm{M}_{\mathrm{st}}(X) = \mathrm{M}_{\mathrm{st}}(Y) = \{Y\}$$

in $\widehat{\mathcal{M}}_{k,r}$ for some $r>0$. We show that we can take r to be 1. There exists a log pair (Z,D) and crepant modifications $f\colon(Z,D)\to X$ and $g^*\colon(Z,D)\to Y$. Since K_X, K_Y and $K_Z+D = f^*K_X$ are all Cartier, we can take r to be 1 when we define $\mathrm{M}_{\mathrm{st}}(X)$, $\mathrm{M}_{\mathrm{st}}(Y)$ and $\mathrm{M}_{\mathrm{st}}(Z,D)$ and prove that they are equal. $\square$

COROLLARY 6.4.9. *Corollary 4.5.5 holds with strong K-equivalence replaced with K-equivalence.*

PROOF. This follows from Proposition 2.5.3. $\square$

6.5. Special local uniformization

To apply stringy motives to minimal log discrepancy in arbitrary characteristic, we show a version of local uniformization, which plays a role of log resolution in characteristic zero.

Let $C \subset \mathrm{J}_\infty X$ be an irreducible ordinary cylinder, let $c \in C$ be its generic point with the corresponding arc $\alpha\colon \mathrm{D}_L \to X$ with L the residue field of c. This arc sends the generic point of D_L to the generic point of X. For, if it was sent to a point $x \in X$ of dimension $\le d-1$, then C would be contained in the negligible subset $\mathrm{J}_\infty \overline{\{x\}}$, where $\overline{\{x\}}$ is the Zariski closure. This is impossible. Therefore, for every modification $Y \to X$, there exists a unique lift

$$\beta\colon \mathrm{D}_L \to Y$$

of α.

LEMMA 6.5.1. *Suppose that the arc α sends the closed point to a point of X of codimension ≥ 2. Then there exists a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that, if $f\colon Y = \mathrm{Bl}_{\mathcal{I}}(X) \to X$ is the associated blowup and β is the lift of α to Y, then we have $\mathfrak{j}_f(\beta) > 0$.*

PROOF. Let $x \in X$ be the image of the closed point D_L and let $a \ge 2$ be its codimension. Let $\kappa(x)$ and $\mathfrak{m}_x$ denote the residue field and the maximal ideal of the local ring $\mathcal{O}_{X,x}$ respectively. Since our base field k is perfect, the extension $\kappa(x)/k$ is separable [**Eis95**, Cor. A1.7] and has transcendental degree $d-a$. The $\kappa(x)$-vector space $\Omega_{\kappa(x)/k}$ has dimension $d-a$ [**Eis95**, Th. 16.14]. Consider the conormal exact sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to \kappa(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k} \to \Omega_{\kappa(x)/k} \to 0.$$

There are elements $x_1, \dots, x_{d-a} \in \mathcal{O}_{X,x}$ and $w_1, w_2, \dots, w_b \in \mathfrak{m}_x$ such that $dx_1, \dots, dx_{d-a}$ map to a basis of $\Omega_{\kappa(x)/k}$ and $dx_1, \dots, dx_{d-a}, dw_1, \dots, dw_b$ map to a basis of $\kappa(x) \otimes_{\mathcal{O}_{X,x}} \Omega_{\mathcal{O}_{X,x}/k}$. By Nakayama's lemma, $dx_1, \dots, dx_{d-a}, dw_1, \dots, dw_b$ generate $\Omega_{\mathcal{O}_{X,x}/k}$. By reordering $x_1, \dots, x_{d-a}$ and $w_1, \dots, w_b$ if necessary, we may assume that $\alpha^\flat \Omega^d_{X/d}$ is generated by the element

$$\alpha^*(dw_1 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l})$$

for some $l \ge a$ such that $w_1, \dots, w_l, x_1, \dots, x_{d-l}$ are algebraically independent over k; the last condition ensures that the above d-form is not a torsion. Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf such that $\mathcal{I}_x = \langle w_1, w_2 \rangle$ and let $f\colon Y \to X$ be the blowup along $\mathcal{I}$. We identify $K(Y)$ with $K(X)$ so that the maps $\alpha^*\colon K(X) \to L(\!(t)\!)$ and $\beta^*\colon K(Y) \to L(\!(t)\!)$ become identical. Let $y \in Y$ be the image of the closed point by β. Either w_1/w_2 or w_2/w_1 is a regular function on an open neighborhood U of y; say $v := w_1/w_2$ is so. Then

$$\begin{aligned}
&dw_1 \wedge dw_2 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l} \\
&= d(vw_2) \wedge dw_2 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l} \\
&= (vdw_2 + w_2 dv) \wedge dw_2 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l} \\
&= w_2 dv \wedge dw_2 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\alpha^\flat \Omega^d_{X/k}\\
&= \langle \alpha^*(dw_1 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l})\rangle\\
&= \langle \beta^*(w_2 dv \wedge dw_2 \wedge \cdots \wedge dw_l \wedge dx_1 \wedge \cdots \wedge dx_{d-l})\rangle\\
&\subset \beta^* w_2 \cdot \beta^\flat \Omega^d_{Y/k}.
\end{aligned}$$

From the definition of x, we have

$$\beta^* w_2 = \alpha^* w_2 \in \alpha^{-1}\mathfrak{m}_x \subset \langle t \rangle \subset L(\!(t)\!).$$

We conclude that

$$\mathrm{j}_f(\beta) = \dim_L \beta^\flat \Omega^d_{Y/k} / \alpha^\flat \Omega^d_{X/k} > 0.$$

□

PROPOSITION 6.5.2 (cf. [**Reg09**, Prop. 3.7 (vii)]). *Suppose that α sends the closed point to a point of positive codimension. Then there exists a normal projective modification $f\colon Y \to X$ such that the lift $\beta\colon \mathrm{D}_L \to Y$ of α sends the closed point of D_L to a point of codimension* 1.

PROOF. Let $x \in X$ be the image of the closed point by α. If this is a point of codimension 1, we are done. Otherwise, consider the projective modification $f\colon X_1 = \mathrm{Bl}_{\mathcal{I}}(X) \to X$ as in the last lemma and let $\alpha_1\colon \mathrm{D}_L \to X_1$ be the lift of α. Let $m := \mathrm{ord}_{\mathcal{I}}(\alpha)$ and let $e := \mathrm{j}_f(\alpha_1)$. Removing the loci with $\mathrm{ord}_{\mathcal{I}} > m$ and $\mathrm{j}_f > e$, we get irreducible cylinders $C' \subset C$ and $C_1 := f_\infty^{-1}(C')$ such that $\mathrm{j}_f|_{C_1}$ has constant value e. From Lemma 5.11.1,

$$\dim \mu_{X_1}(C_1) > \dim \mu_X(C') = \dim \mu_X(C).$$

Note that from Proposition 4.2.1, the generic point of C_1 corresponds to α_1, in particular, it has the same residue field L as the residue field of the generic point c of C. If α_1 sends the closed point to a point $x_1 \in X_1$ of codimension > 1, then we apply the same procedure as above to α_1 and get a projective modification $g\colon X_2 \to X_1$, the lift α_2 of α_1 and an irreducible cylinder $C_2 \subset \mathrm{J}_\infty X_2$ such that the generic point of C_2 corresponds to α_2 and

$$\dim \mu_{X_2}(C_2) > \dim \mu_{X_1}(C_1).$$

We repeat this procedure until we get a point $x_n \in X_n$ of codimension 1. From Corollary 5.4.2, for $m \gg 0$, we have

$$\dim \mu_{X_i}(C_i) \le \dim \pi_m(\mathrm{J}_\infty X) - dm = d.$$

Therefore the procedure stops after finitely many steps and indeed get a point $x_n \in X_n$ of codimension 1. It remans to take the normalization of X_n. □

REMARK 6.5.3. A version of local uniformization says that for a variety X and a valuation v on $K(X)$, there exists a modification $Y \to X$ such that Y is regular at the center of v. This can be regarded as a weak version of resolution of singularities. In positive characteristic, it is still an open problem whether local uniformization in this sense is always possible or not. The above proposition says that local uniformization is valid for the discrete valuation

$$K(X)^* \xrightarrow{\alpha^*} L(\!(t)\!)^* \xrightarrow{\mathrm{ord}} \mathbb{Z}$$

associated to an arc α as above.

COROLLARY 6.5.4. *For an irreducible ordinary cylinder $C \subset \mathrm{J}_\infty X$, there exists a normal modification $f : Y \to X$ and a prime divisor $D \subset Y$ such that $f_\infty\left(\pi_0^{-1}(D_{\mathrm{sm}} \cap Y_{\mathrm{sm}})\right)$ contains the generic point of C.*

PROOF. Let $\alpha : \mathrm{D}_L \to X$ be the arc corresponding to the generic point of C. We take a normal modification $f : Y \to X$ as in the last proposition and let β to be the lift of α to Y. Then the image $y \in Y$ of the closed point by β has codimension one. We see that the prime divisor $D := \overline{\{y\}}$ has the desired property. $\square$

6.6. The minimal log discrepancy via stringy motives

PROPOSITION 6.6.1. *Let (X, D) be a log pair and let $C \subset X$ be a non-empty closed subset. Then*

$$\mathrm{mld}(C; X, D) = d - \dim \mathrm{M}_{\mathrm{st}}(X, D)_C \in \mathbb{Q} \cup \{-\infty\}.$$

Here we follow the convention $\dim \infty_s = s$.

PROOF. Let $f : Y \to X$ be a normal modification such that $f^{-1}(C)$ of pure dimension $d-1$. Any normal modification $Y' \to X$ is dominated by such a normal modification. Let $Y^\circ \subset Y$ be the largest open subset such that Y° is smooth and the subvarieties of Y,

$$f^{-1}(C) \cap Y^\circ \text{ and}$$
$$\left(f^{-1}(C) \cup \mathrm{Supp}\, K_{Y/(X,D)}\right) \cap Y^\circ,$$

are both smooth. Then $Y \setminus Y^\circ$ has codimension ≥ 2. Let us write

$$f^{-1}(C) \cup \mathrm{Supp}\, K_{Y/(X,D)} = \bigcup_{i \in I} E_i$$

with E_i distinct prime divisors on Y°, and write

$$K_{Y/(X,D)}|_{Y^\circ} = \sum_{i \in I} a_i E_i,$$

where a_i are (possibly zero) rational numbers. Note that a_i+1 is the log discrepancy of E_i with respect to (X, D). The restrictions $E_i|_{Y^\circ}$ are mutually disjoint from the smoothness condition. From Proposition 4.7.4, we have

$$\begin{aligned}\dim \mathrm{M}_{\mathrm{st}}(X, D)_C &= \dim \mathrm{M}_{\mathrm{st}}(Y, -K_{Y/(X,D)})_{f^{-1}(C)} \\ &\geq \dim \mathrm{M}_{\mathrm{st}}(Y^\circ, -K_{Y/(X,D)}|_{Y^\circ})_{f^{-1}(C) \cap Y^\circ}.\end{aligned}$$

If $\mathrm{mld}(C; X, D) = -\infty$, then for some f and i, we have $a_i < -1$. For such f, from Proposition 4.7.4,

$$\dim \mathrm{M}_{\mathrm{st}}(Y^\circ, -K_{Y/(X,D)}|_{Y^\circ})_{f^{-1}(C) \cap Y^\circ} = \infty.$$

Thus, in this case, we have $\dim \mathrm{M}_{\mathrm{st}}(X, D)_C = \infty$ and the equality of the proposition holds.

We suppose that $\mathrm{mld}(C; X, D) \geq 0$. Then, for f as above, $a_i \geq -1$ and from Proposition 4.7.4, $\mathrm{M}_{\mathrm{st}}(Y^\circ, -K_{Y/(X,D)}|_{Y^\circ})_{f^{-1}(C) \cap Y^\circ}$ is dimensionally bounded. From Proposition 4.7.5, we have

$$\dim \mathrm{M}_{\mathrm{st}}(Y^\circ, -K_{Y/(X,D)}|_{Y^\circ})_{f^{-1}(C) \cap Y^\circ} = \sup \{d - 1 - a_i \mid i \in K\},$$

where K is the subset of I such that $f^{-1}(C) = \bigcup_{i \in K} D_i$. Since the inequality

$$\dim \mathrm{M}_{\mathrm{st}}(X, D)_C \geq \sup \{d - 1 - a_i \mid i \in K\} \tag{6.6.1}$$

holds for an arbitrary normal modification $Y \to X$, we conclude

$$\dim \mathrm{M}_{\mathrm{st}}(X,D)_C \geq d - \mathrm{mld}(C;X,D).$$

We will show the opposite inequality. Let us write

$$\mathrm{M}_{\mathrm{st}}(X,D)_C = \sum_{i \in I} \int_{A_i} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}}} d\mu_X,$$

where $\mathcal{I} = \mathcal{I}_{X,D,r}$ (see Definition 6.3.6) and A_i are irreducible ordinary cylinders with $\operatorname{ord}_{\mathcal{I}}|_{A_i}$ constant. From Corollary 6.5.4, for each i, there exists a normal modification $f\colon Y \to X$ and an f-ordinary cylinder $B \subset \mathrm{J}_\infty Y^\circ$ such that $f_\infty(B)$ is contained in A_i and contains the generic point of A_i. Here Y° is similarly defined as above. Then

$$\begin{aligned}
\dim \int_{A_i} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}}} d\mu_X &= \dim \int_{f_\infty(B)} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}}} d\mu_X \\
&= \dim \int_B \mathbb{L}^{-\operatorname{ord}_{K_{Y/(X,D)}}} d\mu_Y \\
&\leq \dim \int_{\mathrm{J}_\infty Y^\circ} \mathbb{L}^{-\operatorname{ord}_{K_{Y/(X,D)}}} d\mu_Y \\
&= \dim \mathrm{M}_{\mathrm{st}}(Y^\circ, -K_{Y/(X,D)}|_{Y^\circ})_{f^{-1}(C)\cap Y^\circ} \\
&= d - \inf\{1 + a_i \mid i \in K\} \\
&\leq d - \mathrm{mld}(C;X,D).
\end{aligned}$$

This shows

$$\begin{aligned}
\dim \mathrm{M}_{\mathrm{st}}(X,D)_C &= \sup \dim \int_{A_i} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}}} d\mu_X \\
&\leq d - \mathrm{mld}(C;X,D).
\end{aligned}$$

□

PROPOSITION 6.6.2. *Let (X,D) be a log pair.*

(1) *If $\mathrm{M}_{\mathrm{st}}(X,D)$ converges, then (X,D) is klt. The coverse is also true if (X,D) is K-equivalent to a log pair (Y,E) such that Y is smooth and $\mathrm{Supp}(E)$ is simple normal crossing, in particular, if (X,D) has a log resolution.*
(2) *The following are equivalent:*
   (a) *$\mathrm{M}_{\mathrm{st}}(X,D)$ is dimensionally bounded.*
   (b) *$\dim \mathrm{M}_{\mathrm{st}}(X,D) \leq \dim X$.*
   (c) *(X,D) is log canonical.*

PROOF. (1) First suppose that $\mathrm{M}_{\mathrm{st}}(X,D)$ converges. Let $f\colon Y \to X$ be an arbitrary normal modification and let $E := f^*(K_X + D) - K_Y$ so that $(Y,E) \to (X,D)$ is crepant. Thus $\mathrm{M}_{\mathrm{st}}(Y,E) = \mathrm{M}_{\mathrm{st}}(X,D)$, where $E = f^*(K_X+D) - K_Y$. Let $U \subset Y$ be the largest open subset such that U is smooth and $E|_U$ has simple normal crossing support. Note that $Y \setminus U$ has codimension ≥ 2 in Y and every prime divisor on Y meets U. Since $\mathrm{M}_{\mathrm{st}}(Y,E)$ converges, so does $\mathrm{M}_{\mathrm{st}}(U,E|_U)$. From Proposition 4.7.4, all the coefficients of E are < 1. This is equivalent to that every prime divisor on Y has log discrepancy > 0 with respect to (X,D). Since this is true for every normal modification of X, we conclude that (X,D) is klt.

Next suppose that (X,D) is klt and that (X,D) is K-equivalent to (Y,E) such that Y is smooth and $\mathrm{Supp}(E)$ is simple normal crossing. Then (Y,E) is also klt and all the coefficients of E are < 1. Again from Proposition 4.7.4, $\mathrm{M}_{\mathrm{st}}(X,D) = \mathrm{M}_{\mathrm{st}}(Y,E)$ converges.

(2) Obviously (b) implies (a). A similar argument as above shows (a) implies (c). It remains to show (c) implies (b). Suppose on the contrary that $\dim \mathrm{M}_{\mathrm{st}}(X,D) > \dim X$. Then there exists an irreducible ordinary cylinder $C \subset \mathrm{J}_\infty X$ such that $\mathrm{ord}_{\mathcal{I}_{X,D,r}}$ is constant on C and

$$\dim \int_C \mathbb{L}^{\frac{1}{r}\mathrm{ord}_{\mathcal{I}_{X,D,r}}}\, d\mu_X > \dim X.$$

For such C, we take a normal modification $f\colon Y \to X$ and a prime divisor $F \subset Y$ as in Corollary 6.5.4. Let $U \subset Y_{\mathrm{sm}}$ be a sufficiently small open neighborhood of the generic point of F so that $(U,(1-a)F) \to (X,D)$ is crepant, where $a := a(F;X,D)$. Since $f_\infty(\pi_0^{-1}(F\cap U))$ contains the generic point of C, we have

$$\begin{aligned}\dim \mathrm{M}_{\mathrm{st}}(U,(1-a)F|_U) &= \dim \int_{f_\infty(\pi_0^{-1}(F\cap U))} \mathbb{L}^{\frac{1}{r}\mathrm{ord}_{\mathcal{I}_{X,D,r}}}\, d\mu_X \\ &> \dim \int_C \mathbb{L}^{\frac{1}{r}\mathrm{ord}_{\mathcal{I}_{X,D,r}}} > \dim X.\end{aligned}$$

From Proposition 4.7.4, $a < 0$. Thus (X,D) is not log canonical. □

COROLLARY 6.6.3. *Let (X,D) be a log pair and let $C \subset X$ be a constructible subset such that (X,D) is klt in a neighborhood of C. Suppose that that there exists a resolution $f\colon Y \to X$ such that $K_{Y/(X,D)} = K_Y - f^*(K_X + D)$ has simple normal crossing support and is written as $\sum_{i=1}^l a_i E_i$ with E_i prime divisors and $a_i \neq 0$. Then*

$$\mathrm{M}_{\mathrm{st}}(X,D)_C = \sum_{J\subset I}\{D_I^\circ \cap C\}\prod_{i\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{1+a_i}-1}.$$

6.7. Imperfect fields and non-normal varieties

In this section, we do *not* suppose that the base field k is perfect or that X is normal.

DEFINITION 6.7.1. A *weak log pair* is the pair $(X,\mathcal{L})$ of a k-variety X which is generically smooth over k and an invertible subsheaf of $(\Omega^d_{X/k})^{\otimes r} \otimes_{\mathcal{O}_X} K(X)$ for some $r \in \mathbb{Z}_{>0}$. We call r the *index* of $(X,\mathcal{L})$. A *morphism* $(Y,\mathcal{L}') \to (X,\mathcal{L})$ of weak log pairs is just a morphism $Y \to X$ of k-varieties. We say that a dominant morphism $(Y,\mathcal{L}') \to (X,\mathcal{L})$ is *crepant* if $(\mathcal{L}')^{\otimes r} = \mathcal{L}^{\otimes r'}$ in $(\Omega^d_{X/k})^{\otimes rr'} \otimes_{\mathcal{O}_X} K(X)$, where r' and r are the indices of $(Y,\mathcal{L}')$ and $(X,\mathcal{L})$ respectively.

The sheaf $\mathcal{L}$ plays the role of $\mathcal{O}_X(r(K_X+D))$ in the case of log pairs.

DEFINITION 6.7.2. Let $(X,\mathcal{L})$ be a weak log pair. We define a fractional ideal sheaf $\mathcal{I}_{X,\mathcal{L},r}$ by

$$\mathrm{Im}\left((\Omega^d_{X/k})^{\otimes r} \to (\Omega^d_{X/k})^{\otimes r}\otimes_{\mathcal{O}_X} K(X)\right) = \mathcal{I}_{X,\mathcal{L},r}\cdot\mathcal{L}.$$

We then define

$$\mathrm{M}_{\mathrm{st}}(X,\mathcal{L})_C := \int_{\pi_0^{-1}(C)} \mathbb{L}^{\frac{1}{r}\mathrm{ord}_{\mathcal{I}_{X,\mathcal{L},r}}}\, d\mu_X \in \widehat{\mathcal{M}}_{k,r}\cup\{\infty\}.$$

REMARK 6.7.3. If X is normal and k-smooth in codimension one, then a log pair (X, D) (over a possibly imperfect field k) gives a weak log pair $(X, \mathcal{O}_X(r(K_X+D)))$. Clearly they have the same stringy motive along any constructible subset. If X is smooth and D has simple normal crossing support, then $\mathrm{M}_{\mathrm{st}}(X, D)_C$ is computed in Propositions 4.7.4 and 4.7.5 (even if k is imperfect).

PROPOSITION 6.7.4. *Let $f\colon (Y, \mathcal{L}') \to (X, \mathcal{L})$ be a crepant proper birational morphism of weak log pairs. Let $C \subset X$ be a constructible subset. Then*

$$\mathrm{M}_{\mathrm{st}}(X, \mathcal{L})_C = \mathrm{M}_{\mathrm{st}}(Y, \mathcal{L}')_{f^{-1}(C)}.$$

PROOF. Similar to the proof of Proposition 6.4.6. □

CHAPTER 7

Working over a formal disk

In this chapter, we generalize the theory in earlier chapters to D_k-schemes. Most arguments are parallel to ones in the case of k-schemes. Except a few places where a little caution is required, we omit repeating proofs and just refer to the corresponding results for k-schemes.

7.1. Jet schemes and arc schemes

Definition 7.1.1. A *good* D_k*-scheme* means a D_k-scheme X satisfying the following conditions:

(1) $X \to \mathrm{D}_k$ is flat, of finite type and of pure relative dimension,
(2) there exists an open dense subscheme $U \subset X$ which is smooth over D_k.

In what follows, X and Y denote good D_k-schemes of relative dimension d. A good D_k-scheme has an open dense subscheme which is smooth over D_k. Note that for a D_k-scheme, there are two different notions of D_k-smooth and regular; the former implies the latter, but the converse doesn't hold.

Definition 7.1.2 (cf. Definition 3.2.3). We define the *n-th jet scheme* of a D_k-scheme X, denoted by $\mathrm{J}_n(X/\mathrm{D}_k)$, to be the Weil restriction

$$\mathrm{R}_{\mathrm{D}_{k,n}/\operatorname{Spec} k}(X \times_{\mathrm{D}_k} \mathrm{D}_{k,n}).$$

(In later chapters, we sometimes abbreviate $\mathrm{J}_n(X/\mathrm{D}_k)$ as $\mathrm{J}_n\, X$, if there does not occur any confusion.)

This is by definition a k-scheme (rather than a D_k-scheme) such that for a k-algebra R, we have

$$\begin{aligned}(\mathrm{J}_n(X/\mathrm{D}_k))(R) &= \operatorname{Hom}_{\mathrm{D}_{k,n}}(\mathrm{D}_{R,n}, X \times_{\mathrm{D}_k} \mathrm{D}_{k,n}) \\ &= \operatorname{Hom}_{\mathrm{D}_k}(\mathrm{D}_{R,n}, X).\end{aligned}$$

In particular, k-points of $\mathrm{J}_n(X/\mathrm{D}_k)$ correspond to sections $\mathrm{D}_{k,n} \to X$ of the structure morphism $X \to \mathrm{D}_k$ on the closed subscheme $\mathrm{D}_{k,n} \subset \mathrm{D}_k$.

$$\begin{tikzcd} & X \arrow[d] \\ \mathrm{D}_{k,n} \arrow[ur] \arrow[r, hook] & \mathrm{D}_k \end{tikzcd}$$

From [**BLR90**, Ch. 7, Prop. 5], $\mathrm{J}_n(X/\mathrm{D}_k)$ is of finite type over k. The 0-th jet scheme is the special fiber;

$$\mathrm{J}_0(X/\mathrm{D}_k) = X_0 := X \times_{\mathrm{D}_k} \operatorname{Spec} k.$$

By base change, we can associate the D_k-scheme $W \times_k \mathrm{D}_k$ to each k-scheme W. But not all D_k-schemes are constructed in this way. In this sense, we can think of D_k-schemes as generalization of k-schemes. That being said, jet schemes of D_k-schemes defined above are generalization of jet schemes of k-schemes. More precisely:

LEMMA 7.1.3. *For a scheme W of finite type over k, the n-th jet scheme $\mathrm{J}_n W$ is canonically isomorphic to $\mathrm{J}_n(W \times_k \mathrm{D}_k/\mathrm{D}_k)$.*

PROOF. For a k-algebra R, the R-point sets of $\mathrm{J}_n W$ and $\mathrm{J}_n(W \times_k \mathrm{D}_k/\mathrm{D}_k)$ are both $\mathrm{Hom}_{\mathrm{D}_k}(\mathrm{D}_{R,n}, X)$. Thus the two schemes are identical as functors $\mathbf{Aff}_k^{\mathrm{op}} \to \mathbf{Set}$. $\square$

We can prove basic properties of jet schemes of D_k-schemes in the same way as proving the ones of jet schemes of k-schemes. For instance, we have $\mathrm{J}_n(\mathbb{A}^d_{\mathrm{D}_k}/\mathrm{D}_k) \cong \mathbb{A}_k^{d(n+1)}$ (cf. Lemma 3.2.4). For a closed subscheme $X = V(f_1, \dots, f_l) \subset \mathbb{A}^d_{\mathrm{D}_k}$, we have an explicit description of $\mathrm{J}_n(X/\mathrm{D}_k)$ as in Lemma 3.2.5. For $n' \ge n$, we have the truncation morphism

$$\pi_n^{n'}\colon \mathrm{J}_{n'}(X/\mathrm{D}_k) \to \mathrm{J}_n(X/\mathrm{D}_k).$$

They are affine morphisms (cf. Lemma 3.3.3).

DEFINITION 7.1.4 (cf. Definitions 3.4.1 and 3.4.2). For a k-algebra R, an *arc of X over R* means a D_k-morphism $\mathrm{D}_R \to X$. A *geometric arc of X* means an arc of X over an algebraically closed field. We define the *arc scheme* of X to be

$$\mathrm{J}_\infty(X/\mathrm{D}_k) \cong \varprojlim \mathrm{J}_n(X/\mathrm{D}_k).$$

We denote the morphism $\mathrm{J}_\infty(X/\mathrm{D}_k) \to \mathrm{J}_n(X/\mathrm{D}_k)$ by π_n and call it again by a *truncation morphism.*

In particular, an arc over k is just a section of the structure morphism $X \to \mathrm{D}_k$. For a k-algebra R, R-points of $\mathrm{J}_\infty(X/\mathrm{D}_k)$ are identified with arcs $\mathrm{D}_R \to X$ over R (cf. Lemma 3.4.4).

Two key results, a geometric version of Hensel's lemma (Corollary 3.7.3) and the Greenberg lifting theorem (Proposition 3.7.4) hold also for good D_k-schemes.

7.2. Motivic integration

For a good D_k-scheme X, we can develop motivic integration on $\mathrm{J}_\infty(X/\mathrm{D}_k)$ in the same way as we did for k-schemes. But note that we use the sheaf of differentials Ω_{X/D_k} over D_k rather than $\Omega_{X/k}$. This sheaf is locally free of rank d on an open dense subset, since X is generically D_k-smooth.

For example, we define the *Jacobian ideal sheaf of X,* denoted by $\mathrm{Jac}_{X/\mathrm{D}_k}$, to be the d-th Fitting ideal $\mathrm{Fitt}_d(\Omega_{X/\mathrm{D}_k})$ of Ω_{X/D_k} (cf. Definition 3.6.3) and denote its order function by j_X. For a generically étale morphism $f\colon Y \to X$, we define the *Jacobian order of f at an f-ordinary arc* α to be

$$\mathrm{j}_f(\alpha) := \dim_L \frac{\alpha^\flat \Omega^d_{Y/\mathrm{D}_k}}{(f \circ \alpha)^\flat \Omega^d_{X/\mathrm{D}_k}}$$

(cf. Definition 5.8.2).

Cylinders and negligible subsets are defined in the same way as in Definitions 3.8.1 and 5.6.1 respectively. For an ordinary cylinder $C \subset \mathrm{J}_\infty(X/\mathrm{D}_k)$, we define its measure by $\mu_X(C)$ to be

$$\mu_X(C) := \{\pi_n(C)\}\mathbb{L}^{-nd} \in \widehat{\mathcal{M}}_k \quad (n \gg 0).$$

Then we can define admissible functions

$$h\colon\ \mathrm{J}_\infty(X/\mathrm{D}_k)\supset A\to\frac{1}{r}\mathbb{Z}\cup\{\infty\}$$

as well as the associated integrals

$$\int_A \mathbb{L}^h\,d\mu_X\in\widehat{\mathcal{M}}_{k,r}\cup\{\infty_*\}$$

exactly as in Definition 5.7.1

REMARK 7.2.1. Since jet schemes are k-schemes, measures and integrals above take values in the ring $\widehat{\mathcal{M}}_{k,r}$ constructed from k-schemes rather than the one constructed from $k[\![t]\!]$-schemes.

For a generically étale morphism $f\colon Y\to X$ of good D_k-schemes, we can prove the change of variables formula in the same way as in the case of k-schemes. Although it is exactly the same as before, we restate it here:

THEOREM 7.2.2 (The change of variables formula; cf. Theorem 5.13.2). *Let $f\colon Y\to X$ be a generically étale morphism of good D_k-schemes. Let $B\subset\mathrm{J}_\infty(Y/\mathrm{D}_k)$ be a subset such that $f_\infty|_B$ is almost geometrically injective. Let $h\colon f_\infty(B)\to\frac{1}{r}\mathbb{Z}\cup\{\infty\}$ be an admissible function. Then*

$$\int_{f_\infty(B)}\mathbb{L}^h\,d\mu_X=\int_B\mathbb{L}^{h\circ f_\infty-\mathrm{j}_f}\,d\mu_Y.$$

7.3. Stringy motives

It is also straightforward to generalize the definition of weak log pairs $(X,\mathcal{L})$ (Definition 6.7.1) to the context of D_k-schemes. For a weak log pair $(X,\mathcal{L})$ and for a constructible subset C of the special fiber X_0, we can define the stringy motive along C as

$$\mathrm{M_{st}}(X,\mathcal{L})_C:=\int_{\pi_0^{-1}(C)}\mathbb{L}^{\frac{1}{r}\operatorname{ord}\mathcal{I}_{X,\mathcal{L},r}}\,d\mu_X\in\widehat{\mathcal{M}}_{k,r}\cup\{\infty\}.$$

Here the fractional ideal sheaf $\mathcal{I}_{X,\mathcal{L},r}$ is given by

$$\operatorname{Im}\Big((\Omega^d_{X/\mathrm{D}_k})^{\otimes r}\to(\Omega^d_{X/\mathrm{D}_k})^{\otimes r}\otimes_{\mathcal{O}_X}K(X)\Big)=\mathcal{I}_{X,\mathcal{L},r}\cdot\mathcal{L}.$$

When X is normal and D_k-smooth in codimension one, then a weak log pair corresponds to a log pair (X,D), where D is a $\mathbb{Q}$-divisor such that $K_{X/\mathrm{D}_k}+D$ is $\mathbb{Q}$-Cartier. In this case, we write $\mathrm{M_{st}}(X,\mathcal{L})_C$ also as $\mathrm{M_{st}}(X,D)_C$. In particular, if K_{X/D_k} is $\mathbb{Q}$-Cartier, or equivalently if $\omega^{[r]}_{X/\mathrm{D}_k}:=(\Omega^d_{X/\mathrm{D}_k})^{\vee\vee}$ is invertible for some positive integer r, then

$$\mathrm{M_{st}}(X)_C=\mathrm{M_{st}}(X,0)_C=\mathrm{M_{st}}(X,\omega^{[r]}_{X/\mathrm{D}_k})_C.$$

REMARK 7.3.1. The above definition generalizes stringy motives of weak log pairs over k. To a weak log pair $(W,\mathcal{L})$ over k, we can associate a weak log pair $(W\times_k\mathrm{D}_k,\pi^*\mathcal{L})$ over D_k, where π is the projection $W\times_k\mathrm{D}_k\to W$. For a constructible subset $C\subset W=(W\times_k\mathrm{D}_k)_0$, we have $\mathrm{M_{st}}(W,\mathcal{L})_C=\mathrm{M_{st}}(W\times_k\mathrm{D}_k,\pi^*\mathcal{L})_C$.

LEMMA 7.3.2. *Suppose that X is regular. Let $\gamma\colon\mathrm{D}_L\to X$ be an arc for a separable field extension L/k. Then X is D_k-smooth along the image of γ.*

PROOF. Since L/k is separable, $X_L := X \otimes_k L$ is also regular [**Gro65**, Prop. 6.7.4]. Let $x \in X_L$ be the image of the closed point of D_L by the morphism $\mathrm{D}_L \to X_L$ induced by γ. The map $\gamma^* \colon \mathcal{O}_{X_L,x} \to L[\![t]\!]$ induces a surjection

$$L^d \cong m_x/m_x^2 \twoheadrightarrow \langle t \rangle / \langle t^2 \rangle \cong L.$$

Let $x_1, \dots, x_{d-1} \in m_x$ be generators of the kernel of this map. Then $t, x_1, \dots, x_d$ are a regular system of parameters. Therefore $\widehat{\mathcal{O}_{X_L,x}} \cong L[\![t, x_1, \dots, x_d]\!]$ as $L[\![t]\!]$-algebras, which shows that X_L is D_L-smooth at x. Since being D_L-smooth is an open condition, X_L is D_L-smooth also at the image of the generic point of D_L. The lemma follows again from [**Gro65**, Prop. 6.7.4]. □

PROPOSITION 7.3.3. *Let $C \subset X_0$ be a constructible subset.*

(1) *If X is D_k-smooth, then $\mathrm{M}_{\mathrm{st}}(X)_C = \{C\}$.*
(2) *Suppose that X is regular and that k is perfect. Let $X_{\mathrm{sm}} \subset X$ be the D_k-smooth locus. Then we have $\mathrm{J}_\infty(X/\mathrm{D}_k) = \mathrm{J}_\infty(X_{\mathrm{sm}}/\mathrm{D}_k)$ and $\mathrm{M}_{\mathrm{st}}(X)_C = \{C \cap X_{\mathrm{sm}}\}$.*

PROOF. (1) The function $\frac{1}{r} \operatorname{ord}_{\mathcal{I}_{X,0,r}}$ in the definition of $\mathrm{M}_{\mathrm{st}}(X)_C = \mathrm{M}_{\mathrm{st}}(X,0)_C$ is constantly zero. Therefore

$$\mathrm{M}_{\mathrm{st}}(X)_C = \int_{\pi_0^{-1}(C)} d\mu_X = \mu_X(\pi_0^{-1}(C)).$$

Truncation maps $\pi_n^{n'} \colon \mathrm{J}_{n'}(X/\mathrm{D}_k) \to \mathrm{J}_n(X/\mathrm{D}_k)$ are Zariski locally trivial $\mathbb{A}^d$-bundles. Therefore $\mu_X(\pi_0^{-1}(C)) = \{C\}$.

(2) Lemma 7.3.2 shows that $\mathrm{J}_\infty(X/\mathrm{D}_k) = \mathrm{J}_\infty(X_{\mathrm{sm}}/\mathrm{D}_k)$, which implies

$$\mathrm{M}_{\mathrm{st}}(X)_C = \mathrm{M}_{\mathrm{st}}(X_{\mathrm{sm}})_{C \cap X_{\mathrm{sm}}} = \{C \cap X_{\mathrm{sm}}\}.$$

□

The most important property of stringy motives, invariance under crepant proper birational morphisms, is still valid in the present generalized situation:

PROPOSITION 7.3.4 (cf. Proposition 6.4.6). *Suppose that $f \colon (Y, \mathcal{L}') \to (X, \mathcal{L})$ is a crepant, proper and birational morphism of weak log pairs over D_k. Then*

$$\mathrm{M}_{\mathrm{st}}(X, \mathcal{L})_C = \mathrm{M}_{\mathrm{st}}(Y, \mathcal{L}')_{f^{-1}(C)}.$$

7.4. Explicit formula

Recall that for a smooth k-variety W and a $\mathbb{Q}$-divisor D on W with simple normal crossing support, the stringy motive $\mathrm{M}_{\mathrm{st}}(W, D) := \int_{\pi_0^{-1}(C)} \mathbb{L}^{\operatorname{ord}_D} d\mu_W$ is described by the explicit formula, Proposition 4.7.4 (see also Remark 6.3.7). We now generalize it to D_k-schemes. To do so, we have to be careful about a distinction between regularity and smoothness and one between vertical prime divisors and horizontal prime divisors; a prime divisor on a D_k-scheme is called *vertical* if it is contained in the special fiber, otherwise called *horizontal*.

EXAMPLE 7.4.1. The special fiber X_0 is defined, as a closed subscheme of X, by the ideal sheaf $\mathcal{O}_X \cdot t \subset \mathcal{O}_X$, where t is the given parameter of $\mathrm{D}_k = \operatorname{Spec} k[\![t]\!]$. Therefore the order function $\operatorname{ord}_{X_0} \colon \mathrm{J}_\infty(X/\mathrm{D}_k) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is constantly 1. If X is D_k-smooth, then for each $a \in \mathbb{Q}$, we have

$$\mathrm{M}_{\mathrm{st}}(X, aD)_C = \int_{\pi_0^{-1}(C)} \mathbb{L}^a \, d\mu_X = \mu_X(\pi_0^{-1}(C))\mathbb{L}^a = \{C\}\mathbb{L}^a.$$

Consider a good D_k-scheme X which is regular, a $\mathbb{Q}$-divisor D on X and a constructible subset $C \subset X_0$. Let $X_{\mathrm{sm}} \subset X$ be the D_k-smooth locus. From Lemma, we have $\mathrm{J}_\infty(X/\mathrm{D}_k) = \mathrm{J}_\infty(X_{\mathrm{sm}}/\mathrm{D}_k)$ and

$$\mathrm{M}_{\mathrm{st}}(X, D)_C = \mathrm{M}_{\mathrm{st}}(X_{\mathrm{sm}}, D|_{X_{\mathrm{sm}}})_{C \cap X_{\mathrm{sm}}}.$$

Let us write

$$D|_{X_{\mathrm{sm}}} = \sum_{h \in H} a_h A_h + \sum_{i \in I} b_i B_i,$$

where a_h and b_i are rational numbrers, A_h are vertical prime divsors of X_{sm} and B_i are horizontal prime divisors of X_{sm}. Suppose that $(X_0 \cap X_{\mathrm{sm}}) \cup \bigcup_{i \in I} B_i$ is a simple normal crossing divisor. Namely, each B_i is D_k-smooth and for each geometric point $x \in X(L)$ and for some local coordinates $t, x_1, \dots, x_d \in \mathcal{O}_{X_L, x}$, this divisor is defined by $tx_1 \dots x_l = 0$, $l \le d$ in a neighborhood of x.

Proposition 7.4.2. *With the above notation, if $b_j < 1$ for every j, then we have*

$$\mathrm{M}_{\mathrm{st}}(X, D)_C = \sum_{h \in H} \mathbb{L}^{a_h} \sum_{J \subset I} \{B_J^\circ \cap A_h \cap C\} \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{1 - b_j} - 1}.$$

Proof. By a similar argument, Lemma 4.7.3 is generalied as follows: For $\underline{m} = (m_i) \in (\mathbb{Z}_{\ge 0})^I$ and for any constructible subset $C' \subset X_0$,

$$\mu_X\left(\bigcap_{i \in I} \mathrm{ord}_{B_i}^{-1}(m_i) \cap \pi_0^{-1}(C')\right) = \{B^\circ_{\mathrm{Supp}(\underline{m})} \cap C'\}(\mathbb{L} - 1)^{\sharp \mathrm{Supp}(\underline{m})} \mathbb{L}^{-\sum_{i \in I} m_i}.$$

The same computation as in the proof of Proposition 4.7.4 shows that for each $h \in H$,

$$\mathrm{M}_{\mathrm{st}}\left(X, \sum_{i \in I} b_i B_i\right)_{A_h \cap C} = \sum_{J \subset I} \{B_J^\circ \cap A_h \cap C\} \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{1 - b_j} - 1}.$$

Note that since A_h, $h \in H$ are vertical prime divisors of X_{sm}, they are mutually disjoint. Moreover each A_h is defined by $t = 0$ in a neighborhood of it. Thus ord_{A_h} is constantly 1 on $\pi_0^{-1}(A_h)$, which implies

$$\mathrm{M}_{\mathrm{st}}\left(X, a_h A_h + \sum_{i \in I} b_i B_i\right)_{A_h \cap C} = \mathbb{L}^{a_h} \mathrm{M}_{\mathrm{st}}\left(X, \sum_{i \in I} b_i B_i\right)_{A_h \cap C}.$$

It follows that

$$\begin{aligned} \mathrm{M}_{\mathrm{st}}(X, D)_C &= \mathrm{M}_{\mathrm{st}}(X_{\mathrm{sm}}, D|_{X_{\mathrm{sm}}})_{C \cap X_{\mathrm{sm}}} \\ &= \sum_{h \in H} \mathbb{L}^{a_h} \mathrm{M}_{\mathrm{st}}\left(X, \sum_{i \in I} b_i B_i\right)_{A_h \cap C} \\ &= \sum_{h \in H} \mathbb{L}^{a_h} \sum_{J \subset I} \{B_J^\circ \cap A_h \cap C\} \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{1 - b_j} - 1}. \end{aligned}$$

$\square$

7.5. Mixed characteristics

7.6. p-adic measures

Let K be a non-archimedian local field, that is, a finite extension of $\mathbb{F}_p[\![t]\!]$ or $\mathbb{Z}_p$ and let $\mathcal{O}_K$ be its integer/valuation ring. If X is a smooth $\mathcal{O}_K$-scheme of pure relative dimension d, then the $\mathcal{O}_K$-point set $X(\mathcal{O}_K)$ has a structure of K-analytic manifold. A local generator ω of $\Omega^d_{X/\mathcal{O}_K}$ on an open scheme $U \subset X$ defines a measure ν_ω on $U(\mathcal{O}_K)$ by

$$\nu_\omega(V) := \int_V |\omega| \quad (V \subset U(\mathcal{O}_K)).$$

Local measures defined in this way glue together to give a global measure ν_X on the total space $X(\mathcal{O}_K)$. This measure ν_ω is called a *p-adic or t-adic measure.* It satisfies

$$\nu_X(X(\mathcal{O}_K)) = \sharp X(k)/q^d,$$

where k is the residue field of K and q is its cardinality. We can regard this as an analogue of the fact that $\mu_X(\mathrm{J}_\infty(X/\mathcal{O}_K)) = \{X_0\}$. Indeed, if the motivic measure $\mu_X(\mathrm{J}_\infty(X/\mathcal{O}_K))$ was defined in $\mathcal{M}_k$ rather than in its completion $\widehat{\mathcal{M}}_k$, then the p-adic (or t-adic) measure $\nu_X(X(\mathcal{O}_K))$ is the image of the motivic measure by the point-counting realization $\sharp\colon \mathcal{M}_k \to \mathbb{Q}$ followed by multiplication with q^d.

Unfortunately the realization map $\sharp$ does not extend to a map from the complection $\widehat{\mathcal{M}}_k$ and we cannot get p-adic measures directly from motivic measures, see discussion in [**RÏ1**]. However, we can define the point-counting version of stringy motives by parallel arguments in the context of p-adic measures instead of motivic measures as follows. Let us $(X, \mathcal{L})$ be a weak log pair with $\mathcal{L} \subset \left(\Omega^d_{X/\mathcal{O}_K}\right)^{\otimes r} \otimes_{\mathcal{O}_X} K(X)$. We can define a measure $\nu_\mathcal{L}$ on $X(\mathcal{O}_K)^\circ := X(\mathcal{O}_K) \cap X_{\mathrm{sm}}(K)$ as follows: for a local generator ω of $\mathcal{L}$, we define a local measure ν_ω by

$$\nu_\omega(V) := \int_V |\omega|^{1/r}$$

and define $\nu_\mathcal{L}$ by gluing local measures ν_ω.

REMARK 7.6.1. Cluckers and Loeser [**CL15**, **CL10**, **CL08**] have developped a new model-theoretic framework of motivic integration in charactertic zero and mixed characteristic, which avoids the completion of the Grothendieck ring and thus specializes to p-adic integration. However, in the present book, we stick to the traditional framework, that is, the so-called geometric motivic integration, since positive characterics are one of our main interests and since it appears very laborious to combine the model-theoretic approach with the theory of algebraic stacks.

DEFINITION 7.6.2. Let $(X, \mathcal{L})$ be a weak log pair. For a constructible subset $C \subset X_0$, let $X(\mathcal{O}_K)^\circ_C \subset X(\mathcal{O}_K)^\circ$ be the subset of $\mathcal{O}_K$-points which sends the closed point $\operatorname{Spec} \mathcal{O}_K$ into C. We define the *stringy point count* of $(X, \mathcal{L})$ along C to be

$$\sharp_{\mathrm{st}}(X, \mathcal{L})_C := q^d \cdot \nu_\mathcal{L}(X(\mathcal{O}_K)^\circ_C) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Morally the stringy point count $\sharp_{\mathrm{st}}(X, \mathcal{L})_C$ would be interpreted as the point counting realization of the stringy motive $\mathrm{M}_{\mathrm{st}}(X, \mathcal{L})_C$, which is not rigorously justified because of the completion problem mentioned above.

PROPOSITION 7.6.3. (1) *If X is regular, then* $\sharp_{\mathrm{st}}(X)_C = \sharp(X_{\mathrm{sm}} \cap C)(k)$.
(2) *If* $(Y, \mathcal{L}') \to (X, \mathcal{L})$ *is crepant, then* $\sharp_{\mathrm{st}}(X, \mathcal{L})_C = \sharp_{\mathrm{st}}(Y, \mathcal{L}')_{f^{-1}(C)}$.
(3) *If a log pair* (X, D) *satisfies the situation of Propotision 7.4.2, then*

$$\sharp_{\mathrm{st}}(X, D)_C = \sum_{h \in H} q^{a_h} \sum_{J \subset I} \sharp(B_J^\circ \cap A_h \cap C)(k) \prod_{j \in J} \frac{q-1}{q^{1-b_j}-1}.$$

REMARK 7.6.4. If a weak log pair $(X, \mathcal{L})$ is K-equivalent to a log pair (Y, E) satisfying the situation of Propotision 7.4.2, then $\mathrm{M}_{\mathrm{st}}(X, \mathcal{L}) = \mathrm{M}_{\mathrm{st}}(Y, E)$ lies in

$$R := \mathrm{Im}\left(\mathcal{M}_{k,r}\left[\frac{1}{\mathbb{L}^e - 1} \mid e \in \frac{1}{r}\mathbb{Z}\right] \to \widehat{\mathcal{M}}_{k,r}\right).$$

The point counting realization $\sharp\colon \mathcal{M}_k \to \mathbb{Q}$ extends to a map $\sharp\colon R \to \mathbb{Q}$ and get

$$\sharp(\mathrm{M}_{\mathrm{st}}(X, \mathcal{L})_C) = \sharp_{\mathrm{st}}(X, \mathcal{L})_C.$$

However, to construct such a log pair (Y, E), we would need resolution of singularities for schemes over $\mathcal{O}_K$, which has not been obtained yet.

CHAPTER 8

The McKay correspondence: the tame case

We follow the following convenion on group actions: groups act on schemes from left and on rings and modules from right, unless otherwise noted. When a group G acts on an affine scheme $\operatorname{Spec} R$, the G-action on R is given by $r \cdot g := g^*(r)$, where g^* is the pull-back map $R \to R$ associated to the automorphism $g\colon \operatorname{Spec} R \to \operatorname{Spec} R$.

In this section, G always denotes a finite group. We assume that the base field k contains the l-th roots of unity with l the exponent of G.

8.1. The original McKay correspondence

Consider a finite subgroup $1 \neq G \subset \mathrm{SL}_2(\mathbb{C})$. It acts on $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ say through the action on the linear part $\mathbb{C}[x, y]_1 = \mathbb{C}x \oplus \mathbb{C}y = \mathbb{C}^2$ of $\mathbb{C}[x, y]$ by right multiplication of matrices. Note that since it has finite order, each element $g \in G$ can be diagonalized say to

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix}.$$

Since $\zeta\zeta' = 1$, if $g \neq 1$, then neither ζ or ζ' is eqaul to 1. This shows that the G-action on $\mathbb{A}^2_{\mathbb{C}}/G$ is free and the quotient variety $X := \mathbb{A}^2_{\mathbb{C}}/G = \operatorname{Spec} \mathbb{C}[x, y]^G$ has an isolated singularity at the image of the origin. Singularities appearing by this construction are known by many different names; rational double points, Du Val singularities, Kleinian singularities, ADE singularities, and so on.

The exceptional locus of the minimal resolution $Y \to X$ is a union of projective lines which are normal crossing (for example, see [**Ish18**, Theorem 7.5.1]). The associated *dual graph* has vertices corresponding to these projective lines and edges corresponding to intersection of two projective lines. It is known that the resulting dual graph is one of Dynkin diagrams of types A_n ($n \geq 1$), D_n ($n \geq 4$) and E_n ($n = 6, 7, 8$) (see Figure 8.1.1).

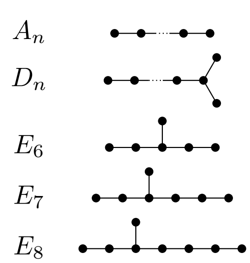

FIGURE 8.1.1. Dynkin diagrams

McKay [**McK80**] observed that the same graph is obtained in a purely representation theoretic way. Let W be the two-dimensional G-representation induced from the inclusion $G \subset \mathrm{SL}_2(\mathbb{C})$ and let $V_1, \ldots, V_l$ be the irreducible G-representations. Here representations are defined over $\mathbb{C}$. For each $i \in \{1, \ldots, l\}$, we can write

$$V_i \otimes W = \bigoplus_{j=1}^{l} V_j^{\oplus n_{ij}}.$$

In our situation, we have that $n_{ij} = n_{ji}$ and $n_{ij} \in \{0, 1\}$. The McKay graph associated to the represetation W has vertices $v_1, \ldots, v_l$ corresponding to irreducible representations $V_1, \ldots, V_l$ respectively. We connect two vertices v_i and v_j by an edge when $n_{ij} = n_{ji} = 1$. The resulting graph is the one of extended Dynkin diagrams (Figure 8.1.2) that corresponds to the Dykin diagram constructed above in terms of the minimal resolution of $\mathbb{A}^2_{\mathbb{C}}/G$. The extra vertex, the circle "$\circ$" in the figure, corresponds to the trivial irreducible representation of dimension one.

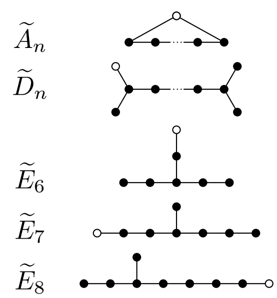

FIGURE 8.1.2. Extended Dynkin diagrams

Let us now recall the following result from representation theory.

LEMMA 8.1.1 ([**CR06**, (27.22)]). *Let G be a finite group and let k be an algebraically closed field such that $\mathrm{char}(k) \nmid \sharp G$. Then the number of irreducible G-representations over k is equal to $\sharp\,\mathrm{Conj}(G)$, where $\mathrm{Conj}(G)$ denotes the set of conjugacy classes of G.*

As an easy consequence of the above observation, we obtain the following proposition, which will be generalized to higher dimensions later:

PROPOSITION 8.1.2. *With notation as above, $\mathrm{e_{top}}(Y) = \sharp\,\mathrm{Conj}(G)$.*

PROOF. Let $E \subset Y$ be the exceptional locus of f. Since $\mathbb{A}^2_{\mathbb{C}} \setminus \{o\} \to X \setminus \{\bar{o}\}$ is an étale finite cover of degree $\sharp G$, we have

$$\begin{aligned} \mathrm{e_{top}}(Y \setminus E) &= \mathrm{e_{top}}(X \setminus \{\bar{o}\}) \\ &= (\sharp G)^{-1} \cdot \mathrm{e_{top}}(\mathbb{A}^2_{\mathbb{C}} \setminus \{o\}) \\ &= (\sharp G)^{-1} \left(\mathrm{e_{top}}(\mathbb{A}^2_{\mathbb{C}}) - \mathrm{e_{top}}(\{o\})\right) \\ &= 0. \end{aligned}$$

Let n be the number of irreducible representations of G. From Lemma 8.1.1, this is also the number of conjugacy classes in G. From McKay's observation above, E has $(n-1)$-irreducible components. Moreover it has $(n-2)$ nodes, corresponding to edges of the Dynkin diagram. Therefore

$$\begin{aligned}\mathrm{e_{top}}(E) &= (n-1)\cdot \mathrm{e_{top}}(\mathbb{P}) - (n-2)\\ &= 2(n-1)-(n-2)\\ &= n.\end{aligned}$$

Thus

$$\mathrm{e_{top}}(Y) = \mathrm{e_{top}}(Y \setminus E) + \mathrm{e_{top}}(E) = n.$$

□

8.2. Pseudo-$\mathbb{A}^n$-bundles

For the rest of this chapter, we work over a field k. Later we will add extra conditions on k when necessary.

For a piecewise trivial $\mathbb{A}^n$-bundle $Y \to X$, we have $\{Y\} = \{X\}\mathbb{L}^n$ in $\mathrm{K}_0(\mathbf{Var}_k)$ and in $\widehat{\mathcal{M}}_k$. In application to situations involving finite group actions such as the McKay correspondenced, we need to have the same equality for more general bundles, that is, pseudo-$\mathbb{A}^n$-bundles.

Definition 8.2.1. A morphism $f\colon Y \to X$ of schemes is called a *universal homeomorphism* if for every morphism $X' \to X$ of schemes, the induced morphism $Y \times_X X' \to X'$ gives a homeomorphism of underlying topological spaces.

Definition 8.2.2. Let $f\colon Y \to X$ be a morphism of k-varieties. We say that f is a *pseudo-$\mathbb{A}^n$-bundle* if for every geometric point $x\colon \operatorname{Spec} L \to X$, there exists a universal homeomorphism $\mathbb{A}^n_L/H \to f^{-1}(x)$ over L, where $\mathbb{A}^n_L/H$ is the quotient variety for some finite group action $H \curvearrowright \mathbb{A}^n_L$. More generally, consider a morphism $f\colon W \to V$ of k-varieties and let $D \subset W$ and $C \subset V$ be constructible subsets with $f(D) \subset C$. The map $f|_D\colon D \to C$ is a *pseudo-$\mathbb{A}^n$-bundle* if there exists a stratification $C = \bigsqcup_{i=1}^{l} C_i$ by locally closed subsets $C_i \subset V$ such that for each i, the preimage $(f|_D)^{-1}(C_i) \subset W$ is a locally closed subset and the morphism $(f|_D)^{-1}(C_i) \to C_i$ is a pseudo-$\mathbb{A}^n$-bundle.

Definition 8.2.3. We denote by $\widehat{\mathcal{M}}'_{k,r}$ the quotient ring of $\widehat{\mathcal{M}}_{k,r}$ modulo the following relation: if a morphism $Y \to X$ of k-varieties is a pseudo-$\mathbb{A}^n$-bundle, then $\{Y\} = \{X\}\mathbb{L}^n$. When $r = 1$, we usually omit the subscript r and simply write $\widehat{\mathcal{M}}'_k$.

Since every universal homeomorphism $Y \to X$ of k-varieties is a pseudo-$\mathbb{A}^0$-bundle, we have $\{Y\} = \{X\}$ in $\widehat{\mathcal{M}}'_k$. The following proposition shows that the new relation imposed in this definition is reasonable.

In what follows, we denote by $\overline{X}$ the base change $X \otimes_k k^{\mathrm{sep}}$ of X to a separable closure k^{sep} of k.

Lemma 8.2.4. *Let G be a finite group acting on $\mathbb{A}^d_k$ and let $X := \mathbb{A}^d_k/G$. Then, for every $i \in \mathbb{Z}$, we have isomorphisms of $\mathfrak{G}_k$-representations:*

$$\mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) \cong \begin{cases}\mathbb{Q}_l(-d) & (i = 2d)\\ 0 & (i \neq 2d)\end{cases}$$

If $k = \mathbb{C}$, we also have isomorphisms of mixed Hodge structures:

$$\mathrm{H}^i_{\mathrm{c}}(X, \mathbb{Q}) \cong \begin{cases}\mathbb{Q}(-d) & (i = 2d)\\ 0 & (i \neq 2d)\end{cases}$$

PROOF. First consider the case $k = \mathbb{C}$. As is well-known, we have

$$\mathrm{H}^i_{\mathrm{c}}(X, \mathbb{Q}) \cong \mathrm{H}^i_{\mathrm{c}}(X, (\pi_* \mathbb{Q}_{\mathbb{C}^d})^G) \cong \mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{\mathbb{C}}, \mathbb{Q})^G.$$

Indeed, this follows from [**Gro57**, p. 202] and the isomorphism of sheaves $\mathbb{Q}_X \cong (\pi_* \mathbb{Q}_{\mathbb{C}^d})^G$. From Lemma 2.3.3 and the fact that the G-action on on $\mathrm{H}^{2d}_{\mathrm{c}}(\mathbb{A}^d_{\mathbb{C}}, \mathbb{Q})$ is trivial, we get that $\mathrm{H}^i_{\mathrm{c}}(X, \mathbb{Q}) = 0$ for $i \neq 2d$ and $\mathrm{H}^{2d}_{\mathrm{c}}(X, \mathbb{Q})$ is one-dimensional. Since $\mathrm{H}^{2d}_{\mathrm{c}}(X, \mathbb{Q}) \to \mathrm{H}^{2d}_{\mathrm{c}}(\mathbb{C}^d, \mathbb{Q})$ is an isomorphism of mixed Hodge structures, we conclude $\mathrm{H}^{2d}_{\mathrm{c}}(X, \mathbb{Q}) \cong \mathbb{Q}(-d)$.

Next consider the case of a general field k. From [**Mil80**, Prop. 11. 8], we have

$$\mathrm{H}^{2d}_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) \cong \mathrm{H}^{2d}_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l) \cong \mathbb{Q}_l(-d).$$

We also have

$$\mathrm{H}^i_{\mathrm{c}}(\overline{X}, f_* f^* \mathbb{Q}_{l,X}) \cong \mathrm{H}^i_{\mathrm{c}}(\overline{X}, f_* \mathbb{Q}_{l, \mathbb{A}^d_{k^{\mathrm{sep}}}}) \cong \mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l).$$

Here the right isomorphism follows from the fact that $\mathrm{R}^i f_! \mathbb{Q}_l = \mathrm{R}^i f_* \mathbb{Q}_l = 0$ for $i \neq 0$ and the spectral sequence $E_2^{i,j} = \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathrm{R}^j f_! \mathbb{Q}_l) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l)$. Let m be the degree of f. From [**MR073**, XVII, 6.2.5 and 6.2.6], the composite map

$$\mathbb{Q}_{l,\overline{X}} \to f_* f^* \mathbb{Q}_{l,\overline{X}} \xrightarrow{\mathrm{Tr}_f} \mathbb{Q}_{l,\overline{X}}$$

of $\mathbb{Q}_l$-sheaves on $\overline{X}$ is multiplication by m. We get maps

$$\mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) \to \mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l) \to \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l)$$

whose composition is multiplication by m. This shows that the map $\mathrm{H}^i_{\mathrm{c}}(\mathbb{A}^d_{k^{\mathrm{sep}}}, \mathbb{Q}_l) \to \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l)$ is surjective. It follows that $\mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) = 0$ for $i \neq 2d$. ☐

LEMMA 8.2.5. *Let $f \colon Y \to X$ be a universal homeomorphism of k-varieties. Then*

$$\mathrm{H}^i_{\mathrm{c}}(\overline{Y}, \mathbb{Q}_l) \cong \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l).$$

If $k = \mathbb{C}$, we also have

$$\mathrm{H}^i_{\mathrm{c}}(Y, \mathbb{Q}) \cong \mathrm{H}^{i+2n}_{\mathrm{c}}(X, \mathbb{Q}).$$

PROOF. The second assertion is clear, as f is a homeomorphism in the ananlytic topology. For the first assertion, $\mathrm{R}^i f_! \mathbb{Q}_{l,\overline{Y}} = 0$ for $i \neq 0$ and from [**MR073**, XVII, 6.2.5], we have a map

$$f_! \mathbb{Q}_{l,\overline{Y}} \to \mathbb{Q}_{l,\overline{X}}.$$

It is easy to see that the last map is an isomorphism. The desired isomorphism follows from the spectral sequence $E_2^{i,j} = \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathrm{R}^j f_* \mathbb{Q}_l) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{c}}(\overline{Y}, \mathbb{Q}_l)$. ☐

LEMMA 8.2.6. *Let $f \colon Y \to X$ be a flat morphism of k-varieties which is a pseudo-$\mathbb{A}^n$-bundle. Then*

$$\mathrm{H}^i_{\mathrm{c}}(\overline{Y}, \mathbb{Q}_l) \cong \mathrm{H}^{i+2n}_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) \otimes \mathbb{Q}_l(-n).$$

Similarly, if $k = \mathbb{C}$ and if the smooth locus of f in Y surjects onto X, then

$$\mathrm{H}^i_{\mathrm{c}}(Y, \mathbb{Q}) \cong \mathrm{H}^{i+2n}_{\mathrm{c}}(X, \mathbb{Q}) \otimes \mathbb{Q}(-n).$$

PROOF. From [**MR073**, XVIII, Th. 2.9], we have the trace map

$$\mathrm{Tr}_f \colon \mathrm{R}^{2n} f_! \mathbb{Q}_{l,\overline{Y}} \to \mathbb{Q}_l(-n)_{\overline{X}}. \tag{8.2.1}$$

For each geometric point x of X, this induces the trace map of $f^{-1}(x)$,

$$\mathrm{H}^{2n}_{\mathrm{c}}(f^{-1}(x), \mathbb{Q}_l) \to \mathbb{Q}_l(-n),$$

which is an isomorphism from [**Mil80**, Lem. 11.3]. It follows that (8.2.1) is an isomorphism. From Lemmas 8.2.4 and 8.2.5, $\mathrm{R}^i f_! \mathbb{Q}_{l,\overline{Y}} = 0$ for $i \neq 2n$. From the spectral sequence

$$E_2^{i,j} = \mathrm{H}^i_{\mathrm{c}}(\overline{X}, \mathrm{R}^j f_! \mathbb{Q}_{l,\overline{Y}}) \Rightarrow \mathrm{H}^i_{\mathrm{c}}(\overline{Y}, \mathbb{Q}_l),$$

we get

$$\mathrm{H}^i_{\mathrm{c}}(\overline{Y}, \mathbb{Q}_l) \cong \mathrm{H}^{i+2n}_{\mathrm{c}}(\overline{X}, \mathrm{R}^{2n} f_! \mathbb{Q}_{l,\overline{Y}}) \cong \mathrm{H}^{i+2n}_{\mathrm{c}}(\overline{X}, \mathbb{Q}_l) \otimes \mathbb{Q}_l(-n).$$

The proof of the second assertion is parallel except that there seems no reference for the trace map corresponding to (8.2.1) in the same generality; we only have a reference in the case of smooth morphisms [**Bil17**, Prop. 4.1.5.4]. We show that the trace map exists in our situation by reducing to the smooth case. Let $U \subset Y$ be the smooth locus of f. From the result mentioned above, we have the trace map $\mathrm{R}^{2n} f_! \mathbb{Q}_U \to \mathbb{Q}(-n)_X$ of mixed Hodge modules. We also have the exact sequence

$$0 = \mathrm{R}^{2n-1} f_! \mathbb{Q}_{Y \setminus U} \to \mathrm{R}^{2n} f_! \mathbb{Q}_U \to \mathrm{R}^{2n} f_! \mathbb{Q}_Y \to \mathrm{R}^{2n} f_! \mathbb{Q}_{Y \setminus U} = 0.$$

Here the two equalities hold, because every fiber of $Y \setminus U \to X$ has dimension $< n$. Thus we have $\mathrm{R}^{2n} f_! \mathbb{Q}_U \cong \mathrm{R}^{2n} f_! \mathbb{Q}_Y$ and get the desired trace morphism $\mathrm{R}^{2n} f_! \mathbb{Q}_Y \to \mathbb{Q}(-n)_X$. The second assertion is then proved by parallel arguments using mixed Hodge modules. □

PROPOSITION 8.2.7. *Let $f\colon Y \to X$ be a pseudo-$\mathbb{A}^n$-bundle of either k-varieties or constructible subsets. Then*

$$\chi_l(Y) = \chi_l(X \times_k \mathbb{A}^d_k) \text{ in } \mathrm{K}_0(\mathbf{Rep}_l(\mathfrak{G}_k)).$$

If $k = \mathbb{C}$, we also have

$$\chi_{\mathrm{Hodge}}(Y) = \chi_{\mathrm{Hodge}}(X \times_k \mathbb{A}^n_k) \text{ in } \mathrm{K}_0(\mathbf{MHS}).$$

PROOF. We first consider the first assertion. From the additivity of χ_l, it suffices to consider the case where f is a morphism of k-varieties. From the generic flatness, we may also assume that f is flat. In this situation, the assertion follows from Lemma 8.2.6. For the second assertion, let $U \subset Y$ be the smooth locus of f. Then $f(U)$ is an open dense subset of X. From Lemma 8.2.6, we have

$$\chi_{\mathrm{Hodge}}(U) = \chi_{\mathrm{Hodge}}(f(U) \times_k \mathbb{A}^n_k).$$

We can show the desired equality by induction. □

COROLLARY 8.2.8. *Consider realization maps* P, E, χ_{Hodge} *and* χ_l *from* $\widehat{\mathcal{M}}_k$ *in Section 2.5 uniquely factor through* $\widehat{\mathcal{M}}'_k$.

PROOF. The uniqueness follows from the fact that $\widehat{\mathcal{M}}'_k$ is a quotient of $\widehat{\mathcal{M}}_k$. If χ denote any of these realization maps, then we claim that for a pseudo-$\mathbb{A}^n$-bundle $Y \to X$, $\chi(\{Y\}) = \chi(\{X\}\mathbb{L}^n)$, which proves the corollary.

For χ_{Hodge} and χ_l, the claim follows from the last proposition. For E, the claim follows from the one for χ_{Hodge}. For P, if k is finitely generated, the claim follows from the one for χ_l. For a general field k, every pseudo-$\mathbb{A}^n$-bundle $Y \to X$ is the base change of a morphism $Y' \to X'$ of k'-varieties such that k' is a finitely generated subfield of k. The morphism $Y' \to X'$ is again a pseudo-$\mathbb{A}^n$-bundle and $\mathrm{P}(\{Y'\}) = \mathrm{P}(\{X'\}\mathbb{L}^n)$. Since the Poincaré polynomial is stable under the bsae change [**Nic11**, Lem. 8.9], we get the equality for $Y \to X$. □

8.3. The motivic McKay correspondence: the tame case

In this section, we generalize Proposition 8.1.2 to higher dimensions under the tameness assumption. Let k be a field of characteristic $p \geq 0$ and let $G \subset \mathrm{GL}_d(k)$ be a finite subgroup. The G-action on the affine space $\mathbb{A}_k^d = \operatorname{Spec} k[x_1, \ldots, x_d]$ is then defined in the same way as in Section 8.1. Let X denote the quotient variety $\mathbb{A}_k^d/G$.

DEFINITION 8.3.1. An element $g \in \mathrm{GL}_d(k)$ of finite order is called *tame* if its order is not divisible by p; we follow the convention that 0 does not divide any positive integer so that everything is tame in characteristic zero. We say that a subgroup $G \subset \mathrm{GL}_d(k)$ and its action on $\mathbb{A}_k^d$ are *tame* if all the elements of G are tame, or equivalently, if $\sharp G$ is not divisible by p.

DEFINITION 8.3.2. An element $g \in G$ is called a *pseudo-reflection* if the fixed point locus $(\mathbb{A}_k^d)^g$ has codimension one. We say that $G \subset \mathrm{GL}_d(k)$ is a *small subgroup* if G has no pseudo-reflection.

LEMMA 8.3.3. *The following are equivalent:*

(1) *G is a small subgroup.*
(2) *The quotient morphism $\mathbb{A}_k^d \to X$ is étale in codimension one.*
(3) *The relative canonical divisor $K_{\mathbb{A}_k^d/X}$ is zero.*

PROOF. The equivalence (1)$\Leftrightarrow$(2) follows from the fact that the ramification locus of π is exactly $\bigcup_{g \in G \setminus \{1\}} (\mathbb{A}_k^d)^g$, the locus of points which are fixed by some nontrivial group element. Since $K_{\mathbb{A}_k^d/X}$ is supported in the ramification locus, we have (2)$\Rightarrow$(3). To show (3)$\Rightarrow$(2), it suffices to show that, outside a closed subset of codimension ≥ 2, $K_{\mathbb{A}_k^d/X}$ coincides with the ramification locus of π . Let $V \subset \mathbb{A}_k^d$ be the preimage of the smooth locus X_{sm}. Note that $X \setminus X_{\mathrm{sm}}$ and $\mathbb{A}_k^d \setminus V$ has codimension ≥ 2 in X and $\mathbb{A}_k^d$ respectively. Indeed, when k is algebraically closed, this follows from the normality of X. For a general field k, we only need to take the base change to an algebraically closure $\overline{k}$. Now $K_{V/X_{\mathrm{sm}}} = K_{\mathbb{A}_k^d/X}|_V$ is the effective divisor defined by the Jacobian ideal Jac_π, which is the 0-th Fitting ideal of $\Omega_{V/X_{\mathrm{sm}}}$. In particular, the support of $K_{V/X_{\mathrm{sm}}}$ is identical to that of $\Omega_{V/X_{\mathrm{sm}}}$. The latter is exactly the ramification locus of $V \to X_{\mathrm{sm}}$. $\square$

REMARK 8.3.4. If G is tame and contained in $\mathrm{SL}_d(k)$, then it is small. For, if $g \in G$ was a pseudo-reflection, then its diagonalization would be of the form $\mathrm{diag}(\zeta, 1, \ldots, 1)$, $\zeta \neq 1$, which is not an element of $\mathrm{SL}_d(k)$. But, there are wild subgroups of $\mathrm{SL}_d(k)$ which are not small. For example, if k has characteristic $p > 0$, then the elements of the subgroup

$$\mathbb{Z}/p\mathbb{Z} \cong \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(k)$$

are pseudo-reflections except the identity matrix.

Let l be the exponent of our finite group G, which is by definition the least common multiple of orders of elements $g \in G$. We suppose that k contains l-th roots of unity and fix a primitive l-th root $\zeta_l \in k$.

DEFINITION 8.3.5. Let $g \in G \subset \mathrm{GL}_d(k)$. We can diagonalize it over k, say into a diagonal matrix $\mathrm{diag}(\zeta_l^{a_1}, \ldots, \zeta_l^{a_d})$ with $0 \leq a_i < l$. Then we define the *age* of g

to be

$$\mathrm{age}(g) := \frac{1}{l}\sum_{i=1}^{d} a_i.$$

Note that the age of g is determined by eigenvalues of g. Thus it is preserved under conjugation in $\mathrm{GL}_d(\bar{k})$.

Remark 8.3.6 (A canonical definition of ages). The age of an element of G depends on the choice of the primitive l-th root ζ_l. If $\zeta_l' = \zeta_l^a$ is another primitive l-th root of unity (a is a poitive integer coprime to l) and if $\mathrm{age}' \colon G \to \mathbb{Q}$ is the age function defined by using ζ_l', then we have $\mathrm{age} \circ \alpha = \mathrm{age}'$, where α is the bijection $G \to G$, $g \mapsto g^a$. The map α induces a bijection $\mathrm{Conj}(G) \to \mathrm{Conj}(G)$ also denoted by α and the equality $\mathrm{age} \circ \alpha = \mathrm{age}'$ hold also on $\mathrm{Conj}(G)$. To define ages more canonically, we can associate the age to an embedding $\iota \colon \mu_m \hookrightarrow G \subset \mathrm{GL}_d(k)$ for a divisor m of l; if V_a denotes the 1-dimensional μ_m-representation given by $\zeta \cdot v = \zeta^a v$ for $\zeta \in \mu_m$ and $v \in V_a$ and if the d-dimensional μ_m-representation induced by ι is isomorphic to $\bigoplus_{i=1}^d V_{a_i}$, then we define

$$\mathrm{age}(\iota) := \frac{1}{l}\sum_{i=1}^{d} a_i.$$

A choice of a primitive l-th root ζ_l gives the primitive m-th root $\zeta_m := \zeta_l^{l/m}$, a generator of μ_m. We have the one-to-one correspondence:

$$\begin{aligned} \{\text{embeddings } \mu_m \hookrightarrow G\} &\leftrightarrow G \\ \iota &\mapsto \iota(\zeta_m) \end{aligned}$$

We easily see that this correspondence preserves ages.

Lemma 8.3.7. *We have that* $\mathrm{age}(g) \in \mathbb{Z}$ *if and only if* $g \in \mathrm{SL}_d(k)$.

Proof. With the above notaion, we have $\det(g) = \zeta_l^{\sum a_i}$. Thus $\det(g) = 1$ if and only if $\sum a_i$ is divisible by l. The latter is equivalent to that $\mathrm{age}(g) \in \mathbb{Z}$. $\square$

Theorem 8.3.8 (The motivic McKay correspondence: the tame case). *Let k be a field and let $G \subset \mathrm{GL}_d(k)$ be a tame finite subgroup with exponent l. Let A be the unique $\mathbb{Q}$-divisor on X such that $\psi^*(K_X + A) = K_{\mathbb{A}^d_k}$, where ψ denotes the morphism $\mathbb{A}^d_k \to X$. Suppose that k contains the l-th roots of unity. Then*

$$\mathrm{M}_{\mathrm{st}}(X, A) = \sum_{[g]\in \mathrm{Conj}(G)} \mathbb{L}^{d-\mathrm{age}(g)} \text{ in } \widehat{\mathcal{M}}'_{k,r}. \tag{8.3.1}$$

Here r is a positive integer such that $\mathrm{age}(g) \in \frac{1}{r}\mathbb{Z}$ for every $g \in G$.

The proof of this theorem will be given in Section 8.7. Note that if G is a small subgroup, then $A = 0$ and we have $\mathrm{M}_{\mathrm{st}}(X, A) = \mathrm{M}_{\mathrm{st}}(X)$. Note also that although the age of each $g \in G$ depends on the choice of the primitive l-th root ζ_l, the right hand side of on this page is independent of the choice. The assumpion on roots of unity is, on one hand, related to the Kummer theory. On the other hand, it is related to the fact that over such a field k, $g \in G$ is diagonalizable. Actually, k is a splitting field for G:

REMARK 8.3.9 (Splitting fields). A G-representation V over a field k is called *absolutely irreducible* if for every extension L/k, the scalar extension $V \otimes_k L$ is irreducible. A field k is a *splitting field for G* if every irreducible G-representation over k is absolutely irreducible. It is known that k is a splitting field for G if and only if every irreducible $\overline{k}$-representation is realizable over k [**CR06**, (70.3)]. From [**Bra45**], if k contains the l-th roots of 1 with l the exponent of G, then k is a splitting field for G. If G is moreover abelian and tame, then every irreducible G-representation over k is of dimension one.

For a finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$, the minimal resolution of $\mathbb{A}^2_{\mathbb{C}}/G$ is crepant (see [**Ish18**, Th. 7.5.1]). In higher dimensions, we use crepant resolutions in place of minimal resolutions. Note that in general, a quotient variety $\mathbb{A}^d_k/G$ may not have a crepant resolution or may have multiple crepant resolutions. The following theorem generalizes Proposition 8.1.2.

THEOREM 8.3.10 ([**Bat99**]). *Let $G \subset \mathrm{SL}_d(\mathbb{C})$ be a finite subgroup and let $X := \mathbb{A}^d_{\mathbb{C}}/G$. Suppose that there exists a crepant resolution $Y \to X$. Then $\mathrm{e}_{\mathrm{top}}(Y) = \sharp\,\mathrm{Conj}(G)$.*

PROOF. We apply Theorem 8.3.8 to the case where $k = \mathbb{C}$ and $G \subset \mathrm{SL}_d(\mathbb{C})$. If $Y \to X$ is a crepant resolution, we obtain

$$\{Y\} = \mathrm{M}_{\mathrm{st}}(X) = \sum_{[g]\in\mathrm{Conj}(G)} \mathbb{L}^{d-\mathrm{age}(g)}. \tag{8.3.2}$$

Taking the E-polynomial realization, we have

$$\mathrm{E}(Y; u, v) = \sum_{[g]\in\mathrm{Conj}(G)} (uv)^{d-\mathrm{age}(g)}$$

and

$$\mathrm{e}_{\mathrm{top}}(Y) = \mathrm{E}(Y; 1, 1) = \sharp\,\mathrm{Conj}(G).$$

□

Theorem 8.3.10 is refined as follows.

THEOREM 8.3.11 ([**IR96**,**Bat99**,**Yas06**]). *In the situation of Theorem 8.3.10, the cohomology groups $\mathrm{H}^{2i+1}(Y,\mathbb{Q})$ of odd degree vanish and*

$$\mathrm{H}^{2i}(Y,\mathbb{Q}) \cong \mathbb{Q}(-i)^{\oplus n_i},$$

where

$$n_i := \sharp\{[g] \in \mathrm{Conj}(G) \mid \mathrm{age}(g) = i\}.$$

SKETCH OF THE PROOF. This basically follows from equality (8.3.2) and the fact that $\mathrm{H}^i_c(Y,\mathbb{Q})$ have pure Hodge structures of weight i. To show the last fact, we first show that the $\mathbb{G}_m$-action on X lifts to Y. Then, using this action, we get a stratification $Y = \bigsqcup Y_i$ where each Y_i has structure of a vector bundle over a smooth proper variety. We see that cohomology groups of Y_i are pure, which implies the desired fact. □

8.4. Twisted arcs

The McKay correspondence can be regarded as the problem of describing invariants of a quotient variety $X = \mathbb{A}^d_k/G$ in terms of the G-action on $\mathbb{A}^d_k$ (without passing to the quotient). The stringy motive $\mathrm{M}_{\mathrm{st}}(X, A)$ (with notation of Theorem 8.3.8) is by definition a motivic integral on the arc space $\mathrm{J}_\infty X$. To describe it in terms of the G-action on the affine space, we would like to lift arcs of X to $\mathbb{A}^d_k$. However we cannot do it in the naive way; an arc $\mathrm{D}_L \to X$ does not generally lift to an arc $\mathrm{D}_L \to \mathbb{A}^d_k$.

$$\begin{tikzcd} & \mathbb{A}^d_k \arrow[d] \\ \mathrm{D}_L \arrow[r] \arrow[ur, dashed, "\nexists"] & X \end{tikzcd}$$

Moreover those arcs which are not liftable to X form a subset of $\mathrm{J}_\infty X$ of nonzero measure. Namely the map $\mathrm{J}_\infty \mathbb{A}^d_k \to \mathrm{J}_\infty X$ is not almost surjective. For this reason, it would be hopeless to express $\mathrm{M}_{\mathrm{st}}(X, A)$ by means of the arc space $\mathrm{J}_\infty \mathbb{A}^d_k$ with the G-action. This is in contrast with the case of proper birational morphisms. If $f\colon Z \to X$ is a proper birational morphism, then $\mathrm{J}_\infty Z \to \mathrm{J}_\infty X$ is almost bijective and a motivic integral on $\mathrm{J}_\infty X$ is transformed to one on $\mathrm{J}_\infty Z$ using the change of variables formula. To settle the issue of non-liftability, we introduce the notion of twisted arcs.

Definition 8.4.1. Suppose that k contains the l-th roots of unity; let $\mu_l \subset k$ be the group consisting of them. For a k-algebra R, let $\mathrm{D}^l_R := \operatorname{Spec} R[\![t^{1/l}]\!]$.

We have a natural μ_l-action on D^l_R. The quotient D^l_R/μ_l is identified with D_R. Let G be a finite group and let V be a k-variety with a G-action.

Definition 8.4.2. For an embedding (an injective homomorphism) $\iota\colon \mu_l \hookrightarrow G$, an *$\iota$-twisted arc* of V over R is an ι-equivariant k-morphism $\mathrm{D}^l_R \to V$; we call it *geometric* when R is an algebraically closed field. We define the *scheme of ι-twisted arcs* of V to be the functor $(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$ mapping $\operatorname{Spec} R$ to the set of ι-twisted arcs of V over R; we denote it by $\mathrm{J}^\iota_\infty V$.

It will turn out in Section 8.5 that this functor is indeed a scheme.

Let $X := V/G$ and let $\pi\colon V \to X$ be the quotient morphism. Given an ι-twisted arc $\mathrm{D}^l_R \to V$, we can take its μ_l-quotient induces $\mathrm{D}_R = \mathrm{D}^l_R/\mu_l \to V/\iota(\mu_l)$. Composing it with $V/\iota(\mu_l) \to X$, we get an arc $\mathrm{D}_R \to X$. This induces a morphism $\mathrm{J}^\iota_\infty V \to \mathrm{J}_\infty X$.

For $g \in G$, let c_g denote the conjugation map $G \to G$, $h \mapsto ghg^{-1}$. For an ι-twisted arc $\gamma\colon \mathrm{D}^l_R \to V$ and for $g \in G$, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{D}^l_R \arrow[r, "\gamma"] \arrow[d, "\zeta"] & V \arrow[r, "g"] \arrow[d, "\iota(\zeta)"] & V \arrow[d, "c_g \circ \iota(\zeta)"] \\ \mathrm{D}^l_R \arrow[r, "\gamma"'] & V \arrow[r, "g"'] & V \end{tikzcd}$$

Thus $g\circ\gamma\colon \mathrm{D}_R^l \to V$ is a $c_g\circ\iota$-twisted arc. Let $\mathrm{Emb}(\mu_l, G)$ be the set of embeddings $\mu_l \hookrightarrow G$. We define a G-action on

$$\coprod_{\iota\in\mathrm{Emb}(\mu_l,G)} \mathrm{J}_\infty^\iota V$$

by $g\cdot\gamma := g\circ\gamma$ and by mapping $\mathrm{J}_\infty^\iota V$ onto $\mathrm{J}_\infty^{c_g\circ\iota} V$.

Let $\mathrm{Conj}(\mu_l, G) = \mathrm{Emb}(\mu_l, G)/G$ be the set of G-conjugacy classes of embeddings $\mu_l \hookrightarrow G$. The stabilzier subgroup of $\iota\in\mathrm{Emb}(\mu_l,G)$ is the centralizer $C_G(\iota) := C_G(\iota(\mu_l))$. Therefore, for each field L, we can identify

$$\left(\coprod_{\iota\in\mathrm{Emb}(\mu_l,G)} (\mathrm{J}_\infty^\iota V)(L)\right)/G = \coprod_{[\iota]\in\mathrm{Conj}(\mu_l,G)} (\mathrm{J}_\infty^\iota V)(L)/C_G(\iota).$$

We see that the map

$$\coprod_{l>0}\coprod_{\iota\in\mathrm{Emb}(\mu_l,G)} \mathrm{J}_\infty^\iota V \to \mathrm{J}_\infty X \tag{8.4.1}$$

is G-invariant and hence induces

$$\coprod_{l>0}\coprod_{[\iota]\in\mathrm{Conj}(\mu_l,G)} (\mathrm{J}_\infty^\iota V)(L)/C_G(\iota) \to (\mathrm{J}_\infty X)(L).$$

The last map is almost bijective. More precisely, let $\mathrm{Ram}\subset V_{\mathrm{D}}$ and $\mathrm{Bra}\subset X_{\mathrm{D}}$ be the ramification and branch loci of $V_{\mathrm{D}}\to X_{\mathrm{D}}$ respectively and let $\mathrm{J}_\infty^{\iota;\circ} V_{\mathrm{D}} := \mathrm{J}_\infty^\iota V_{\mathrm{D}} \setminus \mathrm{J}_\infty^\iota \mathrm{Ram}$ and $\mathrm{J}_\infty^\circ X_{\mathrm{D}} = \mathrm{J}_\infty X_{\mathrm{D}}\setminus \mathrm{J}_\infty \mathrm{Bra}$. They parametrizes ($\iota$-twisted) arcs mapping the generic point of D^l or D into the étale locus of $V_{\mathrm{D}}\to X_{\mathrm{D}}$ in V_{D} or X_{D} respectively.

Proposition 8.4.3. *Let L be an algebraically closed field. Then the map*

$$\coprod_{l>0}\coprod_{[\iota]\in\mathrm{Conj}(\mu_l,G)} (\mathrm{J}_\infty^{\iota,\circ} V)(L)/C_G(\iota) \to (\mathrm{J}_\infty^\circ X)(L)$$

is bijective.

Proof. Let $\gamma\colon \mathrm{D}_L\to X$ be an arc defining a point of $(\mathrm{J}_\infty^\circ X)(L)$. Let E be the normalization of $\mathrm{D}_L\times_X V$. The projection $E\to\mathrm{D}_L$ is a G-cover. Let $E_0\subset E$ be a connected component and let $H\subset G$ be its stabilizer, that is, $H=\{g\in G\mid g(E_0)=E_0\}$. The field extension $F/L(\!(t)\!)$ corresponding to $E_0\to\mathrm{D}_L$ is a Galois extension with Galois group H. If $l := \sharp H$, then there is an $L(\!(t)\!)$-isomorphism $F\cong L(\!(t^{1/l})\!)$, which induces an isomorphism $H\cong\mu_l$. The induced morphism $\mathrm{D}_L^l \xrightarrow{\sim} E_0 \to V$ is an ι-twisted arc, where ι is the composition $\mu_l\xrightarrow{\sim} H\hookrightarrow G$. We see that this ι-twisted arc maps to γ by the map of the proposition. Thus the map of the proposition is surjective.

Next we show the injectivity. Let $\alpha\colon \mathrm{D}_L^l\to V$ be an ι-twisted arc defining a point of $(\mathrm{J}_\infty^{\iota,\circ} V)(L)$ and let $\gamma\colon\mathrm{D}_L\to X$ be the induced arc. Let C be the normalization of $\mathrm{D}_L^l\times_X V$. The projection $C\to\mathrm{D}_L^l$ restricts a G-torsor over $\operatorname{Spec} L(\!(t^{1/l})\!)$, which has a section. Thus this is a trivial G-torsor and C is the disjoint union of $\sharp G$ copies of D_L^l. This shows that there are exactly $\sharp G$ morphisms $\mathrm{D}_L^l\to V$ that induces γ and they are transitively permuted by the G-action on V. Therefore the G-orbit $G\cdot\alpha$ is exactly the fiber of map (8.4.1) over γ, which shows the proposition. $\square$

This proposition gives, up to a negligible subset, a decomposition of the arc space $\mathrm{J}_\infty X$ into finitely many subsets indexed by the finite set $\coprod_{l>0}\mathrm{Conj}(\mu_l, G)$. We will compute the contribution of each stratum to $\mathrm{M}_{\mathrm{st}}(X, A)$. The proposition is also regarded as an analogue of Proposition 4.2.1 for proper birational morphisms.

8.5. Untwisting

We reduce the study of ι-twisted arcs to the one of usual arcs. But, to do so, we need to swtich over from k-varieties to D_k-schemes as discussed in Chapter 7. From now on, we write D_k simply as D. Recall that we can identify $\mathrm{J}_\infty V$ and $\mathrm{J}_\infty X$ respectively with $\mathrm{J}_\infty(V_{\mathrm{D}}/\mathrm{D})$ and $\mathrm{J}_\infty(X_{\mathrm{D}}/\mathrm{D})$. We abbreviate the latters as $\mathrm{J}_\infty V_{\mathrm{D}}$ and $\mathrm{J}_\infty X_{\mathrm{D}}$. We also have $\mathrm{M}_{\mathrm{st}}(X_{\mathrm{D}}) = \mathrm{M}_{\mathrm{st}}(X)$. We define twisted arcs on V_{D} as follows.

Definition 8.5.1. For an embedding $\iota\colon \mu_l \hookrightarrow G$, an *$\iota$-twisted arc* of V_{D} over R is an ι-equivariant D-morphism $\mathrm{D}^l_R \to V_{\mathrm{D}}$; we call it *geometric* when R is an algebraically closed field. We define the *scheme of ι-twisted arcs* as the functor $(\mathbf{Aff}_k)^{\mathrm{op}} \to \mathbf{Set}$ sending $\operatorname{Spec} R$ to the set of *ι-twisted arc* of V_{D} over R; we denote it by $\mathrm{J}^\iota_\infty V_{\mathrm{D}}$.

Clearly $\mathrm{J}^\iota_\infty V = \mathrm{J}^\iota_\infty V_{\mathrm{D}}$. Corresponding to the bijection in Propostion 8.4.3, for an algebraically closed field L, we get a bijection

$$\coprod_{l>0}\ \coprod_{[\iota]\in\mathrm{Conj}(\mu_l,G)} (\mathrm{J}^{\iota,\circ}_\infty V_{\mathrm{D}})(L)/C_G(\iota) \to (\mathrm{J}^\circ_\infty X_{\mathrm{D}})(L), \tag{8.5.1}$$

where the superscript $\circ$ again means restrction to (twisted) arcs sending the generic point to the étale locus of $V_{\mathrm{D}} \to X_{\mathrm{D}}$.

Let us now come back to the situation where V is an affine space $\mathbb{A}^d_k$ and G acts on it linearly. We fix an embedding $\iota\colon \mu_l \hookrightarrow G$. By a suitable coordinate transform, we suppose that the induced μ_l-action on $V = \operatorname{Spec} k[x_1, \dots, x_d]$ is diagonal and write

$$\iota(\zeta) = \mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_d}) \quad (0 \le a_i < l,\ \zeta \in \mu_l).$$

Definition 8.5.2. Let $V^{|\iota|}_{\mathrm{D}}$ be another copy of $\mathbb{A}^d_{\mathrm{D}} = \operatorname{Spec} k[\![t]\!][x_1, \dots, x_d]$ and let

$$V^{\langle\iota\rangle}_{\mathrm{D}} := V^{|\iota|}_{\mathrm{D}} \times_{\mathrm{D}} \mathrm{D}^l = \operatorname{Spec} k[\![t^{1/l}]\!][x_1, \dots, x_d].$$

be its base change to D^l with the canonical morphism $r\colon V^{\langle\iota\rangle}_{\mathrm{D}} \to V^{|\iota|}_{\mathrm{D}}$. We call $V^{|\iota|}_{\mathrm{D}}$ the *untwisting scheme* (of V or V_{D}) with respect to ι and that it is given morphisms below relating it with V_{D}, $V^{\langle\iota\rangle}_{\mathrm{D}}$, and X_{D}.

We define a D-morphism $u\colon V^{\langle\iota\rangle}_{\mathrm{D}} \to V_{\mathrm{D}}$ to be the one corresponding to the following $k[\![t]\!]$-algebra homomorphism:

$$\begin{aligned} u^*\colon k[\![t]\!][x_1, \dots, x_d] &\to k[\![t^{1/l}]\!][x_1, \dots, x_d] \\ x_i &\mapsto t^{a_i/l} x_i \end{aligned}$$

We let $C_G(\iota)$ act on V_{D}, $V^{|\iota|}_{\mathrm{D}}$ and $V^{\langle\iota\rangle}_{\mathrm{D}}$ by scalar extension of the original k-linear action on V.

Lemma 8.5.3. *The morphism u is $C_G(\iota)$-equivariant.*

PROOF. We regard each $g \in G$ as a matrix (g_{ij}) in the obvious way. Then

$$\iota(\zeta)g = (\zeta^{a_i} g_{ij}) \text{ and } g\iota(\zeta) = (\zeta^{a_j} g_{ij}).$$

Thus, for a primitive l-th root ζ of unity, $\iota(\zeta)$ and g are commutative if and only if $g_{ij} = 0$ for every (i, j) with $a_i \neq a_j$. By a similar reasoning, the last condition is equivalent to that $T := \mathrm{diag}(t^{a_1/l}, \dots, t^{a_d/l})$ and g are commutative. We conclude that if $g \in C_G(\iota)$, then T and g are commutative. The linear part $w\colon \bigoplus_i k[\![t]\!]x_i \to \bigoplus_i k[\![t^{1/l}]\!]x_i$ of u^* is given by the right multiplication with T. The commutativity of g and T for $g \in C_G(\iota)$ means that w is $C_G(\iota)$-equivariant. Hence u is also $C_G(\iota)$-equivariant. $\square$

PROPOSITION 8.5.4. *With the above notation, there exist a unique* D*-morphism* $\psi^{|\iota|}\colon V_{\mathrm{D}}^{|\iota|} \to X_{\mathrm{D}}$ *which makes the following diagram commutative:*

(8.5.2)
$$\begin{tikzcd} V_{\mathrm{D}}^{\langle\iota\rangle} \arrow[r, "r"] \arrow[d, "u"'] & V_{\mathrm{D}}^{|\iota|} \arrow[d, "\psi^{|\iota|}"] \\ V_{\mathrm{D}} \arrow[r, "\psi"'] & X_{\mathrm{D}} \end{tikzcd}$$

PROOF. We claim that

$$u^* \left(k[\![t]\!][x_1, \dots, x_d]^G\right) \subset k[\![t]\!][x_1, \dots, x_d].$$

Once this is proved, then $\psi^{|\iota|}$ is defined to be the morphism corresponding to this inclusion. Then we easily see that this satisfies the desired property. To show the claim, it suffices to show

$$u^* \left(k[\![t]\!][x_1, \dots, x_d]^{\iota(\mu_l)}\right) \subset k[\![t]\!][x_1, \dots, x_d].$$

The invariant subring $k[\![t]\!][x_1, \dots, x_d]^{\iota(\mu_l)}$ is generated as a $k[\![t]\!]$-module by those monimials $x_1^{e_1} \cdots x_d^{e_d}$ with $\sum_i e_i a_i$ divisible by l. For such a monomial, we have

$$u^*(x_1^{e_1} \cdots x_d^{e_d}) = t^{(\sum_i e_i a_i)/l} x_1^{e_1} \cdots x_d^{e_d} \in k[\![t]\!][\underline{x}].$$

This shows the latter inclusion above, which in turn shows the claim. $\square$

REMARK 8.5.5. The morphism $V_{\mathrm{D}}^{\langle\iota\rangle} \otimes_{k[\![t^{1/l}]\!]} k(\!(t^{1/l})\!) \to V_{\mathrm{D}} \otimes_{k[\![t]\!]} k(\!(t^{1/l})\!)$ induced by u is an isomorphism. Therefore we have an isomorphism $V_{\mathrm{D}} \otimes_{k[\![t]\!]} k(\!(t^{1/l})\!) \cong V_{\mathrm{D}}^{|\iota|} \otimes_{k[\![t]\!]} k(\!(t^{1/l})\!)$, which is compatible to maps to $X_{\mathrm{D}} \otimes_{k[\![t]\!]} k(\!(t^{1/l})\!)$. In this sense, $\psi^{|\iota|}$ is a twisted form of ψ and they share the same branch locus in X_{D}.

The following result will be necessary in the proof of the motivic McKay correspondence, Theorem 8.3.8.

PROPOSITION 8.5.6. *Following the notation of Theorem 8.3.8, we let A be the $\mathbb{Q}$-divisor on X such that $V \to (X, A)$ is crepant and let A_{D} be its pull-back to X_{D}. Let V_0 denote $V \cong \mathbb{A}_k^d$ regarded as a prime divisor of V_{D}. Then the relative canonical divisor $K_{V_{\mathrm{D}}^{|\iota|}/(X_{\mathrm{D}}, A_{\mathrm{D}})}$ is $\mathrm{age}(\iota)V_0$, where $\mathrm{age}(\iota)$ denotes the age of the embedding ι given in Remark 8.3.6.*

PROOF. By definition, the map $V_{\mathrm{D}} \to (X_{\mathrm{D}}, A_{\mathrm{D}})$ is crepant. From Remark 8.5.5, the map $V_{\mathrm{D}}^{|\iota|} \to (X_{\mathrm{D}}, A_{\mathrm{D}})$ is also crepant on generic fibers (that is, fibers over the generic point $\eta \in \mathrm{D}$). Therefore $K_{V_{\mathrm{D}}^{|\iota|}/(X_{\mathrm{D}},A_{\mathrm{D}})}$ is of the form eV_0 for some $e \in \mathbb{Q}$.

By base changing three schemes in (8.5.2), we get the following diagram.

$$\begin{tikzcd} V_{\mathrm{D}}^{\langle\iota\rangle} \arrow[r, equal] \arrow[d] & V_{\mathrm{D}}^{|\iota|} \times_{\mathrm{D}} \mathrm{D}^l \arrow[d] \\ V_{\mathrm{D}^l} \arrow[r] & X_{\mathrm{D}^l} \end{tikzcd} \tag{8.5.3}$$

Let $V_0' \subset V_{\mathrm{D}}^{\langle\iota\rangle}$ be the special fiber of $V_{\mathrm{D}}^{\langle\iota\rangle} \to \mathrm{D}^l$ regarded as a prime divisor. The relative canonical divisor $K_{V_{\mathrm{D}}^{\langle\iota\rangle}/(X_{\mathrm{D}^l},A_{\mathrm{D}^l})}$ is equal to the pull-back of $K_{V_{\mathrm{D}}^{|\iota|}/(X_{\mathrm{D}},A_{\mathrm{D}})}$ and hence

$$K_{V_{\mathrm{D}}^{\langle\iota\rangle}/(X_{\mathrm{D}^l},A_{\mathrm{D}^l})} = leV_0'.$$

On the other hand, since $V_{\mathrm{D}^l} \to (X_{\mathrm{D}^l}, A_{\mathrm{D}^l})$ is crepant, we have

$$K_{V_{\mathrm{D}}^{\langle\iota\rangle}/(X_{\mathrm{D}^l},A_{\mathrm{D}^l})} = K_{V_{\mathrm{D}}^{\langle\iota\rangle}/V_{\mathrm{D}^l}}.$$

Since the morphism $\mathbb{A}^d_{\mathrm{D}^l} \cong V_{\mathrm{D}}^{\langle\iota\rangle} \to V_{\mathrm{D}^l} \cong \mathbb{A}^d_{\mathrm{D}^l}$ is given by $x_i \mapsto t^{a_i/l}x_i$, the d-form $dx_1 \wedge \cdots \wedge dx_d$ on V_{D^l} is pulled back to $t^{\sum_i a_i/l}dx_1 \wedge \cdots \wedge dx_d$ on $V_{\mathrm{D}}^{\langle\iota\rangle}$. Therefore

$$K_{V_{\mathrm{D}}^{\langle\iota\rangle}/V_{\mathrm{D}^l}} = (\sum_i a_i)V_0'.$$

We get $le = \sum_i a_i$. $\square$

PROPOSITION 8.5.7. *For each k-algebra R, there exists a one-to-one correspondence*

$$(\mathrm{J}^{\iota}_{\infty} V_{\mathrm{D}})(R) \leftrightarrow (\mathrm{J}_{\infty} V_{\mathrm{D}}^{|\iota|})(R) \tag{8.5.4}$$

which is functorial in R, $C_G(\iota)$-equivariant and compatible with the maps

$$\psi_\infty \colon (\mathrm{J}^{\iota}_{\infty} V_{\mathrm{D}})(R) \to (\mathrm{J}_{\infty} X_{\mathrm{D}})(R) \text{ and}$$
$$\psi^{|\iota|}_\infty \colon (\mathrm{J}_{\infty} V_{\mathrm{D}}^{|\iota|})(R) \to (\mathrm{J}_{\infty} X_{\mathrm{D}})(R).$$

Moreover, if we put $\mathrm{J}^{\circ}_{\infty} V_{\mathrm{D}}^{|\iota|} := \mathrm{J}_{\infty} V_{\mathrm{D}}^{|\iota|} \setminus \mathrm{J}_{\infty} Z$ where Z is the non-étale locus of $V_{\mathrm{D}}^{|\iota|} \to X_{\mathrm{D}}$, then the above correspondence restricts to

$$(\mathrm{J}^{\iota}_{\infty} V_{\mathrm{D}})(R)^{\circ} \leftrightarrow (\mathrm{J}_{\infty} V_{\mathrm{D}}^{|\iota|})(R)^{\circ}. \tag{8.5.5}$$

PROOF. Let $\gamma\colon \mathrm{D}^l_R \to V_{\mathrm{D}}$ be a D-morphism. The γ is an ι-twisted arc if and only if $\gamma^*(x_i) \in t^{a_i/l} \cdot R[\![t]\!]$. If this is the case, there exsists a unique D^l-morphism $\tilde\gamma\colon \mathrm{D}^l_R \to V_{\mathrm{D}}^{\langle\iota\rangle}$ such that $\gamma = u \circ \tilde\gamma$, which is given by $\tilde\gamma^*(x_i) = t^{-a_i/l}\gamma^*(x_i) \in R[\![t]\!]$. The composition $r \circ \tilde\gamma$ factors as

$$\mathrm{D}^l_R \to \mathrm{D}_R \xrightarrow{\gamma'} V_{\mathrm{D}}^{|\iota|},$$

where γ' is the arc given by $(\gamma')^*(x_i) = \tilde{\gamma}^*(x_i)$. We define correspondence (8.5.4) by $\gamma \leftrightarrow \gamma'$. Construction gives the following commutative diagram:

(8.5.6)
$$\begin{tikzcd}
\mathrm{D}_R^l \arrow[r] \arrow[d, "\tilde{\gamma}"] \arrow[dd, bend right, "\gamma"'] & \mathrm{D}_R \arrow[d, "\gamma'"] \\
V_{\mathrm{D}}^{\langle\iota\rangle} \arrow[r, "r"] \arrow[d, "u"] & V_{\mathrm{D}}^{|\iota|} \arrow[d, "\psi^{|\iota|}"] \\
V_{\mathrm{D}} \arrow[r, "\psi"'] & X_{\mathrm{D}}
\end{tikzcd}$$

Both $\psi_\infty(\gamma)$ and $\psi_\infty^{|\iota|}(\gamma')$ are identical to the unique morphism $\mathrm{D}_R^l \to X_{\mathrm{D}}$ given by this diagram, which shows that correspondence (8.5.4) is compatible with ψ_∞ and $\psi_\infty^{|\iota|}$. Since $(\mathrm{J}_\infty^{\iota,\circ} V_{\mathrm{D}})(R)$ and $(\mathrm{J}_\infty^\circ V_{\mathrm{D}}^{|\iota|})(R)$ are both the preimages of $(\mathrm{J}_\infty^\circ X_{\mathrm{D}})(R)$, correspondence (8.5.5) holds.

Since u ard r are $C_G(\iota)$-equivariant, a simple diagram chasing shows that the correspondence obtained above is $C_G(\iota)$-equivariant. □

COROLLARY 8.5.8. *The functor* $\mathrm{J}_\infty^\iota V_{\mathrm{D}} \colon (\mathbf{Aff}_k)^\circ \to \mathbf{Set}$ *is a scheme.*

PROOF. The correspondence in the last proposition gives a natural isomorphism of functors $\mathrm{J}_\infty^\iota V_{\mathrm{D}} \cong \mathrm{J}_\infty V_{\mathrm{D}}^{|\iota|}$. Since the right hand side is a scheme, so is the left hand side. □

As a conclusion of this section, for an algebraically closed field L, we obtain the bijection

(8.5.7)
$$\coprod_{l>0} \coprod_{[\iota]\in \mathrm{Conj}(\mu_l, G)} (\mathrm{J}_\infty^\circ V_{\mathrm{D}}^{|\iota|})(L)/C_G(\iota) \to (\mathrm{J}_\infty^\circ X_{\mathrm{D}})(L)$$

which corresponds to (8.5.1). In this map, only ordinary/untwisted arcs are involved. Thus this is very close to the setting of the change of variables formula for D-schemes (Theorem (7.2.2)), except that we take quotients by $C_G(\iota)$. In the next section, we slightly generalized the change of variables formula incorporating so that we can incorporate these $C_G(\iota)$-actions.

8.6. Equivariant motivic integration for untwisted arcs

We now consider an arbitrary good D-scheme X (see Definition 7.1.1) which has an action of a finite group G over D.

DEFINITION 8.6.1. We say that a subset $A \subset \mathrm{J}_\infty X_{\mathrm{D}}$ is *G-invariant* if for every $g \in G$, $g(A) = A$. We say that a function $h \colon A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ on a subset $A \subset \mathrm{J}_\infty X_{\mathrm{D}}$ is G-invariant if A is G-invariant and for every g, $h = h \circ g$.

For a G-invariant ordinary cylinder C, from Lemma 3.8.2, its image $\pi_n(C)$ by a truncation map is a G-invariant constructible subset. The quotient $\pi_n(C)/G$ is then defined to be the image of $\pi_n(C)$ in the quotient scheme $(\mathrm{J}_n X_{\mathrm{D}})/G$, which is a constructible subset.

REMARK 8.6.2. If $\pi_n(C)$ is a locally closed subset say with reduced structure, then there exists a universal homeomorphism from the quotient scheme $\pi_n(C)/G$

to its image in $(\mathrm{J}_n X_{\mathrm{D}})/G$, but it may not be an isomorphism. As we will work in $\widehat{\mathcal{M}}'_k$, the difference by a universal homeomorphism does not matter.

Definition 8.6.3. For a G-invariant ordinary cylinder $C \subset |\mathrm{J}_\infty X_{\mathrm{D}}|$, we define

$$\mu_{X,G}(C) := \{\pi_n(C)/G\}\mathbb{L}^{-dn} \in \widehat{\mathcal{M}}'_k \quad (n \gg 0).$$

Here d denotes the relative dimension of X over D.

From Lemma 5.5.3, for $n \gg 0$, the map $\pi_{n+1}(C) \to \pi_n(C)$ is a piecewise trivial $\mathbb{A}^d$-bundle. It follows that the map $\pi_{n+1}(C)/G \to \pi_n(C)/G$ is a pseudo-$\mathbb{A}^d$-bundle, which in turn shows that $\mu_{X,G}(C)$ is well-defined.

Lemma 8.6.4. *Let*

$$h\colon \mathrm{J}_\infty(X/\mathrm{D}) \supset A \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$$

*be an admissible function which is G-invariant. Then there exists a stratification $A = \bigsqcup_{i\in I} A_i \sqcup N$ into countably many ordinary G-*invariant *cylinders A_i, $i \in I$ and a G-*invariant *negligible subset N such that for every i, the restriction $f|_{A_i}$ is constant with value different from ∞.*

Proof. From the definition of admissible functions (Definition 5.7.1), there exists a stratification $A = \bigsqcup_{i\in\mathbb{N}} A_i \sqcup N$ satisfying the above conditions except the one that A_i and N be G-invariant. We modify A_i's inductively into G-invariant ones. First we replace A_0 with $\bigcup_{g\in G} g(A_0)$, then replace A_i, $i \neq 0$, with $A_i \setminus A_0$. This operation makes A_0 to be G-invariant, keeping A_i's disjoint and their union unchanged. Applying the same operation inductively, we can make all the A_i to be G-invariant. Lastly we replace N with $N \setminus \bigcup_{i\in\mathbb{N}} A_i$ to get the desired stratification. □

Definition 8.6.5. With the notation of the above lemma, we can define the motivic integral associated to h with respect to $\mu_{X,G}$ as follows:

$$\int_A \mathbb{L}^h \, d\mu_{X,G} := \sum_i \mu_{X,G}(A_i)\mathbb{L}^{h(A_i)} \in \widehat{\mathcal{M}}'_{k,r} \cup \{\infty_*\}.$$

We can show that the above integral is independent of the choice of stratification $A = \bigsqcup_{i\in I} A_i \sqcup N$ in the same way as in the non-equivariant case.

Remark 8.6.6. We may write the above integral also as

$$\left(\int_A \mathbb{L}^h \, d\mu_X\right)/G.$$

Here we define $\int_A \mathbb{L}^h \, d\mu_X$ in the G-equivariant version G-$\widehat{\mathcal{M}}_{k,r}$ of $\widehat{\mathcal{M}}_{k,r}$, lifting the one defined in $\widehat{\mathcal{M}}_{k,r}$. The quotient above is obtained by sending it by the "taking quotients" map

$$\bullet/G\colon G\text{-}\widehat{\mathcal{M}}_{k,r} \to \widehat{\mathcal{M}}'_{k,r},\ \{X\} \mapsto \{X/G\}.$$

To prove the change of variables formula in the equivariant setting, we first show the following slight generalization of Lemma 5.9.3.

Lemma 8.6.7 (Fiber inclusion lemma; the equivariant case). *Let Y and X be good* D*-schemes with G-actions and let $f\colon Y \to X$ be a generically étale and G-equivariant* D*-morphism. Let L be a field and let $\beta, \beta' \in (\mathrm{J}_\infty Y)(L)$. Let $n \in \mathbb{Z}_{\geq 0}$*

and suppose that $f_n\pi_n(\beta)$ *and* $f_n\pi_n(\beta')$ *are in the same* G*-orbit. Writing* $e := \mathfrak{j}_f(\beta)$, $e_X := \mathfrak{j}_X(f_\infty(\beta))$ *and* $e_Y := \mathfrak{j}_Y(\beta)$*, we suppose that* $n \geq \max\{2e+e_Y, e_X\}$*. Suppose also that* $\beta, \beta' \in C(L)$ *for a* G*-invariant cylinder* C *of level* $n-e$ *and that the map*

$$C(L)/G \to f_\infty(C(L))/G$$

induced by f_∞ *is injective. Then* $\pi_{n-e}(\beta)$ *and* $\pi_{n-e}(\beta')$ *are in the same* G*-orbit.*

Proof. Replacing β with an element in the same G-orbit, we may suppose that $f_n\pi_n(\beta) = f_n\pi_n(\beta')$. For $\alpha := f_\infty(\beta')$, let $\gamma \in (\mathrm{J}_\infty Y)(L)$ as in Lemma 5.9.1. Since C is a cylinder of level $n-e$ and $\pi_{n-e}(\beta) = \pi_{n-e}(\gamma)$, we see that $\gamma \in C(L)$. Since $C(L)/G \to f_\infty(C(L))/G$ is injective, the equality $f_\infty(\gamma) = f_\infty(\beta')$ implies that γ and β' are in the same G-orbit, which shows the lemma. $\square$

Lemma 8.6.8 (The $\mathbb{A}^e$-fibration lemma; the equivariant case). *Let* $f\colon Y \to X$ *be a generically étale and* G*-equivariant morphism of good* D*-schemes. Let* $n, e, e_X, e_Y \in \mathbb{Z}_{\geq 0}$ *be such that* $n \geq \max\{2e+e_Y, e_X\}$*. Let* $C \subset \mathrm{J}_\infty Y$ *be a* G*-invariant cylinder of level* $n-e$ *such that* $\mathfrak{j}_f|_C \leq e$, $(\mathfrak{j}_X \circ f_\infty)|_C \leq e_X$ *and* $\mathfrak{j}_Y|_C \leq e_Y$*. Suppose that the map* $C/G \to f_\infty(C)/G$ *is geometrically injective. Then the map*

$$\pi_n(C)/G \to f_n(\pi_n(C))/G$$

is a pseudo-$\mathbb{A}^e$*-bundle. In particular,*

$$\mu_{Y,G}(C) = \mu_{X,G}(f_\infty(C))\mathbb{L}^e.$$

Proof. From Lemma 5.10.1, $f_\infty(C)$ is a cylinder of level n. Since only geometric fibers are concerned, by the usual base change argument, we may assume that k is algebraically closed and it suffices to consider fibers over k-points. Let $\beta \in C(k)$ and let $\alpha \in f_\infty(C)(k)$ be its image. Let H be the fiber of $\pi_n(C) \to f_n(\pi_n(C))$ over $\alpha_n = \pi_n(\alpha)$ and let $F^\flat$ be the fiber of $\pi_n(C) \to \pi_{n-e}(C)$ over $\pi_{n-e}(\beta)$. From Lemma 8.6.7, we have $H \subset \bigcup_{g\in G} g(F^\flat)$. Let $H' := H \cap F^\flat$. By the same argument as one in the proof of 5.11.1, we can identify $F^\flat(k)$ with $\mathrm{Hom}_{k[\![t]\!]}(\beta^\flat\Omega_{Y/k[\![t]\!]}, \mathfrak{t}_{n+1}^{n-e+1})$ and $H'(k) \subset F^\flat(k)$ with the following linear subspace

$$\mathrm{Ker}\left(\mathrm{Hom}_{k[\![t]\!]}(\beta^\flat\Omega_{Y/k[\![t]\!]}, \mathfrak{t}_{n+1,k}^{n-e+1}) \to \mathrm{Hom}_{k[\![t]\!]}((f\circ\beta)^\flat\Omega_{X/k[\![t]\!]}, \mathfrak{t}_{n+1,k}^{n-e+1})\right) \cong k^{\oplus e}.$$

This shows that H' is an e-dimensional linear subspace of $F^\flat \cong \mathbb{A}_k^{de}$. Since $H = \bigcup_{g\in G} g(H')$, if $G' \subset G$ denotes the stablizer of H', then we have an injection

$$\mathbb{A}_k^e/G' \cong H'/G' \to \pi_n(C)/G$$

onto H/G (as a subset of $\pi_n(C)/G$; cf. Remark 8.6.2). We conclude that $\pi_n(C)/G \to f_n(\pi_n(C))/G$ is a pseudo-$\mathbb{A}^e$-bundle. It follows

$$\begin{aligned}\mu_{Y,G}(C) &= \{\pi_n(C)/G\}\mathbb{L}^{-nd}\\ &= \{f_n(\pi_n(C))/G\}\mathbb{L}^{-nd+e}\\ &= \mu_{X,G}(f_\infty(C))\mathbb{L}^e.\end{aligned}$$

$\square$

Now, by using Lemma 8.6.8, the change of variables formula in the present equivariant situation is proved in the same way as was Theorem 5.13.2.

THEOREM 8.6.9 (The change of variables formula; the equivariant case). *Let G be a finite group and let Y and X be good* D*-schemes with G-actions. Let $f: Y \to X$ be a generically étale and G-equivariant* D*-morphism. Let $B \subset \mathrm{J}_\infty Y$ be a G-invariant subset such that the map $B/G \to f_\infty(B)/G$ is almost geometrically injective; namely, there exists G-invariant negligible subsets $N \subset \mathrm{J}_\infty Y$ and $M \subset \mathrm{J}_\infty X$ such that $(B \setminus N)/G \to (f_\infty(B) \setminus M)/G$ is geometrically bijective. Let $h: f_\infty(B) \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$ be a G-invariant admissible function. Then*

$$\int_{f_\infty(B)} \mathbb{L}^h \, d\mu_{X,G} = \int_B \mathbb{L}^{h \circ f_\infty - \mathrm{j}_f} \, d\mu_{Y,G} \in \widehat{\mathcal{M}}'_{k,r} \cup \{\infty_*\}.$$

REMARK 8.6.10. As a morphism $f: Y \to X$ in the above theorem, we are mainly interested in one such that the G-action on X is trivial. In this case, $\mu_{X,G}$ is the same as μ_X except that it takes values in $\widehat{\mathcal{M}}'_{k,r}$ rather than $\widehat{\mathcal{M}}_{k,r}$. Thus we may denote $\mu_{X,G}$ also by μ_X and write the change of variables formula as

$$\int_{f_\infty(B)} \mathbb{L}^h \, d\mu_X = \int_B \mathbb{L}^{h \circ f_\infty - \mathrm{j}_f} \, d\mu_{Y,G} = \left(\int_B \mathbb{L}^{h \circ f_\infty - \mathrm{j}_f} \, d\mu_Y \right) / G.$$

For the right equality, see Remark 8.6.6.

8.7. Proof of the tame motivic McKay correspondence

In this section, we prove Theorem 8.3.8. We write $V = \mathbb{A}^d_k$, which is given with the G-action. By definition, $X = V/G$. From Remark 7.3.1, we have $\mathrm{M}_{\mathrm{st}}(X, A) = \mathrm{M}_{\mathrm{st}}(X_{\mathrm{D}}, A_{\mathrm{D}})$, where A_{D} is the pull-back of A by the projection $X_{\mathrm{D}} \to X$. Thus we can write

$$\mathrm{M}_{\mathrm{st}}(X, A) = \int_{\mathrm{J}_\infty X_{\mathrm{D}}} \mathbb{L}^{\frac{1}{r} \operatorname{ord}_{\mathcal{I}}} \, d\mu_{X_{\mathrm{D}}}, \tag{8.7.1}$$

where $\mathcal{I}$ denotes the ideal sheaf $\mathcal{I}_{X_{\mathrm{D}}, A_{\mathrm{D}}, r}$ (see Section 7.3).

By untwisting, we get geometric bijection 8.5.7

$$\coprod_{[\iota]} \mathrm{J}^\circ_\infty V_{\mathrm{D}}^{|\iota|} / C_G(\iota) \to \mathrm{J}^\circ_\infty X_{\mathrm{D}},$$

where $[\iota]$ runs over G-conjugacy classes of embeddings $\mu_l \hookrightarrow G$ (without l being fixed). Let $K_\iota := K_{V_{\mathrm{D}}^{|\iota|}/X_{\mathrm{D}}}$. Applying Lemma 6.4.5 to the crepant map $\psi^{|\iota|}: (V_{\mathrm{D}}^{|\iota|}, -K_\iota) \to (X_{\mathrm{D}}, A_{\mathrm{D}})$, we get

$$\frac{1}{r} \operatorname{ord}_{\mathcal{I}_{X_{\mathrm{D}}, A_{\mathrm{D}}, r}} \circ \psi^{|\iota|}_\infty - \mathrm{j}_{\psi^{|\iota|}} = \frac{1}{r} \operatorname{ord}_{\mathcal{I}_{V_{\mathrm{D}}^{|\iota|}, -K_\iota, r}} = -\operatorname{ord}_{K_\iota}.$$

From the change of variables formula, we have

$$\int_{\mathrm{J}_\infty X_{\mathrm{D}}} \mathbb{L}^{\frac{1}{r} \operatorname{ord}_{\mathcal{I}}} \, d\mu_{X_{\mathrm{D}}} = \sum_{[\iota]} \int_{\mathrm{J}_\infty V_{\mathrm{D}}^{|\iota|}} \mathbb{L}^{-\operatorname{ord}_{K_\iota}} \, d\mu_{V_{\mathrm{D}}^{|\iota|}, C_G(\iota)}. \tag{8.7.2}$$

From Proposition 8.5.6, K_ι is the special fiber multiplied with $\mathrm{age}(\iota)$. Therefore its order function is constant $\mathrm{age}(\iota)$ (see Example 7.4.1). It follows that

$$\begin{aligned}\int_{\mathrm{J}_\infty V_{\mathrm{D}}^{|\iota|}} \mathbb{L}^{-\operatorname{ord}_{K_\iota}} d\mu_{V_{\mathrm{D}}^{|\iota|}, C_G(\iota)} &= \mu_{V_{\mathrm{D}}^{|\iota|}, C_G(\iota)}(\mathrm{J}_\infty V_{\mathrm{D}}^{|\iota|})\mathbb{L}^{-\operatorname{age}(\iota)} \\ &= \{(\mathrm{J}_0 V_{\mathrm{D}}^{|\iota|})/G\}\mathbb{L}^{-\operatorname{age}(\iota)} \\ &= \{\mathbb{A}_k^d/G\}\mathbb{L}^{-\operatorname{age}(\iota)} \\ &= \mathbb{L}^{d-\operatorname{age}(\iota)}.\end{aligned} \tag{8.7.3}$$

Note that the last equality follows from the definition of $\widehat{\mathcal{M}}'_{k,r}$. Combining (8.7.1), (8.7.2) and (8.7.3) gives

$$\mathrm{M}_{\mathrm{st}}(X, A) = \sum_{[\iota]} \mathbb{L}^{d-\operatorname{age}(\iota)}.$$

Theorem (8.3.8) is now obtained by rewriting the right hand side as a sum over conjugacy claases in G acording to Remark 8.3.6.

CHAPTER 9

The McKay correspondence: the wild case

In this chapter, we discuss generalization of the McKay correspondence to the wild case. We postpone the rigorous proofs of main results until ??. Instead, we focus on understanding key ideas on why these results should be expected.

We keep denoting the characteristic of our field k by p.

9.1. G-covers of the formal disk

Definition 9.1.1. Let G be a finite group. A *G-cover* of $\mathrm{D}_k^* = \operatorname{Spec} k(\!(t)\!)$ means an (étale) G-torsor E^* over D_k^*. A *G-cover* of D_k means the integral closure of D_k in some G-cover E^* of D_k^*. Equivalently, a G-cover of D_k^* is a regular scheme E given with a G-action and a flat, finite, and G-invariant morphism $E \to \mathrm{D}_k$ that induces an isomorphism $E/G \cong \mathrm{D}_k$.

There is an obvious one-to-one correspondence betweeen G-covers of D_k^* and G-covers of D_k. Regarding the McKay correspondence in the tame case, μ_l-covers,

$$\mathrm{D}_k^l = \operatorname{Spec} k[\![t^{1/l}]\!] \to \mathrm{D}_k = \operatorname{Spec} k[\![t]\!],$$

for positive integer l with $p \nmid l$ played important roles. It is because, if k is algebraically closed and if G is a tame finite group, then every G-cover of D_k is induced from the μ_l-cover $\mathrm{D}_k^l \to \mathrm{D}_k$ for some l via an injection $\mu_l \hookrightarrow G$. In particular, there are only finitely many G-covers, which are in one-to-one correspondence with elements of $\coprod_{p \nmid l} \operatorname{Conj}(\mu_l, G)$. If we drop the tameness condition, this is no longer true, as the following example shows.

Example 9.1.2 (Artin-Schreier extensions of $k(\!(t)\!)$). The Artin-Schreier theory says that $\mathbb{Z}/p\mathbb{Z}$-covers of D_k are parametrized by the quotient set $k(\!(t)\!)/\wp(k(\!(t)\!))$. Here $\wp$ is the selfmap of $k(\!(t)\!)$ given by $f \mapsto f^p - f$. If k is algebraically closed, then the composite map

$$\bigoplus_{l>0,\, p\nmid l} k \cdot t^{-l} \hookrightarrow k(\!(t)\!) \twoheadrightarrow k(\!(t)\!)/\wp(k(\!(t)\!))$$

is bijective. It follows that there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-covers of D_k, which are parametrized by the infinite dimensional affine space $\bigoplus_{l>0,\, p\nmid l} k \cdot t^{-l}$. Since the trivial cover $\mathrm{D}_k \amalg \cdots \amalg \mathrm{D}_k \to \mathrm{D}_k$ is the only non-connected $\mathbb{Z}/p\mathbb{Z}$-cover of D_k, there are infinitely many connected $\mathbb{Z}/p\mathbb{Z}$-covers.

Definition 9.1.3 (Working definition of G-Cov(D)). A *P-moduli space of G-covers of the formal disk*, denoted by G-Cov(D), is a k-scheme $X = \coprod_{i \in I} X_i$ that is the disjoint union of countably many affine k-schemes $X_i = \operatorname{Spec} R_i$, $i \in I$, that is given, for each $i \in I$, a finite-type affine k-scheme $Y_i = \operatorname{Spec} S_i$, a G-torsor

$\mathcal{E}_i \to \mathrm{D}^*_{S_i} = \operatorname{Spec} S_i(\!(t)\!)$ and a surjective morphism $f_i \colon Y_i \twoheadrightarrow X_i$, and such that for every algebraically closed field F/k, the induced map

$$\coprod_{i \in I} Y_i(F) \to \{G\text{-torsors over } \mathrm{D}^*_F\}$$

uniquely factors as

$$\coprod_{i \in I} Y_i(F) \xrightarrow{f_i(F)} X(F) \xrightarrow{\text{bij.}} \{G\text{-torsors over } \mathrm{D}^*_F\}.$$

Roughly, the above definition says that geometric points of G-Cov(D) parametrizes G-torsors over D^*_F and there exists a versal family $\coprod \mathcal{E}_i \to \coprod \mathrm{D}^*_{S_i}$ of G-torsors over D^*. Note that a P-moduli space G-Cov(D) (according to the working definition) is not unique. Indeed, stratifying one component X_i as $X_i = \coprod_j X_{ij}$ by finitely many subschemes X_{ij} gives a new P-moduli space. If $X_i \to X_i^{(p)}$ denotes the relative Frobenius, then replacing X_i with $X_i^{(p)}$ also gives a new P-moduli space. However, for two P-moduli spaces, say G-Cov(D)$_1$ and G-Cov(D)$_2$, there exists a unique P-isomorphism G-Cov(D)$_1 \to G$-Cov(D)$_2$ that preserves correspondences between geometric points and G-torsors.

THEOREM 9.1.4. *For any finite group G, a P-moduli space G-Cov(D) exists.*

EXAMPLE 9.1.5. For $G = \mathbb{Z}/p\mathbb{Z}$, we can put

$$G\text{-Cov(D)} = \coprod_{j>0,\, p\nmid j} \mathbb{G}_{m,k} \times \mathbb{A}_k^{j-1-\lfloor j/p \rfloor}.$$

The j-component $\mathbb{G}_{m,k} \times \mathbb{A}_k^{j-1-\lfloor j/p \rfloor}$ corresponds to the set

$$\left\{ f \in \bigoplus_{l>0,\, p\nmid l} k \cdot t^{-l} \mid \operatorname{ord}(f) = -j \right\}.$$

9.2. P-moduli space

DEFINITION 9.2.1. A morphism of k-schemes $Y \to X$ is a *sur covering* if it is locally of finite presentation and surjective.

We denote by $\mathbf{GP}_k$ the category of geometric poits $\operatorname{Spec} F \to \operatorname{Spec} k$ of of $\operatorname{Spec} k$, which is a full subcategory of $\mathbf{Aff}_k$. In what follows, we often identify a k-scheme X with the associated (contravariant) functor $\mathbf{Aff}_k \to \mathbf{Set}$. For a functor $F \colon \mathbf{Aff}_k \to \mathbf{Set}$, we denote by $F|_{\mathbf{GP}_k}$ its restriction to $\mathbf{GP}_k$.

DEFINITION 9.2.2. For k-schemes X and Y, a *P-morphism* $f \colon Y \to X$ is a morphism $f \colon Y|_{\mathbf{GP}_k} \to X|_{\mathbf{GP}_k}$ of functors $\mathbf{GP}_k \to \mathbf{Set}$ such that there exist a k-scheme Z, a sur covering $g \colon Z \to X$ and a morphism $h \colon Z \to Y$ that make the following diagram commutative:

$$\begin{tikzcd} & Z|_{\mathbf{GP}_k} \arrow[dl, "h|_{\mathbf{GP}_k}"'] \arrow[dr, "g|_{\mathbf{GP}_k}"] & \\ Y|_{\mathbf{GP}_k} \arrow[rr, "f"'] & & X|_{\mathbf{GP}_k} \end{tikzcd}$$

We denote by $\mathrm{Hom}_k^P(Y,X)$ the set of P-morphisms $Y \to X$. A P-morphism $f\colon Y \to X$ is a *P-isomorphism* if there exists a P-morphism $g\colon X \to Y$ such that $f \circ g = \mathrm{id}_{X|_{\mathbf{GP}_k}}$ and $g \circ f = \mathrm{id}_{Y|_{\mathbf{GP}_k}}$.

We easily see that the composition of P-morphisms is again a P-morphism.

PROPOSITION 9.2.3. *Let X and Y be k-schemes locally of finite type. A P-morphism $f\colon Y \to X$ is a P-isomorphism if and only if there exist universally bijective finite-type morphisms $Z \to Y$ and $Z \to X$ of k-schemes such that the diagram in the last definition is commutative.*

PROOF. ??? $\square$

DEFINITION 9.2.4. We denote by P-$\mathbf{Sch}_k$ the category whose objects are k-schemes and morphisms are P-morphisms. We mean by a *P-scheme* a k-scheme regarded as an object of P-$\mathbf{Sch}_k$.

DEFINITION 9.2.5. For a k-scheme X, we define a functor $X^P\colon \mathbf{Aff}_k \to \mathbf{Set}$ by

$$X^P(Y) := \mathrm{Hom}_k^P(Y,X).$$

Note that for k-schemes X and Y, the set $\mathrm{Hom}(Y^P, X^P)$ of morphisms $Y^P \to X^P$ of functors is canonically identified with $\mathrm{Hom}_k^P(Y,X)$.

DEFINITION 9.2.6. Let $\mathcal{F}\colon \mathbf{Aff}_k \to \mathbf{Set}$ be a contravariant functor. A *P-moduli space* of $\mathcal{F}$ is a k-scheme X given with a morphism $f\colon \mathcal{F} \to X^P$ of functors such that

1. (1) for every algebraically closed field F/k, $f(F)\colon \mathcal{F}(F) \to X^P(F)$ is bijective,
2. (2) for any morphism $g\colon \mathcal{F} \to Y^P$ with Y a k-scheme, there exists a unique morphism $h\colon X^P \to Y^P$ of functors such that $g = h \circ f$.

It is clear that a P-moduli space is unique up to unique P-isomorphism.

THEOREM 9.2.7. *The functor*

$$\mathcal{C}_G\colon \mathbf{Aff}_k \to \mathbf{Set},\ \mathrm{Spec}\, R \mapsto \{G\textit{-torsors over } \mathrm{Spec}\, R(\!(t)\!)\}/\cong$$

has a P-moduli space that is the disjoint union of countably many affine k-schemes of finite type.

DEFINITION 9.2.8. We denote a P-moduli space of the last theorem by G-Cov(D).

9.3. Twisted arcs

Let V be a good D-scheme (see Definition 7.1.1) given with an action of a finite group G. We suppose that there exists a G-invariant open dense subscheme $U \subset V$ on which G acts freely. Let $X := V/G$ be the associated quotient scheme and let $\pi\colon V \to X$ be the quotient morphism.

DEFINITION 9.3.1. For a G-cover $E \to \mathrm{D}$, an *E-twisted arc* of V is a G-equivariant D-morphism $E \to V$. Let $(\mathrm{J}^E V)(k)$ be the set of E-twisted arcs of V.

REMARK 9.3.2 (Relation to ι-twisted arc). Suppose that G is tame and k is algebraically close. A connected component E_0 of every G-cover $E \to \mathrm{D}$ is isomorphic to $\mathrm{D}^l = \operatorname{Spec} k[\![t^{1/l}]\!]$ as a cover of D. Choosing an isomorphism $E_0 \cong \mathrm{D}^l$ defines an injection $\iota\colon \mu_l \hookrightarrow G$. For an E-twisted arc $E \to V$, the composition

$$\mathrm{D}^l \to E_0 \hookrightarrow E \to V$$

is an ι-twisted arc. Thus, in the tame case, ι-twisted arc anre E-twisted arc are essentially the same notion. However, simplicty of the group μ_l and its action on D^l is an advantage of considering ι-twisted arcs.

For an E-twisted arc $\gamma\colon E \to V$, the induced morphism

$$\pi_\infty(\gamma)\colon \mathrm{D} = E/G \to V/G = X$$

is an arc of X. Thus we get a map

$$\pi_\infty\colon \coprod_E (\mathrm{J}_\infty^E V)(k) \to (\mathrm{J}_\infty X)(k),\ \gamma \mapsto \pi_\infty(\gamma),$$

where E runs over all G-covers of D (modulo isomorphisms). If $\alpha\colon E \to E$ is an automorphism of E as a G-cover of D (that is, a G-equivariant D-isomorphism), then $\pi_\infty(\gamma) = \pi_\infty(\gamma \circ \alpha)$. Thus π_∞ factors through a map

$$\coprod_{E \to \mathrm{D}} (\mathrm{J}_\infty^E V)(k)/\operatorname{Aut}(E) \to (\mathrm{J}_\infty X)(k), \tag{9.3.1}$$

where $E \to \mathrm{D}$ runs over the G-covers of D.

Conversely, suppose that we are given an arc $\beta\colon \mathrm{D} \to X$ is an arc sending the generic point η_{D} into the étale locus of π. If E denotes the normalization of $\mathrm{D} \times_{\beta, X, \pi} V$, then the narual morphism $E \to \mathrm{D}$ is a G-cover and the natural morphism $\gamma\colon E \to V$ is an E-twisted arc such that $\pi_\infty(\gamma) = \beta$.

$$\begin{tikzcd}
E \arrow[r] \arrow[rr, bend left, "\gamma"] & \mathrm{D} \times_X V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\
 & \mathrm{D} \arrow[r, "\beta = \pi_\infty(\gamma)"'] & X
\end{tikzcd}$$

Moreover, the induced twisted arc γ is unique modulo automorphisms of E. Roughly speaking, the above arguments show the claim that map (9.3.1) is "almost bijective." To make this claim precise, we need to consider geometric points rather than k-points and G-covers of D_F for algebraically closed fields F, and define a motivic measure on the space of twisted arcs. This will be treated in a later section in a more general context by using stacks.

9.4. Hom schemes

We keep the notation of the last section. We construct the untwisting scheme $V^{|E|}$ that plays the same role as $V^{|\iota|}$ considered in Chapter 8. Construction is more intrinsic than the one in the tame case and uses the Hom scheme.

DEFINITION 9.4.1. Let S be a scheme and let X and Y be S-scheme. Suppose that X is flat and proper over S and that Y is finitely presented over S. The *Hom* *scheme* $\underline{\mathrm{Hom}}_S(X,Y)$ is defined to be an S-scheme representating the following functor:

$$\begin{aligned} \mathbf{Sch}_S &\to \mathbf{Set} \\ T &\mapsto \mathrm{Hom}_T(X \times_S T, Y \times_S T) \end{aligned}$$

Note that $\mathrm{Hom}_T(X \times_S T, Y \times_S T)$ is identified with $\mathrm{Hom}_S(X \times_S T, Y)$. When X and Y have G-actions, we can also define the *G-equivariant Hom scheme* $\underline{\mathrm{Hom}}_S^G(X,Y)$ representing the functor:

$$\begin{aligned} \mathbf{Sch}_S &\to \mathbf{Set} \\ T &\mapsto \mathrm{Hom}_T^G(X \times_S T, Y \times_S T) \end{aligned}$$

It is known that the Hom scheme exists and is locally of finite presentation over S . When X and Y have G-actions, we can define the G-action on $\underline{\mathrm{Hom}}_S(X,Y)$ by $g \cdot f := g \circ f \circ g^{-1}$. The G-equivariant Hom scheme $\underline{\mathrm{Hom}}_S^G(X,Y)$ is the fixed locus of this action, which is a closed subscheme of $\underline{\mathrm{Hom}}_S(X,Y)$.

Remark 9.4.2. Note that, for an S-scheme W with an automorphism $g\colon W \to W$, the fixed point locus W^g fits into the cartesian diagram:

$$\begin{tikzcd} W^g \arrow[r, hook] \arrow[d, hook] & W \arrow[d, "\Delta"] \\ W \arrow[r, "(\mathrm{id},\alpha)"'] & W \times_S W \end{tikzcd}$$

This shows that W^g is a closed subscheme of W. (Recall that every scheme is, by assumption, separated and the diagonal morphism is a closed immersion.) The G-fixed locus W^G for an action of a finite group G is given as the intersection $\bigcap_{g \in G} W^g$.

Lemma 9.4.3. *If X is finite over S, then $\underline{\mathrm{Hom}}_S(X,Y)$ is of finite presentation over S.*

Proof. We prove this only when Y is quasi-projective over S. For the general case, we refer the reader to [**Yas19**]. The problem is local on S. So we may suppose that S is connected. Then $X \to S$ has constant rank say r. We choose an immersion $X \times Y \hookrightarrow \mathbb{P}_S^n$. Sending a morphism $f\colon X \times_S T \to Y \times_S T$ to the graph

$$\Gamma_f \subset (X \times_S T) \times_T (Y \times_S T) = (X \times_S Y) \times T$$

defines an immersion

$$\underline{\mathrm{Hom}}_S(X,Y) \hookrightarrow \mathrm{Hilb}_r(\mathbb{P}_S^n/S).$$

Here the subscript r means that this is the Hilbert scheme associated to the constant polynomial r. Since $\mathrm{Hilb}_r(\mathbb{P}_S^n/S)$ is of finite presentation over S, so is $\underline{\mathrm{Hom}}_S(X,Y)$. $\square$

Proposition 9.4.4. *Suppose that X and Y have G-actions and that $X \to S$ is a G-torsor.*

(1) *There is a natural isomorphism*

$$\underline{\mathrm{Hom}}_S^G(X,Y) \cong (X \times_S Y)/G,$$

where G acts on $X \times_S Y$ diagonally.

(2) *If* $S = \operatorname{Spec} F$ *for a field* F*, then* $\underline{\mathrm{Hom}}^G_{\operatorname{Spec} F}(X,Y)$ *is a twisted form of* Y*. Namely,* $\underline{\mathrm{Hom}}^G_{\operatorname{Spec} F}(X,Y) \otimes_F \overline{F}$ *and* $Y \otimes_F \overline{F}$ *are isomorphic over* $\overline{F}$*.*

PROOF. (1) A G-equivariant morphism $f\colon X \to Y$ induces the G-equivariant morphism

$$X \xrightarrow{(\mathrm{id},f)} X \times_S Y,$$

which, in turn, induces

$$S = X/G \to (X \times_S Y)/G.$$

Thus, we get a map of sets of S-points,

$$\underline{\mathrm{Hom}}^G_S(X,Y)(S) \to ((X \times_S Y)/G)(S). \tag{9.4.1}$$

Conversely, for a morphism $S \to (X \times_S Y)/G$, let

$$X' := S \times_{(X \times_S Y)/G} (X \times_S Y).$$

Since $X \times_S Y \to (X \times_S Y)/G$ is a G-torsor, so is $X' \to S$. The natural morphisms $X' \to X$ and $X' \to Y$ are G-equivariant S-morphisms. In particular, $X' \to X$ is a G-equivariant morphism of G-torsors, and hence an isomorphism. We get a G-equivariant morphism

$$X \xrightarrow{\sim} X' \to Y.$$

This construction gives the inverse of map (9.4.1). We get a one-to-one correspondence

$$\underline{\mathrm{Hom}}^G_S(X,Y)(S) \leftrightarrow ((X \times_S Y)/G)(S).$$

Moreover, this correspondence is functorial; for each S-scheme T, we can construct a one-to-one correspondence of T-points. Thus, $\underline{\mathrm{Hom}}^G_S(X,Y)$ and $(X \times_S Y)/G$ are naturally isomorphic functors, and hence isomorphic as S-schemes.

(2) We have

$$\underline{\mathrm{Hom}}^G_{\operatorname{Spec} F}(X,Y) \otimes_F \overline{F} = \underline{\mathrm{Hom}}^G_{\operatorname{Spec} \overline{F}}(X_{\overline{F}}, Y_{\overline{F}}) \cong (X_{\overline{F}} \times_{\overline{F}} Y_{\overline{F}})/G.$$

Since $X \otimes_F \overline{F}$ is a trivial G-torsor over X, we have $(X_{\overline{F}} \times_{\overline{F}} Y_{\overline{F}})/G \cong Y_{\overline{F}}$. $\square$

LEMMA 9.4.5. *Suppose that* G *transitively acts on the connected components of* X*. Let* X_0 *be a connected component of* X *and let* $H \subset G$ *be its stabilizer. Then we have a canonical isomorphism* $\underline{\mathrm{Hom}}^G_S(X,Y) \cong \underline{\mathrm{Hom}}^H_S(X_0,Y)$*.*

PROOF. We choose a subset $\{g_0 = 1, g_1, \dots, g_m\}$ such that $X = \coprod_i g_i(X_0)$. For an S-scheme T and a G-equivariant morphism $X \times_S T \to Y$, the restriction $X_0 \times_S T \to Y$ is H-equivariant. Conversely, an H-equivariant morphism $X_0 \times_S T \to Y$ uniquely extends to a G-equivariant morphism $\gamma\colon X \times_S T \to Y$; we define it so that its restriction to $g_i(X_0) \times_S T$ is $g_i \circ \gamma \circ g_i^{-1}$. It is straightforward to check that the constructed morphism is indepedent of the choice of $\{g_0, \dots, g_m\}$ and G-equivariant. These constructions give a natural isomorphism of two functors

$$\underline{\mathrm{Hom}}^G_S(X,Y), \underline{\mathrm{Hom}}^H_S(X_0,Y)\colon \mathbf{Sch}_S \to \mathbf{Set}.$$

$\square$

9.5. Untwisting revisited

We follow the notation of Section 9.3.

DEFINITION 9.5.1. For a G-cover $E \to \mathrm{D}$, we define the *untwisting scheme* $V^{|E|}$ of V with respect to E to be the closure of $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V) \times_{\mathrm{D}} \mathrm{D}^*$ in $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V)$ given with the reduced structure.

REMARK 9.5.2. From Lemma 9.4.5, if $E_0 \subset E$ is a connected componet with stabilizer H and if we define the untwisting scheme $V^{|E_0|}$ with respect to the induced H-action on V, then $V^{|E|} \cong V^{|E_0|}$. It is sometimes helpful to reduce to the case where E is connected by using this fact.

LEMMA 9.5.3. *The untwisting scheme is a good* D*-scheme.*

PROOF. From Lemma 9.4.3, $V^{|E|}$ is a finite type D-scheme. From Proposition 9.4.4, the generic fiber $(V^{|E|})_\eta$ is of pure dimension and has an open dense subscheme that is smooth over D^*. From construction, $V^{|E|}$ is of pure relative dimension and flat over D, and contains an open dense subscheme that is smooth over D. $\square$

The universal morphism $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V) \times_{\mathrm{D}} E \to V$, which is defined over D and G-equivariant, restricts to $u\colon V^{|E|} \times_{\mathrm{D}} E \to V$. Since $V^{|E|} \to \mathrm{D}$ is flat, we have $(V^{|E|} \times_{\mathrm{D}} E)/G = V^{|E|}$. Therefore, we get the natural morphiosm

$$\pi^{|E|}\colon V^{|E|} = (V^{|E|} \times_{\mathrm{D}} E)/G \to V/G = X.$$

This is $\mathrm{Aut}(E)$-invariant with respect to the natural $\mathrm{Aut}(E)$-action on $V^{|E|}$. These morphisms form the following commutative diagram:

$$\begin{tikzcd} V^{|E|} \times_{\mathrm{D}} E \arrow[r, "r_E"] \arrow[d, "u_E"'] & V^{|E|} \arrow[d, "\pi^{|E|}"] \\ V \arrow[r, "\pi"'] & X \end{tikzcd} \tag{9.5.1}$$

Here r_E denotes the projection. The restrcition of $\pi^{|E|}$ to generic fibers,

$$(\pi^{|E|})_\eta\colon (V^{|E|})_\eta = \underline{\mathrm{Hom}}_{\mathrm{D}^*}^{G}(E^*, V_\eta) \to X_\eta,$$

can be expressed as a natural transform of funtors as follows:

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathrm{D}^*}^{G}(E^*, V_\eta)(T) \to X_\eta(T) \\ &\quad (E^* \times_{\mathrm{D}^*} T \to V_\eta) \mapsto (T = (E^* \times_{\mathrm{D}^*} T)/G \to V_\eta/G = X_\eta) \end{aligned}$$

This expression implies that $(\pi^{|E|})_\eta$ is a twisted form π_η (cf. Proposition 9.4.4). In particular, $(\pi^{|E|})_\eta$ is a generically étale finite morphism whose branch locus is the same as the one of π_η. In summary, diagram (9.5.1) has the following properties:

PROPOSITION 9.5.4. *Morphisms r_E and π are finite. Morphisms $(r_E)_\eta$, $(\pi^{|E|})_\eta$, $(u_E)_\eta$, and π_η are all finite and generically étale. Morphisms $(\pi^{|E|})_\eta$ and π_η are twisted forms of each other.*

REMARK 9.5.5. In the tame case, we can similarly define a morphism $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V) \to X$ as the natural transform that sends a G-equivariant D-morphism $E \times_{\mathrm{D}} T \to V$ to the induced morphism

$$T = (E \times_{\mathrm{D}} T)/G \to V/G = X.$$

In the wild case, the equality $T = (E \times_{\mathrm{D}} T)/G$ holds only when $T \to \mathrm{D}$ is flat. Therefore, the above construction of the morphism $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V) \to X$ is not valid, and taking the flat variant $V^{|E|}$ of $\underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V)$ is necessary.

PROPOSITION 9.5.6. *There exists a natural one-to-one correspondence*

$$(\mathrm{J}^E V)(k) \leftrightarrow (\mathrm{J}\, V^{|E|})(k)$$

that is compatible with the map $\pi_\infty \colon (\mathrm{J}^E V)(k) \to (\mathrm{J}_\infty X)(k)$ *defined in Section 9.3 and the map* $(\pi^{|E|})_\infty \colon (\mathrm{J}\, V^{|E|})(k) \to (\mathrm{J}_\infty X)(k)$ *derived from* $\pi^{|E|}$*. Moreover, the correspondence is compatible with the* $\mathrm{Aut}(E)$*-action on both sides.*

PROOF. By construction, we have the natural one-to-one correspondences:

$$(\mathrm{J}^E V)(k) \leftrightarrow \underline{\mathrm{Hom}}_{\mathrm{D}}^{G}(E, V)(\mathrm{D}) \leftrightarrow (\mathrm{J}\, V^{|E|})(k)$$

Let $\alpha \colon E \to V$ be a twisted arc and let $\beta \colon \mathrm{D} \to V^{|E|}$ be the corresponding arc. The arc $\pi_\infty(\alpha)$ is the morphism

$$\mathrm{D} = E/G \to V/G = X$$

induced by α. Let us consider the restriction of β to D^*, $\beta_\eta \colon \mathrm{D}^* \to (V^{|E|})_\eta$, which corresponds to $\alpha_\eta \colon E^* \to V_\eta$. From the description of $(\pi^{|E|})_\eta$ as a natural transform of functors, the morphism $(\pi^{|E|})_\eta \circ \beta_\eta \colon \mathrm{D}^* \to X$ is the morphism

$$\mathrm{D}^* = E^*/G \to V_\eta/G = X_\eta$$

induced by α_η. This shows that the arcs $\pi_\infty(\alpha)$ and $(\pi^{|E|})_\infty(\beta)$ coincide when restricted to D^*. Since X is separated, they coincide without restriction. It is straightforward to check that the correspondence with the $\mathrm{Aut}(E)$-actions. $\square$

REMARK 9.5.7. The correspondence of the propostioin induces a map

$$\coprod_{E \in G\text{-}\mathrm{Cov}(\mathrm{D})(k)} (\mathrm{J}\, V^{|E|})(k)/\,\mathrm{Aut}(E) \to (\mathrm{J}\, X)(k).$$

If k is algebraically closed or if we consider all geometric points, then this map becomes almost bijective in a justifiable sense.

The following lemma is useful to reduce problems to the case where E is connected.

LEMMA 9.5.8. *Let* E_0 *be a connected component of* E *and let* $H \subset G$ *be its stabilizer. We have a canonical isomorphism*

$$V^{|E|} \cong V^{|E_0|}.$$

PROOF. We have $\square$

9.6. A rough sketch of the wild McKay correspondence

Suppose now that V is normal and given a $\mathbb{Q}$-divisor B with $K_X + B$ being $\mathbb{Q}$-Cartier and that B is stable under the G-action. Then X has a unique $\mathbb{Q}$-divisor A such that $K_X + A$ is $\mathbb{Q}$-Cartier and $(V, B) \to (X, A)$ is crepant. Indeed, if $K_{V/X}$ is the relative canonical divisor (that is, the ramification divisor), then A is determined by the equality

$$\pi^* A = \Delta + K_{V/X}.$$

See [**Kol13**, pages 64–65]. For a G-cover $\mathrm{E} \to \mathrm{D}$, let $V^{|E|,\nu}$ denote the normalization of $V^{|E|}$. It has a unique $\mathbb{Q}$-divisor $B^{|E|}$ such that $K_{V^{|E|,\nu}} + A$ is $\mathbb{Q}$-Cartier and $(V^{|E|,\nu}, B^{|E|}) \to (X, A)$ is crepant.

We now mimic the proof of the tame McKay correspondence in Section 8.7. From Lemma 6.4.5, we have

$$\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,A,r}} \circ (\pi^{|E|,\nu})_\infty - \mathrm{j}_{\pi^{|E|,\nu}} = \frac{1}{r}\operatorname{ord}_{\mathcal{I}_{V^{|E|,\nu},B^{|E|},r}}.$$

Here $\pi^{|E|,\nu}$ denotes the morphism $V^{|E|,\nu} \to X$. From Theorem 8.6.9,

$$\begin{aligned}\int_{(\pi^{|E|,\nu})_\infty(\mathrm{J}_\infty V^{|E|,\nu})} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,A,r}}}\, d\mu_X &= \int_{\mathrm{J}_\infty V^{|E|}} \mathbb{L}^{\frac{1}{r}\operatorname{ord}_{\mathcal{I}_{X,A,r}}}\, d\mu_{X,\mathrm{Aut}(E)} \\ &= \mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})/\operatorname{Aut}(E).\end{aligned}$$

From the "almost bijection" in Remark 9.5.7, we see that $\mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})/\operatorname{Aut}(E)$ is the contribution of E to the stringy motive $\mathrm{M}_{\mathrm{st}}(X,A)$. Similarly, if $C \subset X_0$ is a constructible subset of $X_0 \otimes_{k[\![t]\!]} k$, then the contribution of E to $\mathrm{M}_{\mathrm{st}}(X,A)_C$ is $\mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})_{(\pi^{|E|,\nu})^{-1}(C)}$. Therefore, we can expect:

THE WILD MCKAY CORRESPONDENCE (A NAIVE FORMULATION). *We have:*

$$\mathrm{M}_{\mathrm{st}}(X,A)_C = \text{“}\int_{G\text{-}\mathrm{Cov(D)}} \mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})_{(\pi^{|E|,\nu})^{-1}(C)}/\operatorname{Aut}(E)\text{”}$$

In particulaf, for $C = X_0$, we have:

$$\mathrm{M}_{\mathrm{st}}(X,A) = \text{“}\int_{G\text{-}\mathrm{Cov(D)}} \mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})/\operatorname{Aut}(E)\text{”}$$

REMARK 9.6.1. We do not know whether the map

$$G\text{-}\mathrm{Cov(D)}(k) \to \widehat{\mathcal{M}}'_{k,r},\ E \mapsto \mathrm{M}_{\mathrm{st}}(V^{|E|,\nu}, B^{|E|})_{(\pi^{|E|,\nu})^{-1}(C)}/\operatorname{Aut}(E)$$

extends to a locally constructible function on G-Cov(D). Therefore, we need some effort to define the above integrals over G-Cov(D) rigorously.

REMARK 9.6.2. If we use weak log pairs rather than usual log pairs, we can treat the case where X is non-normal and avoid taking the normalization of $V^{|E|}$. If $(X, \mathcal{L})$ is a weak log pair, then for each G-cover $E \to \mathrm{D}$, we can define a natural structure of weak log pair $(V^{|E|}, \mathcal{L}^{|E|})$ on the untwisting scheme $V^{|E|}$. Then, the wild McKay correspondence is then naively formulated as:

$$\mathrm{M}_{\mathrm{st}}(X,\mathcal{L})_C = \text{“}\int_{G\text{-}\mathrm{Cov(D)}} \mathrm{M}_{\mathrm{st}}(V^{|E|}, \mathcal{L}^{|E|})_{(\pi^{|E|})^{-1}(C)}/\operatorname{Aut}(E)\text{”}$$

9.7. Linear actions: tuning modules and v-functions

We now focus on the case where $V = \mathbb{A}^d_{\mathrm{D}}$ and G acts on it linearly. We can write $V = \operatorname{Spec}(\operatorname{Sym}(M))$, where M is a free $k[\![t]\!]$-module of rank d with a G-action and $\operatorname{Sym}(M)$ is the associated symmetric algebra having the induced G-action, which is identified with $k[\![t]\!][x_1, \ldots, x_d]$ if we fix a basis $x_1, \ldots, x_d$ of M.

DEFINITION 9.7.1. Let $E = \operatorname{Spec} O_E \to \mathrm{D}$ be a G-cover. We define the *tuning module* of E to be

$$\Xi_E := \operatorname{Hom}^G_{k[\![t]\!]}(M, O_E),$$

the module of G-equivariant $k[\![t]\!]$-module homomorphisms $M \to O_E$.

LEMMA 9.7.2. *The module Ξ_E is a saturated $k[\![t]\!]$-submodule of $\mathrm{Hom}_{k[\![t]\!]}(M, O_E)$. (Here the term "saturated" means that $\mathrm{Hom}_{k[\![t]\!]}(M, O_E)/\Xi_E$ is a torsion-free $k[\![t]\!]$-module.)*

PROOF. The tuning module Ξ_E is the kernel of the $k[\![t]\!]$-linear map

$$\mathrm{Hom}_{k[\![t]\!]}(M, O_E) \to \mathrm{Hom}_{k[\![t]\!]}(M, O_E)^{\oplus G},\ \alpha \mapsto (\alpha \circ g - g \circ \alpha)_{g \in G}. \tag{9.7.1}$$

Therefore it is a $k[\![t]\!]$-submodule of $\mathrm{Hom}_{k[\![t]\!]}(M, O_E)$. To show the saturatedness, we need to show that for $\alpha \in \mathrm{Hom}_{k[\![t]\!]}(M, O_E)$ and $0 \neq f \in k[\![t]\!]$, if $f\alpha \in \Xi_E$, then $\alpha \in \Xi_E$. Indeed, if $f\alpha \in \Xi_E$, then for every $m \in M$ and $g \in G$, we have the following equalities of elements of $k[\![t]\!]$:

$$\begin{aligned}
f(\alpha(mg)) &= (f\alpha)(mg) && \text{(definition of } f\alpha) \\
&= ((f\alpha)(m))g && (f\alpha \in \Xi_E) \\
&= (f(\alpha(m)))g && \text{(definition of } f\alpha) \\
&= f((\alpha(m))g) && (g \text{ is } k[\![t]\!]\text{-linear})
\end{aligned}$$

Since $k[\![t]\!]$ is an integral domain, we have that $\alpha(mg) = (\alpha(m))g$. This means that α is G-equivariant, and hence $\alpha \in \Xi_E$. $\square$

LEMMA 9.7.3. *For a flat $k[\![t]\!]$-algebra R, we have natural isomorphism:*

$$\Xi_E \otimes_{k[\![t]\!]} R \cong \mathrm{Hom}_L^G(M \otimes_{k[\![t]\!]} R, O_E \otimes_{k[\![t]\!]} R) \cong \mathrm{Hom}_{k[\![t]\!]}^G(M, O_E \otimes_{k[\![t]\!]} R)$$

PROOF. The right isomorphism is obvious. Applying $- \otimes_{k[\![t]\!]} R$ to the exact seqeunce

$$0 \to \Xi_E \to \mathrm{Hom}_{k[\![t]\!]}(M, O_E) \xrightarrow{\text{map (9.7.1)}} \mathrm{Hom}_{k[\![t]\!]}(M, O_E)^{\oplus G},$$

we get the exact sequence

$$0 \to \Xi_E \otimes_{k[\![t]\!]} R \to \mathrm{Hom}_{k[\![t]\!]}(M, O_E) \to \mathrm{Hom}_{k[\![t]\!]}(M, O_E \otimes_{k[\![t]\!]} R)^{\oplus G}.$$

This shows the desired isomorphism. $\square$

LEMMA 9.7.4. *The tuning module Ξ_E is a free $k[\![t]\!]$-module of rank d.*

PROOF. Since $\mathrm{Hom}_{k[\![t]\!]}(M, O_E)$ is a torsion-free $k[\![t]\!]$-module, so is Ξ_E. Since $k[\![t]\!]$ is a princial ideal domain, Ξ_E is a free $k[\![t]\!]$-module. It remains to compute the rank. We take a field extension $L/k(\!(t)\!)$ trivializing the G-torsor $E^* \to \mathrm{D}^*$. Namely, $O_E \otimes_{k[\![t]\!]} L$ is isomorphic to $L^{\oplus G}$ as a $k[\![t]\!]$-algebra and G acts on it by permutation. Since L is a flat $k[\![t]\!]$-algebra, we see

$$\Xi_E \otimes_{k[\![t]\!]} L \cong \mathrm{Hom}_L^G(M \otimes_{k[\![t]\!]} L, O_E \otimes_{k[\![t]\!]} L) \cong \mathrm{Hom}_L^G(M \otimes_{k[\![t]\!]} L, L^{\oplus G}).$$

The last module is explicitly presented as

$$\{(\phi \circ g)_{g \in G} \colon M \otimes_{k[\![t]\!]} L \to L^{\oplus G} \mid \phi \in \mathrm{Hom}_L(M \otimes_{k[\![t]\!]} L, L)\},$$

and hence isomorphic to $\mathrm{Hom}_L(M \otimes_{k[\![t]\!]} L, L) \cong L^{\oplus d}$. Thus

$$\mathrm{rank}_{k[\![t]\!]} \Xi_E = \mathrm{rank}_L(\Xi_E \otimes_{k[\![t]\!]} L) = d.$$

$\square$

PROPOSITION 9.7.5. *We have natural isomorphisms:*

$$V^{|E|} \cong \mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee)) \cong \mathbb{A}_{\mathrm{D}}^d$$

PROOF. We first observe that for a flat $k[\![t]\!]$-algebra R, we have functorial one-to-one correspondences:

$$\begin{aligned}\underline{\mathrm{Hom}}^G_{\mathrm{D}}(E,V)(R) &\leftrightarrow \mathrm{Hom}^G_{k[\![t]\!]}(M, O_E \otimes_{k[\![t]\!]} R)\\ &\leftrightarrow \mathrm{Hom}^G_{k[\![t]\!]}(M, O_E) \otimes_{k[\![t]\!]} R\\ &\leftrightarrow \Xi_E^{\vee\vee} \otimes R\\ &\leftrightarrow (\mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee))(R)\end{aligned}$$

Letting R vary over $k(\!(t)\!)$-algebras, we see

$$\underline{\mathrm{Hom}}^G_{\mathrm{D}}(E,V)_\eta \cong \mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee))_\eta.$$

The canonical $\mathrm{Sym}(\Xi_E^\vee)$-point of $\mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee))$ corresponds to a morphism

$$\mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee)) \to \underline{\mathrm{Hom}}^G_{\mathrm{D}}(E,V),$$

which is the unique extension of the isomorphism of generic fibers. The image of the last morphism is contained in $V^{|E|}$. We construct the inverse $V^{|E|} \to \mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee))$ as follows. We take an affine open covering $V^{|E|} = \bigcup \mathrm{Spec}\, S_i$. The above correspondences give morphisms

$$\mathrm{Spec}\, S_i \to \mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee)),$$

which glue together to define $V^{|E|} \to \mathrm{Spec}(\mathrm{Sym}(\Xi_E^\vee))$. □

In particular, $V^{|E|}$ is normal and $V^{|E|,\nu} = V^{|E|}$.

DEFINITION 9.7.6. For a G-cover $E \to \mathrm{D}$, we define

$$\mathbf{v}_\rho(E) := \frac{1}{\sharp G}\ell_{k[\![t]\!]}\left(\frac{\mathrm{Hom}_{k[\![t]\!]}(M, O_E)}{O_E \cdot \Xi_E}\right) \in \frac{1}{\sharp G}\mathbb{Z}_{\ge 0},$$

where $\ell_{k[\![t]\!]}$ denotes the length of a $k[\![t]\!]$-module.

LEMMA 9.7.7. *Let k'/k be a field extension and let $E \to \mathrm{D}$ be a G-cover. Let $\rho_{k'}$ be the G-representation over $k'[\![t]\!]$ induced by ρ and let $E_{k'} \to \mathrm{D}_{k'}$ be the induced G-cover. Then*

$$\mathbf{v}_\rho(E) = \mathbf{v}_{\rho_{k'}}(E_{k'}).$$

PROOF. We see that

$$\frac{\mathrm{Hom}_{k'[\![t]\!]}(M \otimes_{k[\![t]\!]} k'[\![t]\!], O_E \otimes_{k[\![t]\!]} k'[\![t]\!])}{O_{E_{k'}} \cdot \Xi_{E_{k'}}} \cong \frac{\mathrm{Hom}_{k[\![t]\!]}(M, O_E)}{O_E \cdot \Xi_E} \otimes_{k[\![t]\!]} k'[\![t]\!].$$

Therefore, the $k'[\![t]\!]$-length of the left module is equal to the $k[\![t]\!]$-length of

$$\mathrm{Hom}_{k[\![t]\!]}(M, O_E)/O_E\Xi_E.$$

□

PROPOSITION 9.7.8. *Suppose that V has the zero boundary divisor $B = 0$. Then the induced divisor $B^{|E|}$ on $V^{|E|}$ is $\mathbf{v}_\rho(E) \cdot (V^{|E|})_0$, where $(V^{|E|})_0$ is the special fiber of $V^{|E|}$ and regarded as a prime divisor.*

PROOF. From Lemma 9.7.7, we may assume that k is algebraically closed. From Remark 9.5.2, we may suppose that E is connected. It suffices to show that $V^{|E|} \times_{\mathrm{D}} E \to V$ and $V^{|E|} \times_{\mathrm{D}} E \to (V^{|E|}, \mathbf{v}(E) \cdot (V^{|E|})_0)$ have the same relative canonical divisor. The different $\mathfrak{d} \subset O_E$ of the extension $O_E/k[\![t]\!]$ is, by definition, the annihilator ideal of $\Omega_{O_E/k[\![t]\!]}$. We have

$$K_{V^{|E|}\times_{\mathrm{D}}E/V} = K_{V^{|E|}\times_{\mathrm{D}}E/V\times_{\mathrm{D}}E} + \mathrm{div}(\mathfrak{d}),$$

where $\mathrm{div}(\mathfrak{d})$ denotes the effective divisor on $V^{|E|}\times_{\mathrm{D}} E$ defined by $\mathfrak{d}$. The E-morphism $V^{|E|}\times_{\mathrm{D}} E \to V\times_{\mathrm{D}} E$ corresponds to the natural O_E-module homomophism

$$\alpha\colon M\otimes_{k[\![t]\!]} O_E \to \mathrm{Hom}_{k[\![t]\!]}(\mathrm{Hom}^G_{k[\![t]\!]}(M, O_E), O_E)$$

between two free O_E-modules of rank d. The divisor $K_{V^{|E|}\times_{\mathrm{D}}E/V\times_{\mathrm{D}}E}$ is the effective divisor defined by the determinant of this map. Thus

$$K_{V^{|E|}\times_{\mathrm{D}}E/V} = \mathrm{div}(\det(\alpha)) + \mathrm{div}(\mathfrak{d}).$$

On the other hand, we have

$$\begin{aligned} K_{V^{|E|}\times_{D}E/(V^{|E|},\mathbf{v}(E)\cdot(V^{|E|})_0)} &= K_{V^{|E|}\times_{D}E/V^{|E|}} + \mathbf{v}(E)\mathrm{div}(t). \\ &= \mathrm{div}(\mathfrak{d}) + \mathbf{v}(E)\mathrm{div}(t). \end{aligned}$$

It remains to show the equality

$$\mathrm{div}(\det(\alpha)) = \mathbf{v}(E)\mathrm{div}(t)$$

of divisors on $V^{|E|}\times_{\mathrm{D}} E$. The O_E-dual of α is identical to the natural injection

$$\mathrm{Hom}^G(M, O_E)\otimes_{O_E} O_E \hookrightarrow \mathrm{Hom}_{k[\![t]\!]}(M, O_E),$$

which has the same determinant as α does. Therefore,

$$\mathbf{v}(E) = \ell_{k[\![t]\!]}(O_E/(\det(\alpha)))/\sharp G.$$

Let $u \in O_E$ be a uniformizer. Since $tO_E = (u^{\sharp G})$,

$$\mathbf{v}(E)\mathrm{div}(t) = \ell_{k[\![t]\!]}(O_E/(\det(\alpha)))\cdot \mathrm{div}(u) = \mathrm{div}(\det(\alpha)).$$

□

In particular, the proposition shows that $\mathbf{v}_\rho(E) \neq \infty$. We can define a function

$$\mathbf{v}_\rho\colon G\text{-Cov}(\mathrm{D}) \to \frac{1}{\sharp G}\mathbb{Z}_{\geq 0} \tag{9.7.2}$$

as follows. For a point $b \in G$-Cov(D), we take a geometric point $b'\colon \operatorname{Spec} F \to G$-Cov(D) mapping to b. If $E \to \mathrm{D}_F$ is the G-cover associated to b', then we define

$$\mathbf{v}_\rho(b) := \mathbf{v}_{\rho_F}(E).$$

From Lemma 9.7.7, this is indepedent of the choice of the geometric point.

Definition 9.7.9. We call function (9.7.2) the *v-function* associated to the representation ρ.

The following result is useful to compute the v-function.

Remark 9.7.10. Choosing a basis of M, let us identify $\mathrm{Hom}_{k[\![t]\!]}(M, \mathcal{O}_E)$ with $\mathcal{O}_E^{\oplus d}$. There exist two G-actions on $\mathrm{Hom}_{k[\![t]\!]}(M, \mathcal{O}_E) = \mathcal{O}_E^{\oplus d}$. One is induced by the action on M and the other is the diagonal action $\mathcal{O}_E^{\oplus d}$ derived from the G-action on $\mathcal{O}_E$. The tuning module Ξ_E consists of those elements on which the two actions coincide:

$$\Xi_E = \{\alpha \in \mathrm{Hom}_{k[\![t]\!]}(M, \mathcal{O}_E) \mid \forall g\in G,\ \alpha\cdot g = \alpha * g\},$$

where $\cdot$ and $*$ denote these two actions. This presentation of tuning module is closer to the ones in [**Yas17**, **WY15**].

Proposition 9.7.11. *The v-function is locally constructible. (Postpone the proof)*

From the proposition, there is a stratification $G\text{-Cov}(\mathrm{D}) = \bigsqcup_{i \in I} C_i$ by at most countably many constructible sets C_i such that for each i, $\mathbf{v}_\rho|_{C_i}$ is constant. The integral $\int_{G\text{-Cov}(\mathrm{D})} \mathbb{L}^{d-\mathbf{v}_\rho}$ is then defined as the (possibly diviergent) sum

$$\sum_{i \in I} \{C_i\} \mathbb{L}^{d-\mathbf{v}_\rho(C_i)}.$$

Theorem 9.7.12. *Let $\rho\colon G \curvearrowright V = \mathbb{A}^d_{\mathrm{D}}$ be a representation over $k[\![t]\!]$, let $X := V/G$ and let A be the boundary divisor on X such that $V \to (X, A)$ is crepant. Then*

$$\mathrm{M}_{\mathrm{st}}(X, A) = \int_{G\text{-Cov}(\mathrm{D})} \mathbb{L}^{d-\mathbf{v}_\rho}.$$

Definition 9.7.13. Let $o \in V(k)$ be the origin in the special fiber and let $\overline{o} \in X(k)$ be its image. We define

$$\mathbf{d}_\rho(E) := \dim(u_E)^{-1}(o) = \dim(\pi^{|E|})^{-1}(\overline{o}).$$

Note that the right equality follows from Proposition 9.5.4. Like $\mathbf{v}_\rho$, the above definition gives a function

$$\mathbf{d}_\rho\colon G\text{-Cov}(\mathrm{D}) \to \{0, 1, \dots, d\} \subset \mathbb{Z}.$$

It will turn out that this is locally constructible (??).

Theorem 9.7.14. *We keep the above notation. Then*

$$\mathrm{M}_{\mathrm{st}}(X, A)_{\overline{o}} = \int_{G\text{-Cov}(\mathrm{D})} \mathbb{L}^{\mathbf{d}_\rho - \mathbf{v}_\rho}.$$

Remark 9.7.15. If $V \to X$ is étale in codimension one, then $A = 0$ and we have

$$\mathrm{M}_{\mathrm{st}}(X) = \int_{G\text{-Cov}(\mathrm{D})} \mathbb{L}^{d-\mathbf{v}_\rho} \text{ and } \mathrm{M}_{\mathrm{st}}(X)_{\overline{o}} = \int_{G\text{-Cov}(\mathrm{D})} \mathbb{L}^{\mathbf{d}_\rho - \mathbf{v}_\rho}.$$

If moreover there exists a proper birational morphism $f\colon Y \to X$ with Y regular, then we have $\mathrm{M}_{\mathrm{st}}(X) = \{Y_0 \cap Y_{\mathrm{sm}}\}$ and $\mathrm{M}_{\mathrm{st}}(X)_{\overline{o}} = \{f^{-1}(\overline{o}) \cap Y_{\mathrm{sm}}\}$, where Y_{sm} denotes the D-smooth locus of Y. See Proposition 7.3.3.

9.8. The tame case revisited

In general, we have the one-to-one correspondence:

$$\{G\text{-covers of D}\} \leftrightarrow \{\text{continuous homomorphisms } \Gamma_{k(\!(t)\!)} \to G\}/G$$

Here the G-action on the set of continuous homomorphisms is the one induced from the conjugation action of G on itself. If G is tame, then the right side is unchanged by replacing $\Gamma_{k(\!(t)\!)}$ with its maximal tame quotient $\Gamma^t_{k(\!(t)\!)}$. If k is algebraically closed, then only finite tame Galois extensions of $k(\!(t)\!)$ are of the form $k(\!(t^{1/n})\!)/k(\!(t)\!)$, $p \nmid n$. Therefore,

$$\Gamma^t_{k(\!(t)\!)} = \mathrm{Gal}(k(\!(t^{1/n})\!)/k(\!(t)\!)) \cong \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z} \cong \prod_{l \neq p} \mathbb{Z}_l,$$

where n runs over positive integers coprime to p and l runs over prime numbers different from p. The right isomorphism is canonical. Let us fix the left one, which amounts to choosing a compatible system of primitive n-th roots $\zeta_n \in k$ of unity. Then, $\Gamma^t_{k(\!(t)\!)}$ has a topological generator γ corresponding to $1 \in \mathbb{Z} \subset \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z}$. Therefore, we get the following one-to-one correspondence:

$$\begin{aligned} \{\text{continuous homomorphisms } \Gamma_{k(\!(t)\!)} \to G\}/G &\leftrightarrow \mathrm{Conj}(G) \\ [\phi] &\mapsto [\phi(\gamma)] \end{aligned}$$

Concretely, if $g \in G$ is an element of order n, then a conjugacy class $[g] \in \mathrm{Conj}(G)$ corresponds to a G-cover $E \to \mathrm{D}$ having a connected component $E_0 = \operatorname{Spec} k[\![t^{1/n}]\!]$ on which the subgroup $\langle g \rangle \subset G$ acts by $t^{1/n}g = \zeta_n t^{1/n}$.

Let n be the exponent of G, the least positive integer such that for every $g \in G$, $g^n = 1$. Suppose that k contains all the n-th roots of unity. For each $l \mid n$, we choose a primitive l-th root of unity, $\zeta_l \in k$, such that, if $l \mid l'$, then $(\zeta_l)^{l/l'} = \zeta_{l'}$. As a pseudo-moduli space G-$\mathrm{Cov}(\mathrm{D})$, we can take

$$G\text{-}\mathrm{Cov}(\mathrm{D}) = \coprod_{[g] \in \mathrm{Conj}(G)} \operatorname{Spec} k.$$

Let $g \in G$ be an element of order l. The component associated to its conjugacy class $[g]$ corresponds to the G-cover $E \to \mathrm{D}$ that has a connected component $E_0 = \operatorname{Spec} k[\![t^{1/l}]\!]$ with the action of $\langle g \rangle \subset G$ given by $t^{1/l}g = \zeta_l t^{1/l}$.

9.9. The point-counting version and mass formulae

9.10. The case $G = \mathbb{Z}/p\mathbb{Z}$

9.11. The case $G = \mathbb{Z}/p^n\mathbb{Z}$

9.12. More examples

9.13. Non-linear actions

9.14. Duality

CHAPTER 10

Deligne-Mumford stacks

In this chapter, we give a brief introduction to Deligne-Mumford stacks (DM stacks for short) as well as prepare general results on DM stacks for later use.

Throughout the chapter, k denotes a fixed base ring. Symbols in bold type such as **Set** and $\mathbf{Sch}_k$ usually mean categories. When $\mathbf{C}$ is a category, the notation $c \in \mathbf{C}$ means that c is an object of $\mathbf{C}$. When we describe a functor from a category to another, we usually describe only the correspondence of objects and write the functor, for example, as

$$\mathbf{C} \to \mathbf{D},\ c \mapsto d,$$

like a map of sets. For the correspondence of morphisms is often obvious.

10.1. Motivation

Deligne-Mumford stacks (DM stacks for short) are generalization of schemes. This notion originates in a study of moduli of curves by Deligne and Mumford [**DM69**]. In a general moduli problem, one seeks for a moduli space/scheme M parametrizing objects in question. In algebraic geometry, we sometimes has even a *fine* moduli scheme, that is, a scheme X whose associated functor

$$h_X\colon (\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set},\ Y \mapsto \mathrm{Hom}_{\mathbf{Sch}_k}(Y, X)$$

is isomorphic to the moduli functor

$$(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set},\ Y \mapsto \{\text{family of objects over } Y\}/\cong.$$

The Hilbert scheme parametrizing closed subscheme of a given projective variety is a typical example of fine moduli scheme.

However, when objects have non-trivial automorphisms, a fine moduli scheme usually does not exit. For automorphisms often give rise to two distinct families over the same scheme Y which are fiberwise isomorphic. If there was a fine moduli scheme M, then the two families induce two maps $Y \rightrightarrows M$ and lead to the contradiction that the two maps are both distinct and identical. The problem is that in the moduli functor, we discard information of isomorphisms by considering sets of objects *modulo isomorphisms*. In the theory of stacks, we consider instead categories of objects. Accordingly, each point of stack has an automorphism group. In case of DM stacks, it is a finite group. Very rouhghly, one may imagine a DM stack as something like a scheme each point of which is equipped with a finite group.

Besides the moduli problem, the study of varieties with quotient singularities motivates the study of DM stacks. For such a variety X, there exists a *smooth* DM stack $\mathcal{X}$ and a morphism $\mathcal{X} \to X$ which is a bijection on geometric points. Thus $\mathcal{X}$ is a smooth analogue of X and often has nicer behavior than X. Therefore one can first study $\mathcal{X}$ and then extract information of X from it. The McKay correspondence, one of the main themes of this book, may be viewed as relation between the geometry of the variety X and the geometry of the DM stack $\mathcal{X}$ mixed with algebraic aspects of finite group actions.

10.2. Categories fibered in groupoids

From the Yoneda lemma below, the category of k-schemes is embeded into the category of functors $(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set}$. According to this fact, we often identify a scheme X with the associated functor

$$h_X \colon (\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set},\ S \mapsto \mathrm{Hom}_{\mathbf{Sch}_k}(S, X) = X(S).$$

THEOREM 10.2.1 (The Yoneda lemma). *Let* $\mathbf{C}$ *be a category and let* $\mathbf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ *be the category of functors* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$*. Then the functor*

$$\mathbf{C} \to \mathbf{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}),\ c \mapsto h_c$$

is a fully faithful embedding. Here h_c *denotes the functor* $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$*,* $d \mapsto \mathrm{Hom}_{\mathbf{C}}(d, c)$*.*

Roughly speaking, the theory of stacks replaces the category of sets which appeared above with the larger category of groupoids.

DEFINITION 10.2.2. A *groupoid* is a category whose morphisms are all isomorphisms.

EXAMPLE 10.2.3 (A set as a groupoid). Every set S is regarded as a groupoid; objects of this groupoid are elements of S, the identity morphism is attached to each object and they are the only morphisms of this groupoid.

EXAMPLE 10.2.4 (A group as a groupoid). Every group is regarded as a groupoid; it has a unique object and the automorphism group of this object is G.

Stacks can be regarded as functors $(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Groupoid}$; the target is the "category" of groupoids. But, groupoids form a 2-category rather than a genuine category and functors $(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Groupoid}$ should be, in fact, pseudo-functors. The standard definition of stacks uses categories fibered in groupoids, a notion essentially equivalent to pseudo-functors (see **[Vis05]**).

We now fix a category $\mathbf{S}$.

DEFINITION 10.2.5. A *category fibered in groupoids over* $\mathbf{S}$ is a category $\mathcal{X}$ endowed with a functor $\pi \colon \mathcal{X} \to \mathbf{S}$ which satisfies the following conditions:

(1) For every morphism $f \colon T \to S$ in $\mathbf{S}$ and an object $x \in \mathcal{X}$ with $\pi(x) = S$, there exists a morphism $\phi \colon y \to x$ in $\mathcal{X}$ with $\pi(\phi) = f$.
(2) Let $\phi \colon y \to x$ and $\chi \colon z \to x$ be morphisms in $\mathcal{X}$ and let $\overline{\psi} \colon \pi(z) \to \pi(y)$ be a morphism in $\mathbf{S}$ such that $\pi(\phi) \circ \overline{\psi} = \pi(\chi)$. Then there exists a unique morphism $\psi \colon z \to y$ in $\mathcal{X}$ such that $\phi \circ \psi = \chi$ and $\pi(\psi) = \overline{\psi}$ (see the following diagram).

$$\begin{tikzcd}
z \arrow[r, dotted, "\exists!\,\psi"] \arrow[d, mapsto] \arrow[rr, bend left, "\chi"] & y \arrow[r, "\phi"] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
\pi(z) \arrow[r, "\overline{\psi}"] \arrow[rr, bend right, "\pi(\chi)"'] & \pi(y) \arrow[r, "\pi(\phi)"] & \pi(x)
\end{tikzcd}$$

Condition (2) shows that if $\phi'\colon y' \to x$ is another morphism as in Condition (1), then there exists a unique isomorphism $y' \to y$ mapping to the identity morphism of T. We call y as in Condition (1) the *pullback of* X *by* f and denote it by f^*x or x_T.

DEFINITION 10.2.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be categories fibered in groupoids over $\mathbf{S}$ and let $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ be their functors to $\mathbf{S}$ respectively. A *morphism from* $\mathcal{Y}$ *to* $\mathcal{X}$ is a functor $F\colon \mathcal{Y} \to \mathcal{X}$ with $\pi_{\mathcal{X}} \circ F = \pi_{\mathcal{Y}}$ (this is the strict equality rather than an isomorphism of functors). We say that a morphism $\mathcal{Y} \to \mathcal{X}$ is an *isomorphism* if it is an equivalence.

Note that categories fibered in groupoids over $\mathbf{S}$ and their morphisms form a 2-category. It means that there are morphisms between morphisms. Morphisms betwen objects are called 1-morphisms and morphisms between 1-morphisms are called 2-morphisms. Morphisms defined above of categories fibered in groupoids are 1-morphisms, while 2-morphisms between them are natural transforms of functors.

DEFINITION 10.2.7. Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbf{S}$ and let $S \in \mathbf{S}$. The *fiber of* $\mathcal{X}$ *over* S*,* denoted by $\mathcal{X}(S)$, is the subcategory of $\mathcal{X}$ consisting of objects and morphisms that map to S and id_S respectively.

LEMMA 10.2.8. *Every fiber* $\mathcal{X}(S)$ *is a groupoid.*

PROOF. Let $\phi\colon y \to x$ be a morphism in $\mathcal{X}(S)$ and let $\chi := \mathrm{id}_X$ and $\overline{\psi} := \mathrm{id}_S$. Condition (2) of Definition 10.2.5 shows that there exists a morphism ψ such that $\phi \circ \psi = \mathrm{id}_X$ and $\pi(\psi) = \mathrm{id}_S$. Namely ϕ has a right inverse ψ in $\mathcal{X}(S)$. Similarly, ψ has a right inverse ϕ'. Since ψ has both a left inverse ϕ and a right inverse ϕ', we have that $\phi = \phi'$ and that ψ and ϕ are inverses to each other. □

REMARK 10.2.9. For a category $\mathcal{X}$ fibered in groupoids over $\mathbf{S}$, the assignment $S \mapsto \mathcal{X}(S)$ gives a pseudo-functor from $\mathbf{S}$ to the 2-category of groupoids (see [**Vis05**, Section 3.1.2]).

REMARK 10.2.10 (Fiberwise description of categories fibered in groupoids). In a concrete example of categories fibered in groupoids, we often have caonical pullback functors $\mathcal{X}(S) \to \mathcal{X}(T)$, $x \mapsto x_T$. Then morphisms $y \to x$ over $T \to S$ bijectively correspond to morphisms $y \to x_T$ in the fiber $\mathcal{X}(T)$ to the canonical pullback x_T. In that case, we describe only fibers $\mathcal{X}(S)$ to define the category fibered in groupoids $\mathcal{X}$.

EXAMPLE 10.2.11 (Functors as categories fibered in groupoids). We can associate a category $\mathcal{X}_h$ fibered in groupoids to a functor $h\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. The fiber $\mathcal{X}_h(S)$ over $S \in \mathbf{S}^{\mathrm{op}}$ is the set $h(S)$ regarded as a groupoid. For two functors $h, h'\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$, there is a one-to-one correspondence between natural transforms $h' \to h$ and morphisms $\mathcal{X}_{h'} \to \mathcal{X}_h$. In other words, the functor $h \mapsto \mathcal{X}_h$ is fully faithful. (Strictly speaking, morphisms $\mathcal{X}_{h'} \to \mathcal{X}_h$ form a category, but it is a groupoid associated to a set and has only identity morphisms as morphisms. Thus we can safely regard it as a set.) Thanks to this fact, we often identify h with $\mathcal{X}$.

EXAMPLE 10.2.12 (Schemes as categories fibered in groupoids). In the last example, let $\mathbf{S} = \mathbf{Sch}_k$ and let h be the functor h_X associated to a k-scheme X. Namely $h(S) = \mathrm{Hom}(S, X)$. Then the associated category $\mathcal{X}_h$ fibered in groupoids is nothing but the category $\mathbf{Sch}_X$ of X-schemes endowed with the natural functor $\mathbf{Sch}_X \to \mathbf{Sch}_k$. In summary, a k-scheme X is identified with the category $\mathbf{Sch}_X$ fibered in groupoids over $\mathbf{Sch}_k$.

EXAMPLE 10.2.13 (Moduli stack of curves). Let g be a non-negative integer. The moduli stack $\mathcal{M}_g$ of cuvers of genus g is defined to be a category fibered in groupoids over $\mathbf{Sch}_k$ as follows. An object of $\mathcal{M}_g(S)$ is a smooth projective curve C over S of genus g and a morphism in $\mathcal{M}_g(S)$ is an isomorphism over S.

10.3. Grothendieck topologies and sites

The functor associated to a scheme is not a mere functor but a sheaf for several Grothendieck topogologies. Similarly, a stack is not a mere category fibered in groupoids but satisfy some "sheaf conditions." We first recall Grothendieck topologies.

DEFINITION 10.3.1. A *Grothendieck topology* on a category $\mathbf{S}$ is a datum of associating to each object $S \in \mathbf{S}$ a collection of families $(U_i \to S)_{i \in I}$ of morphisms, called *coverings* of X, satisfying the following conditions:

(1) For every isomorphism $T \to S$, $(T \to S)$ is a covering of X.
(2) If $(U_i \to S)_{i\in I}$ is a covering, then for any morphism $T \to S$, the fiber products $U_i \times_S T$ exist and the induced family $(U_i \times_S T \to T)_{i\in I}$ is a covering.
(3) If $(U_i \to S)_{i\in I}$ is a covering and if $(V_{ij} \to U_i)_{j\in J_i}$, $i \in I$ are coverings, then the induced family $(V_{ij} \to S)_{i\in I, j\in I_j}$ is a covering.

A category endowed with a Grothendieck topology is called a *site*.

EXAMPLE 10.3.2. Let X be a topological space and let $\mathcal{X}$ be the category in which objects are open subsets of X and morphisms are inclusion maps. Associating to each open subset $U \subset X$ the set of inclusion maps $(U_i \hookrightarrow U)_{i\in I}$ with $U = \bigcup_{i\in I} U_i$, we obtain a Grothendieck topology on $\mathcal{X}$.

EXAMPLE 10.3.3. For a scheme S, the *Zariski topology* on $\mathbf{Sch}_S$ associates to an S-scheme X the set of open immersions $(U_i \to X)_{i\in I}$ whose images over X. This is a Grothendieck topology on $\mathbf{Sch}_S$.

EXAMPLE 10.3.4. For a scheme S, the *étale topology* on $\mathbf{Sch}_S$ associates to a k-scheme X the set of étale morphisms $(U_i \to X)_{i\in I}$ whose images cover X. This is also a Grothendieck topology of $\mathbf{Sch}_S$ and our canonical choice.

DEFINITION 10.3.5. Let $\mathbf{S}$ be a site. A functor $F\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ is said to be a *sheaf* if for every covering $(U_i \to S)_{i\in I}$, the diagram

$$F(S) \xrightarrow{q} \prod_{i\in I} F(U_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{(i,j)\in I^2} F(U_i \times_S U_j)$$

is an equalizer diagram; namely q is injective and

$$\mathrm{Im}(q) = \left\{ x \in \prod_{i\in I} F(U_i) \mid p_1^*(x) = p_2^*(x) \right\}.$$

When $\mathbf{S}$ is the site associated to a topological space (Example 10.3.2), the above definition coincides with the one of sheaves on topological spaces. The injectivity of q says that a section of F on S is determined by its "restrictions" to U_i's and the other condition says that sections on $U_i's$ can be glued to one on S if and only if the sections on U_i and U_j "restricts" to the same section on $U_i \times_S U_j$ (a counterpart of the intersection $U_i \cap U_j$ of two open subsets in the Grothendieck topology).

10.4. Stacks

In what follows, for any scheme S, we regard $\mathbf{Sch}_S$ as a site by giving it the étale topology (Example 10.3.4). Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbf{Sch}_k$.

Definition 10.4.1. For $S \in \mathbf{S}$ and $x, y \in \mathcal{X}(S)$, we define a functor

$$\underline{\mathrm{Iso}}_S(x, y) \colon (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set},\, T \mapsto \mathrm{Iso}_{\mathcal{X}(T)}(x_T, y_T),$$

where x_T and y_T are respectively the pullbacks of x and y to T and $\mathrm{Iso}_{\mathcal{X}(T)}(x_T, y_T)$ is the set of isomorphisms $x_T \to y_T$ in the groupoid $\mathcal{X}(T)$.

Definition 10.4.2. Let $(U_i \to S)_{i \in I}$ be a covering in $\mathbf{Sch}_k$. A *descent datum in $\mathcal{X}$ with respect to this covering* consists of objects $x_i \in \mathcal{X}(U_i)$, $i \in I$ and isomorphisms $\phi_{ij} \colon p_1^* x_i \to p_2^* x_j$ in $\mathcal{X}(U_i \times_S U_j)$ for $(i, j) \in I^2$ satisfying the cocycle condition

$$\widetilde{\phi_{ik}} = \widetilde{\phi_{jk}} \circ \widetilde{\phi_{ij}} \text{ in } \mathcal{X}(U_i \times_S U_j \times_S U_k)$$

for $(i, j, k) \in I^3$, where $\widetilde{\phi_{ik}}$ is the pullback of ϕ_{ik} by the projection $U_i \times_S U_j \times_S U_k \to U_i \times_S U_k$ and similarly for $\widetilde{\phi_{jk}}$ and $\widetilde{\phi_{ij}}$.

Example 10.4.3 (A descent datum induced by an object). Each object $x \in \mathcal{X}(S)$ induces a descent datum in $\mathcal{X}$ with respect to any covering $(U_i \to S)_{i \in I}$ given by pullbacks x_{U_i} and canonical isomorphisms $\psi_{ij} \colon p_1^* x_{U_i} \to p_2^* x_{U_j}$.

Definition 10.4.4. We say that a descent datum (x_i, ϕ_{ij}) in $\mathcal{X}$ with respect to $(U_i \to S)_{i \in I}$ is *effective* if there exists an object $x \in \mathcal{X}(S)$ such that (x_i, ϕ_{ij}) is isomorphic to the descent datum (x_{U_i}, ψ_{ij}) induced from x as in Example 10.4.3, that is, there exist isomorphisms $x_i \to x_{U_i}$ compatible with ϕ_{ij} and ψ_{ij}.

In the analogy to the classical topology, a descent datum corresponds to a gluing datum of objects over U_i relative to some open covering $S = \bigcup_i U_i$ and its effectivity means that we can indeed glue these objects to get an object on S.

Definition 10.4.5. We keep the notation. We say that $\mathcal{X}$ is a *stack over k* or *k-stack* if the following conditions hold:

(1) For $S \in \mathbf{Sch}_k$ and $x, y \in \mathcal{X}(S)$, the functor $\underline{\mathrm{Iso}}_S(x, y) \colon (\mathbf{Sch}_S)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf on the site $\mathbf{Sch}_S$.
(2) For every covering $(U_i \to S)_{i \in I}$ in $\mathbf{Sch}_k$, every descent datum in $\mathcal{X}$ with respect to $(U_i \to S)_{i \in I}$ is effective.

A *morphism* between k-stacks is defined to be a morphism as categories fibered in groupoids over $\mathbf{Sch}_k$. We often abbreviate a k-stack to a stack, when there is no confusion.

Remark 10.4.6. The two conditions in the above definition are equivalent to the following single condition: for each covering $(U_i \to S)_{i \in I}$, the functor $\mathcal{X}(S) \to \mathcal{X}((U_i \to S)_{i \in I})$, where $\mathcal{X}((U_i \to S)_{i \in I})$ denotes the category of descent data with respect to this covering.

EXAMPLE 10.4.7. Let X be a k-scheme. The category $\mathbf{Sch}_X$ fibered in groupoids over $\mathbf{Sch}_k$ (Example 10.2.12) is a stack.

REMARK 10.4.8 (Working over $\mathbf{Aff}_k$). We defined stacks as categories fibered in groupoids over $\mathbf{Sch}_k$ satisfying some condition. However it is sometimes more convenient to use the category $\mathbf{Aff}_k$ of affine schemes instead, of which we will take advantage in later chapters. Since every scheme S admits an étale covering $(S_i \to S)_i$ with S_i affine, the fiber $\mathcal{X}(S)$ of a stack $\mathcal{X}$ over a scheme S is recovered from the fibers $\mathcal{X}(S_i)$ and functors among them. Thus we do not lose anything by restricting the base category to $\mathbf{Aff}_k$.

10.5. Fiber products and schematic morphisms

As schemes are Zariski locally affine schemes, DM stacks are étale locally schemes. Namely every DM stack $\mathcal{X}$ admits an étale surjective morphism $U \to \mathcal{X}$ from a scheme U. But what does it mean that a morphism $U \to \mathcal{X}$ from a scheme to a stack is étale and surjective? To make this precise, we need the notions of fiber products and schematic morphisms.

First recall that the fiber product $Y \times_X Z$ of schemes is defined as a functor by

$$(Y \times_X Z)(S) = Y(S) \times_{X(S)} Z(S) := \{(y,z) \in Y(S) \times Z(S) \mid f(y) = g(z)\}.$$

DEFINITION 10.5.1. Let $f\colon \mathcal{Y} \to \mathcal{X}$ and $g\colon \mathcal{Z} \to \mathcal{X}$ be functors of groupoids. The *fiber product* $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} = \mathcal{Y} \times_{f,\mathcal{X},g} \mathcal{Z}$ is a category defined as follows. An object is a triple (y, z, α) where y and z are objects of $\mathcal{Y}$ and $\mathcal{Z}$ respectively and α is an isomorphism $f(y) \to g(z)$ in $\mathcal{X}$. A morphism $(y', z', \alpha') \to (y, z, \alpha)$ is the pair (ϕ, ψ) of morphisms $\phi\colon y' \to y$ and $\psi\colon z' \to z$ such that the following diagram is commutative.

$$\begin{tikzcd}
f(y') \arrow[r, "f(\phi)"] \arrow[d, "\alpha"'] & f(y) \arrow[d, "\alpha'"] \\
g(z') \arrow[r, "g(\psi)"'] & g(z)
\end{tikzcd}$$

DEFINITION 10.5.2. Let $f\colon \mathcal{Y} \to \mathcal{X}$ and $g\colon \mathcal{Z} \to \mathcal{X}$ be morphisms of categories fibered in groupoids over $\mathbf{Sch}_k$. We define the *fiber product* (also called the *2-fiber product*) $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} = \mathcal{Y} \times_{f,\mathcal{X},g} \mathcal{Z}$ to be a category fibered in groupoids over $\mathbf{Sch}_k$ given by $(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z})(S) := \mathcal{Y}(S) \times_{\mathcal{X}(S)} \mathcal{Z}(S)$.

A fiber product of stacks is again a stack. When $\mathcal{X} = \mathbf{Sch}_k = \operatorname{Spec} k$, we denote $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ simply by $\mathcal{Y} \times_k \mathcal{Z}$ or $\mathcal{Y} \times \mathcal{Z}$. In this case, the third entry α of a triple (y, z, α) as above is always the identity map. Thus $(\mathcal{Y} \times_k \mathcal{Z})(S)$ is identified with the product category $\mathcal{Y}(S) \times \mathcal{Z}(S)$.

We have the forgetting functors (or projections)

$$p_{\mathcal{Y}}\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y} \text{ and } p_{\mathcal{Z}}\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}.$$

Moreover there exists a canonical isomorphism $\theta\colon f \circ p_{\mathcal{Y}} \xrightarrow{\sim} g \circ p_{\mathcal{Z}}$ of functors given by

$$\theta(Y, Z, \alpha) = \alpha\colon f(Y) \xrightarrow{\sim} g(Z).$$

Thus we get the 2-commutative diagram:

$$\begin{tikzcd} \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \arrow[r, "p_{\mathcal{Z}}"] \arrow[d, "p_{\mathcal{Y}}"'] & \mathcal{Z} \arrow[d, "g"] \\ \mathcal{Y} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\theta"] & \mathcal{X} \end{tikzcd}$$

The fiber product has the desired universal propery. Let $\mathcal{T}$ be another k-stack and let $a\colon \mathcal{T} \to \mathcal{Y}$ and $b\colon \mathcal{T} \to \mathcal{Z}$ be morphisms and let $\gamma\colon f \circ a \xrightarrow{\sim} g \circ b$ be an isomorphism of functors. Then there exists the canonical morphism $c\colon \mathcal{T} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ sending $T \in \mathcal{T}$ to $(a(T), b(T), \gamma(T))$.

Definition 10.5.3. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. We say that f is *schematic* if for every morphism $T \to \mathcal{X}$ from a scheme T, the fiber product $T \times_{\mathcal{X}} \mathcal{Y}$ is (isomorphic to) a scheme. Let $\mathbf{P}$ be a property of morphisms of schemes which is stable under base change and of local nature on the target for the étale topology (e.g. unramified, étale, smooth, finite, affine, surjective, being a closed immersion, proper). We say that a schematic morphism $f\colon \mathcal{Y} \to \mathcal{X}$ *has the property* $\mathbf{P}$ if for every morphism $T \to \mathcal{X}$ from a k-scheme, the base change $f_T\colon T \times_{\mathcal{X}} \mathcal{Y} \to T$ has the property $\mathbf{P}$.

10.6. DM stacks and algebraic spaces

For a stack $\mathcal{X}$, the *diagonal morphism*

$$\Delta = \Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$$

is given by sending an object X to the pair (X, X).

Lemma 10.6.1. *Let $\mathcal{X}$ be a stack such that $\Delta_{\mathcal{X}}$ is schematic. Then any morphism $V \to \mathcal{X}$ from a scheme V is schematic.*

Proof. We need to show that for any morphism $S \to \mathcal{X}$ from a scheme, the fiber product $S \times_{\mathcal{X}} V$ is a scheme. This follows from

$$S \times_{\mathcal{X}} V \cong (S \times V) \times_{\mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

and the assumption that Δ is schematic. □

Definition 10.6.2. A k-stack $\mathcal{X}$ is called a *DM stack (over k)* if the following conditions hold:

(1) The diagonal morphism $\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is schematic and finite.
(2) There exists a morphism $V \to \mathcal{X}$ from a scheme which is étale and surjective. (Such a scheme V or a morphism $V \to \mathcal{X}$ is called an *atlas* of $\mathcal{X}$.)

Note that from the first condition and Lemma 10.6.1, the second condition makes sense.

Remark 10.6.3. Our definition of DM stacks is equivalent to the one of *separated* DM stacks for the usual terminology. According to the usual definition, the diagonal morphism $\Delta_{\mathcal{X}}$ of a DM stack is only supposed to be representable, a weaker condition than schematic. Then a DM stack is separated if the diagonal morphism is also proper. As mentioned above, One can show that the diagonal morphism of a DM stack is formally unramified [**Ols16**, Th. 8.3.3]. Being representable, formally unramified and proper implies being schematic and finite.

DEFINITION 10.6.4. An *algebraic space* is defined to be a DM stack $\mathcal{X}$ such that every fiber $\mathcal{X}(S)$ is equivalent to a set, or equivalently such that $\mathcal{X}$ is isomorphic to the category fibered in groupoid associated to a functor $(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set}$.

It is clear from definition that every scheme is an algebraic space and every algebraic space is a DM stack.

REMARK 10.6.5. For an algebraic space $\mathcal{X}$, there exists an algebraic space $\mathcal{X}'$ isomorphic to $\mathcal{X}$ such that every fiber $\mathcal{X}'(S)$ is a genuine set rather than a groupoid equivalent to a set. Indeed, we can define $\mathcal{X}'$ by puttin $\mathcal{X}'(S)$ to be the set of isomorphism classes in $\mathcal{X}(S)$. Therefore, when talking about an algebraic space $\mathcal{X}$, we basically suppose that fibers $\mathcal{X}(S)$ are sets.

The following notion is a slight generalization of representable morphisms.

DEFINITION 10.6.6. A morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of categories fibered in groupoids is called *representable (by algebraic spaces)* if for every morphism $T \to \mathcal{X}$ from an algebraic space, the fiber product $T \times_{\mathcal{X}} \mathcal{Y}$ is an algebraic space.

Definitions of properties of schematic morphisms (Definition 10.5.3) are generalized to representable morphisms.

REMARK 10.6.7. Usually one defines algebraic spaces and representable morphisms before defining DM stacks and assumes the diagonal morphism of a DM stak only to be representable by algebraic spaces (for instance, see [**Ols16**]). Since we restricted ourselves to *separated* DM stacks (see Remark 10.6.3), we were able to avoid the use of representable morphisms in the definition of DM stacks.

We now define various properties of DM stacks and one of their morphisms. We first define local properties of DM stacks.

DEFINITION 10.6.8. Let $\mathbf{P}$ be a property of schemes which is local in the étale topology (e.g. locally of finite type, locally smooth, normal, regular, local complete intersection, klt). We say that a DM stack $\mathcal{X}$ *has property* $\mathbf{P}$ if some (equivalently every) atlas V of $\mathcal{X}$ has property $\mathbf{P}$.

Since algebraic spaces are by definition DM stacks, the above definition is valid also for algebraic spaces.

DEFINITION 10.6.9. Let $\mathbf{P}$ be a property of morphisms of schemes which is stable under base change and of local nature on the target for the étale topology (e.g. unramified, étale, smooth, finite, affine, surjective, being a closed immersion, proper). We say that a representable morphism $f\colon \mathcal{Y} \to \mathcal{X}$ *has the property* $\mathbf{P}$ if for every morphism $T \to \mathcal{X}$ from a k-scheme, the base change $f_T\colon T \times_{\mathcal{X}} \mathcal{Y} \to T$ has the property $\mathbf{P}$.

DEFINITION 10.6.10. We say that a DM stack $\mathcal{X}$ is *quasi-compact* if there exists an atlas $V \to \mathcal{X}$ such that V is a quasi-compact scheme. We say that $\mathcal{X}$ is *of finite type* (resp. of finite presentation, Noetherian) if it is locally of finite type (resp. locally of finite presentation, locally Noetherian) and quasi-compact.

Finally we define immersions and substacks.

DEFINITION 10.6.11. A morphism $\iota\colon \mathcal{Y} \to \mathcal{X}$ of DM stacks is called an *open (resp. closed, locally closed) immersion* if it is schematic (or equivalently representable) and an open (resp. closed, locally closed) immersion in the sense of

Definition 10.5.3. An *open (resp. closed, locally closed) substack* of a DM stack $\mathcal{X}$ is an equivalence class of open (resp. closed, locally closed) immersions to $\mathcal{X}$. Here two immersions $\iota' \colon \mathcal{Y}' \to \mathcal{X}$ and $\iota \colon \mathcal{Y} \to \mathcal{X}$ are equivalent if there exists an isomorphism $f \colon \mathcal{Y}' \to \mathcal{Y}$ with $\iota \circ f \cong \iota'$. By abuse of terminology and notation, we say that given an immersion $\mathcal{Y} \to \mathcal{X}$, we say that $\mathcal{Y}$ is a (open, closed or locally closed) substack of $\mathcal{X}$ and write $\mathcal{Y} \subset \mathcal{X}$.

10.7. Points and their automorphism groups

Recall that for schemes S and X, an S-point of X just means a morphism $S \to X$. On the other hand, when we regard X as a functor $(\mathbf{Sch}_k)^{\mathrm{op}} \to \mathbf{Set}$, an S-point of X is an element of the set $X(S)$.

Definition 10.7.1. For a scheme S and a DM stack $\mathcal{X}$, an *S-point* of $\mathcal{X}$ means a morphism $S \to \mathcal{X}$. When $S = \operatorname{Spec} R$ for a ring R, S-points are also called *R-points*. A *geometric point* of $\mathcal{X}$ is a K-point for some algebraically closed field K.

As in the case of schemes, giving an S-point is equivalent to giving an object of the groupoid $\mathcal{X}(S)$. Indeed, to a morphism

$$\phi \colon S = \mathbf{Sch}_S \to \mathcal{X},$$

we associate the object $\phi(S \xrightarrow{\mathrm{id}} S)$. Conversely, to an object $x \in \mathcal{X}(S)$, we define a morphism:

$$\begin{aligned} \mathbf{Sch}_S &\to \mathcal{X} \\ (f \colon T \to S) &\mapsto f^* x \end{aligned}$$

Tautologically, S-points form the groupoid $\mathcal{X}(S)$ rather than a set and each S-point x comes with the *automorphism group*

$$\operatorname{Aut}(x) := \operatorname{Aut}_{\mathcal{X}(S)}(x) = \underline{\operatorname{Aut}}_S(x)(S),$$

which is also called the *stabilizer.*

Proposition 10.7.2. *Let $\mathcal{X}$ be a DM stack $\mathcal{X}$, let $S \in \mathbf{Sch}_k$ and let $x, y \in \mathcal{X}(S)$. Then the sheaf $\underline{\operatorname{Iso}}_S(x,y)$ on the site $\mathbf{Sch}_S$ is a finite and unramified S-scheme. In particular, the automorphism group scheme $\underline{\operatorname{Aut}}_S(x)$ is a finite and unramified group scheme over S.*

Proof. From the definition of fiber products,

$$\underline{\operatorname{Iso}}_S(x,y) \cong S \times_{(x,y), \mathcal{X} \times_k \mathcal{X}, \Delta} \mathcal{X}.$$

Therefore the first condition in Definition 10.6.2 is equivalent to saying that the sheaf $\underline{\operatorname{Iso}}_S(x,y)$ on the site $\mathbf{Sch}_S$ is a finite S-scheme. From [**Ols16**, Th. 8.3.3], $\Delta_{\mathcal{X}}$ is unramified, and so is $\underline{\operatorname{Iso}}_S(x,y)$. Thus the first assertion of the proposition holds. The second assertion follows from the first. □

Definition 10.7.3. The *point set* of a DM stack $\mathcal{X}$, denoted by $|\mathcal{X}|$, is the set of equivalence classes of geometric points of $\mathcal{X}$. Here two geometric points $\operatorname{Spec} K_1 \to \mathcal{X}$ and $\operatorname{Spec} K_2 \to \mathcal{X}$ are *equivalent* if and only if there exist a third geometric point $\operatorname{Spec} K_3 \to \mathcal{X}$ and morphisms $\operatorname{Spec} K_3 \to \operatorname{Spec} K_i$ $(i = 1, 2)$ fitting into the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} K_3 \arrow[r] \arrow[d] \arrow[dr] & \operatorname{Spec} K_2 \arrow[d] \\
\operatorname{Spec} K_1 \arrow[r] & \mathcal{X}
\end{tikzcd}$$

The point set of a DM stack is equipped with a topology as follows:

Definition 10.7.4. For a DM stack, we can define a topology on $|\mathcal{X}|$, called the *Zariski topology,* as follows: an open subset of $|\mathcal{X}|$ is the point set $|\mathcal{U}|$ of an open substack $\mathcal{U} \subset \mathcal{X}$. We call

If two geometric points x_1 and x_2 of $\mathcal{X}$ defines the same point of $|\mathcal{X}|$, then we have $\operatorname{Aut}(x_1) \cong \operatorname{Aut}(x_2)$. Threfore we can talk about the automorphism group $\operatorname{Aut}(x)$ of a point $x \in |\mathcal{X}|$. We can characterize algebraic spaces among DM stacks by looking only at geometric points:

Proposition 10.7.5 ([**Con07**, Th. 2.2.5]). *Let $\mathcal{X}$ be a DM stack. The $\mathcal{X}$ is an algebraic space if and only if every gemetric point x of $\mathcal{X}$ has the trivial automorphism group.*

The following is the relative version of the above proposition:

Proposition 10.7.6. *A morphism of DM stacks $f\colon \mathcal{Y} \to \mathcal{X}$ is representable if and only if for every geometric point y of $\mathcal{Y}$, the induced map $\operatorname{Aut}(y) \to \operatorname{Aut}(f(y))$ is injective.*

Proof. The if part: Consider a fiber product $S \times_{\mathcal{X}} \mathcal{Y}$ with S an algebraic space. For an algebraically closed field K, the automorphism group of $(s, y, \alpha) \in (S \times_{\mathcal{X}} \mathcal{Y})(K)$ consists of pairs $(\phi, \psi) \in \operatorname{Aut}(s) \times \operatorname{Aut}(y)$ comaptible with α. But, since $\operatorname{Aut}(s) = 1$, if $\operatorname{Aut}(y) \to \operatorname{Aut}(f(y))$ is injective, then ϕ should be the identity. Thus (s, y, α) has the trivial automorphism group. From Proposition 10.7.6, $S \times_{\mathcal{X}} \mathcal{Y}$ is an algebraic space and hence f is representable.

The only if part: If $\operatorname{Aut}(y) \to \operatorname{Aut}(f(y))$ is not injective for some $y \in \mathcal{Y}(K)$, then the induced K-point of $\operatorname{Spec} K \times_{\mathcal{X}} \mathcal{Y}$ has the non-trivial group $\operatorname{Ker}(\operatorname{Aut}(y) \to \operatorname{Aut}(f(y)))$ as its automorphism group. Again from Proposition 10.7.6, $\operatorname{Spec} K \times_{\mathcal{X}} \mathcal{Y}$ is not an algebraic space. Thus f is not representable. □

10.8. Inertia stacks

Since points of a DM stack have automorphisms, it is natural to consider a stack parametrizing pairs of points and there automorphisms.

Definition 10.8.1. The *inertia stack* $\mathrm{I}\,\mathcal{X}$ of a DM stack $\mathcal{X}$ is a category fibered in groupoids over $\mathbf{Sch}_k$ defined as follows. An object of $(\mathrm{I}\,\mathcal{X})(S)$ is the pair (x, α) of $x \in \mathcal{X}(S)$ and $\alpha \in \operatorname{Aut}_{\mathcal{X}(S)}(x)$. A morphism $(y, \beta) \to (x, \alpha)$ in $(\mathrm{I}\,\mathcal{X})(S)$ is a morphism $\phi\colon y \to x$ in $\mathcal{X}(S)$ with $\alpha \circ \phi = \phi \circ \beta$.

We have the forgetting morphism $\mathrm{I}\,\mathcal{X} \to \mathcal{X}$, $(x, \alpha) \mapsto x$ and the section $\mathcal{X} \to \mathrm{I}\,\mathcal{X}$, $x \mapsto (x, \mathrm{id}_x)$. Note that the fiber of the map $|\mathrm{I}\,\mathcal{X}| \to |\mathcal{X}|$ of point sets over $x \in |\mathcal{X}|$ is not the automorphism group $\operatorname{Aut}(x)$ but the set $\operatorname{Conj}(\operatorname{Aut}(x))$ of conjugacy classes in $\operatorname{Aut}(x)$. For pairs (x, α) and (x, β) are isomorphic if and only if α and β are conjugate in $\operatorname{Aut}(x)$.

LEMMA 10.8.2. *There exists also an isomorphism*

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times_k \mathcal{X}, \Delta} \mathcal{X} \cong \mathrm{I}\,\mathcal{X}$$

which is compatible with the first projection $\mathcal{X} \times_{\Delta, \mathcal{X} \times_k \mathcal{X}, \Delta} \mathcal{X}$ *and the morphism* $\mathrm{I}\,\mathcal{X} \to \mathcal{X}$.

PROOF. An object of the left side is the triple $(x, x', (\alpha_1, \alpha_2))$ where α_1 and α_2 are isomorphisms $x \to x'$. The isomorphism of the lemma is given by sending this triple to the pair $(x, \alpha_2^{-1} \circ \alpha_1)$. □

From the lemma, we have:

COROLLARY 10.8.3. *The inertia stack* $\mathrm{I}\,\mathcal{X}$ *of a DM stack* $\mathcal{X}$ *is a DM stack.*

10.9. Coarse moduli spaces

For a DM stack $\mathcal{X}$, there always exists the algebraic space that approximates $\mathcal{X}$ the best in a certain sense. This algebraic space is the coarse moduli space of $\mathcal{X}$. Its precise definition is as follows:

DEFINITION 10.9.1. Let $\mathcal{X}$ be a DM stack. A *coarse moduli space* of $\mathcal{X}$ is an algebraic space X given with a morphism $f\colon \mathcal{X} \to X$ such that

(1) for every algebraically closed field K, the map $\mathcal{X}(K)/\cong \to X(K)$ of K-point sets is bijective, and
(2) for every morphism $g\colon \mathcal{X} \to Z$ to an algebraic space, there exists a unique morphism $h\colon X \to Z$ with $f \circ g = h$.

Clearly a coarse moduli space is unique up to unique isomorphism and always exists:

THEOREM 10.9.2 ([**KM97**, **Con**, **Ryd13**]). *Let* $\mathcal{X}$ *be a DM stack. Then there exists a corase moduli space* $\pi\colon \mathcal{X} \to X$. *Moreover, if* $Y \to X$ *is a flat morphism of algebraic spaces, then* Y *is a coarse moduli space of* $Y \times_X \mathcal{X}$.

10.10. Quotient stacks

Quotient stacks are the most basic examples of DM stacks as well as the most important for our purpose. Suppose that a finite group G acts on an algebraic space V. The quotient V/G always exists as an algebraic space. Its main properites are:

(1) The canonical morphism $V \to V/G$ is the universal G-invariant morphism from V in the category of algebraic spaces.
(2) For each algebraically closed field, the induced map $V(K)/G \to (V/G)(K)$ is bijective.

If V is a scheme and every G-orbit is contained in an affine open subset, then the quotient V/G is a scheme. The morphism $V \to V/G$ factors as

$$V \to [V/G] \to V/G$$

through the quotient stack $[V/G]$. The quotient stack is a hybrid of the G-space V and the quotient space V/G; its local structure is close to the G-space V and its global structure is close to V/G. We give the precise definition of quotient stacks below, slightly generalizing to the case where G is an étale finite group scheme. We first need to define G-torsors.

DEFINITION 10.10.1. Let G be an étale finite group scheme. A *G-torsor* (also called a *principal G-bundle*) is a morphism $T \to S$ of schemes given with a G-action on T which satisfies the following conditions:

(1) The morphism $T \to S$ is G-invariant.
(2) There exists an étale covering $(S_i \to S)_i$ such that for each i, we have a G-equivariant S_i-isomorphism $T_{S_i} := T \times_S S_i \xrightarrow{\sim} G \times S_i$.

We may rephrase the definition as follows: a G-torsor is an G-equivariant morphism $T \to S$ which looks like the trivial torsor $G \times S \to S$ étale locally on the base scheme S.

DEFINITION 10.10.2. Let V be an algebraic space endowed with an action of an étale finite group scheme G. The *quotient stack* $[V/G]$ is a category fibered in groupoids over $\mathbf{Sch}_k$ defined as follows. An object of the fiber groupoid $[V/G](S)$ is the pair $(T \to S, T \to V)$ of a G-torsor $T \to S$ and a G-equivariant morphism $T \to V$. A morphism $(T' \to S, T' \to V) \to (T \to S, T \to V)$ in $[V/G](S)$ is a G-equivariant S-isomorphism $T' \to T$ compatible with to V.

$$\begin{tikzcd} T' \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & T \arrow[r] \arrow[d] & V \\ S \arrow[r] & S & \end{tikzcd}$$

The quotient stack is a DM stack [**Ols16**, Section 8.4.1]. The canonical morphism $V \to [V/G]$ sends an S-point $S \to V$ of V to the induced pair

$$(G \times S \to S, G \times S \to V)$$

of the trivial torsor $G \times S \to S$ and the unique equivariant morphism $G \times S \to V$ extending the given morphism $S = \{1\} \times S \to V$. The morphism $V \to [V/G]$ is an atlas of $[V/G]$. The morphism $[V/G] \to V/G$ sends the pair $(T \to S, T \to V)$ to the induced S-point $S = T/G \to V/G$. By this morphism, V/G is a coarse moduli space of $[V/G]$.

For an algebraically closed field K, any G-torsor over $\operatorname{Spec} K$ is trivial. Therefore a geometric point $\operatorname{Spec} K \to [V/G]$ corresponds to a G-equivariant morphism $\operatorname{Spec} K \times G \to V$. This implies that isomorphism classes of K-points of $[V/G]$ correspond to G-orbits in $V(K)$.

LEMMA 10.10.3. *Let $x\colon \operatorname{Spec} K \to [V/G]$ be a geometric point and let $v\colon \operatorname{Spec} K \to V$ be any lift of x with the stabilizer subgroup $\operatorname{Stab}(v) \subset G$. Then $\operatorname{Aut}(x) \cong \operatorname{Stab}(v)$.*

PROOF. The automorphism group of the trivial G-torsor $G \times \operatorname{Spec} K \to \operatorname{Spec} K$ is identified with G. Indeed, if the neutral component $\{1\} \times \operatorname{Spec} K$ maps to $\{g\} \times \operatorname{Spec} K$ by an G-equivariant automorphism, then a component $\{h\} \times \operatorname{Spec} K$ maps to $\{hg\} \times \operatorname{Spec} K$. Thus this automorphism is given by the right multiplication with g. Let $w\colon G \times \operatorname{Spec} K \to V$ be the G-equivariant morphism extending $v\colon \operatorname{Spec} K = \{1\} \times \operatorname{Spec} K \to V$. The automorphism of the trivial G-torsor $G \times \operatorname{Spec} K \to \operatorname{Spec} K$ corresponding to g is compatible with w if and only if $g \in \operatorname{Stab}(v)$. This proves the lemma. $\square$

The lemma shows that $[V/G]$ is an algebraic space if and only if the G-action on V is free. If this is the case, we have $[V/G] \cong V/G$.

10.11. Étale groupoid schemes

Since a stack is a category fibered in groupoid, it contains infinite data. To express a stack by "finite data," we can use a *groupoid scheme.* It is a tuple $(M, O, s, t, \epsilon, i, m)$ satisfying several compatibility conditions. Here M and O are schemes and the other entries are morphisms as follows:

source: $s\colon M \to O$
target: $t\colon M \to O$
identity: $\epsilon\colon O \to M$
inverse: $i\colon M \to M$
composition: $m\colon M \times_{s,O,t} M \to M$

An example of compatibilty conditions is the associativity saying that the two compositions

$$M \times_{s,O,t} M \times_{s,O,t} M \overset{m\times\mathrm{id}}{\underset{\mathrm{id}\times m}{\rightrightarrows}} M \times_{s,O,t} M \xrightarrow{m} M$$

are the same. For the other conditions, we refer the reader to [**LMB00**, 2.4.3] or [**Ols16**, Section 3.4.4]. We usually denote a groupoid scheme as $M \rightrightarrows O$, omitting morphisms ϵ, i, m. We call a groupoid scheme *étale* if the morphisms s and t are étale.

To an étale groupoid scheme $M \rightrightarrows O$, we can construct a DM stack denoted by $[M \rightrightarrows O]$ or $[O/M]$. First we define a category fibered in groupoids $[O/M]'$ such that the fiber $[O/M]'(S)$ over a scheme S has $O(S)$ as the set of objects and $M(S)$ has the set of morphisms whose sources and targets are specified by s and t. This becomes a so-called prestack. The DM stack $[O/M]$ is obtained by stackifying this (a similar operation to the sheafification of a presheaf).

For a scheme V with an action of an étale group scheme G, we can define an étale groupoid scheme $G \times V \rightrightarrows V$ in which the morphisms s and t are the projection and the G-action. The associated stack $[G \times V \rightrightarrows V]$ is isomorphic to the quotient stack $[V/G]$ defined in the last section [**LMB00**, (3.4.3)].

For a DM stack $\mathcal{X}$ with an atlas V, we have the associated étale groupoid scheme $V \times_{\mathcal{X}} V \rightrightarrows V$, where the two projections are the source and target. We have $\mathcal{X} \cong [V \times_{\mathcal{X}} V \rightrightarrows V]$ [**LMB00**, (3.8) and (4.3)].

10.12. Quasi-coherent sheaves

Let $\mathcal{X}$ be a DM stack.

Definition 10.12.1. The *small étale site* of $\mathcal{X}$, denoted by $\mathcal{X}_{\acute{e}t}$, has atlases $V \to \mathcal{X}$ as its objects. A morphism from $b\colon V \to \mathcal{X}$ to $a\colon U \to \mathcal{X}$ is the pair (f, α) of a morphism $f\colon V \to U$ and an isomorphism $\alpha\colon b \xrightarrow{\sim} a \circ f$. Namely this is a morphism $b \to a$ in $\mathcal{X}$ with a and b regarded as objects of $\mathcal{X}$. A collection of morphisms, $((V_i \to \mathcal{X}) \to (U \to \mathcal{X}))_i$, is a covering if the induced collection of scheme morphisms $(V_i \to U)_i$ is an étale covering.

We define define quasi-coherent sheaves on $\mathcal{X}$ as sheaves on the site $\mathcal{X}_{\mathrm{\acute{e}t}}$. We firs define the structure sheaf:

Definition 10.12.2. The *structure sheaf* $\mathcal{O}_{\mathcal{X}}$ is the sheaf on $\mathcal{X}_{\mathrm{\acute{e}t}}$ such that $\mathcal{O}_{\mathcal{X}}(V \to \mathcal{X}) = \mathcal{O}_V(V)$ and the map $\mathcal{O}_U(U) \to \mathcal{O}_V(V)$ associated to a pair (f, α) as above is the usual pullback map f^*.

If $V \to \mathcal{X}$ is an atlas, then restriction of $\mathcal{O}_{\mathcal{X}}$ to $V_{\text{ét}}$ coincides with the structure sheaf $\mathcal{O}_V$ of V.

DEFINITION 10.12.3. A *quasi-coherent sheaf* on $\mathcal{X}$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}_{\text{ét}}$ such that for each atlas $V \to \mathcal{X}$, the induced sheaf $\mathcal{F}_{V_{\text{ét}}}$ on $V_{\text{ét}}$ is a quasi-coherent $\mathcal{O}_V$-module.

EXAMPLE 10.12.4. For a morphism $\mathcal{Y} \to \mathcal{X}$ of DM stacks, we can define a quasi-coherent sheaf $\Omega_{\mathcal{Y}/\mathcal{X}}$ on $\mathcal{Y}$, called the *sheaf of differentials.* For an étale morphism $V \to \mathcal{Y}$ from a scheme, if $V \to \mathcal{Y} \to \mathcal{X}$ factors through an étale morphism $U \to \mathcal{X}$, then we have $(\Omega_{\mathcal{Y}/\mathcal{X}})_{V_{\text{ét}}} \cong \Omega_{V/U}$. The restriction $(\Omega_{\mathcal{Y}/\mathcal{X}})_{W_{\text{ét}}}$ for a general étale morphism $W \to \mathcal{X}$ is determined by descent along an étale morphism $V \to W$ such that the composition $V \to W \to \mathcal{X}$ satisfies the above condition.

To a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra $\mathcal{A}$, we can define the relative spectrum $\underline{\mathrm{Spec}}_{\mathcal{X}}\, \mathcal{A}$ as in the case of schemes. The canonical morphism $\underline{\mathrm{Spec}}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X}$ is representable and affine. This construction gives an equivalence between the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras and the category of DM stacks representable and affine over $\mathcal{X}$ [**Ols16**, Th. 10.2.4].

10.13. Local structure of DM stacks

The following result says that every DM stack is locally a quotient stack.

PROPOSITION 10.13.1 ([**AV02**, Lem. 2.2.3], [**Ols16**, Th. 11.3.1]). *Let $\mathcal{X}$ be a DM stack and let X be its coarse moduli space. There exists an étale covering $(X_i \to X)_i$ such that for every i, the fiber product $X_i \times_X \mathcal{X}$ is isomorphic to the quotient stack $[U_i/G_i]$ associated to an action of a finite group G_i on a scheme U_i.*

Of course, we may also say that every DM stack is locally a scheme because of the existence of an atlas. But an atlas $V \to \mathcal{X}$ does not preserve automorphism groups of geometric points, unless $\mathcal{X}$ is an algebraic space. The morphism $\coprod_i [U_i/G_i] \to \mathcal{X}$ obtained from the above proposition is not only representable, étale and surjective, but also *stabilizer-preserving*:

DEFINITION 10.13.2. A morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of DM stacks is *stabilizer-preserving* if for every geometric point y of $\mathcal{Y}$, the map $\mathrm{Aut}(y) \to \mathrm{Aut}(f(y))$ is bijective.

The morphism $\coprod_i [U_i/G_i] \to \mathcal{X}$ is stabilizer-preserving, because stabilizer-preserving morphisms are stable under the base change and the morphism $\coprod_i X_i \to X$ is obviously stabilizer-preserving. The above proposition follows from the following similar lemma:

10.14. Hom stacks

A Hom stack parametrizes morphisms between two stacks. This notion will play an important role later in developing motivic integration over DM stacks.

DEFINITION 10.14.1. Let $\mathcal{S}$ be a DM stack and let $\mathcal{Y}, \mathcal{X}$ be DM stacks over $\mathcal{S}$ (that is, given with a morphism to $\mathcal{S}$). We define a category fibered in groupoids over $\mathcal{S}$, denoted by $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})$ (resp. $\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})$) as follows. Let S be a k-scheme and let $\sigma \in \mathcal{S}(S)$ be an object corresponding to a morphism $\sigma\colon S \to \mathcal{S}$. The fiber $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})(\sigma)$ (resp. $\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})(\sigma)$) over σ is the groupoid

$$\mathrm{Hom}_S(\mathcal{Y} \times_{\mathcal{S},\sigma} S, \mathcal{X} \times_{\mathcal{S},\sigma} S)$$

consisting of S-morphisms (resp. representable S-morphisms) from $\mathcal{Y} \times_{\mathcal{S},\sigma} S$ to $\mathcal{X} \times_{\mathcal{S},\sigma} S$, that is, it has S-morphisms (functors) $\mathcal{Y} \times_{\mathcal{S},\sigma} S \to \mathcal{X} \times_{\mathcal{S},\sigma} S$ as objects and natural transforms between them (which are necessarily invertible) as morphisms.

In general, if $\mathcal{U} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{S}$ are both categories fibered in groupoids, then the composition $\mathcal{U} \to \mathcal{S}$ is also a category fibered in groupoids [**Sta20**, tag 09WW]. This shows that Hom stacks are categories fibered in groupoids over $\mathbf{Sch}_k$.

Proposition 10.14.2 ([**Ols07**, **Ols06**, **Yas19**]). *Let $\mathcal{S}$ be a DM stack of finite type and let $\mathcal{Y}, \mathcal{X}$ be DM stack of finite type over $\mathcal{S}$. Suppose that there exists a finite, étale and surjective morphism $\mathcal{U} \to \mathcal{Y}$ such that the composition $\mathcal{U} \to \mathcal{Y} \to \mathcal{S}$ is representable, flat and finite. Then $\underline{\mathrm{Hom}}_{\mathcal{S}}(\mathcal{Y}, \mathcal{X})$ and $\underline{\mathrm{Hom}}_{\mathcal{S}}^{\mathrm{rep}}(\mathcal{Y}, \mathcal{X})$ are DM stacks of finite type.*

CHAPTER 11

Untwisted arcs

In this chapter, we generalize further motivic integration over D_k-schemes into two directions. First we allow the target to be a DM stack. Second we replace the base field k with a DM stack, considering a family of DM stacks over D_k parameterized by a DM stack.

the We generalize the motivic integration to DM stacks in two steps. Firstly we develop the theory for untwisted arcs; the only target space is generalized to DM stacks, but the source of an arc remains to be the scheme $\mathrm{D}_R = \operatorname{Spec} R[\![t]\!]$. Secondly we develop the theory for twisted arcs, here the source also becomes a DM stack. The first step is rather straightforward, which we demonstrate in this chapter.

From this chapter on, we use the category of affine schemes $\mathbf{Aff}_k$ as the base category of stacks. Throughout this chapter, we denote by $\mathcal{X}$ and $\mathcal{Y}$ good D_k-stacks of relative dimension d, which are defined as follows.

Definition 11.0.1. A *good* D_k*-stack* means a reduced DM stack $\mathcal{X}$ over D_k satisfying the following conditions:

(1) $\mathcal{X} \to \mathrm{D}_k$ is flat, of finite type and of pure relative dimension,
(2) the generic fiber $\mathcal{X}_\eta$ is geometrically reduced.

We denote by f a generically étale D_k-morphism $\mathcal{Y} \to \mathcal{X}$.

11.1. Untwisted jets and arcs

Definition 11.1.1. We define an *untwisted* n*-jet of* $\mathcal{X}$ *over* R to be a D_k-morphism $\mathrm{D}_{R,n} \to \mathcal{X}$. We define the *stack of untwisted* n*-jets of* $\mathcal{X}$, denoted by $\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$, to be the fibered category over $\mathbf{Aff}_k$ such that $(\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k))(R) = \mathrm{Hom}_{\mathrm{D}_k}(\mathrm{D}_{R,n}, \mathcal{X})$.

Lemma 11.1.2. *The fibered category* $\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$ *is a DM stack of finite type over* k*.*

Proof. Let $\mathcal{X}_n := \mathcal{X} \times_{\mathrm{D}_k} \mathrm{D}_{k,n}$. Giving an untwisted n-jet of $\mathcal{X}$ over R is equivalent to giving an R-morphism $\mathrm{D}_{R,n} \to \mathcal{X}_n \otimes_k R$ such that the composition

$$\mathrm{D}_{R,n} \to \mathcal{X}_n \otimes_k R \to \mathrm{D}_{R,n}$$

is the identity morphism. This shows that we have an isomorphism

$$\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \cong \underline{\mathrm{Hom}}_k(\mathrm{D}_{k,n}, \mathcal{X}_n) \times_{\underline{\mathrm{Hom}}_k(\mathrm{D}_{k,n}, \mathrm{D}_{k,n})} \operatorname{Spec} k.$$

The lemma follows from Proposition 10.14.2. $\square$

For $n = 0$, we have $\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \cong \mathcal{X}_0$, the special fiber of $\mathcal{X} \to \mathrm{D}_k$.

Lemma 11.1.3. *If* $f\colon \mathcal{Y} \to \mathcal{X}$ *is a representable étale morphism, then*

$$\mathrm{J}_n(\mathcal{Y}/\mathrm{D}_k) \cong \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \times_{\mathcal{X}_0} \mathcal{Y}_0.$$

Proof. Giving an object of the right side over $\operatorname{Spec} R$ is equivalent to giving a 2-commutative diagram consisting of D_k-morphisms and a 2-isomorphism of the form:

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathrm{D}_{n,R} \arrow[r] \arrow[ur, Rightarrow] & \mathcal{X} \end{tikzcd}$$

Such a digram determines a morphism $\operatorname{Spec} R \to \mathrm{D}_{n,R} \times_{\mathcal{X}} \mathcal{Y}$. Since $\mathrm{D}_{n,R} \times_{\mathcal{X}} \mathcal{Y} \to \mathrm{D}_{n,R}$ is an étale morphism of schemes, the morphism $\operatorname{Spec} R \to \mathrm{D}_{n,R} \times_{\mathcal{X}} \mathcal{Y}$ extends to a section $\mathrm{D}_{n,R} \to \mathrm{D}_{n,R} \times_{\mathcal{X}} \mathcal{Y}$. Composing it with the projection $\mathrm{D}_{n,R} \times_{\mathcal{X}} \mathcal{Y}$, we get a D_k-morphism $\mathrm{D}_{n,R} \to \mathcal{Y}$, that is, an object of $(\mathrm{J}_n(\mathcal{Y}/\mathrm{D}_k))(R)$. This construction defines a morphism $\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \times_{\mathcal{X}_0} \mathcal{Y}_0 \to \mathrm{J}_n(\mathcal{Y}/\mathrm{D}_k)$.

Conversely, given a D_k-morphism $\mathrm{D}_{n,R} \to \mathcal{Y}$, we get a 2-commutative diagram of the above form. This gives a quasi-converse $\mathrm{J}_n(\mathcal{Y}/\mathrm{D}_k) \to \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \times_{\mathcal{X}_0} \mathcal{Y}_0$ to the above functor. □

For each $n \geq 0$, we have a *truncation morphism*

$$\pi_n^{n+1} \colon \mathrm{J}_{n+1}(\mathcal{X}/\mathrm{D}_k) \to \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$$

which sends an $(n+1)$-jet $\mathrm{D}_{n+1,R} \to \mathcal{X}$ to the composition $\mathrm{D}_{n,R} \hookrightarrow \mathrm{D}_{n+1,R} \to \mathcal{X}$. This morphism is unique up to unique 2-isomorphism, but not strictly unique as we have room to choose pullback functors $\mathcal{X}(\mathrm{D}_{n+1,R}) \to \mathcal{X}(\mathrm{D}_{n,R})$. We fix one for each n. For two integers $n' \geq n \geq 0$, we define $\pi_n^{n'} \colon \mathrm{J}_{n'}(\mathcal{X}/\mathrm{D}_k) \to \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$ as the composition $\pi_n^{n+1} \circ \cdots \circ \pi_{n'-2}^{n'-1} \circ \pi_{n'-1}^{n'}$. Thus we obtain the inverse system $(\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k), \pi_n^{n'})$ of DM stacks.

Corollary 11.1.4. *For integers $n' \geq n \geq 0$, the natural morphism $\mathrm{J}_{n'}(\mathcal{X}/\mathrm{D}_k) \to \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$ is schematic and affine.*

Proof. Let $V \to \mathcal{X}$ be an atlas. From Lemma 11.1.3, we have the following 2-commutative diagram:

$$\begin{tikzcd} \mathrm{J}_{n'}(V/\mathrm{D}_k) \arrow[r] \arrow[d] & \mathrm{J}_{n'}(\mathcal{X}/\mathrm{D}_k) \arrow[d] \\ \mathrm{J}_n(V/\mathrm{D}_k) \arrow[r] & \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k) \end{tikzcd}$$

Moreover the horizontal arrows are étale and surjective. Since the left vertical arrow is an affine morphism of schemes, the right one is representable and affine. Hence it is schematic as well. □

Definition 11.1.5. The limit stack $\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k) = \varprojlim \mathrm{J}_n(\mathcal{X}/\mathrm{D}_k)$ is defined as follows. An object of the fiber $(\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(\operatorname{Spec} R)$ is the sequence $(\alpha_n)_{n\geq 0}$ of objects $\alpha_n \in (\mathrm{J}_n(\mathcal{X}/\mathrm{D}_k))(\operatorname{Spec} R)$ with $\pi_n^{n+1}(\alpha_{n+1}) = \alpha_n$. A morphism $(\beta_n)_{n\geq 0} \to (\alpha_n)_{n\geq 0}$ in $(\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(\operatorname{Spec} R)$ is the sequence $(\phi_n)_{n\geq 0}$ of morphisms $\phi_n \colon \beta_n \to \alpha_n$ with $\pi_n^{n+1}(\phi_{n+1}) = \phi_n$. We call $\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k)$ the *stack of untwisted arcs of $\mathcal{X}$*. We define an *untwisted arc of $\mathcal{X}$ over R* to be a D_k-morphism $\mathrm{D}_R \to \mathcal{X}$. When R is an algebraically closed field, we call it a *geometric untwisted arc.*

We can see that this is indeed a stack over $\mathbf{Aff}_k$.

Lemma 11.1.6. *The stack $\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k)$ is a DM stack.*

PROOF. For an atlas $V \to \mathcal{X}$, we have $\mathrm{J}_\infty(V/\mathrm{D}_k) \cong \mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k) \times_{\mathcal{X}_0} V_0$, which is a scheme. Thus $\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k) \to \mathcal{X}_0$ is representable and $\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k)$ is a DM stack. □

LEMMA 11.1.7. *For an algebraically closed field K, the fiber $(\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(K)$ is identified with the category of untwisted arcs $\mathrm{D}_K \to \mathcal{X}$.*

PROOF. Let $\mathbf{C}_{\mathcal{X},K}$ denote the category of untwisted arcs $\mathrm{D}_K \to \mathcal{X}$. We have a functor $\mathbf{C}_{\mathcal{X},K} \to (\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(K)$ sending an arc $\alpha\colon \mathrm{D}_K \to \mathcal{X}$ to the sequence of indued jets $\alpha_n\colon \mathrm{D}_{n,K} \to \mathcal{X}$. We show that this is fully faithful and essentially surjective, which implies the lemma.

Fully faithful: Isomorphisms from an arc β to another α in the category $\mathbf{C}_{\mathcal{X},K}$ correspond to sections of $\underline{\mathrm{Iso}}_{\mathrm{D}_K}(\beta,\alpha) \to \mathrm{D}_K$. On the other hand, the isomorphisms from (β_n) to (α_n) in $(\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(K)$ correspond to compatible sequences of sections of $\underline{\mathrm{Iso}}_{\mathrm{D}_{n,K}}(\beta_n,\alpha_n) \to \mathrm{D}_{n,K}$. These two sets of isomorphisms correspond to each other from the universal property of the completion of a ring. Thus the above functor is fully faithful.

Essentially surjective: Let $V \to \mathcal{X}$ be an atlas. The functor $(\mathrm{J}_\infty(V/\mathrm{D}_k))(K) \to (\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k))(K)$ is essentially surjective. The functor $\mathbf{C}_{V,K} \to (\mathrm{J}_\infty(V/\mathrm{D}_k))(K)$ is also essentially injective. This shows the desired essential surjectivity. □

The following corollary is a direct consequence of this lemma:

COROLLARY 11.1.8. *The point set $|\mathrm{J}_\infty(\mathcal{X}/\mathrm{D}_k)|$ is identified with the set of equivalence classes of geometric untwisted arcs of $\mathcal{X}$. Here we define two geometric untwisted arcs $\alpha_i\colon \mathrm{D}_{K_i} \to \mathcal{X}$, $i \in \{1,2\}$ to be* equivalent *if there exists a third one $\alpha_3\colon \mathrm{D}_{K_3} \to \mathcal{X}$ fitting into a 2-commutative diagram of D_k-morphisms of the form:*

$$\begin{tikzcd}
\mathrm{D}_{K_3} \arrow[r] \arrow[d] \arrow[dr, "\alpha_3"] & \mathrm{D}_{K_2} \arrow[d, "\alpha_2"] \\
\mathrm{D}_{K_1} \arrow[r, "\alpha_1"'] & \mathcal{X}
\end{tikzcd}$$

CHAPTER 12

Twisted formal disks

12.1. Twisted formal disks

Definition 12.1.1. Let K be an algebraically closed field. A *twisted formal disk over* K is a DM stack $\mathcal{E}$ endowed with a morphism $\mathcal{E} \to \mathrm{D}_K$ such that

(1) $\mathcal{E}$ is regular,
(2) the morphism $\mathcal{E} \to \mathrm{D}_K$ is a coarse moduli space and birational (that is, $\mathcal{E} \times_{\mathrm{D}_K} \mathrm{D}_K^* \to \mathrm{D}_K^*$ is an isomorphism).

Definition 12.1.2. A *Galois cover* of D_K means an integral regular D_K-scheme E such that $E^* := E \times_{\mathrm{D}_K} \mathrm{D}_K^* \to \mathrm{D}_K^*$ is a finite étale Galois cover.

For a Galois cover E, if $E^* = \operatorname{Spec} L$, then $L/K(\!(t)\!)$ is a finite Galois extension. Conversely, given a finite Galois extension $L/K(\!(t)\!)$, we can construct a Galois cover of D_K as the integral closure of D_K in L.

For a Galois cover $E \to \mathrm{D}_K$ with the Galois group G, the quotient stack $[E/G]$ is a twisted formal disk. Conversely any twisted formal disk is of this form:

Lemma 12.1.3. *Let $E \to \mathcal{E}$ be an atlas and let E_0 be a connected component such that the closed point of $|\mathcal{E}|$ is in the image of $|E_0|$. Then the composition $E_0 \to \mathcal{E} \to \mathrm{D}_K$ is a Galois cover and if G denotes its Galois group, then $\mathcal{E} \cong [E_0/G]$. Moreover this Galois cover is uniquely determined from $\mathcal{E}$ up to isomorphism.*

Proof. We redefine E to be E_0. Then E is an integral regular scheme. Since the first projection $E \times_{\mathcal{E}} E \to E$ is étale and finite, there exists an isomorphism $E \times_{\mathcal{E}} E \cong E \times G$ with G a finite set which is compatible with the first projections. The groupoid scheme structure of $E \times_{\mathcal{E}} E \rightrightarrows E$ gives G a structure of a group and makes the second projection $E \times G = E \times_{\mathcal{E}} E \to E$ a group action. Thus

$$\mathcal{E} \cong [E \times_{\mathcal{E}} E \rightrightarrows E] \cong [E/G].$$

Let $E^* := E \times_{\mathrm{D}_K} \mathrm{D}_K^*$. The morphism $E^* \to \mathrm{D}_K^*$ is étale and $E^*/G = \mathrm{D}_K^*$. Thus $E \to \mathrm{D}_K$ is a Galois cover. If $E' \to \mathcal{E}$ is another integral atlas, then every connected component of $E \times_{\mathcal{E}} E'$ maps isomorphically onto both E and E'. Thus we get an isomorphism $E \cong E'$ compatible with the morphisms to D_K. $\square$

These results are summarized in the following proposition.

Proposition 12.1.4. *For each algebraically closed field K, the above construction gives the following one-to-one correspondences:*

$$\begin{aligned} &\{\textit{finite Galois extensions of } K(\!(t)\!)\}/\cong \\ &\leftrightarrow \{\textit{Galois covers of } \mathrm{D}_K\}/\cong \\ &\leftrightarrow \{\textit{twisted formal disks over } K\}/\cong \end{aligned}$$

12.2. A pseudo-universal family of twisted formal disks

DEFINITION 12.2.1. A *finite affine atlas* of a DM stack just means an atlas $V \to \mathcal{X}$ such that V is an affine scheme and $V \to \mathcal{X}$ is a finite morphism.

Let Φ be a DM stack with a finite affine atlas $V = \operatorname{Spec} R \to \mathcal{X}$. Then $V \times_{\mathcal{X}} V$ is also an affine scheme, say $\operatorname{Spec} S$. We get the groupoid scheme $\operatorname{Spec} S \rightrightarrows \operatorname{Spec} R$. This induces the groupoid scheme $\operatorname{Spec} S[\![t]\!] \rightrightarrows \operatorname{Spec} R[\![t]\!]$

DEFINITION 12.2.2. With the above notation, we define the *formal disk over* Φ*,* denoted by D_Φ, to be the stack $[\operatorname{Spec} S[\![t]\!] \rightrightarrows \operatorname{Spec} R[\![t]\!]]$ associated to the above groupoid scheme. Similarly we define the *punctured formal disk over* Φ*,* denoted by D^*_Φ, to be $\mathrm{D}_\Phi \times_{\mathrm{D}_k} \mathrm{D}^*_k$. We also define $\mathrm{D}_{n,\Phi} := \mathrm{D}_\Phi \times_{\mathrm{D}_k} \mathrm{D}_{n,k}$. (Equivalently we may define D^*_Φ and $\mathrm{D}_{n,\Phi}$ as the stacks assocated to the groupoid schemes $\operatorname{Spec} S(\!(t)\!) \rightrightarrows \operatorname{Spec} R(\!(t)\!)$ and $\operatorname{Spec} S[t]/(t^{n+1}) \rightrightarrows \operatorname{Spec} R[t]/(t^{n+1})$.) More generally, if Φ_i, $i \in I$ are countably many DM stacks with finite affine atlases and $\Phi = \coprod_{i \in I} \Phi_i$, then we define $\mathrm{D}_\Phi := \coprod_{i\in I} \mathrm{D}_{\Phi_i}$ and similarly for D^*_Φ and $\mathrm{D}_{n,\Phi}$.

THEOREM 12.2.3. *There exist countably many DM stacks Φ_i, $i \in I$ of finite type with finite affine atlases and a morphism $\mathcal{E} \to \mathrm{D}_\Phi$ of DM stacks with $\Phi := \coprod_{i\in I} \Phi_i$ such that:*

(1) *For each geometric point $\phi\colon \operatorname{Spec} K \to \Phi$, $\mathcal{E}_\phi := \mathcal{E} \times_{\mathrm{D}_\Phi} \mathrm{D}_K$ is a twisted formal disk over K.*
(2) *For each algebraically closed field K, the map*

$$\begin{aligned} \Phi(K)/{\cong} &\to \{\text{twisted formal disks over } K\}/{\cong} \\ \phi &\mapsto \mathcal{E}_\phi \end{aligned}$$

is bijective.

The rest of this chapter is devoted to the proof of this theorem.

DEFINITION 12.2.4. We call $\mathcal{E} \to \mathrm{D}_\Phi$ as above a *pseudo-universal family of twisted formal disks.*

12.3. Galoisian group schemes

In order to prove Theorem 12.2.3, we first construct the moduli stack of all possible Galois groups.

DEFINITION 12.3.1. A finite group G is called *Galoisian* if there exists a finite Galois extension $L/K(\!(t)\!)$ such that K is an algebraically closed field and the Galois group $\operatorname{Gal}(L/K(\!(t)\!))$ is isomorphic to G.

It is known that if $p > 0$, every Galoisian group is isomorphic to the semidirect product $H \rtimes C$ of a p-group and a tame cyclic group C [**Ser79**, pp. 67–68]. If $p = 0$, Galoisian groups are exactly finite cylic groups. Quotients and subgroups of a Galoisian group is again Galoisian.

DEFINITION 12.3.2. A finite étale group scheme G over a scheme S is called *Galoisian* if for every geometric point $s\colon \operatorname{Spec} K \to S$, the fiber $G_s = G \times_S \operatorname{Spec} K$ is a Galoisian finite group. For a Galoisian finite group G, a finite étale group scheme $H \to S$ is called *G-Galoisian* if every geometric fiber H_s is isomorphic to G.

DEFINITION 12.3.3. We define the *moduli stack of Galoisian group schemes*, denoted by $\mathcal{A}$, as follows. An object of a fiber $\mathcal{A}(S)$ is a Galoisian group scheme $G \to S$. A morphism from $H \to S$ to $G \to S$ in $\mathcal{A}(S)$ is an isomorphism $H \to G$ of group schemes over S. We call $\mathcal{A}$ the *moduli stack of Galoisian group schemes.* For the isomorphism class $[G]$ of a Galoisian finite group G, we define $\mathcal{A}_{[G]}$ to be its full subcategory consisting of G-Galoisian group schemes.

It is straightforward to show that $\mathcal{A}$ is a stack. Let us denote by GalGps a representative set of isomorphism classes of Galoisian finite groups. We have

$$\mathcal{A} = \coprod_{G \in \mathrm{GalGps}} \mathcal{A}_{[G]}.$$

If we fix a Galoisian group and an algebraically closed field K, then all G-Galoisian group scheme over K are isomorphic one another and have $\mathrm{Aut}(G)$ as their automorphism groups. Thus we may guess that the stack $\mathcal{A}_{[G]}$ would be $\mathrm{B}(\mathrm{Aut}\, G)$. This is indeed true:

LEMMA 12.3.4. *For a Galoisian finite group G, we have an isomorphism*

$$\mathcal{A}_{[G]} \cong \mathrm{B}(\mathrm{Aut}\, G)$$

such that the constant group scheme G over k corresponds to the standard morphism $\mathrm{Spec}\, k \to \mathrm{B}(\mathrm{Aut}\, G)$.

PROOF. In **[Yas19]**, we proved this by explicit construction of funtors which are pseudo-inverses to each others. We give a different and shorter proof here. Consider the fiber product $\mathrm{Spec}\, k \times_{\mathcal{A}_{[G]}} \mathrm{Spec}\, k$ given by the morphism $\mathrm{Spec}\, k \to \mathcal{A}_{[G]}$ associated to the constant group scheme G over k. We claim that this stack $\mathrm{Spec}\, k \times_{\mathcal{A}_{[G]}} \mathrm{Spec}\, k$ is isomorphic to the constant group scheme $\mathrm{Aut}\, G$ over k. Indeed, an object of this fiber product over an affine scheme S is the triple $(S \to \mathrm{Spec}\, k, S \to \mathrm{Spec}\, k, \alpha\colon G_S \to G_S)$, where the first two entries are the structure morphism and the last one is an isomorphism of group schemes over S by the definition of fiber products. Thus the fiber $(\mathrm{Spec}\, k \times_{\mathcal{A}_{[G]}} \mathrm{Spec}\, k)(S)$ is canonically equivalent to $(\mathrm{Aut}\, G)(S)$. Thus $\mathcal{A}_{[G]}$ is is isomorphic to the stack associated the groupoid scheme $\mathrm{Aut}\, G \rightrightarrows \mathrm{Spec}\, k$, that is, the classifying stack $\mathrm{B}(\mathrm{Aut}\, G) = [\mathrm{Spec}\, k / \mathrm{Aut}\, G]$. □

The moduli stack $\mathcal{A}$ has the universal family

$$\mathcal{G} \to \mathcal{A}$$

of Galoisian group schemes. An object of $\mathcal{G}(S)$ is the pair $(G \to S, \sigma)$ of a Galoisian group scheme $G \to S$ and a section $\sigma \in G(S)$. A morphism $(G' \to S, \sigma') \to (G \to S, \sigma)$ in $\mathcal{G}(S)$ is a morphism $G' \to G$ of group schemes over S compatible with $\sigma' \mapsto \sigma$. For a Galoisian group scheme $G \to S$ corresponding to $S \to \mathcal{A}$, we have

$$\mathcal{G} \times_{\mathcal{A}} S \cong G.$$

In particular, $\mathcal{G} \to \mathcal{A}$ is schematic, étale and finite.

12.4. The Artin-Schreier theory

For each k-algebra R, we have the Artin-Schreier exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0$$

of sheaves on the small étale site of $\operatorname{Spec} R$ [**Mil80**, p. 67]. Here $\wp$ is the Artin-Schreier map which maps a section s to $s^p - s$. This induces an exact sequence of cohomology groups

$$\mathrm{H}^0_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_a) \xrightarrow{\wp} \mathrm{H}^0_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_a) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_a).$$

The group $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/p\mathbb{Z})$ parametrizes isomorphism classes $\mathbb{Z}/p\mathbb{Z}$-torsors over $\operatorname{Spec} R$ We also have

$$\mathrm{H}^i_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_a) = \mathrm{H}^i(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = \begin{cases} R & (i = 0) \\ 0 & (i \neq 0). \end{cases}$$

Here the higher cohomology groups vanish, since $\operatorname{Spec} R$ is affine [**Gro61**, Th. 1.3.1] (cf. [**Har77**, III, Th. 3.5]). Thus we have a one-to-one correspondence:

$$R/\wp(R) \leftrightarrow \{\mathbb{Z}/p\mathbb{Z}\text{-torsors over } \operatorname{Spec} R\}/\cong \tag{12.4.1}$$

This maps the class of $f \in R$ to the torsor $\operatorname{Spec} R[X]/(X^p - X - f)$. The (right) action of $\mathbb{Z}/p\mathbb{Z}$ is given by the (left) action

$$a \cdot X = X + a \quad (a \in \mathbb{Z}/p\mathbb{Z}).$$

To categorify this correspondence, we define a cagegory $\mathbf{B}_R$ as follows.

Definition 12.4.1. An object of $\mathbf{B}_R$ is an element $r \in R$. A morphism $r' \to r$ is an element $b \in R$ such that $b - b^p = r' - r$. For morphisms $c\colon r'' \to r'$ and $b\colon r' \to r$, the composition $b \circ c\colon r'' \to r$ is the sum $b + c$.

Indeed, since

$$(b + c) - (b + c)^p = (b - b^p) + (c - c^p) = r'' - r,$$

the sum $b + c$ is a morphism $r'' \to r$. In the category $\mathbf{B}_R$, an element $r \in R$ is isomorphic to its p-th power r^p by the isomorphism $r\colon r \xrightarrow{\sim} r^p$.

For a morphism $b\colon r' \to r$, we have the isomorphism

$$\alpha_b\colon \operatorname{Spec} R[X]/(X^p - X - r') \to \operatorname{Spec} R[X]/(X^p - X - r)$$

of torsors given by $\alpha_b^*(X) = X + b$.

Lemma 12.4.2. *The map*

$$\begin{aligned} \operatorname{Hom}_{\mathbf{B}_R}(r', r) &\to \operatorname{Iso}_R\left(\operatorname{Spec}\frac{R[X]}{X^p - X - r'}, \operatorname{Spec}\frac{R[X]}{X^p - X - r}\right) \\ b &\mapsto \alpha_b \end{aligned}$$

is bijective.

Proof. The map extends to the map

$$\operatorname{Spec} R[X]/(X^p - X - (r' - r)) \to \underline{\operatorname{Iso}}_R\left(\operatorname{Spec}\frac{R[X]}{X^p - X - r'}, \operatorname{Spec}\frac{R[X]}{X^p - X - r}\right)$$

of functors $\mathbf{Aff}_R \to \mathbf{Set}$ so that the original map is the one evaluated at $\operatorname{Spec} R$. Moreover these schemes are $\mathbb{Z}/p\mathbb{Z}$-torsors over $\operatorname{Spec} R$ and the map is $\mathbb{Z}/p\mathbb{Z}$-equivariant. Since every equivariant morphism of torsors is an isomorphism, the above map of functors is an isomorphism. It follows that the map of the lemma is bijective. $\square$

The lemma shows the following proposition, which categorifies the correspondence (12.4.1):

PROPOSITION 12.4.3. *The functor*

$$\mathbf{B}_R \to \mathrm{B}_R(\mathbb{Z}/p\mathbb{Z}) = \{\mathbb{Z}/p\mathbb{Z}\text{-}torsors\ over\ \operatorname{Spec} R\}$$

that sends an object r to $\operatorname{Spec} R[X]/(X^p - X - r)$ *and a morphism $b\colon r' \to r$ to α_b is an equivalence.*

DEFINITION 12.4.4. For a finite étale group scheme G over k, we denote by Δ_G the stack of G-torsors over D^*. More precisely, we define this stack so that the fiber $\Delta_G(\operatorname{Spec} R)$ is $(\mathrm{B}\,G)(\mathrm{D}^*_R)$, the groupoid of G-torsors over $\mathrm{D}^*_R = \operatorname{Spec} R(\!(t)\!)$.

Thanks to Proposition 12.4.3, we may identify $(\mathrm{B}(\mathbb{Z}/p\mathbb{Z}))(\operatorname{Spec} R)$ with $\mathbf{B}_R$ and $\Delta_G(\operatorname{Spec} R)$ with $\mathbf{B}_{R(\!(t)\!)}$. We give an explicit description of $\Delta_{\mathbb{Z}/p\mathbb{Z}}$. Let $N := \{n \in \mathbb{Z}_{>0} \mid p \nmid n\}$ and let $\mathbb{A}_k^{\oplus N}$ be the functor $\mathbf{Aff}_k^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\mathbb{A}_k^{\oplus N}(\operatorname{Spec} R) := R^{\oplus N} = \{(r_n)_{n\in N} \in R^N \mid r_n = 0 \text{ for all but finitely many } n\}.$$

This is isomorphic to the inductive limit $\varinjlim \mathbb{A}_k^m$ of affine spaces $\mathbb{A}_k^m$, $m \geq 0$ with respect to standard embedings $\mathbb{A}_k^m \hookrightarrow \mathbb{A}_k^{m+1}$ as transition maps. An object of $\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ over $\operatorname{Spec} R$ is a pair $((r_n)_{n\in N}, r_0)$ with $r_n \in R$; we denote it also as $(r_n)_{n\in N\cup\{0\}}$ or simply (r_n). For two objects (r_n) and (s_n) over R, if $r_n = s_n$ for every $n > 0$, then an R-isomorphism $(r_n) \to (s_n)$ is an element $u_0 \in R$ such that $u_0 - u_0^p = r_n - s_n$. If $r_n \neq s_n$ for some $n > 0$, then (r_n) and (s_n) are not isomorphic.

DEFINITION 12.4.5. For each $m \in \mathbb{Z}_{\geq 0}$, we define a morphism $\psi_m\colon \mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \to \Delta_{\mathbb{Z}/p\mathbb{Z}}$ as follows; for an object $(r_n)_{n\in N\cup\{0\}}$ of $\mathbb{A}_k^{\oplus N} \times \mathrm{B}\,G$,

$$\psi_m((r_n)) := \sum_{n\in N\cup\{0\}} r_n t^{-np^m},$$

and for a morphism $u_0\colon (r_n) \to (s_n)$,

$$\psi_m(u_0) = u_0.$$

Let $F\colon \mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ be the Frobenius morphism sending (r_n) to (r_n^p). Then we have functorial isomorphisms

$$\psi_m\left((r_n)\right) : \psi_m\left((r_n)\right) \xrightarrow{\sim} \psi_{m+1} \circ F\left((r_n)\right),$$

which gives an isomorphism $\psi_m \xrightarrow{\sim} \psi_{m+1} \circ F$. We get a morphism

$$\psi_\infty\colon \left(\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})\right)^{\mathrm{iper}} \to \Delta_{\mathbb{Z}/p\mathbb{Z}},$$

where $(-)^{\mathrm{iper}}$ denotes the inductive perfection . Note that since $\mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ is perfect, we have

$$\left(\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})\right)^{\mathrm{iper}} \cong (\mathbb{A}_k^{\oplus N})^{\mathrm{iper}} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z}).$$

THEOREM 12.4.6. *The above morphism ψ_∞ is an isomorphism.*

PROOF. It suffices to show that this functor ψ_∞ is essentially surjective and fully faithful.

Essential surjectivity: Let us take an arbitrary k-algebra R and an arbitrary $f = \sum f_i t^i \in R(\!(t)\!)$, which is an object of $\Delta_{\mathbb{Z}/p\mathbb{Z}}(\operatorname{Spec} R) = \mathbf{B}_{R(\!(t)\!)}$. We write $f_+ := \sum_{i>0} f_i t^i$ and $f_- := \sum_{i\leq 0} f_i t^i$ so that $f = f_+ + f_-$. From Lemma 12.4.7, there exists a unique $g \in t \cdot R[\![t]\!]$ with $\wp(g) = f_+$. This g is an isomorphism $f_- \xrightarrow{\sim} f$ in $\mathbf{B}_{R(\!(t)\!)}$. Therefore we may and will suppose that $f = f_-$, that is, f has no term of positive degree. We claim that replacing terms of f, we get an isomorphic Laurent polynomial $h = \sum_{i \le 0} h_i t^i$ whose nozero terms all have degrees with the same p-adic (additive) valuation, say m, except the zeroth term h_0. This is equivalent to saying that h is in the image of ψ_m. Thus the claim implies that ψ_∞ is essentially surjective. The claim follows from the fact that if $f = a + b$, then $f \cong a^p + b$ and we can replace a term with its p-th power without changing the isomorphism class.

Faithfullness: Let $(r_n), (s_n)$ be two objects of $\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ over $\operatorname{Spec} R$ such that $r_n = s_n$ for $n > 0$ and let $u_0, v_0 \in R$ be two isomorphisms $r_0 \xrightarrow{\sim} s_0$ in $(\mathrm{B}(\mathbb{Z}/p\mathbb{Z}))(\operatorname{Spec} R)$. These elements u_0, v_0 also give R-isomorphisms $(r_n) \xrightarrow{\sim} (s_n)$ in $\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$. Suppose that $\psi_m(u_0) = \psi_m(v_0)$ for some m. By definition, $u_0^m = v_0^m$ in R. Therefore $F^m(u_0) = F^m(v_0)$ in $\mathrm{B}(\mathbb{Z}/p\mathbb{Z})$. Since $\mathrm{B}\,G$ is perfect, $u_0 = v_0$. This shows the faithfullness of ψ_∞.

Fullness: Let (s_n) and (r_n) be two objects of $\mathbb{A}_k^{\oplus N} \times \mathrm{B}(\mathbb{Z}/p\mathbb{Z})$ over $\operatorname{Spec} R$. Let $m \in \mathbb{Z}_{\ge 0}$ and let $b\colon \psi_m((s_n)) \to \psi_m((r_n))$ be an isomorphism in $\Delta_{\mathbb{Z}/p\mathbb{Z}}(\operatorname{Spec} R)$ for some m. Namely b is an element of $R(\!(t)\!)$ such that

$$b - b^p = \sum_{n \in N \cup \{0\}} (s_n - r_n) t^{-np^m}.$$

We will show the claim that for $e \gg 0$, $b^{p^e} \in R$. From Lemma 12.4.7, b has the trivial positive-degree part. Let $e \gg 0$ such that every nonzero coefficient of $c := b^{p^e}$ is not nilpotent (nilpotent coefficiets are killed by being raised to the p^e-th power). We get the equality

$$c - c^p = \sum_n (s_n - r_n)^{p^e} t^{-np^{m+e}}. \tag{12.4.2}$$

Writing $c = \sum c_i t^i$, we define the set,

$$M_c := \{v_p(i) \mid i < 0 \text{ and } c_i \ne 0\} \subset \mathbb{Z}_{\ge 0}.$$

Here v_p is the additive normalized p-adic valuation. If the claim is false, then this set is not empty. Then M_{c-c^p} contains at least two distinct numbers $\max M_c$ and $\max M_c + 1$. But the corresponding set for the right side of (12.4.2) contains at most one number $m + e$. This is a contradiction. We have proved the claim.

The claim shows that for $e \gg 0$, b^{p^e} gives an isomorphism $F^e((s_n)) \xrightarrow{\sim} F^e((r_n))$ as well as an isomorphism

$$\psi_{m+e}((s_n)) = \psi_m((s_n))^{p^e} \xrightarrow{\sim} \psi_m((r_n))^{p^e} = \psi_{m+e}((r_n)).$$

This shows that ψ_∞ is full. $\square$

LEMMA 12.4.7. *For each $f \in t \cdot R[\![t]\!]$, there exists a unique $g \in t \cdot R[\![t]\!]$ with $\wp(g) = f$.*

PROOF. For power series $f = \sum f_i t^i$ and $g = \sum g_i t^i$ in $t \cdot R[\![t]\!]$, the equality $\wp(g) = f$ is equivalent to the system of countable equaltities

$$g_{i/p}^p - g_i = f_i \quad (i \in \mathbb{Z}_{>0}).$$

Here we follow the convention that if i/p is not an integer, then $g_{i/p} = 0$. Given f, the unque solution is inductively given by $g_i := g_{i/p}^p - f_i$. $\square$

LEMMA 12.4.8. *Let $f = \sum_{n \le 0} f_n t^{-np^m} \in R(\!(t)\!)$. Suppose that every nozero coefficient f_n of f is non-nilpotent. Suppose also that $f_n \ne 0$ for some $n > 0$. Then there is no $b \in R(\!(t)\!)$ such that $b - b^p = f$.*

Proof. On the contrary, suppose that there exists $b \in R(\!(t)\!)$ such that $b - b^p = f$. Raising b and f to the p^n-th power for $n \gg 0$, we may suppose that every nonzero coefficient of b is not nilpotent. Then

$$\operatorname{ord} b^p = p \operatorname{ord} b = \operatorname{ord} f =: l,$$

here the order of a Laurent polynomial is defined to be the least degree of nonzero terms. Then ord □

Lemma 12.4.9. *Let R be a ring. The natual maps*

$$\begin{aligned} &\{\textit{open and closed subsets of } \operatorname{Spec} R\} \\ &\to \{\textit{open and closed subsets of } \operatorname{Spec} R[\![t]\!]\} \\ &\to \{\textit{open and closed subsets of } \operatorname{Spec} R(\!(t)\!)\} \end{aligned}$$

are bijective.

Proof. Open and closed subsets of $\operatorname{Spec} R$ correspond to idempotents of R. It suffices to show that every idempotent of $R[\![t]\!]$ belongs to R and every idempotent of $R(\!(t)\!)$ belongs to R.

The first map: On the contrary, suppose that there exists an idempotent r in $R[\![t]\!] \setminus R$. For $n \gg 0$, its image $\bar{r}$ in $R[\![t]\!]/(t^n)$ is still an idempotent which does not belong to R. But, since $\operatorname{Spec} R \to \operatorname{Spec} R[\![t]\!]/(t^n)$ is a homeomorphism, this is impossible, a contradiction.

The second map: Suppose that the map is not bijective for some ring R. We choose a pair (R, f) of a ring R and an idempotent $f \in R(\!(t)\!) \setminus R[\![t]\!]$ such that $\operatorname{ord} f$ attains the maximum, which is a negative integer. If $c \in R$ is the coefficient of $t^{\operatorname{ord} f}$ in f, then $c^2 = 0$. The image $\bar{f}$ of f in $(R/c)(\!(t)\!)$ is an idempotent of order $> \operatorname{ord} f$. Therefore $\bar{f}$ belongs to $(R/(c))[\![t]\!]$. Since the map $R[\![t]\!] \to R/(c)[\![t]\!] = R[\![t]\!]/(c)$ induces a bijection of idempotents, $\bar{f}$ belongs to $R[\![t]\!]$, a contradiction. □

Corollary 12.4.10. *Let B be a finite set and let $\underline{B}$ be the associated sheaf on $\mathbf{Aff}_k$. Namely $\underline{B}(\operatorname{Spec} R)$ is the set of locally constant maps $\operatorname{Spec} R \to B$. Then the maps*

$$\underline{B}(\operatorname{Spec} R) \to \underline{B}(\operatorname{Spec} R[\![t]\!]) \to \underline{B}(\operatorname{Spec} R(\!(t)\!))$$

are bijective.

Proof. The set $\underline{B}(\operatorname{Spec} R)$ correspond to a finite stratification $\operatorname{Spec} R = \bigsqcup_{b \in B} U_b$ by open and closed subsets U_b indexed by B. Thus the corollary is a direct consequence of the above lemma. □

APPENDIX A

This is an appendix

A.1. Quotients of schemes by finite group actions

A.2. Descent

Consider an fpqc (that is, faithfully flat and quasi-compact) morphism $p\colon S' \to S$ of schemes. We put $S'' := S' \times_S S'$ and $S''' := S' \times_S S' \times_S S'$. Let $p_i\colon S'' \to S'$ be the i-th projection and let $p_{ij}\colon S''' \to S''$ be the projection to the i-th and j-th entries.

Definition A.2.1 (Descent data). Let $\mathcal{F}'$ be a quasi-coherent sheaf on S'. A *descent datum on* $\mathcal{F}'$ (with respect to p) is an isomorphism $\phi\colon p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ satisfying the cocycle condition

$$p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi.$$

Similarly, let X' be an S'-scheme. A *descent datum on* X' is an S''-isomorphism $\phi\colon p_1^*X' \to p_2^*X'$ satisfying the cocycle condition

$$p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi.$$

Here $p_?^*$ means the base change by $p_?$.

For a quasi-coherent sheaf $\mathcal{F}$ on S, the pull-back $p^*\mathcal{F}$ has the natural descent datum. Similarly for p^*X of an S-scheme X.

Remark A.2.2 (The case of Galois covering). Suppose that p is a finite étale Galois cover of integral schemes with $\mathrm{Gal}(S'/S) = G$. Then we can identify S'' with $G \times S'$ so that p_1 corresponds to the projection $G \times S' \to S'$ and p_2 corresponds to the G-action $G \times S' \to S'$. Now giving an isomorphism $\phi\colon p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ is equivalent to giving an isomorphism $\phi_g\colon \mathcal{F}' \to g^*\mathcal{F}'$ for each $g \in G$. The cocycle condition means that

$$\phi_{hg} = g^*(\phi_h) \circ \phi_g.$$

Thus giving a quasi-coherent sheaf on S' with a descent datum is equivalent to giving a G-equivariant quasi-coherent sheaf on S'. Similarly, giving an S'-scheme with a descent datum is equivalent to giving a G-equivariant S'-scheme. See [**BLR90**, pp. 139-141] for details.

Proposition A.2.3 (fpqc descent; [**BLR90**, Section 6.1, Theorems 4 and 6], [**Gro65**, Propositions 2.5.2]). *The functor $\mathcal{F} \mapsto p^*\mathcal{F}$ gives an equivalence from the category of quasi-coherent sheaves on S to the category of quasi-coherent sheaves on S' with descent data. Moreover, $\mathcal{F}$ is locally free if and only if so is $p^*\mathcal{F}$.*

(2) *Suppose that S and S' are affine. Then the functor $X \mapsto p^*X$ gives an equivalence from the category of affine S-schemes to the category of affine S'-schemes with descent data.*

From Remark A.2.2, we get the following corollary in the case of Galois covers:

COROLLARY A.2.4 (Galois descent). *Suppose that p is a finite étale Galois cover of integral schemes with* $\mathrm{Gal}(S'/S) = G$.

(1) *The functor $\mathcal{F} \mapsto p^*\mathcal{F}$ gives an equivalence from the category of quasi-coherent sheaves on S to the category of G-equivariant quasi-coherent sheaves on S'.*
(2) *The functor $X \mapsto p^*X$ gives an equivalence from the category of S-schemes to the category of G-equivariant S'-schemes.*

COROLLARY A.2.5 (Galois descent for vector bundles). *Let X be an S-scheme. Suppose that p is a finite étale Galois cover of integral schemes with* $\mathrm{Gal}(S'/S) = G$. *Suppose also that the pull-back p^*X has a structure of a G-equivariant vector bundle with respect to the natural G-action. Then X has a structure of a vector bundle. In particular, if p is the morphism* $\mathrm{Spec}\, L \to \mathrm{Spec}\, K$ *associated to a finite Galois extension of fields, then X is isomorphic to an affine space $\mathbb{A}^n_K$.*

PROOF. The vector bundle p^*X with the descent datum corresponds to a locally free sheaf $\mathcal{F}'$ with a descent datum. From Proposition A.2.3 and Corollary A.2.4, there exists a locally free sheaf $\mathcal{F}$ which induces $\mathcal{F}'$ with the descent datum via p^*. It is straightforward to see that the vector bundle Y associated to $\mathcal{F}$ induces p^*X toghether with the given descent datum. Again from Corollary A.2.4, X is S-isomorphic to Y. We have proved the first assertion. In the situation of the second assertion, since $S = \mathrm{Spec}\, K$ is a point, the locally free sheaf $\mathcal{F}$ is in fact free and the associated vector bundle Y is trivial. $\square$

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