Computing F-blowups

Takehiko Yasuda

Osaka University

Computational workshop on Frobenius singularities and invariants

Ann Arbor, June 1st, 2012

Plan of the Talk

1. Definition of F-blowups and known results
2. How to compute F-blowups: the toric case
3. How to compute F-blowups: the general case$\leftarrow$ the main part, based on a joint work with Nobuo Hara and Tadakazu Sawada

Definition of F-blowups

- $char(k) = p > 0$, $k = \bar{k}$ (easily generalized to perfect fields)
- X: a (singular) variety over k.
- $x \in X \rightsquigarrow$ inf. nbhd $x^{[p^e]} \subset X$, defined by $\mathfrak{m}_x^{[p^e]} \subset \mathcal{O}_X$.

Definition

$FB_e(X) := \overline{\{x^{[p^e]} \mid x \in X_{sm}\}}^{Zar} \subset Hilb_{p^{e \cdot dim X}}(X)$. (the e-th F-blowup of X)

- $[Z] \in FB_e(X) \leftrightarrow$ 0-dim'l sub. $Z \subset X$ with $Supp(Z) = pt$
- $\exists$ natural map $FB_e(X) \to X$, $[Z] \mapsto Supp(Z)$.
- projective and birational, an isom. over X_{sm}

Basic Properties

- a sequence of blowups:

$$\begin{array}{ccccccc} FB_0(X) = X & \overset{\nexists}{- - -} & FB_1(X) & \overset{\nexists}{- - -} & FB_2(X) & & \cdots \\ \downarrow & \swarrow & & \swarrow & & & \\ X & & & & & & \end{array}$$

- X is F-pure $\Rightarrow \exists\ FB_{e+1}(X) \to FB_e(X)$

Note

$FB_{e+e'}(X) \neq FB_{e'}(FB_e(X))$

Known Results

- X: F-reg. surf. $\Rightarrow$ For $e \gg 0$, $FB_e(X)$ is the min. res. [Hara]
- This is NOT true for non-F-reg. surf. [Hara, Sawada, Y]
- $X = \mathbb{A}^d / G$, $p \nmid \sharp G \Rightarrow$ For $e \gg 0$, $FB_e(X) \cong Hilb^G(\mathbb{A}^d)$. [Y, Toda-Y]

Summary

Every variety in char. p has canonical blowups $FB_e(X)$.

Sometimes this gives a nice resolution in one step.

How to Compute F-blowups: the Toric Case

- $N = \mathbb{Z}^d$, $M := N^\vee$
- $\sigma \subset N_{\mathbb{R}}$: a strongly convex d-dimensional cone
- $X = Spec\, k[M \cap \sigma^\vee]$: affine toric variety

Fact

The normalization of $FB_e(X)$ corresponds to the Groebner fan of some ideal I_e.

The Groebner Fan

- $I \subset k[M \cap \sigma^{\vee}]$: an ideal
- $w \in \sigma$: an interior point $\rightsquigarrow$ $w : M \cap \sigma^{\vee} \to \mathbb{R}_{\geq 0}$
- $\mathbb{R}$-grading of $k[M \cap \sigma^{\vee}]$ w.r.t. w
- $w \in int\, \sigma \mapsto$ initial ideal $In_w(I) \subset k[M \cap \sigma^{\vee}]$

Definition

The Groebner fan Δ of I:

- $|\Delta| = \sigma$,
- $In_w(I) = In_v(I) \Leftrightarrow w, v \in rel.int.\, \tau$ for the same $\tau \in \Delta$.

The ideal I_e

- $1 \in T = Spec\, k[M] \subset X$: the unit point of the open torus
- $1^{[p^e]} \subset T$ is defined by

$$\langle x_1 - 1, \ldots, x_d - 1 \rangle^{[p^e]} = \left\langle x_1^{p^e} - 1, \ldots, x_d^{p^e} - 1 \right\rangle \subset k[x_1^{\pm}, \ldots, x_d^{\pm}]$$

- $1^{[p^e]} \subset X$ is defined by

$$I_e := \left\langle x_1^{p^e} - 1, \ldots, x_d^{p^e} - 1 \right\rangle_{k[x_1^{\pm}, \ldots, x_d^{\pm}]} \cap k[M \cap \sigma^{\vee}]$$

Fact

For an affine toric variety X, the normalization of $FB_e(X)$ corresponds to the Groebner fan of I_e.

Note

We can also describe unnormalized $FB_e(X)$ with initial ideals.

Macaulay2 Computation

The toric case

The computation of Groebner fans are implemented only for polynomial rings (as far as I know).

Example

- The A_{13}-singularity $X = (xy - z^{14} = 0) \subset \mathbb{A}^3$ in char. 3
- Compute $FB_4(X)$ with Macaulay2 by using the embedding $X \subset \mathbb{A}^3$ and a Groebner fan for $k[x, y, z]$ as follows:

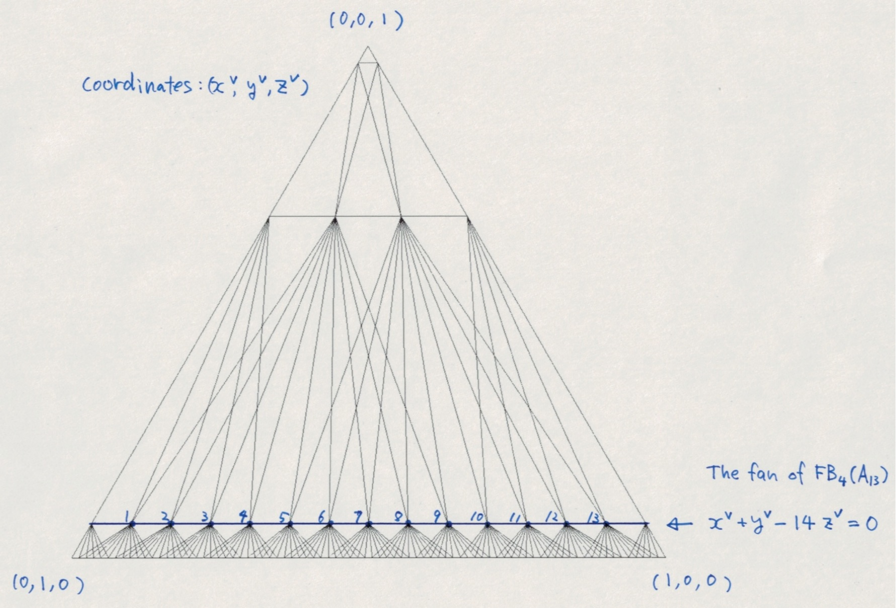
Macaulay2 Computation

The toric case: Command lines

```
i1 : loadPackage "gfanInterface";

i2 : S = ZZ/3[x,y,z];

i3 : I = ideal(x*y-z^14); -- A_13-singularity

i4 : J = I + ideal(x^81-1,y^81-1,z^81-1); -- the [3^4]-nbhd of the unit pt

i5 : render J -- render the Groebner fan of J
```

Macaulay2 Computation

Picture: The fan of $FB_4(A_{13})$



How to Compute F-blowups: the General Case

- X: a variety
- M: a coherent $\mathscr{O}_X$-module

Definition

A modification $f : Y \to X$ is the blowup at M if:

- $(f^* M)/tors$ is locally free,
- f is universal for this property.

Example

The Nash blowup of X is the blowup at $\Omega_{X/k}$.

Fact

$FB_e(X)$ is the blowup at $F^e_* \mathscr{O}_X$.

Two Parts of Computation

1. Compute blowups at modules.
2. Compute $F^e_* \mathscr{O}_X$.

Computing Blowups at Modules

Expressing them as blowups at ideals

- $X = Spec\, R$ (Noetherian)
- M: f.g. R-module of rank r
- $R^n \xrightarrow{A} R^m \to M \to 0$: free presen. by a $m \times n$-matrix A
- $\exists$ $m \times (m - r)$-submatrix A' such that

  $$M' := coker(R^{m-r} \xrightarrow{A'} R^m)$$

  has rank r
- $I \subset R$: the ideal generated by $(m - r)$-minors of A'

Fact [Villamayor, J. Alg.]

$Bl_M(X) = Bl_I(X)$

Computing Blowups at Modules

A recipe

1. Given $R^n \xrightarrow{A} R^m \to M \to 0$,
2. determine a submatrix A', then
3. compute the ideal I generated by minors of A', then
4. compute the Rees algebra, $Rees(I) = \bigoplus_{i \geq 0} I^i$. (with Macaulay2, for instance)
5. We get $Bl_M(X) = Proj\, Rees(I)$.

Computing $F_*^e \mathscr{O}_X$

In precise, we compute the following:

Input:

- $S = \mathbb{F}_p[x_1, \dots, x_n]$, $R = S/\langle f_1, \dots, f_l \rangle$
- R-module M with a presentation,

$$R^n \xrightarrow{A} R^m \to M \to 0, A = (\overline{a_{ij}}), a_{ij} \in S.$$

Output:

- a presentation of $F_*^e M$,

$$R^s \xrightarrow{B} R^t \to F_*^e M \to 0,$$

with explicit B.

How to compute $F_*^e M$

An algorithm

Input: $S = \mathbb{F}_p[x_1, \dots, x_n]$, $R = S/\langle f_1, \dots, f_l \rangle$, $M = coker(R^n \xrightarrow{A} R^m)$.

Algorithm:

1. Compute an S-presentation of M,

$$S^a \to S^b \to M \to 0.$$

2. Compute an S-presentation of $F_*^e M$,

$$S^c \xrightarrow{B} S^d \to F_*^e M \to 0, B = (b_{ij}), b_{ij} \in S.$$

   (explained in the next slide)

3. By $- \otimes_S R$, get an R-presenation of $F_*^e M$,

$$R^c \xrightarrow{\bar{B}} R^d \to F_*^e M \to 0, \bar{B} = (\overline{b_{ij}}).$$

How to compute $F^e_* M$ for a S-module M

The monomial case

The key case:

- $x^a = x_1^{a_1} \cdots x_n^{a_n} \in S$: a monomial
- $(\bigstar)$: $S \xrightarrow{x^a} S \to M \to 0$

$(\bigstar)$ is regarded as a presen. of $F^e_* M$, looked throught $F^e : S \to S$:

$$\bigoplus_b S \cdot [x^b] \xrightarrow{\phi} \bigoplus_b S \cdot [x^b] \to F^e_* M \to 0,$$

where $b = (b_1, \ldots, b_n)$ with $0 \leq b_i < q := p^e$. Explicitly:

$$\phi([x^b]) = x^{(a+b) \div q} [x^{(a+b) \% q}],$$

$(a+b) \div q$, $(a+b)\%q$: the elementwise quotient and remainder

How to compute $F_*^e M$ for a S-module M

The general case

The monomial case:

$$S \xrightarrow{x^a} S \to M \to 0 \qquad \rightsquigarrow \qquad S^{q^n} \xrightarrow{\phi_a} S^{q^n} \to F_*^e M \to 0$$

The polynomial case: For $f = \sum c_a x^a$, put $\phi_f := \sum c_a \phi_a$ and

$$S \xrightarrow{f} S \to M \to 0 \qquad \rightsquigarrow \qquad S^{q^n} \xrightarrow{\phi_f} S^{q^n} \to F_*^e M \to 0$$

The matrix case: For $A = (a_{ij})$, ϕ_A is a block matrix with blocks $\phi_{a_{ij}}$ and

$$S^s \xrightarrow{A} S^t \to M \to 0 \qquad \rightsquigarrow \qquad S^{sq^n} \xrightarrow{\phi_A} S^{tq^n} \to F_*^e M \to 0$$

Macaulay2 Computation

The 1st F-blowup of a simple elliptic singularity

```
i1 : load "MyPackage.m2"

i2 : R = ZZ/2[x,y,z]/ideal(y^2+x^3+x*y*z+z^6); -- a simple ell. sing., type \tilde{E}_8

i3 : isBlowupNormal fBlowup(R,1)

o3 = false  -- FB_1 (R) is non-normal

i4 : fBlSpecial(R,1)  -- the exceptional locus

                               ZZ
                               --[w , w , w , w , w ]
                                2  0   1   2   3   4
o4 = ------------------------------------------------------------------------------------
                                                           2
     (0, 0, 0, 0, 0, w , 0, w w  + w w , w w  + w w  + w , w w  + w w , w w  + w w )
                      3      2 3    1 4   0 3    1 4    4   1 2    2 4   1 3    3 4

o4 : QuotientRing

i5 : decompose ideal oo

o5 = {ideal (w , w , w ), ideal (w , w , w )}  -- the exceptional locus = two P^1's
              4   3   1            4   3   2
```

References

Toric: Yasuda, *Universal flattening of Frobenius*, Amer. J. Math, 2012

General: Sawada, Hara & Yasuda, *F-blowups of normal surface singularities*, to appear in Algebra & Number Theory, arXiv:1108.1840

Slides and Codes

I have put slides of this talk and some Macaulay2 functions at my homepage:

http://takehikoyasuda.jimdo.com/