A note on Markov type constants*†

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Abstract

We prove that, if a geodesic metric space has Markov type 2 with constant 1, then it is an Alexandrov space of nonnegative curvature. The same technique provides a lower bound of the Markov type 2 constant of a space containing a tripod or a branching point.

1 Introduction

Rademacher type and cotype are fundamental and significant tools in the local theory of Banach spaces. In recent years, there has been an increasing interest in their nonlinearizations. Enflo [En] first gave a generalized notion of type for general metric spaces which is now called Enflo type. After that, Ball [Ba] introduced another kind of nonlinear type, called Markov type, based on a different idea. Markov type has found deep applications in the extension problem of Lipschitz maps ([Ba], [NPSS], [MN1]) as well as in the theory of bi-Lipschitz embeddings of finite metric spaces or graphs ([LMN], [BLMN], [NPSS], [MN1]). Roughly speaking, upper bounds of the Markov type constant (K in (1.1)) relate to the Lipschitz extension problem, while lower bounds are useful at controling the distortion of bi-Lipschitz embeddings into ℓ_p-spaces. We also refer to [MN2] for recent striking progress on nonlinear cotype and its applications.

A metric space X has Enflo type 2 if there is a constant $K \geq 1$ such that the inequality

$$\sum_{\varepsilon \in \{-1,1\}^N} d(x_\varepsilon, x_{-\varepsilon})^2 \leq K^2 \sum_{\varepsilon \sim \varepsilon'} d(x_\varepsilon, x_{\varepsilon'})^2$$

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holds for $N \in \mathbb{N}$ and $(x_\varepsilon)_{\varepsilon \in \{-1,1\}^N} \in X^{2^N}$, where $\varepsilon \sim \varepsilon'$ if $\sum_{i=1}^N |\varepsilon_i - \varepsilon'_i| = 2$. Markov type 2 is defined by the inequality

$$\mathbb{E}[d(Z_0, Z_l)^2] \le K^2 l \mathbb{E}[d(Z_0, Z_1)^2], \tag{1.1}$$

where $\{Z_l\}_{l \in \mathbb{N} \cup \{0\}}$ is a reversible, stationary Markov chain on the N-point state space identified with $(x_i)_{i=1}^N \in X^N$. Markov type seems stronger than Enflo type because it concerns general Markov chains, and it is indeed the case if we admit a constant multiplication of K. However, if we also care about the value of K, then a difference between them arises. We clarify such a difference by comparing these nonlinear types with curvature conditions in metric (Riemannian) geometry.

Our main result (Theorem 2.5) asserts that, if a geodesic metric space has Markov type 2 with $K = 1$, then it is an Alexandrov space of nonnegative curvature. The converse is also true in a certain weak sense. This result makes an interesting contrast to the fact that a geodesic metric space has Enflo type 2 with $K = 1$ if and only if it is a CAT(0)-space (Proposition 2.2). The proof of Theorem 2.5 relies on Sturm's characterization of Alexandrov spaces in [St]. Theorem 2.5 implicitly means that the negative curvature makes the Markov type 2 constant K worse. Actually, our technique also gives a lower bound of the Markov type 2 constant of a space containing an N-pod or, more generally, a branching point (Theorem 3.1).

The article is organized as follows. We start with the study of nonlinear types and curvature bounds, and prove our main theorem in Section 2. Then we use the same technique to obtain lower bounds of the Markov type 2 constants of certain spaces in Section 3. In Section 4, we discuss how to construct spaces having Markov type by gluing.

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2 Nonlinear types and curvature bounds

A Banach space $(V, \|\cdot\|)$ is said to have *Rademacher type* 2 if there is a constant $K \ge 1$ such that, for any $N \in \mathbb{N}$ and $(v_i)_{i=1}^N \in V^N$, it holds that

$$\frac{1}{2^N} \sum_{(\varepsilon_i) \in \{-1,1\}^N} \left\| \sum_{i=1}^N \varepsilon_i v_i \right\|^2 \le K^2 \sum_{i=1}^N \|v_i\|^2. \tag{2.1}$$

The parallelogram identity shows that Hilbert spaces have Rademacher type 2 with $K = 1$. Conversely, if $(V, \|\cdot\|)$ has Rademacher type 2 with $K = 1$, then (2.1) yields the parallelogram identity and hence $(V, \|\cdot\|)$ is a Hilbert space. We will study a similar problem in a nonlinear setting.

Before it, we recall two curvature conditions (see [BH] and [BBI]). A metric space (X, d) is said to be *geodesic* if every two points $x, y \in X$ can be joined by a rectifiable curve whose length coincides with $d(x,y)$. A rectifiable curve $\gamma:[0,1]\longrightarrow X$ is called a *minimal geodesic* if it is globally minimizing and has a constant speed. A geodesic metric space is called a CAT(0)-*space* if we have

$$d\big(x,\gamma(1/2)\big)^2 \le \frac{1}{2}d(x,y)^2+\frac{1}{2}d(x,z)^2-\frac{1}{4}d(y,z)^2$$

for any $x,y,z\in X$ and any minimal geodesic $\gamma:[0,1]\longrightarrow X$ from y to z. If the reverse inequality

$$d\big(x,\gamma(1/2)\big)^2 \ge \frac{1}{2}d(x,y)^2+\frac{1}{2}d(x,z)^2-\frac{1}{4}d(y,z)^2$$

holds for any x,y,z and γ, then we say that (X,d) is an *Alexandrov space of nonnegative curvature*. A complete, simply-connected Riemannian manifold is a CAT(0)-space if and only if its sectional curvature is nonpositive everywhere. Similarly, a complete Riemannian manifold is an Alexandrov space of nonnegative curvature if and only if its sectional curvature is nonnegative.

2.1 Enflo type and nonpositive curvature

We proceed to compare the above curvature bounds with nonlinear notions of type.

Definition 2.1 (Enflo type, [En]) A metric space (X,d) is said to have *Enflo type* 2 if there is a constant $K\ge 1$ such that, for any $N\in\mathbb{N}$ and $(x_\varepsilon)_{\varepsilon\in\{-1,1\}^N}\in X^{2^N}$, we have

$$\sum_{\varepsilon\in\{-1,1\}^N} d(x_\varepsilon,x_{-\varepsilon})^2 \le K^2\sum_{\varepsilon\sim\varepsilon'} d(x_\varepsilon,x_{\varepsilon'})^2,$$

where $\varepsilon=(\varepsilon_i)_{i=1}^N$ and $\varepsilon\sim\varepsilon'$ holds if $\sum_{i=1}^N|\varepsilon_i-\varepsilon'_i|=2$.

By choosing $x_\varepsilon=\sum_{i=1}^N\varepsilon_i v_i$ in a Banach space, we find that Enflo type 2 implies Rademacher type 2 with the same constant K. The converse is only partially known (see [NS]). Fundamental examples of spaces with Enflo type 2 are 2-uniformly smooth Banach spaces and their nonlinearizations (see [Oh]). Combining known results, we observe the following:

Proposition 2.2 *For a metric space (X,d), the following are equivalent*:

(i) *(X,d) has Enflo type* 2 *with $K=1$.*

(ii) *For any four points $w,x,y,z\in X$, it holds that*

$$d(w,y)^2+d(x,z)^2\le d(w,x)^2+d(x,y)^2+d(y,z)^2+d(z,w)^2.$$

If (X,d) is geodesic, then (i) *and* (ii) *are also equivalent to*:

(iii) *(X,d) is a* CAT(0)*-space.*

Proof. The implication from (i) to (ii) is trivial and the converse follows by induction (see, e.g., [Oh]). The equivalence between (ii) and (iii) in a geodesic metric space is established in [BN2] (see also [Sa] for a recent short alternative proof). □

The condition (ii) is also called the *roundness* 2 (cf. [BL, Chapter 17]). We refer to [FLS] for another interesting characterization of CAT(0)-spaces in terms of Busemann's nonpositive curvature together with the Ptolemy inequality.

2.2 Markov type and nonnegative curvature

Next we are concerned with Markov type introduced by Ball. Given $N \in \mathbb{N}$, we consider a stationary, reversible Markov chain on the state space $\{1, 2, \dots, N\}$ with stationary distribution $(\pi_i)_{i=1}^N$ and transition matrix $A = (a_{ij})_{i,j=1}^N$. That is to say, we require

$$0 \le \pi_i, a_{ij} \le 1, \quad \sum_{i=1}^N \pi_i = 1, \quad \sum_{j=1}^N a_{ij} = 1, \quad \pi_i a_{ij} = \pi_j a_{ji} \tag{2.2}$$

for all $i, j = 1, 2, \dots, N$. The third and fourth inequalities guarantee the stationariness and reversibility of the Markov chain. We identify the state space $\{1, 2, \dots, N\}$ with a sequence $(x_i) \in X^N$ in a metric space. For $l \in \mathbb{N}$, we set $A^l = (a_{ij}^{(l)})_{i,j=1}^N$ and

$$\mathcal{E}(l) := \sum_{i,j=1}^N \pi_i a_{ij}^{(l)} d(x_i, x_j)^2.$$

Definition 2.3 (Markov type, [Ba]) A metric space (X, d) is said to have *Markov type* 2 if there is a constant $K \ge 1$ such that, given any $N \in \mathbb{N}$, $(x_i) \in X^N$, $(\pi_i)_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfying (2.2), we have $\mathcal{E}(l) \le K^2 l \mathcal{E}(1)$ for all $l \in \mathbb{N}$. The least such a constant K is denoted by $M_2(X)$ and called the *Markov type* 2 *constant* of X.

As straightforward consequences of the definition, Markov type is preserved under some deformations.

Example 2.4 (a) (ℓ_2-products) Let X be the ℓ_2-product of two metric spaces X_1 and X_2. Then we find $M_2(X) \le \max\{M_2(X_1), M_2(X_2)\}$.

(b) (Bi-Lipschitz embeddings) If there is a bi-Lipschitz embedding $f : X \longrightarrow Y$, then we have $M_2(X) \le \mathrm{Lip}(f)\mathrm{Lip}(f^{-1})M_2(Y)$. Here $\mathrm{Lip}(f)$ stands for the Lipschitz constant of f and the quantity $\mathrm{Lip}(f)\mathrm{Lip}(f^{-1})$ is called the *distortion* of f. In particular, for any subset $X \subset Y$, it holds that $M_2(X) \le M_2(Y)$.

(c) (Gromov-Hausdorff limits) If a sequence of (pointed) metric spaces $\{X_i\}_{i \in \mathbb{N}}$ converges to a (pointed) metric space X in the sense of the (pointed) Gromov-Hausdorff convergence, then we have $M_2(X) \le \liminf_{i \to \infty} M_2(X_i)$.

It is known that Markov type implies Enflo type up to a constant multiplication of K ([NS, Proposition 1]). Naor, Peres, Schramm and Sheffield [NPSS] provide many important examples of spaces having Markov type 2, such as 2-uniformly smooth Banach spaces, trees and complete, simply-connected Riemannian manifolds with pinched negative sectional curvature. In [Oh], the first author shows that Alexandrov spaces of nonnegative curvature have Markov type 2. The following theorem asserts that the converse is also true in a certain sense.

Theorem 2.5 *For a metric space (X,d), the following are equivalent*:

(i) *For any $N \in \mathbb{N}$, $(x_i) \in X^N$, $(\pi_i)_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ satisfying* (2.2)*, we have* $\mathcal{E}(2) \le 2\mathcal{E}(1)$.

(ii) *For any $N \in \mathbb{N}$, $(x_i) \in X^N$, $y \in X$ and $(\lambda_i) \in [0,1]^N$ with $\sum_{i=1}^N \lambda_i = 1$, we have*

$$\sum_{i,j=1}^N \lambda_i \lambda_j d(x_i,x_j)^2 \le 2\sum_{i=1}^N \lambda_i d(x_i,y)^2.$$

If (X,d) is geodesic, then (i) *and* (ii) *are also equivalent to*:

(iii) *(X,d) is an Alexandrov space of nonnegative curvature.*

In particular, if a geodesic metric space (X,d) has Markov type 2 *with $M_2(X) = 1$, then it is an Alexandrov space of nonnegative curvature.*

Proof. The equivalence between (ii) and (iii) in a geodesic metric space is shown in [St]. The implication from (ii) to (i) is due to [Oh].

Assume (i) and take $(x_i) \in X^N$, $y \in X$ and $(\lambda_i) \in [0,1]^N$ with $\sum_{i=1}^N \lambda_i = 1$. Given $\varepsilon > 0$, we consider the Markov chain on $\{0,1,\dots,N\}$ defined by

$$\begin{aligned}
&\pi_0 = \varepsilon, \quad \pi_i = (1-\varepsilon)/N \quad \text{for } 1 \le i \le N,\\
&a_{00} = 0, \quad a_{0i} = \lambda_i, \quad a_{i0} = \varepsilon\lambda_i N/(1-\varepsilon) \quad \text{for } 1 \le i \le N,\\
&a_{ij} = \begin{cases} 1 - (\varepsilon\lambda_i N)/(1-\varepsilon) & \text{for } i = j \ge 1, \\ 0 & \text{for } i,j \ge 1,\ i \ne j. \end{cases}
\end{aligned}$$

Note that this Markov chain is reversible and stationary. We regard $x_0 = y$ and calculate

$$\begin{aligned}
\mathcal{E}(1) &= \sum_{i=1}^N \varepsilon\lambda_i \{d(y,x_i)^2 + d(x_i,y)^2\} = 2\varepsilon \sum_{i=1}^N \lambda_i d(x_i,y)^2,\\
\mathcal{E}(2) &= \sum_{i=1}^N \varepsilon\lambda_i \left(1 - \frac{\varepsilon\lambda_i N}{1-\varepsilon}\right) \{d(y,x_i)^2 + d(x_i,y)^2\} + \sum_{i,j=1}^N \varepsilon\lambda_i\lambda_j d(x_i,x_j)^2\\
&= 2\varepsilon \sum_{i=1}^N \lambda_i \left(1 - \frac{\varepsilon\lambda_i N}{1-\varepsilon}\right) d(x_i,y)^2 + \varepsilon \sum_{i,j=1}^N \lambda_i\lambda_j d(x_i,x_j)^2.
\end{aligned}$$

Then we apply (i) and find that

$$\sum_{i,j=1}^N \lambda_i\lambda_j d(x_i,x_j)^2 \le 2\sum_{i=1}^N \lambda_i \left(1 + \frac{\varepsilon\lambda_i N}{1-\varepsilon}\right) d(x_i,y)^2.$$

Letting ε go to zero, we obtain (ii) and complete the proof. $\square$

Remark 2.6 (a) By comparing Proposition 2.2 and Theorem 2.5, it seems that Enflo and Markov types behave like opposite curvature bounds. Nevertheless, as we mentioned, Markov type implies Enflo type up to a constant multiplication of K. One interpretation of this phenomenon is to think of both curvature bounds as kinds of nonlinear 2-uniform smoothness (see [Oh] for more details).

(b) It is unclear if $\mathcal{E}(2) \le 2\mathcal{E}(1)$ implies $M_2 = 1$. We know only that a simple inductive technique due to Naor and Peres gives $M_2 \le 1 + \sqrt{2}$ (see [Oh]). As far as the authors know, even whether $M_2(\mathbb{S}^1) = 1$ or not is open.

Since a Banach space is a CAT(0)-space or an Alexandrov space of nonnegative curvature if and only if it is a Hilbert space, we immediately observe the following:

Corollary 2.7 *Let $(V, \|\cdot\|)$ be a Banach space. Then the following are equivalent*:

(i) $(V, \|\cdot\|)$ *has Enflo type* 2 *with* $K = 1$.

(ii) $(V, \|\cdot\|)$ *has Markov type* 2 *with* $K = 1$.

(iii) $(V, \|\cdot\|)$ *is a Hilbert space.*

Proof. We deduce from Proposition 2.2 the equivalence between (i) and (iii). The implication from (ii) to (iii) follows from Theorem 2.5.

Let $(V, \|\cdot\|)$ be a Hilbert space. Then $M_2(V) = 1$ can be seen as follows. As we treat only Markov chains on finite state spaces in Definition 2.3, it suffices to show $M_2(\mathbb{R}^n) = 1$ for all $n \in \mathbb{N}$. Moreover, $M_2(\mathbb{R}^n) = 1$ is a consequence of $M_2(\mathbb{R}) = 1$ because Markov type descends to ℓ_2-products (see Example 2.4(a)). We can prove $M_2(\mathbb{R}) = 1$ by fundamental functional analysis as in [NPSS, Section 4] (see also [Ba, Proposition 1.4]). □

3 Lower bounds for N-pods

By using the Markov chain considered in the proof of Theorem 2.5, we observe that embedded N-pods or, more generally, branching points make the Markov type 2 constant worse. It should be compared with a lower bound $M_2(\Gamma_3) \ge \sqrt{3}$ for the 3-regular tree Γ_3 obtained in [NPSS, Section 5.1] (see also [LMN, Proposition 2.2]).

Theorem 3.1 *Let (X, d) be a metric space.*

(i) *Assume that, for some $N \ge 3$ and $r > 0$, there are $\{x_i\}_{i=1}^N \subset X$ and $y \in X$ satisfying $d(y, x_i) = r$ for $1 \le i \le N$ and $d(x_i, x_j) = 2r$ for $1 \le i < j \le N$. Then we have $M_2(X) \ge \sqrt{(3N-2)/2N}$.*

(ii) *If there are $\{x_i\}_{i=1}^3 \subset X$, $y \in X$ and $r, \delta > 0$ such that $d(y, x_i) = r$ for $1 \le i \le 3$, $d(x_1, x_2) = d(x_1, x_3) = 2r$ and that $d(x_2, x_3) \ge r\delta$, then we have $M_2(X) \ge \sqrt{1 + \delta^2/32}$.*

Proof. (i) Define the Markov chain on $\{0, 1, \ldots, N\}$ just as in the proof of Theorem 2.5 with $\lambda_i = 1/N$. Then the same calculation yields that

$$\mathcal{E}(1) = \frac{2\varepsilon}{N} \sum_{i=1}^{N} d(x_i, y)^2 = 2\varepsilon r^2,$$

$$\mathcal{E}(2) = \frac{2\varepsilon}{N}\left(1 - \frac{\varepsilon}{1-\varepsilon}\right) \sum_{i=1}^{N} d(x_i, y)^2 + \frac{\varepsilon}{N^2} \sum_{i,j=1}^{N} d(x_i, x_j)^2$$
$$= \frac{2(1-2\varepsilon)}{1-\varepsilon}\varepsilon r^2 + \frac{N-1}{N}\varepsilon(2r)^2.$$

Hence we have

$$M_2(X)^2 \geq \frac{\mathcal{E}(2)}{2\mathcal{E}(1)} = \frac{1-2\varepsilon}{2(1-\varepsilon)} + \frac{N-1}{N}.$$

Letting ε tend to zero gives the required estimate.

(ii) We again consider the Markov chain as in the proof of Theorem 2.5 with $N = 3$, $\lambda_1 = 1/2$ and $\lambda_2 = \lambda_3 = 1/4$. Then we see that $\mathcal{E}(1) = 2\varepsilon r^2$ and

$$\mathcal{E}(2) = \left\{\left(1 - \frac{3\varepsilon}{2(1-\varepsilon)}\right) + \left(1 - \frac{3\varepsilon}{4(1-\varepsilon)}\right)\right\}\varepsilon r^2 + \frac{4}{8}\varepsilon(2r)^2 + \frac{2}{16}\varepsilon(r\delta)^2$$
$$= \left(4 + \frac{\delta^2}{8} - \frac{9\varepsilon}{4(1-\varepsilon)}\right)\varepsilon r^2.$$

Therefore we obtain $M_2(X)^2 \geq 1 + \delta^2/32$. □

Denote by $\mathbb{H}^n(\kappa)$ the n-dimensional hyperbolic space of constant sectional curvature $\kappa < 0$. Then we deduce from Theorem 3.1(i) that $M_2(\mathbb{H}^n(\kappa)) \geq \sqrt{3/2}$, for $\mathbb{H}^n(\kappa)$ contains an arbitrarily good bi-Lipschitz copy of an N-pod in terms of the distortion (see Example 2.4(b)).

4 Gluing constructions

We finally discuss how to construct spaces with Markov type by gluing.

Proposition 4.1 *Let $\{(X_m, d_m)\}_{m=0}^{M}$ be a family of metric spaces with* $\max_m M_2(X_m) \leq K$ *for some $K \geq 1$. Fix $(x'_m) \in (X_0)^M$ and $x_m \in X_m$ for $1 \leq m \leq M$, and consider the space $Y = (\bigsqcup_{m=0}^{M} X_m)/\sim$, where $x'_m \sim x_m$ for $1 \leq m \leq M$. Define the distance d_Y on Y by*

$$d_Y := d_m \quad \text{on } X_m,\ 0 \leq m \leq M,$$
$$d_Y(y, z) := d_0(y, x'_m) + d_m(x_m, z) \quad \text{for } y \in X_0,\ z \in X_m,\ m \geq 1,$$
$$d_Y(y, z) := d_m(y, x_m) + d_0(x'_m, x'_n) + d_n(x_n, z)$$
$$\text{for } y \in X_m,\ z \in X_n,\ m, n \geq 1,\ m \neq n.$$

Then (Y, d_Y) has Markov type 2 with $M_2(Y) \leq \sqrt{3}K$.

Proof. Take $N \in \mathbb{N}$, $(\pi_i)_{i=1}^N$ and $A = (a_{ij})_{i,j=1}^N$ as in (2.2), and choose $(y_i) \in Y^N$ with $y_i \in X_m$ for $i_{m-1}+1 \le i \le i_m$ (where we put $i_{-1} = 0$). Then we find, by the definition of d_Y,

$$\begin{aligned}
\mathcal{E}(l) &= \sum_{i,j=1}^{N} \pi_i a_{ij}^{(l)} d_Y(y_i, y_j)^2 \\
&= \sum_{m=0}^{M} \sum_{i,j=i_{m-1}+1}^{i_m} \pi_i a_{ij}^{(l)} d_m(y_i, y_j)^2 \\
&\quad + 2 \sum_{i=1}^{i_0} \sum_{m=1}^{M} \sum_{j=i_{m-1}+1}^{i_m} \pi_i a_{ij}^{(l)} \big\{ d_0(y_i, x'_m) + d_m(x_m, y_j) \big\}^2 \\
&\quad + \sum_{m=1}^{M} \sum_{i=i_{m-1}+1}^{i_m} \sum_{n \neq 0,m} \sum_{j=i_{n-1}+1}^{i_n} \pi_i a_{ij}^{(l)} \big\{ d_m(y_i, x_m) + d_0(x'_m, x'_n) + d_n(x_n, y_j) \big\}^2 \\
&\le 3 \bigg[\sum_{i,j=1}^{i_0} \pi_i a_{ij}^{(l)} d_0(y_i, y_j)^2 + 2 \sum_{i=1}^{i_0} \sum_{m=1}^{M} \sum_{j=i_{m-1}+1}^{i_m} \pi_i a_{ij}^{(l)} d_0(y_i, x'_m)^2 \\
&\quad + \sum_{m=1}^{M} \sum_{i=i_{m-1}+1}^{i_m} \sum_{n=1}^{M} \sum_{j=i_{n-1}+1}^{i_n} \pi_i a_{ij}^{(l)} d_0(x'_m, x'_n)^2 \bigg] \\
&\quad + 3 \sum_{m=1}^{M} \bigg[\sum_{i,j=i_{m-1}+1}^{i_m} \pi_i a_{ij}^{(l)} d_m(y_i, y_j)^2 + 2 \sum_{i=i_{m-1}+1}^{i_m} \sum_{n \neq m} \sum_{j=i_{n-1}+1}^{i_n} \pi_i a_{ij}^{(l)} d_m(y_i, x_m)^2 \bigg].
\end{aligned}$$

We denote the right-hand side of the above inequality by $3F(l)$ and deduce from our hypothesis $\max_m M_2(X_m) \le K$ that $F(l) \le K^2 l F(1)$. Therefore we have

$$\mathcal{E}(l) \le 3F(l) \le 3K^2 l F(1) \le 3K^2 l \mathcal{E}(1).$$

□

We remark that d_Y is geodesic if every d_m is geodesic. It is also possible to show the proposition by regarding $(X_0 \sqcup X_1)/\sim$ as a subset of the ℓ_1-product of X_0 and X_1 and repeating this procedure. If we use the ℓ_2-product instead, then the resulting metric space (Y, d'_Y) satisfies $M_2(Y, d'_Y) \le K$. However, (Y, d'_Y) is not geodesic even if every d_m is geodesic.

In Proposition 4.1, $M_2(Y, d_Y)$ can be larger than $\max_m M_2(X_m)$ in accordance with Theorem 3.1. For instance, $(\mathbb{R}^{n+1}, d)$ with

$$d\big((x,y),(x',y')\big) := \begin{cases} |y - y'| & \text{if } x = x' \ (\in \mathbb{R}^n), \\ |y| + \|x - x'\| + |y'| & \text{if } x \neq x' \end{cases}$$

(where $\|\cdot\|$ stands for the standard Euclidean norm of $\mathbb{R}^n$) satisfies $M_2(\mathbb{R}^{n+1}, d) \le \sqrt{3}$ by Proposition 4.1 with $K = 1$, and $M_2(\mathbb{R}^{n+1}, d) \ge \sqrt{5/4}$ by Theorem 3.1 with $N = 4$.

References

[Ba] K. Ball, *Markov chains, Riesz transforms and Lipschitz maps*, Geom. Funct. Anal. **2** (1992), 137–172.

[BLMN] Y. Bartal, N. Linial, M. Mendel and A. Naor, *On metric Ramsey-type phenomena*, Ann. of Math. (2) **162** (2005), 643–709.

[BL] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis. Vol. 1, American Mathematical Society, Providence, RI, 2000.

[BN1] I. D. Berg and I. G. Nikolaev, *On a distance characterization of A. D. Aleksandrov spaces of nonpositive curvature*, Dokl. Math. **75** (2007), 336–338 (translated from Russian original in Dokl. Akad. Nauk. **414** (2007), 10–12).

[BN2] I. D. Berg and I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata **133** (2008), 195–218.

[BH] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, 1999.

[BBI] D. Burago, Yu. Burago and S. Ivanov, A course in metric geometry, American Mathematical Society, Providence, RI, 2001.

[En] P. Enflo, *On infinite-dimensional topological groups*, Séminaire sur la Géométrie des Espaces de Banach (1977–1978), Exp. No. 10–11, 11 pp., École Polytech. Palaiseau, 1978.

[FLS] T. Foertsch, A. Lytchak and V. Schroeder, *Nonpositive curvature and the Ptolemy inequality*, Int. Math. Res. Not. 2007, Art. ID rnm100, 15pp.

[LMN] N. Linial, A. Magen and A. Naor, *Girth and Euclidean distortion*, Geom. Funct. Anal. **12** (2002), 380–394.

[MN1] M. Mendel and A. Naor, *Some applications of Ball's extension theorem*, Proc. Amer. Math. Soc. **134** (2006), 2577–2584.

[MN2] M. Mendel and A. Naor, *Metric cotype*, Ann. of Math. **168** (2008), 247–298.

[NPSS] A. Naor, Y. Peres, O. Schramm and S. Sheffield, *Markov chains in smooth Banach spaces and Gromov hyperbolic metric spaces*, Duke Math. J. **134** (2006), 165–197.

[NS] A. Naor and G. Schechtman, *Remarks on non linear type and Pisier's inequality*, J. Reine Angew. Math. **552** (2002), 213–236.

[Oh] S. Ohta, *Markov type of Alexandrov spaces of nonnegative curvature*, to appear in Mathematika.

[Sa] T. Sato, *An alternative proof of Berg and Nikolaev's characterization of* CAT(0)*-spaces via quadrilateral inequality*, preprint (2008).

[St] K.-T. Sturm, *Metric spaces of lower bounded curvature*, Exposition. Math. **17** (1999), 35–47.