Products, cones, and suspensions of spaces with the measure contraction property*†

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Abstract

This article concerns several geometric properties of metric measure spaces satisfying the measure contraction property (MCP) which can be considered as a generalized notion of lower Ricci curvature bounds. We prove that the MCP of spaces descends to their products and Euclidean cones. We also show that a positively curved space in terms of the MCP with a maximal diameter can be represented as the spherical suspension of some topological measure space.

1 Introduction

The measure contraction property (MCP for short) of metric measure spaces has been introduced independently by the author [O] and Sturm [S2] as a candidate of a generalized notion of lower Ricci curvature bounds in Riemannian geometry. (Related properties had been proposed in [CC] and [G].) Metric spaces with lower or upper 'sectional' curvature bounds had been already formulated (Alexandrov spaces and CAT-spaces) and deeply studied from various viewpoints. As for the sectional curvature, since it is a quantity defined for each two-dimensional subspaces, we need only to treat geodesic triangles. In particular, we do not care for the dimension of the entire space. However, in the case of the Ricci curvature, the dimension plays an essential role (consider, for example, the Bishop-Gromov volume comparison theorem), so that we need two parameters, $K, N \in \mathbb{R}$ with $N \geq 1$, corresponding to the Ricci curvature and the dimension, respectively. More precisely, as spaces with a uniform lower Ricci curvature bound can collapse to a lower dimensional space with respect to the measured Gromov-Hausdorff convergence, the parameter N plays a role of an upper bound of the dimension. Namely, for $K, N \in \mathbb{R}$ with $N \geq 1$, the (K, N)-MCP intuitively means that 'Ricci curvature $\geq K$ and dimension $\leq N$'.

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On one hand, the MCP is a natural generalization of the Bishop inequality, on the other hand, it is a relaxed version of the Curvature-Dimension condition which is also introduced in [S2] (see also [LV1] and [LV2]). We refer to [S1] and [LV1] for the infinite-dimensional Curvature-Dimension condition (i.e., $N = \infty$) and [vR] and [LV2] for related works concerning the Poincaré inequality. It is known that the generalizations of the Bishop-Gromov volume comparison theorem as well as the Bonnet-Myers theorem hold true under the MCP, and that the MCP is preserved under the measured Gromov-Hausdorff convergence (see [O] and [S2]).

In this article, we study some fundamental geometric properties of the MCP. We will prove that the product of metric measure spaces (X_i, μ_i), $i = 1, 2$, satisfying the (K_i, N_i)-MCP, respectively, satisfies the $(\min\{K_1, K_2\}, N_1 + N_2)$-MCP (Proposition 3.3), and that the Euclidean cone of a metric measure space satisfying the $(N-1, N)$-MCP satisfies the $(0, N+1)$-MCP (Theorem 4.2). There the reduction lemma (Lemma 2.6) and the perturvation lemma (Lemma 4.1) play key roles. We also consider a metric measure space satisfying the (K, N)-MCP for $K > 0$ and $N > 1$ and attaining the maximal diameter $\pi\sqrt{(N-1)/K}$ derived from the Bonnet-Myers theorem, and show that such a space is represented as the spherical suspension of some topological measure space (Theorem 5.6).

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2 The measure contraction property

A metric space (X, d_X) is called a *length space* if it satisfies $d_X(x, y) = \inf_\gamma \ell(\gamma)$ for all $x, y \in X$, where $\ell(\gamma)$ denotes the length of γ and the infimum is taken over all rectifiable curves γ from x to y. If, for every $x, y \in X$, there exists a curve γ which satisfies $d_X(x, y) = \ell(\gamma)$, then we say that (X, d_X) is *geodesic*. Note that, if a length space is complete and locally compact, then it is geodesic. A rectifiable curve γ in a metric space (X, d_X) is called a *geodesic* if it is locally minimizing and has a constant speed. A geodesic $\gamma : [0, l] \longrightarrow X$ is said to be *minimal* if it satisfies $\ell(\gamma) = d_X(\gamma(0), \gamma(l))$. By taking a reparametrization of a curve which attains the distance, every two points in a geodesic metric space are joined by a (not necessarily unique) minimal geodesic.

Throughout this article, without otherwise indicated, let (X, d_X) be a complete length space, and let μ be a Borel measure on X such that $0 < \mu(B(x, r)) < \infty$ holds for every $x \in X$ and $r > 0$, where $B(x, r)$ denotes the open ball with center $x \in X$ and radius $r > 0$. The closed ball with center $x \in X$ and radius $r > 0$ is denoted by $\overline{B}(x, r)$. Henceforce, we denote $d_X(x, y)$ by $|x - y|_X$ for $x, y \in X$, and write simply X instead of (X, d_X).

As in [LV1], let $\Gamma = \Gamma(X)$ be the set of minimal geodesics, say $\gamma : [0, 1] \longrightarrow X$, in X and define the evaluation map $e_t : \Gamma \longrightarrow X$ by $e_t(\gamma) := \gamma(t)$ for each $t \in [0, 1]$. We regard Γ as a subset of the set of Lipschitz maps $\mathrm{Lip}([0, 1], X)$ with the uniform topology. A *dynamical transference plan* Π is a Borel probability measure on Γ. For $K \in \mathbb{R}$, we define the function $\mathbf{s}_K$ on $[0,\infty)$ (on $[0,\pi/\sqrt{K})$ if $K>0$) by

$$\mathbf{s}_K(t) := \begin{cases} (1/\sqrt{K})\sin(\sqrt{K}t) & \text{if } K>0, \\ t & \text{if } K=0, \\ (1/\sqrt{-K})\sinh(\sqrt{-K}t) & \text{if } K<0. \end{cases}$$

We also define, as in [S2],

$$\varsigma_{K,N}^{(t)}(d) := t\left\{\frac{\mathbf{s}_K(td/\sqrt{N-1})}{\mathbf{s}_K(d/\sqrt{N-1})}\right\}^{N-1}$$

if $N>1$, and $\varsigma_{K,1}^{(t)} := t$ if $K \le 0$ and $N=1$.

Definition 2.1 For $K,N \in \mathbb{R}$ with $N>1$, or with $K\le 0$ and $N=1$, a metric measure space (X,μ) is said to satisfy the (K,N)-*measure contraction property* (the (K,N)-*MCP* for short) if, for every point $x\in X$ and measurable set $A\subset X$ (provided that $A\subset B(x,\pi\sqrt{(N-1)/K})$ if $K>0$) with $0<\mu(A)<\infty$, there exists a dynamical transference plan $\Pi=\Pi_{x,A}$ satisfying the following:

(1) We have $(e_0)_*\Pi = \delta_x$ and $(e_1)_*\Pi = \mu(A)^{-1}\cdot\mu|_A$ as measures;

(2) For every $t\in[0,1]$,
$$d\mu \ge (e_t)_*\big[\varsigma_{K,N}^{(t)}\big(\ell(\gamma)\big)\mu(A)\,d\Pi(\gamma)\big] \tag{2.1}$$
holds as measures on X.

Geometrically, the inequality (2.1) is a natural generalization of the classical Bishop inequality ([BC, §11.10, Corollary 3]) under a lower Ricci curvature bound. Actually, an n-dimensional, complete Riemannian manifold M with $n\ge 2$ satisfies the (K,n)-MCP if and only if $\mathrm{Ric}_M \ge K$ holds ([O]). Furthermore, an n-dimensional, complete Alexandrov space with curvature $\ge K$ equipped with the n-dimensional Hausdorff measure satisfies the $((n-1)K,n)$-MCP ([O]).

Remark 2.2 We mention that the (K,N)-MCP has an essentially similar form as that defined by Sturm in [S2]. Only a difference is the *symmetricity* which is crucial in a construction of a Dirichlet form. Moreover, the (K,N)-MCP can be regarded as a weak version of the *Curvature-Dimension condition* also introduced in [S2]. Even in the Riemannian case, it is known that there is a gap between them. Typically, for an n-dimensional, complete Riemannian manifold M with $n\ge 2$ and $N\in\mathbb{R}$, the condition '$\mathrm{Ric}_M \ge K$ and $n\le N$' implies the (K,N)-MCP, but the converse is not true in general, while that condition is equivalent to the Curvature-Dimension condition $\mathrm{CD}(K,N)$.

We recall some results on the (K,N)-MCP obtained in [O].

Lemma 2.3 (i) *The (K,N)-MCP of (X,μ) implies the (K',N')-MCP for all $K'\le K$ and $N'\ge N$.*

(ii) *If (X,d_X,μ) satisfies the (K,N)-MCP and if $a,b>0$, then the scaled metric measure space $(X,a\cdot d_X,b\cdot\mu)$ satisfies the $(K/a^2,N)$-MCP.*

Next two theorems generalize the Bishop-Gromov volume comparison theorem and the Bonnet-Myers theorem. For $x \in X$ and $r > 0$, we define $S(x,r) := \{y \in X \mid |x-y|_X = r\}$.

Theorem 2.4 ([O]) *Let (X,μ) be a metric measure space satisfying the (K,N)-MCP. Then, for any $x \in X$, the function*

$$\mu\big(B(x,r)\big) \Big/ \bigg\{ \int_0^r \mathbf{s}_K\bigg(\frac{s}{\sqrt{N-1}}\bigg)^{N-1} ds \bigg\}$$

is monotone non-increasing in $r \in (0,\infty)$ ($r \in (0, \pi\sqrt{(N-1)/K})$ if $K > 0$). In particular, the Hausdorff dimension of X is less than or equal to N.

Theorem 2.5 ([O]) *If a metric measure space (X,μ) satisfies the (K,N)-MCP for some $K > 0$ and $N > 1$, then we have $\operatorname{diam} X \le \pi\sqrt{(N-1)/K}$. Moreover, for any $x \in X$, the set $S(x, \pi\sqrt{(N-1)/K})$ consists of at most one point.*

As a corollary to Theorem 2.4, the (K,N)-MCP implies the (local) doubling condition. Namely, for any $R > 0$, $r \in (0,R]$, and $x \in X$, we have

$$\frac{\mu(B(x,r))}{\mu(B(x,r/2))} \le C_{K,N,R},$$

where $C_{K,N,R} < \infty$ is a constant depending only on K, N, and R. The doubling condition implies that every bounded closed ball in X is totally bounded. Therefore, if X is complete, then it is proper (i.e., all bounded closed sets are compact) and hence geodesic.

We end this section with a useful reduction lemma. For $x \in X$ and $0 < r < R < \infty$, we set $\operatorname{rad}(x) := \sup_{y \in X} |x-y|_X$ and $A(x;r,R) := B(x,R) \setminus B(x,r)$.

Lemma 2.6 *Let (X,μ) be a metric measure space. Assume that, for any $x \in X$, there exist a monotone increasing sequence $\{r_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} r_i = \operatorname{rad}(x)$, a sequence of measurable sets $\{W_i\}_{i=1}^{\infty}$ with $A(x;r_{i-1},r_i) \subset W_i$, and a dynamical transference plan $\Pi_i = \Pi_{x,W_i}$ for which Definition 2.1(1),(2) hold. Here we put $r_0 := 0$ and $A(x;r_0,r_1) := B(x,r_1)$. Then (X,μ) satisfies the (K,N)-MCP.*

Proof. Fix a point $x \in X$ and a measurable set $A \subset X$ with $0 < \mu(A) < \infty$. For a sequence $\{r_i\}_{i=1}^{\infty}$ as in the hypothesis, we define $A_i := A \cap A(x;r_{i-1},r_i)$. For each $i \in \mathbb{N}$, define $\Pi'_i := \{\mu(W_i)/\mu(A)\}\Pi_i|_{\{e_1 \in A_i\}}$ and put $\Pi_{x,A} := \sum_{i=1}^{\infty} \Pi'_i$. Note that, if $i \ne j$, then

$$\mu\big(\operatorname{supp}[(e_t)_*\Pi_i] \cap \operatorname{supp}[(e_t)_*\Pi_j]\big) \le \mu\big(A(x;tr_{i-1},tr_i) \cap A(x;tr_{j-1},tr_j)\big) = 0$$

for all $t \in [0,1]$. Thus $\Pi_{x,A}$ satisfies Definition 2.1(1), (2). □

The lemma above helps us to reduce a class of sets in considering the MCP to a smaller one, such as product sets in a product space (see Section 3). We do not know whether this kind of reduction also works for the Curvature-Dimension condition or not.

3 Products

In this section, we consider the product of metric measure spaces satisfying the MCP. By virtue of Lemma 2.6, our discussion is based only on calculations. Compare this with [S1, Proposition 4.16]. We first prove a lemma which is deduced from the fact that, for a Riemannian manifold M, $\mathrm{Ric}_M \ge K$ implies $\mathrm{Ric}_{M\times\mathbb{R}^k}(\xi,\xi) \ge K$ for unit vectors $\xi \in SM \subset S(M\times\mathbb{R}^k)$.

Lemma 3.1 *For $t \in (0,1)$ and $d > 0$ ($d \in (0, \pi\sqrt{(N-1)/K})$ if $K > 0$), the function*

$$N \longmapsto t^{-N}\varsigma_{K,N}^{(t)}(d) = \left\{\frac{\mathbf{s}_K(td/\sqrt{N-1})}{t\mathbf{s}_K(d/\sqrt{N-1})}\right\}^{N-1}$$

is monotone non-increasing in $N \in (1,\infty)$.

Proof. Given $1 < N < M$, we need to prove that

$$\frac{\{t^{-1}\mathbf{s}_K(td/\sqrt{N-1})\}^{N-1}}{\{t^{-1}\mathbf{s}_K(td/\sqrt{M-1})\}^{M-1}} \ge \frac{\mathbf{s}_K(d/\sqrt{N-1})^{N-1}}{\mathbf{s}_K(d/\sqrt{M-1})^{M-1}}. \tag{3.1}$$

If $K = 0$, then it is clear, so that we may assume $K = 1$ or -1. As the proofs are common, we put $K = 1$ in the following.

Denote by $f(t)$ the left hand side of (3.1). Then we have

$$\begin{aligned} f'(t) &= t^{M-N-1}\frac{\sin(td/\sqrt{N-1})^{N-1}}{\sin(td/\sqrt{M-1})^{M-1}}\left[td\left\{\sqrt{N-1}\cot\left(\frac{td}{\sqrt{N-1}}\right)\right.\right. \\ &\qquad \left.\left. -\sqrt{M-1}\cot\left(\frac{td}{\sqrt{M-1}}\right)\right\} + (M-N)\right] \\ &= t^{M-N+1}d^2\frac{\sin(td/\sqrt{N-1})^{N-1}}{\sin(td/\sqrt{M-1})^{M-1}}\left[\left\{\frac{\sqrt{N-1}}{td}\cot\left(\frac{td}{\sqrt{N-1}}\right) - \frac{N-1}{t^2d^2}\right\}\right. \\ &\qquad \left. -\left\{\frac{\sqrt{M-1}}{td}\cot\left(\frac{td}{\sqrt{M-1}}\right) - \frac{M-1}{t^2d^2}\right\}\right]. \end{aligned}$$

Thus the following claim (with $s = td/\sqrt{N-1}$) implies $f'(t) \le 0$ and completes the proof.

Claim 3.2 *For $s \in (0,\pi)$, put $g(s) := s^{-1}\cot s - s^{-2}$. Then we have $g'(s) \le 0$.*

To prove the claim, we calculate

$$\begin{aligned} g'(s) &= -\frac{1}{s^2}\cot s - \frac{1}{s}\frac{1}{\sin^2 s} + \frac{2}{s^3} \\ &= \frac{1}{s^3\sin^2 s}(-s\cos s\sin s - s^2 + 2\sin^2 s) =: \frac{g_1(s)}{s^3\sin^2 s}, \\ g_1'(s) &= 3\cos s\sin s + 2s\sin^2 s - 3s, \\ g_1''(s) &= 4\sin s(s\cos s - \sin s) \le 0. \end{aligned}$$

Thus we obtain $g'(s) \le 0$. □

Proposition 3.3 *If (X_i, μ_i) satisfies the (K_i, N_i)-MCP for $i = 1, 2$, then their product $(X_1 \times X_2, \mu_1 \times \mu_2)$ satisfies the $(\min\{K_1, K_2\}, N_1 + N_2)$-MCP.*

Proof. By Lemma 2.3(i), we may assume $K_1 = K_2$. For simplicity, put $X := X_1 \times X_2$, $\mu := \mu_1 \times \mu_2$, $K := K_1 = K_2$, and $N := N_1 + N_2$. Fix a point $x = (x_1, x_2) \in X$ and measurable sets $A_i \subset X_i$ with $0 < \mu_i(A_i) < \infty$ for $i = 1, 2$, and set $A := A_1 \times A_2 \subset X$. By the hypothesis, for each $i = 1, 2$, there exists a dynamical transference plan $\Pi_i = \Pi_{x_i, A_i}$ for which Definition 2.1(1), (2) hold. We shall show that $\Pi := \Pi_1 \times \Pi_2$ gives a required dynamical transference plan between x and A.

We first observe that

$$\begin{aligned}(e_0)_*\Pi &= \big((e_0)_*\Pi_1\big) \times \big((e_0)_*\Pi_2\big) = \delta_{x_1} \times \delta_{x_2} = \delta_x,\\ (e_1)_*\Pi &= \big((e_1)_*\Pi_1\big) \times \big((e_1)_*\Pi_2\big)\\ &= (\mu_1(A_1)^{-1} \cdot \mu_1|_{A_1}) \times (\mu_2(A_2)^{-1} \cdot \mu_2|_{A_2}) = \mu(A)^{-1} \cdot \mu|_A.\end{aligned}$$

We admit the following claim and will prove it afterward.

Claim 3.4 *For any $d_i > 0$ ($d_i \in (0, \pi\sqrt{(N_i - 1)/K})$ if $K > 0$) and $t \in [0, 1]$, we have*

$$\varsigma_{K,N_1}^{(t)}(d_1) \cdot \varsigma_{K,N_2}^{(t)}(d_2) \ge \varsigma_{K,N}^{(t)}(d),$$

where we set $d := \sqrt{d_1^2 + d_2^2}$.

We remark that, if $K > 0$, then

$$d^2 = d_1^2 + d_2^2 < \pi^2 \frac{N_1 + N_2 - 2}{K} < \pi^2 \frac{N - 1}{K}.$$

By Claim 3.4, we immediately obtain, for any $t \in (0, 1)$,

$$\begin{aligned}d\mu &= d\mu_1 \times d\mu_2\\ &\ge \big[(e_t)_*\{\varsigma_{K,N_1}^{(t)}\big(\ell(\gamma_1)\big)\mu_1(A_1)\, d\Pi_1(\gamma_1)\}\big] \times \big[(e_t)_*\{\varsigma_{K,N_2}^{(t)}\big(\ell(\gamma_2)\big)\mu_2(A_2)\, d\Pi_2(\gamma_2)\}\big]\\ &= (e_t)_*\{\varsigma_{K,N_1}^{(t)}\big(\ell(p_1 \circ \gamma)\big)\varsigma_{K,N_2}^{(t)}\big(\ell(p_2 \circ \gamma)\big)\mu(A)\, d\Pi(\gamma)\}\\ &\ge (e_t)_*\{\varsigma_{K,N}^{(t)}\big(\ell(\gamma)\big)\mu(A)\, d\Pi(\gamma)\}.\end{aligned}$$

Here we denote by $p_i : X \longrightarrow X_i$, $i = 1, 2$, the projection map. By Lemma 2.6, this completes the proof. □

Now we give a proof of Claim 3.4. As the claim clearly holds if $K = 0$, we need to treat the cases of $K = 1$ or -1. Since their proofs are completely common, we consider only the case of $K = 1$.

We first suppose that $d_1/\sqrt{N_1 - 1} = d_2/\sqrt{N_2 - 1}$. Then we observe

$$\frac{d^2}{N-1} = \frac{d_1^2 + d_2^2}{N-1} = \frac{1}{N-1}\left(1 + \frac{N_2 - 1}{N_1 - 1}\right) d_1^2 = \frac{1}{N-1}\frac{N-2}{N_1 - 1} d_1^2 < \frac{d_1^2}{N_1 - 1}.$$

Thus Lemma 3.1 yields that, by putting $N_1' := (N-1)d_1^2 d^{-2} + 1 > N_1$,

$$\frac{d_1^2}{N_1'-1} = \frac{d^2}{N-1},$$
$$t^{-N_1}\varsigma_{K,N_1}^{(t)}(d_1) \geq t^{-N_1'}\varsigma_{K,N_1'}^{(t)}(d_1) = t^{-N_1'+1}\{t^{-1}\varsigma_{K,N}^{(t)}(d)\}^{(N_1'-1)/(N-1)}.$$

Similarly, we have $\varsigma_{K,N_2}^{(t)}(d_2) \geq t^{N_2-N_2'+1}\{t^{-1}\varsigma_{K,N}^{(t)}(d)\}^{(N_2'-1)/(N-1)}$, where we set $N_2' := (N-1)d_2^2 d^{-2} + 1$. Note that $N_1' + N_2' = N+1$. By combining these, we obtain

$$\varsigma_{K,N_1}^{(t)}(d_1)\varsigma_{K,N_2}^{(t)}(d_2) \geq t^{N-(N_1'+N_2')+1}\varsigma_{K,N}^{(t)}(d) = \varsigma_{K,N}^{(t)}(d).$$

We next assume $d_1/\sqrt{N_1-1} < d_2/\sqrt{N_2-1}$. Fix d and consider $d_2 = \sqrt{d^2-d_1^2}$ as a function of d_1. We set

$$\begin{aligned} F(d_1) &:= \log[\varsigma_{K,N_1}^{(t)}(d_1)\varsigma_{K,N_2}^{(t)}(d_2)] \\ &= 2\log t + (N_1-1)\left\{\log\left[\sin\left(\frac{td_1}{\sqrt{N_1-1}}\right)\right] - \log\left[\sin\left(\frac{d_1}{\sqrt{N_1-1}}\right)\right]\right\} \\ &\quad + (N_2-1)\left\{\log\left[\sin\left(\frac{td_2}{\sqrt{N_2-1}}\right)\right] - \log\left[\sin\left(\frac{d_2}{\sqrt{N_2-1}}\right)\right]\right\}. \end{aligned}$$

The discussion in the previous paragraph guarantees that the following claim implies the Claim 3.4.

Claim 3.5 *If $d_1/\sqrt{N_1-1} < d_2/\sqrt{N_2-1}$, then we have $F'(d_1) \leq 0$.*

We calculate

$$\begin{aligned} F'(d_1) = td_1&\left\{\frac{\sqrt{N_1-1}}{d_1}\cot\left(\frac{td_1}{\sqrt{N_1-1}}\right) - \frac{\sqrt{N_2-1}}{d_2}\cot\left(\frac{td_2}{\sqrt{N_2-1}}\right)\right\} \\ &- d_1\left\{\frac{\sqrt{N_1-1}}{d_1}\cot\left(\frac{d_1}{\sqrt{N_1-1}}\right) - \frac{\sqrt{N_2-1}}{d_2}\cot\left(\frac{d_2}{\sqrt{N_2-1}}\right)\right\}. \end{aligned}$$

Note that, by putting $\theta := d_1/\sqrt{N_1-1}$ and $\eta := d_2/\sqrt{N_2-1}$, it suffices to see the following to prove $F'(d_1) \leq 0$.

Claim 3.6 *For $0 < \theta < \eta < \pi$, the function*

$$f(t) := t\{\theta^{-1}\cot(t\theta) - \eta^{-1}\cot(t\eta)\}$$

is monotone non-decreasing in $t \in (0,1)$.

We calculate

$$f'(t) = t\left\{\frac{\cos(t\theta)\sin(t\theta) - t\theta}{t\theta\sin^2(t\theta)} - \frac{\cos(t\eta)\sin(t\eta) - t\eta}{t\eta\sin^2(t\eta)}\right\}.$$

To show $f'(t) \geq 0$, it is sufficient to show the following.

Claim 3.7 *The function*

$$f_1(\theta) := \frac{\cos\theta\sin\theta - \theta}{\theta\sin^2\theta}$$

is monotone non-increasing in $\theta \in (0, \pi)$.

We can prove the claim above by calculations as follows.

$$f_1'(\theta) = \frac{1}{\theta^2\sin^3\theta}(-\cos\theta\sin^2\theta - \theta\sin\theta + 2\theta^2\cos\theta) =: \frac{f_2(\theta)}{\theta^2\sin^3\theta}.$$

If $\theta \in (\pi/2, \pi)$, then

$$f_2(\theta) \le -\cos\theta\sin^2\theta + \theta^2\cos\theta = \cos\theta(\theta^2 - \sin^2\theta) \le 0.$$

For $\theta \in (0, \pi/2]$, we have

$$\begin{aligned} f_2'(\theta) &= -3\cos^2\theta\sin\theta + 3\theta\cos\theta - 2\theta^2\sin\theta, \\ f_2''(\theta) &= 9\cos\theta\sin^2\theta - 7\theta\sin\theta - 2\theta^2\cos\theta \\ &= 7\sin\theta(\cos\theta\sin\theta - \theta) + 2\cos\theta(\sin^2\theta - \theta^2) \le 0. \end{aligned}$$

This completes the proof of Claim 3.4, and hence that of Proposition 3.3.

4 Euclidean cones

Our object in this section is the Euclidean cone. For a metric measure space (X, μ) and $N \in [1, \infty)$, its N-*Euclidean cone* (Y, ν) is a metric measure space defined as follows:

(i) $Y := X \times [0, \infty)/X \times \{0\}$;

(ii) For $(x, s), (x', t) \in X \times [0, \infty)$,

$$|(x, s) - (x', t)|_Y := \left[s^2 + t^2 - 2st\cos(\min\{|x - x'|_X, \pi\})\right]^{1/2};$$

(iii) $d\nu := d\mu \times r^N dr$.

We first recall a lemma in [O] concerning what happens when we perturb $t \in [0, 1]$ in (2.1).

Lemma 4.1 ([O]) *Let* (X, μ) *satisfy the* (K, N)*-MCP and, for* $0 \le r < r' \le \infty$*, let* $\tau : (r, r') \longrightarrow (0, 1]$ *be a* C^1*-function satisfying* $\tau'(l)l + \tau(l) > 0$ *for all* $l \in (r, r')$*. Then we have, for any point* $x \in X$*, any measurable set* $A \subset A(x; r, r')$ *with* $0 < \mu(A) < \infty$*, and for* $\Pi = \Pi_{x,A}$ *as in Definition* 2.1*,*

$$d\mu \ge (e_\tau)_* \left[\frac{\tau'(\ell(\gamma))\ell(\gamma) + \tau(\ell(\gamma))}{\tau(\ell(\gamma))} \varsigma_{K,N}^{(\tau(\ell(\gamma)))}\big(\ell(\gamma)\big)\mu(A)\, d\Pi(\gamma)\right] \tag{4.1}$$

as measures. Here $e_\tau : \Gamma \longrightarrow X$ *denotes a map defined by* $e_\tau(\gamma) := e_{\tau(\ell(\gamma))}(\gamma)$*. We similarly have*

$$d\mu \ge (e_\tau)_* \left[\frac{\tau'(\ell(\gamma))\ell(\gamma) + \tau(\ell(\gamma))}{\tau(\ell(\gamma))} \varsigma_{K,N}^{(\tau(\ell(\gamma)))}\big(\ell(\gamma)\big)\mu(A)\, d\Pi(\gamma)\right]$$

if τ *satisfies* $\tau'(l)l + \tau(l) < 0$ *for all* $l \in (r, r')$*.*

Theorem 4.2 *If a metric measure space (X,μ) satisfies the $(N-1,N)$-MCP, then its N-Euclidean cone (Y,ν) satisfies the $(0,N+1)$-MCP.*

Proof. Take a point $y_0=(x_0,r_0)\in Y$, a measurable set $A\subset X$ with $0<\mu(A)<\infty$, positive numbers $a,b>0$ with $a<b$, and put $W:=A\times(a,b)\subset Y$. For a minimal geodesic $\gamma:[0,1]\longrightarrow X$ with $\gamma(0)=x_0$ and $r>0$, we define a minimal geodesic $\Phi(\gamma,r)$ in Y by

$$\Phi(\gamma,r):[0,1]\ni t\longmapsto \big(\gamma(s(t,\ell(\gamma),r)),\rho(t,\ell(\gamma),r)\big)\in Y,$$

where $s\in[0,1]$ and $\rho\in[0,\infty)$ are chosen so as to satisfy

$$|y_0-\Phi(\gamma,r)(t)|_Y=t|y_0-(\gamma(1),r)|_Y,$$

that is,

$$\begin{aligned}\rho^2&=(1-t)r_0^2+tr^2-(1-t)t\{r_0^2+r^2-2r_0r\cos\ell(\gamma)\}\\&=(1-t)^2r_0^2+t^2r^2+2(1-t)tr_0r\cos\ell(\gamma),\end{aligned}\tag{4.2}$$

$$\begin{aligned}\cos\big(s\ell(\gamma)\big)&=\frac{1}{2r_0\rho}\big\{r_0^2+\rho^2-t^2\big(r_0^2+r^2-2r_0r\cos\ell(\gamma)\big)\big\}\\&=\frac{1}{2r_0\rho}\{2(1-t)r_0^2+2tr_0r\cos\ell(\gamma)\}\\&=\frac{1}{\rho}\{(1-t)r_0+tr\cos\ell(\gamma)\},\end{aligned}\tag{4.3}$$

$$\sin\big(s\ell(\gamma)\big)=\big\{1-\cos^2\big(s\ell(\gamma)\big)\big\}^{1/2}=\frac{tr}{\rho}\sin\ell(\gamma).\tag{4.4}$$

Thus we obtain a map

$$\Phi:\{\gamma\in\Gamma(X)\,|\,\gamma(0)=x_0\}\times[0,\infty)\longrightarrow\{\gamma\in\Gamma(Y)\,|\,\gamma(0)=y_0\}.$$

We push-forward a dynamical transference plan $\Pi=\Pi_{x,A}$ as in Definition 2.1 by using Φ as follows:

$$d\Xi:=\left\{\int_a^b r^N\,dr\right\}^{-1}\Phi_*\big(r^N\,d\Pi dr|_{(a,b)}\big).$$

Note that Ξ is a probability measure on $\Gamma(Y)$ and that

$$\begin{aligned}(e_0)_*\Xi&=\delta_{y_0},\\(e_1)_*(d\Xi)&=\left\{\mu(A)\int_a^b r^N\,dr\right\}^{-1}d\mu|_A\times r^N dr|_{(a,b)}=\nu(W)^{-1}\,d\nu|_W.\end{aligned}$$

We shall show that, for all $t\in(0,1)$,

$$d\nu\geq(e_t)_*(t^{N+1}\nu(W)\,d\Xi).$$

We remark that r and s can be regarded as at most two-valued functions of $t,\ell(\gamma)$ and ρ, and that

$$(e_t\circ\Phi)(\gamma,r_\pm(t,\ell(\gamma),\rho))=(e_{s_\pm(t,\ell(\gamma),\rho)}(\gamma),\rho).$$

Here we denote by (r_+, s_+) the solution of (4.2)–(4.4) with $tr \geq -(1-t)r_0 \cos \ell(\gamma)$. Similarly, (r_-, s_-) is the solution with $tr < -(1-t)r_0 \cos \ell(\gamma)$ if it exists. By using (4.2)–(4.4), we calculate

$$
\begin{aligned}
2\rho &= 2t^2 r \frac{\partial r}{\partial \rho} + 2(1-t)tr_0 \frac{\partial r}{\partial \rho} \cos \ell(\gamma), \\
\frac{\partial r}{\partial \rho} &= \frac{\rho}{t^2 r + (1-t)tr_0 \cos \ell(\gamma)}, \\
0 &= 2t^2 r \frac{\partial r}{\partial \ell(\gamma)} + 2(1-t)tr_0 \frac{\partial r}{\partial \ell(\gamma)} \cos \ell(\gamma) - 2(1-t)tr_0 r \sin \ell(\gamma), \\
\frac{\partial r}{\partial \ell(\gamma)} &= \frac{(1-t)r_0 r \sin \ell(\gamma)}{tr + (1-t)r_0 \cos \ell(\gamma)}, \\
\frac{\partial (s\ell(\gamma))}{\partial \ell(\gamma)} &= \frac{1}{\cos(s\ell(\gamma))} \frac{\partial \sin(s\ell(\gamma))}{\partial \ell(\gamma)} \\
&= \frac{1}{\cos(s\ell(\gamma))} \left\{ \frac{tr}{\rho} \cos \ell(\gamma) + \frac{t}{\rho} \frac{\partial r}{\partial \ell(\gamma)} \sin \ell(\gamma) \right\} \\
&= \frac{tr}{\rho \cos(s\ell(\gamma))} \left\{ \cos \ell(\gamma) + \frac{(1-t)r_0 \sin^2 \ell(\gamma)}{tr + (1-t)r_0 \cos \ell(\gamma)} \right\} \\
&= \frac{tr}{\rho \cos(s\ell(\gamma))} \frac{tr \cos \ell(\gamma) + (1-t)r_0}{tr + (1-t)r_0 \cos \ell(\gamma)} \\
&= \frac{tr}{tr + (1-t)r_0 \cos \ell(\gamma)}.
\end{aligned}
$$

Therefore we have, by Lemma 4.1 with $\tau(l) = s_+(t, l, \rho)$ or $s_-(t, l, \rho)$ for each fixed t

and ρ,

$$
\begin{aligned}
&(e_t)_*(t^{N+1}\nu(W)\,d\Xi) \le t^{N+1}(e_t \circ \Phi)_*(r^N \mu(A)\,d\Pi dr) \\
&= t^{N+1}(e_{s_+})_* \left[\left\{ \frac{\rho}{t} \frac{\sin(s_+\ell(\gamma))}{\sin \ell(\gamma)} \right\}^N \frac{\partial r_+}{\partial \rho} \mu(A)\,d\Pi \right] d\rho \\
&\quad - t^{N+1}(e_{s_-})_* \left[\left\{ \frac{\rho}{t} \frac{\sin(s_-\ell(\gamma))}{\sin \ell(\gamma)} \right\}^N \frac{\partial r_-}{\partial \rho} \mu(A)\,d\Pi \right] d\rho \\
&= t(e_{s_+})_* \left[\rho^N \left\{ \frac{\sin(s_+\ell(\gamma))}{\sin \ell(\gamma)} \right\}^N \frac{\rho}{t^2 r_+ + (1-t)tr_0 \cos \ell(\gamma)} \mu(A)\,d\Pi \right] d\rho \\
&\quad - t(e_{s_-})_* \left[\rho^N \left\{ \frac{\sin(s_-\ell(\gamma))}{\sin \ell(\gamma)} \right\}^N \frac{\rho}{t^2 r_- + (1-t)tr_0 \cos \ell(\gamma)} \mu(A)\,d\Pi \right] d\rho \\
&= (e_{s_+})_* \left[\left\{ \frac{\sin(s_+\ell(\gamma))}{\sin \ell(\gamma)} \right\}^{N-1} \frac{tr_+}{tr_+ + (1-t)r_0 \cos \ell(\gamma)} \rho^N \mu(A)\,d\Pi \right] d\rho \\
&\quad - (e_{s_-})_* \left[\left\{ \frac{\sin(s_-\ell(\gamma))}{\sin \ell(\gamma)} \right\}^{N-1} \frac{tr_-}{tr_- + (1-t)r_0 \cos \ell(\gamma)} \rho^N \mu(A)\,d\Pi \right] d\rho \\
&= (e_{s_+})_* \left[\frac{\partial(s_+\ell(\gamma))}{\partial \ell(\gamma)} \left\{ \frac{\sin(s_+\ell(\gamma))}{\sin \ell(\gamma)} \right\}^{N-1} \rho^N \mu(A)\,d\Pi \right] d\rho \\
&\quad - (e_{s_-})_* \left[\frac{\partial(s_-\ell(\gamma))}{\partial \ell(\gamma)} \left\{ \frac{\sin(s_-\ell(\gamma))}{\sin \ell(\gamma)} \right\}^{N-1} \rho^N \mu(A)\,d\Pi \right] d\rho \\
&\le \rho^N \, d\mu d\rho = d\nu.
\end{aligned}
$$

By Lemma 2.6, this completes the proof. □

5 Spaces with maximal diameters

Let (X, μ) be a metric space equipped with a Borel regular measure and satisfying the (K, N)-MCP for some $K > 0$ and $N > 1$. Then Theorem 2.5 asserts that $\operatorname{diam} X \le \pi\sqrt{(N-1)/K}$. In the present section, we study the situation that the equality holds. In the Riemannian case, it is known that X is isometric to a sphere. Although it is not the case of our generalized situation (consider orbifolds, convex subsets of a sphere, etc.), we will show that (X, μ) is a spherical suspension of some topological measure space under an assumption on the structure of the cut locus. We refer to [F] and [M] for the fundamentals of the general measure theory.

By Lemma 2.3, we may assume $K = N - 1$. We suppose $\operatorname{diam} X = \pi$ and take two points $x_N, x_S \in X$ with $|x_N - x_S|_X = \pi$. By the second assertion of Theorem 2.5, we know $S(x_N, \pi) = \{x_S\}$ and $S(x_S, \pi) = \{x_N\}$. We first prove a lemma which is easily observed from the discussions in [O].

Lemma 5.1 *For any $\alpha, \beta \in [0, \pi]$ with $\alpha < \beta$, we have*

$$
\mu\big(A(x_N; \alpha, \beta)\big) = \mu\big(A(x_S; \alpha, \beta)\big) = \frac{\int_\alpha^\beta \sin^{N-1}\theta\,d\theta}{\int_0^\pi \sin^{N-1}\theta\,d\theta} \mu(X).
$$

Proof. Without loss of generality, we may assume $\alpha = 0$. It follows from Theorem 2.4 that

$$\mu\big(B(x_N,\beta)\big) \geq \frac{\int_0^\beta \sin^{N-1}\theta\,d\theta}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\mu(X),$$
$$\mu\big(B(x_S,\pi-\beta)\big) \geq \frac{\int_0^{\pi-\beta} \sin^{N-1}\theta\,d\theta}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\mu(X).$$

Therefore we obtain

$$\begin{aligned}\mu\big(B(x_N,\beta)\big) &\geq \frac{\int_0^\beta \sin^{N-1}\theta\,d\theta}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\mu(X)\\ &= \mu(X) - \frac{\int_0^{\pi-\beta} \sin^{N-1}\theta\,d\theta}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\mu(X)\\ &\geq \mu(X) - \mu\big(B(x_S,\pi-\beta)\big) \geq \mu\big(B(x_N,\beta)\big).\end{aligned}$$

This completes the proof. □

As a corollary to the lemma above, we see the following.

Lemma 5.2 *For every point $z \in X$, we have $|x_N - z|_X + |x_S - z|_X = \pi$. In particular, there exists a minimal geodesic from x_N to x_S passing through z.*

Proof. Suppose that there exists a point $z \in X \setminus \{x_N, x_S\}$ satisfying

$$|x_N - z|_X + |x_S - z|_X > \pi.$$

Note that it implies that, for some $r \in (0,\pi)$, it holds that

$$\mu\big(B(x_N,r)\big) + \mu\big(B(x_S,\pi-r)\big) = \mu\big(B(x_N,r) \cup B(x_S,\pi-r)\big) < \mu(X).$$

This contradicts to Lemma 5.1. □

Throughout the rest of this section, we suppose

$$\mathrm{Cut}(x_N) \setminus \{x_S\} = \mathrm{Cut}(x_S) \setminus \{x_N\} = \emptyset. \tag{5.1}$$

Here we define the *cut locus* $\mathrm{Cut}(x)$ of a point $x \in X$ as the set of points $z \in X$ such that there exists at least two distinct minimal geodesics between x and z.

Remark 5.3 The structure of the cut locus in this sense is studied in [vR]. By the discussion in [vR, §3.3], we observe that $\mathrm{Cut}(x_S) \setminus \{x_N\} = \emptyset$ if there is no pair of minimal geodesics $\gamma_1, \gamma_2 : [0,\ell] \longrightarrow X$ such that $\gamma_1(0) = \gamma_2(0) = x_N$, $\gamma_1(\ell) \neq \gamma_2(\ell)$, and that $\gamma_1(t) = \gamma_2(t)$ for some $t > 0$. This 'non-branching' condition holds true for Riemannian manifolds as well as Alexandrov spaces with lower curvature bounds. However, the MCP alone is not sufficient to avoid the branchings of geodesics, for ℓ_1^n and ℓ_∞^n satisfy the $(0,n)$-MCP. Moreover, a sufficiently small ball in ℓ_1^n can satisfy the $(n,n+1)$-MCP (see [S2]). Nevertheless, it still remains a possibility that the $(N-1,N)$-MCP together with the maximal diameter condition implies (5.1).

We define Y as the set of unit speed minimal geodesics $\eta : [0, \pi] \longrightarrow X$ from x_N to x_S equipped with a topology induced from the supremum distance

$$|\eta_1 - \eta_2|_Y := \sup_{0 \le \theta \le \pi} |\eta_1(\theta) - \eta_2(\theta)|_X$$

for $\eta_1, \eta_2 \in Y$.

Lemma 5.4 *For all $z \in X \setminus \{x_N, x_S\}$, there exists a unique $\eta \in Y$ which passes z. Furthermore, for all $\theta_0 \in (0, \pi)$, the mapping*

$$Y \ni \eta \longmapsto \eta(\theta_0) \in S(x_N, \theta_0)$$

is homeomorphic. Here we equip $S(x_N, \theta_0)$ with a topology induced from d_X.

Proof. By the assumption (5.1), for every $z \in X \setminus \{x_N, x_S\}$, we find two unique minimal geodesics $\eta_i : [0, d_i] \longrightarrow X$, $i = 1, 2$, with $\eta_1(0) = x_N$, $\eta_1(d_1) = \eta_2(0) = z$, and $\eta_2(d_2) = x_S$, where we set $d_1 := |x_N - z|_X$ and $d_2 := |x_S - z|_X$. It is easy to see that the curve $\eta : [0, \pi] \longrightarrow X$ defined by $\eta(\theta) := \eta_1(\theta)$ for $\theta \in [0, d_1]$ and by $\eta(\theta) := \eta_2(\theta - d_1)$ for $\theta \in [d_1, \pi]$ gives a required unique element in Y.

Next we fix $\theta_0 \in (0, \pi)$ and put

$$\Phi : Y \ni \eta \longmapsto \eta(\theta_0) \in S(x_N, \theta_0).$$

Note that the bijectivity of Φ follows from the first part of this lemma, and that Φ is clearly continuous by the definition of the topology of Y. To show the continuity of the inverse map Φ^{-1}, we suppose that there are a sequence $\{\eta_i\}_{i=1}^{\infty} \subset Y$ and an element $\eta \in Y$ such that $\eta_i(\theta_0) \to \eta(\theta_0)$ in $S(x_N, \theta_0)$ but $\eta_i \not\to \eta$ in Y. Namely, we find $\theta \in (0, \pi)$ for which $\limsup_{i \to \infty} |\eta_i(\theta) - \eta(\theta)|_X =: a > 0$. Take a subsequence $\{\eta_j\}$ of $\{\eta_i\}$ such that $\lim_{j \to \infty} |\eta_j(\theta) - \eta(\theta)|_X = a$. However, by the Arzela-Ascoli theorem, we can choose a subsequence $\{\eta_k\}$ of $\{\eta_j\}$ and $\eta' \in Y$ such that $\eta_j \to \eta' \in Y$. Then we have $\eta'(\theta_0) = \eta(\theta_0)$ and $\eta'(\theta) \neq \eta(\theta)$, so that $\eta(\theta_0) \in \mathrm{Cut}(x_N)$ if $\theta \in (0, \theta_0)$ and $\eta(\theta_0) \in \mathrm{Cut}(x_S)$ if $\theta \in (\theta_0, \pi)$. This contradicts to the assumption (5.1), and hence there is no such $\{\eta_i\}$ and η. Therefore Φ^{-1} is continuous. □

We define a map $\Psi : Y \times [0, \pi] \longrightarrow X$ by $\Psi(\eta, \theta) := \eta(\theta)$ and a measure ν on Y by, for a set $W \subset Y$,

$$\nu(W) := \left\{ \int_0^{\pi} \sin^{N-1} \theta \, d\theta \right\}^{-1} \mu\big(\Psi(W \times [0, \pi])\big).$$

Note that the countable sub-additivity of ν follows from that of μ.

Lemma 5.5 *The measure ν is regular, that is, for any set $W \subset Y$, there exists a ν-measurable set $W_0 \supset W$ with $\nu(W_0) = \nu(W)$.*

Proof. Fix a set $W \subset Y$ and put $A := \Psi(W \times [0, \pi])$. It follows from the regularity of μ that we can choose a μ-measurable set $A_0 \supset A$ such that $\mu(A_0) = \mu(A)$. We put

$$W_0 := \{\eta \in Y \mid \eta([0, \pi]) \subset A_0\}.$$

Clearly we have $W_0 \supset W$. To prove the ν-measurability of W_0, we take an arbitrary subset $W' \subset Y$. We set $c := \{\int_0^\pi \sin^{N-1}\theta\, d\theta\}^{-1}$ for simplicity and observe

$$\begin{aligned}
\nu(W') &= c\mu\big(\Psi(W' \times [0,\pi])\big) \\
&= c\big\{\mu\big(\Psi(W' \times [0,\pi]) \setminus A_0\big) + \mu\big(\Psi(W' \times [0,\pi]) \cap A_0\big)\big\} \\
&\geq c\big\{\mu\big(\Psi(W' \times [0,\pi]) \setminus A\big) - \mu(A_0 \setminus A) + \mu\big(\Psi([W' \cap W_0] \times [0,\pi])\big)\big\} \\
&\geq c\big\{\mu\big(\Psi([W' \setminus W_0] \times [0,\pi])\big) + \mu\big(\Psi([W' \cap W_0] \times [0,\pi])\big)\big\} \\
&= \nu(W' \setminus W_0) + \nu(W' \cap W_0).
\end{aligned}$$

Hence W_0 is ν-measurable. □

Now we consider the spherical suspension (SY, ω) of (Y, ν) as a topological measure space, that is, $SY := Y \times [0,\pi]/\sim$, where $(\eta, 0) \sim (\xi, 0)$ and $(\eta, \pi) \sim (\xi, \pi)$ for every $\eta, \xi \in Y$, equipped with a measure $d\omega := d\nu \times (\sin^{N-1}\theta\, d\theta)$ and with a quotient topology induced from the product topology of $Y \times [0,\pi]$. We remark that, as both ν and θ are regular, so is ω. Clearly Ψ induces an homeomorphism from SY to X, we again denote this map by Ψ.

Theorem 5.6 *Let (X, μ) be a compact metric measure space satisfying the (K, N)-MCP for some $K > 0$ and $N > 1$, and assume* (5.1). *If* $\operatorname{diam} X = \pi\sqrt{(N-1)/K}$ *holds, then there exists a topological measure space (Y, ν) such that (X, μ) is the spherical suspension of (Y, ν) as a topological measure space. More precisely, the homeomorphism $\Psi : (SY, \omega) \longrightarrow (X, \mu)$ constructed as above is measure preserving.*

Proof. We can suppose $K = N - 1$. It suffices to show that $\omega(V) = \mu(\Psi(V))$ holds for every set $V \subset SY$. We first treat a product set $V = W \times [a, b]$ for $W \subset Y$ and $0 \leq a \leq b \leq \pi$. As μ is a regular measure, we find a μ-measurable set $A_0 \supset \Psi(W \times [0,\pi])$ such that $\mu(A_0) = \mu(\Psi(W \times [0,\pi]))$. We define a map $\Phi : X \setminus \{x_S\} \times (0,1] \longrightarrow X$ as, for each $\eta \in Y$, $\Phi(\eta(\theta), t) := \eta(t\theta)$. Then a similar discussion as in the proof of Theorem 2.4 yields that

$$\mu\big(\Phi(A_0, t)\big) = \frac{\int_0^{\pi t} \sin^{N-1}\theta\, d\theta}{\int_0^\pi \sin^{N-1}\theta\, d\theta}\mu(A_0).$$

We remark that, by our construction, $\Phi(x, t) \in \Psi(W \times [0, \pi t]) \subset A_0$ for μ-a.e. $x \in A_0$. Therefore we obtain

$$\begin{aligned}
\mu\big(\Psi(V)\big) &\leq \mu\big(\Phi(A_0, b/\pi) \setminus \Phi(A_0, a/\pi)\big) \\
&= \frac{\int_a^b \sin^{N-1}\theta\, d\theta}{\int_0^\pi \sin^{N-1}\theta\, d\theta}\mu(A_0) \\
&= \left\{\int_0^\pi \sin^{N-1}\theta\, d\theta\right\}^{-1} \mu\big(\Phi(W \times [0,\pi])\big) \int_a^b \sin^{N-1}\theta\, d\theta \\
&= \nu(W) \cdot \phi([a,b]) = \omega(V),
\end{aligned}$$

where we set $d\phi := \sin^{N-1}\theta\, d\theta$ on $[0, \pi]$. This implies $\mu \leq \Psi_*\omega$ on X. Moreover, by the definition of ν, we see

$$\mu(X) = \nu(Y) \cdot \phi([0,\pi]) = \omega(SY) \geq \mu(X),$$

and it yields $\mu(X) = \omega(SY)$. For any set $V \subset SY$, we take a ω-measurable set $V_0 \supset V$ with $\omega(V_0) = \omega(V)$ and observe

$$\begin{aligned}\mu(X) &\le \mu\big(\Psi(V)\big) + \mu\big(X \setminus \Psi(V)\big) \le \omega(V) + \omega(SY \setminus V)\\ &= \omega(V_0) + \omega(SY \setminus V_0) = \omega(SY) = \mu(X).\end{aligned}$$

Therefore we obtain $\omega(V) = \mu(\Psi(V))$ and it completes the proof. □

References

[BC] R. L. Bishop and R. J. Crittenden, Geometry of manifolds, Academic Press, New York-London, 1964.

[CC] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below.* I, J. Differential Geom. **46** (1997), 37–74.

[F] H. Federer, Geometric measure theory, Springer-Verlag New York Inc., New York, 1969.

[G] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser Boston, Inc., Boston, MA, 1999.

[LV1] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, to appear in Ann. of Math.

[LV2] J. Lott and C. Villani, *Weak curvature conditions and functional inequalities*, J. Funct. Anal. **245** (2007), 311–333.

[M] P. Mattila, Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability, Cambridge University Press, Cambridge, 1995.

[O] S. Ohta, *On the measure contraction property of metric measure spaces*, to appear in Comment. Math. Helv.

[vR] M.-K. von Renesse, *On local Poincaré via transportation*, to appear in Math. Z.

[S1] K.-T. Sturm, *On the geometry of metric measure spaces*, Acta Math. **196** (2006), 65–131.

[S2] K.-T. Sturm, *On the geometry of metric measure spaces.* II, Acta Math. **196** (2006), 133–177.