Extending Lipschitz and Hölder maps between metric spaces*†

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Abstract

We introduce a stochastic generalization of Lipschitz retracts, and apply it to the extension problems of Lipschitz, Hölder, large-scale Lipschitz and large-scale Hölder maps into barycentric metric spaces. Our discussion gives an appropriate interpretation of a work of Lee and Naor.

1 Introduction

The extendability of Lipschitz maps is one of the central topics in the theory of Banach spaces. The question is, given two Banach spaces Y and Z, whether an arbitrary L-Lipschitz map $f : X \longrightarrow Z$ from a subset $X \subset Y$ can be extended to a CL-Lipschitz map $\tilde{f} : Y \longrightarrow Z$, as well as the estimate of the constant C. For example, the asymptotic behavior of C as the dimensions of Y and Z are increasing, and the relationship between C and other invariants (e.g., the modulus of convexity or smoothness) are important problems.

As Lipschitz maps make sense between general metric spaces, it is natural and interesting to ask the same question for nonlinear metric spaces Y and Z. The most fundamental result is McShane's classical lemma which asserts that $C = 1$ if Z is the real line, and there are rather recent contributions from the viewpoints of metric geometry and (the nonlinearization of) the geometry of Banach spaces (see [LS], [LPS], [Ba], [NPSS], [O2] etc.). Also the extension problems for Hölder maps and large-scale Lipschitz maps receive independent interests (see [Na] and [La]).

Recently, Lee and Naor [LN] have made a deep progress which has an impact on both the linear and nonlinear settings. They are concerned with another aspect of the extension problem by fixing a certain X and letting Y be arbitrary, while Z is a Banach space or a β-barycentric metric space. Then they construct a stochastic decomposition

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of Y with respect to X by adopting ideas coming from combinatorics and theoretical computer science, and use it to construct a gentle partition of unity of Y with respect to X. From a gentle partition of unity to an extension of a Lipschitz map is the last and easiest step. According to this strategy, they improved several known results and also obtained a number of new results.

In this article, we provide a general method for extending maps between certain classes of metric spaces. It is simple enough to be applicable to all the extension problems for Lipschitz, Hölder, large-scale Lipschitz and large-scale Hölder maps. The essential idea has already appeared behind the discussion around the last (i.e., from a gentle partition to an extension) step in [LN]. We will present it in an appropriate form.

We briefly explain our idea. Given metric spaces X, Z and Y containing X, we assume the existence of a Lipschitz map ρ from Y to the space of probability measures $\mathcal{P}(X)$ on X, and also assume the existence of a Lipschitz map $c : \mathcal{P}(Z) \longrightarrow Z$. Then the required extension of an arbitrary map $f : X \longrightarrow Z$ is given by $\widetilde{f} := c \circ f_* \circ \rho : Y \longrightarrow Z$, where $f_* : \mathcal{P}(X) \longrightarrow \mathcal{P}(Z)$ is the push-forward map. Here the map ρ is thought of as a generalization of Lipschitz retractions, and c maps a measure to its barycenter. We can construct ρ from a gentle partition of unity, and hence we obtain a rich family of examples of the source space X (and Y) through Lee and Naor's constructions of gentle partitions of unity. Examples of the target space Z are Banach spaces, CAT(0)-spaces and 2-uniformly convex metric spaces.

The organization of the article is as follows: In Section 2, we recall some classes of maps between metric spaces, as well as geometric structures on spaces of probability measures on metric spaces. Section 3 contains the extension lemma. We enumerate examples of the source space and the target space in Sections 4 and 5, respectively.

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2 Preliminaries

2.1 Lipschitz maps and generalizations

We recall four classes of maps in order to fix notations. Let (X, δ_X) and (Z, δ_Z) be spaces equipped with certain symmetric functions

$$\delta_X : X \times X \longrightarrow [0, \infty), \qquad \delta_Z : Z \times Z \longrightarrow [0, \infty).$$

They are not necessarily distance functions. For a nonnegative constant $L \geq 0$, a map $f : X \longrightarrow Z$ is said to be *L-Lipschitz* if

$$\delta_Z\big(f(x), f(y)\big) \leq L \cdot \delta_X(x, y)$$

holds for all $x, y \in X$. The Lipschitz condition is extended in two distinct directions.

First, for $L \geq 0$ and $\alpha \in (0, 1]$, we say that a map $f : X \longrightarrow Z$ is (L, α)-*Hölder* if we have

$$\delta_Z\big(f(x), f(y)\big) \leq L \cdot \delta_X(x, y)^\alpha$$

for all $x, y \in X$. The $(L, 1)$-Hölder condition is nothing but the L-Lipschitz condition. Secondly, a map $f : X \longrightarrow Z$ is said to be (L, ε)-*Lipschitz* for $L, \varepsilon \geq 0$ if

$$\delta_Z\big(f(x), f(y)\big) \leq L \cdot \delta_X(x, y) + \varepsilon$$

holds for all $x, y \in X$. The case where $\varepsilon = 0$ reduces to the L-Lipschitz condition. We remark that the (L, ε)-Lipschitz condition says nothing about the behavior of f in a small scale. For instance, f is not necessarily continuous in terms of δ_X and δ_Z.

Besides them, it is convenient for the later use to consider also an (L, α, ε)-*Hölder* map $f : X \longrightarrow Z$ for $L, \varepsilon \geq 0$ and $\alpha \in (0, 1]$ which satisfies

$$\delta_Z\big(f(x), f(y)\big) \leq L \cdot \delta_X(x, y)^\alpha + \varepsilon$$

for all $x, y \in X$.

2.2 Geometric structures on spaces of probability measures

Geometric (e.g., distance) structures on a space of probability measures on a metric space are key ingredients of this article. There are many such structures known to play important roles in probability theory and statistics. (There is a list in [Ra].) We treat two of them in this article. We refer to [Du] for a fundamental knowledge of probability theory.

Let (X, d_X) be a metric space. We denote by $\mathcal{P}(X)$ the set of Borel probability measures on X, equipped with an equivalence relation such that $\mu \sim \nu$ holds if we have $\mu(U) = \nu(U)$ for every Borel set $U \subset X$. In addition, $\mathcal{P}_\infty(X) \subset \mathcal{P}(X)$ stands for the set of probability measures with bounded support. Given two probability measures $\mu, \nu \in \mathcal{P}(X)$, we can take the *Hahn-Jordan decomposition* of their difference

$$\mu - \nu = [\mu - \nu]_+ - [\mu - \nu]_-.$$

Note that $[\nu - \mu]_+ = [\mu - \nu]_-$ as measures. The *total variation measure* for $\mu - \nu$ is defined by

$$|\mu - \nu| := [\mu - \nu]_+ + [\mu - \nu]_- = [\mu - \nu]_+ + [\nu - \mu]_+,$$

and the *total variation* of $\mu - \nu$ is $|\mu - \nu|(X) = 2[\mu - \nu]_+(X) = 2[\nu - \mu]_+(X)$. With these notations, we define a function $\delta^V_\infty : \mathcal{P}_\infty(X) \times \mathcal{P}_\infty(X) \longrightarrow [0, \infty)$ by, for $\mu, \nu \in \mathcal{P}_\infty(X)$,

$$\delta^V_\infty(\mu, \nu) := \frac{1}{2} \operatorname{diam}\big(\operatorname{supp}(\mu + \nu)\big) \cdot |\mu - \nu|(X). \tag{2.1}$$

The underlying space (X, d_X) is isometrically embedded in $(\mathcal{P}_\infty(X), \delta^V_\infty)$ by associating a point $x \in X$ with a Dirac measure $\delta_x \in \mathcal{P}_\infty(X)$ at x. We remark that δ^V_∞ does not satisfy the triangle inequality in most cases. For instance, put $X = \mathbb{R}$, $\mu = (1/2)\cdot\delta_{-1} + (1/2)\cdot\delta_{-\varepsilon}$, $\nu = (1/2)\cdot\delta_{\varepsilon} + (1/2)\cdot\delta_1$ and $\omega = (1/2)\cdot\delta_{-\varepsilon} + (1/2)\cdot\delta_{\varepsilon}$ for $\varepsilon \in (0,1)$. Then we find

$$\delta^V_\infty(\mu,\nu) = \frac{1}{2}\cdot 2\cdot 2 = 2, \quad \delta^V_\infty(\mu,\omega) + \delta^V_\infty(\omega,\nu) = \left\{\frac{1}{2}\cdot(1+\varepsilon)\cdot 1\right\}\cdot 2 = 1+\varepsilon < 2.$$

We next recall another, more sophisticated geometric structure. For $p \in [1,\infty)$, define $\mathcal{P}_p(X) \subset \mathcal{P}(X)$ as the set of probability measures with finite moments of order p, that is, $\mu \in \mathcal{P}_p(X)$ if we have

$$\int_X d_X(x,y)^p \, d\mu(y) < \infty$$

for some (and hence all) point $x \in X$. We remark that $\mathcal{P}_\infty(X) \subset \mathcal{P}_p(X)$ for any $p \in [1,\infty)$. Given $\mu,\nu \in \mathcal{P}(X)$, a probability measure $q \in \mathcal{P}(X\times X)$ is called a *coupling* of μ and ν if we have

$$q(U\times X) = \mu(U), \qquad q(X\times U) = \nu(U)$$

for all Borel set $U \subset X$. We denote by $\Pi(\mu,\nu) \subset \mathcal{P}(X\times X)$ the set of all couplings of μ and ν. For $\mu,\nu \in \mathcal{P}_p(X)$, we define

$$\delta^W_p(\mu,\nu) := \inf_{q\in\Pi(\mu,\nu)} \left(\int_{X\times X} d_X(x,y)^p \, dq(x,y)\right)^{1/p}. \tag{2.2}$$

Note that $\delta^W_p(\mu,\nu) \in [0,\infty)$ and (X,d_X) is again isometrically embedded in $(\mathcal{P}_p(X),\delta^W_p)$ through the correspondence between $x \in X$ and $\delta_x \in \mathcal{P}_p(X)$. As

$$q := \varphi_*(\mu - [\mu-\nu]_+) + \frac{2}{|\mu-\nu|(X)}\cdot\big([\mu-\nu]_+ \times [\nu-\mu]_+\big)$$

is a coupling of μ and ν, where $\varphi : X \longrightarrow X\times X$ denotes the diagonal map $\varphi(x) = (x,x)$, we observe that $\delta^W_1(\mu,\nu) \le \delta^V_\infty(\mu,\nu)$. We remark that $\mu - [\mu-\nu]_+ = \nu - [\nu-\mu]_+$. On the other hand, $\delta^W_p(\mu,\nu) \le \delta^V_\infty(\mu,\nu)$ does not hold true for $p > 1$ (see Example 4.10 below). If (X,d_X) is separable and complete, then δ^W_p turns out to be a separable and complete distance on $\mathcal{P}_p(X)$ and is called the (L_p)-*Wasserstein distance* or *Kantorovich-Rubinstein distance* (see [Du], [Ra], [RR] and [Vi]). However, our usage of δ^W_p is only a subsidiary one, so we do not really need such a property. All we need is the following.

Lemma 2.1 *Let (X,d_X) and (Z,d_Z) be metric spaces, $f : X \longrightarrow Z$ be a Borel measurable map and $\delta_{\mathcal{P}_p} = \delta^W_p$ or δ^V_∞. If f is L-Lipschitz, (L,α)-Hölder, (L,ε)-Lipschitz or (L,α,ε)-Hölder, then so is the induced push-forward map $f_* : (\mathcal{P}_p(X),\delta_{\mathcal{P}_p}) \longrightarrow (\mathcal{P}_p(Z),\delta_{\mathcal{P}_p})$.*

Proof. Note that the (L,α,ε)-Hölder situation covers everything. We first consider the case of δ^W_p. Given $\mu,\nu \in \mathcal{P}_p(X)$, fix a coupling $q \in \mathcal{P}(X\times X)$ of μ and ν. Then we observe that a measure $(f \times f)_*q \in \mathcal{P}(Z \times Z)$ provides a coupling of $f_*\mu$ and $f_*\nu$. Hence we have, if f is (L, α, ε)-Hölder,

$$\begin{aligned}
\delta_p^W(f_*\mu, f_*\nu) &\le \Big(\int_{Z\times Z} d_Z(w,z)^p \, d[(f\times f)_*q](w,z) \Big)^{1/p} \\
&= \Big(\int_{X\times X} d_Z\big(f(x), f(y)\big)^p \, dq(x,y) \Big)^{1/p} \\
&\le \Big(\int_{X\times X} \{L \cdot d_X(x,y)^\alpha + \varepsilon\}^p \, dq(x,y) \Big)^{1/p} \\
&\le L \cdot \Big(\int_{X\times X} d_X(x,y)^{\alpha p} \, dq(x,y) \Big)^{1/p} + \varepsilon \\
&\le L \cdot \Big(\int_{X\times X} d_X(x,y)^{p} \, dq(x,y) \Big)^{\alpha/p} + \varepsilon.
\end{aligned}$$

Here the last inequality follows from the Hölder inequality. By taking the infimum over all couplings q of μ and ν, we obtain $\delta_p^W(f_*\mu, f_*\nu) \le L \cdot \delta_p^W(\mu,\nu)^\alpha + \varepsilon$. We remark that, by letting $\nu = \delta_x$ for some $x \in X$, we also deduce that $\delta_p^W(f_*\mu, \delta_{f(x)}) < \infty$, and hence $f_*\mu \in \mathcal{P}_p(Z)$. This completes the proof for δ_p^W.

We next treat δ_∞^V. For $\mu \in \mathcal{P}_\infty(X)$, we observe that

$$\operatorname{diam}(\operatorname{supp} f_*\mu) \le \operatorname{diam}(\operatorname{supp}\mu)^\alpha + \varepsilon < \infty,$$

and hence $f_*\mu \in \mathcal{P}_\infty(Z)$. Note also that $|f_*\mu - f_*\nu|(Z) \le |\mu-\nu|(X) \le 2$ for $\mu, \nu \in \mathcal{P}_\infty(X)$. Thus we see, if f is (L, α, ε)-Hölder,

$$\begin{aligned}
\delta_\infty^V(f_*\mu, f_*\nu) &= \frac{1}{2} \operatorname{diam}\big(\operatorname{supp}(f_*\mu + f_*\nu)\big) \cdot |f_*\mu - f_*\nu|(Z) \\
&\le \frac{1}{2}\{L \cdot \operatorname{diam}\big(\operatorname{supp}(\mu+\nu)\big)^\alpha + \varepsilon\} \cdot |\mu-\nu|(X) \\
&\le L \cdot \Big(\frac{1}{2} \operatorname{diam}\big(\operatorname{supp}(\mu+\nu)\big) \cdot |\mu-\nu|(X) \Big)^\alpha + \frac{|\mu-\nu|(X)}{2}\varepsilon \\
&\le L \cdot \delta_\infty^V(\mu,\nu)^\alpha + \varepsilon.
\end{aligned}$$

□

Remark 2.2 We remark that, once $\delta_{\mathcal{P}_p}$ is chosen, then it is fixed during the argument. Thus the conclusion of Lemma 2.1 concerns $f_* : (\mathcal{P}_p(X), \delta_p^W) \longrightarrow (\mathcal{P}_p(Z), \delta_p^W)$ or $f_* : (\mathcal{P}_\infty(X), \delta_\infty^V) \longrightarrow (\mathcal{P}_\infty(Z), \delta_\infty^V)$, but does not include $f_* : (\mathcal{P}_p(X), \delta_p^W) \longrightarrow (\mathcal{P}_\infty(Z), \delta_\infty^V)$. The same remark will be applied throughout the article.

3 An extension lemma

This section is concerned with a general strategy for extending maps between metric spaces. We take the ideas in [LPS] and [LN] into account.

3.1 Stochastic Lipschitz retracts

Let (Y, d_Y) be a metric space. A subset $X \subset Y$ is called a *σ-Lipschitz retract* of Y if there is a σ-Lipschitz map $\rho : Y \longrightarrow X$ which is the identity on X. Moreover, if a metric space (X, d_X) is a σ-Lipschitz retract of every metric space containing it, then we call X an *absolute σ-Lipschitz retract*. As is comprehensively surveyed in [BL, Chapters 1, 2], there is a strong connection between Lipschitz retracts and the Lipschitz extension problem. For instance, given a metric space (X, d_X), following three conditions are equivalent (cf. [BL, Proposition 1.2]):

(i) X is an absolute σ-Lipschitz retract.

(ii) For any metric space Y containing X and for any metric space Z, every L-Lipschitz map $f : X \longrightarrow Z$ can be extended to a σL-Lipschitz map $\tilde{f} : Y \longrightarrow Z$.

(iii) For any metric space Y and its subset $Z \subset Y$, every L-Lipschitz map $f : Z \longrightarrow X$ can be extended to a σL-Lipschitz map $\tilde{f} : Y \longrightarrow X$.

We introduce a generalization of Lipschitz retracts from a stochastic viewpoint.

Definition 3.1 ($\delta_{\mathcal{P}_p}$-stochastic Lipschitz retracts) Let (Y, d_Y) be a metric space, $X \subset Y$ be its subset and $\delta_{\mathcal{P}_p} = \delta_p^W$ or δ_∞^V. We say that X is a *$\delta_{\mathcal{P}_p}$-stochastic σ_r-Lipschitz retract* of Y if there is a σ_r-Lipschitz map $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$ with $\sigma_r \geq 1$ such that $\rho(x) = \delta_x$ for all $x \in X$. Then the map ρ is called a *stochastic σ_r-Lipschitz retraction.*

A metric space (X, d_X) is called a *$\delta_{\mathcal{P}_p}$-absolute stochastic σ_r-Lipschitz retract* if it is a $\delta_{\mathcal{P}_p}$-stochastic σ_r-Lipschitz retract of every metric space containing it.

In this generalized context, a usual Lipschitz retract can be regarded as a special case where the image of ρ is included in $X \subset \mathcal{P}_p(X)$, namely $\rho(y)$ is a Dirac measure for every $y \in Y$. We obtain an analogue of Lipschitz retracts as follows.

Proposition 3.2 *Given a metric space (X, d_X) and $\delta_{\mathcal{P}_p} = \delta_p^W$ or δ_∞^V, following three conditions are equivalent*:

(i) *X is a $\delta_{\mathcal{P}_p}$-absolute stochastic σ_r-Lipschitz retract.*

(ii) *For any metric space Y containing X and for any metric space Z, every L-Lipschitz map $f : X \longrightarrow Z$ can be extended to a $\sigma_r L$-Lipschitz map $\tilde{f} : Y \longrightarrow (\mathcal{P}_p(Z), \delta_{\mathcal{P}_p})$.*

(iii) *For any metric space Y and its subset $Z \subset Y$, every L-Lipschitz map $f : Z \longrightarrow X$ can be extended to a $\sigma_r L$-Lipschitz map $\tilde{f} : Y \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$.*

Proof. It is easy to see that either (ii) or (iii) implies (i) by taking $X = Z$ and letting f be the identity map on X.

(i) $\Rightarrow$ (ii) Let $\rho : Y \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$ be a stochastic σ_r-Lipschitz retraction. Then it immediately follows from Lemma 2.1 that $\tilde{f} := f_* \circ \rho : Y \longrightarrow (\mathcal{P}_p(Z), \delta_{\mathcal{P}_p})$ is $\sigma_r L$-Lipschitz. By construction, $\tilde{f}$ extends f.

(i) $\Rightarrow$ (iii) Recall that X is isometrically embedded in the space $\ell_\infty(X)$ of Borel measurable, bounded functions on X. On one hand, by assumption, there is a stochastic

σ_r-Lipschitz retraction $\rho : \ell_\infty(X) \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$ which is the identity on X. On the other hand, just like McShane's lemma, an L-Lipschitz map $f : Z \longrightarrow X$ is extended to an L-Lipschitz map $F : Y \longrightarrow \ell_\infty(X)$ by

$$[F(y)](x) := \inf_{z \in Z} \big\{ [f(z)](x) + L \cdot d_Y(z, y) \big\},$$

where we regard $[f(z)]$ as an element in $\ell_\infty(X)$. By putting $\tilde{f} := \rho \circ F$, we complete the proof. $\square$

Remark 3.3 We can describe the absolute stochastic Lipschitz retract more intrinsically by using the injective hull of X (see [Is], [BL] and [Ko]). A metric space (X, d_X) is said to be *injective* if it is an absolute 1-Lipschitz retract. Given a metric space (X, d_X), an *injective hull* (or an *injective envelope*) of X is an injective metric space εX together with an isometric embedding $\psi : X \longrightarrow \varepsilon X$ such that there is no proper injective subset of εX containing $\psi(X)$. It is known that such a space exists and is unique upto an appropriate equivalent relation, that is, given two injective hulls (E, ψ_E) and (F, ψ_F) of X, there exists an isometry $i : E \longrightarrow F$ such that $i \circ \psi_E = \psi_F$.

In [Ko], Kozdoba introduced a quantity $I(X)$ as the infimum of constants $\sigma \geq 1$ for which there is an σ-Lipschitz map $\Psi : \varepsilon X \longrightarrow \mathcal{F}(X)$ with $\Psi \circ \psi = \phi$ on X. Here $\mathcal{F}(X)$ stands for the *free Banach space* associated to X and $\phi : X \longrightarrow \mathcal{F}(X)$ is the isometric embedding. Roughly speaking, $\mathcal{F}(X)$ is a vector space of signed measures with separable range equipped with the L_1-Wasserstein norm. He showed that $I(X)$ is coincide with the infimum of constants $\sigma \geq 1$ such that every L-Lipschitz map $f : X \longrightarrow Z$ into an arbitrary Banach space Z can be extended to a σL-Lipschitz map $\tilde{f} : Y \longrightarrow Z$ for every Y containing X. Then he investigated the behavior of $I(X)$ by using these two characterizations.

Back to our context, we observe that (X, d_X) is a $\delta_{\mathcal{P}_p}$-absolute stochastic σ_r-Lipschitz retract if and only if there is a σ_r-Lipschitz map $\Psi : \varepsilon X \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$ with $\Psi \circ \psi(x) = \delta_x$ for all $x \in X$. Thus Kozdoba's result corresponds to the δ_1^W-case of Lemma 3.5 below.

3.2 Barycentric metric spaces

We consider a kind of dual condition of being a stochastic Lipschitz retract by using a barycenter (also called a center of mass or a center of gravity). We refer to [St] and references therein for more detailed treatment of this concept.

Definition 3.4 ($\delta_{\mathcal{P}_p}$-barycenters) Let $\delta_{\mathcal{P}_p} = \delta_p^W$ or δ_∞^V. A metric space (Z, d_Z) is said to be $\delta_{\mathcal{P}_p}$*-barycentric* if there is a β_c-Lipschitz map $c : (\mathcal{P}_p(Z), \delta_{\mathcal{P}_p}) \longrightarrow (Z, d_Z)$ with $\beta_c \geq 1$ such that $c(\delta_z) = z$ for all $z \in Z$. Then we call $c(\mu) \in Z$ a *barycenter* of $\mu \in \mathcal{P}_p(Z)$.

We remark that the barycentric property in [LN] corresponds to our δ_∞^V-barycentric property, and that the condition (28) in [LN] lies between the δ_1^W- and δ_∞^V-barycentric properties (or, to be more precise, amounts to the δ_1'-barycentric property, see (4.4) below).

3.3 An extension lemma

The following extension lemma is the essence of [LN, Lemmas 2.1, 6.1].

Lemma 3.5 (Extension lemma) *Let (Y, d_Y) and (Z, d_Z) be metric spaces, $X \subset Y$ be a closed subset, and let $\delta_{\mathcal{P}_p} = \delta_p^W$ or δ_∞^V. Assume that X is a $\delta_{\mathcal{P}_p}$-stochastic σ_r-Lipschitz retract of Y and that Z is $\delta_{\mathcal{P}_p}$-barycentric. Then we have the following*:

(i) *Every L-Lipschitz map $f : X \longrightarrow Z$ is extended to a $\sigma_r \beta_c L$-Lipschitz map $\tilde{f} : Y \longrightarrow Z$.*

(ii) *Every (L, α)-Hölder map $f : X \longrightarrow Z$ is extended to a $(\sigma_r^\alpha \beta_c L, \alpha)$-Hölder map $\tilde{f} : Y \longrightarrow Z$.*

(iii) *Every Borel measurable and (L, ε)-Lipschitz map $f : X \longrightarrow Z$ is extended to a $(\sigma_r \beta_c L, \beta_c \varepsilon)$-Lipschitz map $\tilde{f} : Y \longrightarrow Z$.*

(iv) *Every Borel measurable and (L, α, ε)-Hölder map $f : X \longrightarrow Z$ is extended to a $(\sigma_r^\alpha \beta_c L, \alpha, \beta_c \varepsilon)$-Hölder map $\tilde{f} : Y \longrightarrow Z$.*

In particular, if a metric space (X, d_X) is a $\delta_{\mathcal{P}_p}$-absolute stochastic Lipschitz retract, then each of three extensions above can be performed for every metric space (Y, d_Y) containing X.

Proof. It is sufficient to treat the case of (L, α, ε)-Hölder maps. Let $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_p(X), \delta_{\mathcal{P}_p})$ be a stochastic Lipschitz retraction. We define a map $\tilde{f} : Y \longrightarrow Z$ by, for each $y \in Y$,

$$\tilde{f}(y) := c\big(f_*[\rho(y)]\big).$$

Then clearly $\tilde{f}$ extends f and it follows from Lemma 2.1 that, for any $x, y \in Y$,

$$\begin{aligned} d_Z\big(\tilde{f}(x), \tilde{f}(y)\big) &\le \beta_c \cdot \delta_{\mathcal{P}_p}\big(f_*[\rho(x)], f_*[\rho(y)]\big) \le \beta_c \big\{ L \cdot \delta_{\mathcal{P}_p}\big(\rho(x), \rho(y)\big)^\alpha + \varepsilon \big\} \\ &\le \sigma_r^\alpha \beta_c L \cdot d_Y(x, y)^\alpha + \beta_c \varepsilon. \end{aligned}$$

□

4 Examples: Source spaces

In this section, we give examples of spaces which are adopted as the source space in Lemma 3.5. Most fundamental examples are usual Lipschitz retracts, such as projections to factors from a product of metric spaces or the 'nearest point map' to a closed convex subset in a CAT(0)-space (cf. [St]). They are $\delta_{\mathcal{P}_p}$-stochastic Lipschitz retracts for all $\delta_{\mathcal{P}_p} = \delta_p^W$ and δ_∞^V. Beyond them, for $p = 1, \infty$ (i.e., $\delta_{\mathcal{P}_p} = \delta_1^W, \delta_\infty^V$), we obtain surprisingly rich families of (absolute) stochastic Lipschitz retracts through a work of Lee and Naor [LN]. The case of $p \in (1, \infty)$ is more restrictive and we know only almost trivial examples at present.

4.1 $p = 1, \infty$

We recall two kinds of gentle partitions of unity introduced in [LN].

Definition 4.1 (K-gentle partitions of unity) Let (Y, d_Y) be a metric space, $X \subset Y$ be a closed subset, and let (Ω, ω) be a measure space. For $K \geq 1$, a function $\Psi : \Omega \times Y \longrightarrow [0, \infty)$ is called a *K-gentle partition of unity with respect to X* if the following hold:

(1) For every $x \in X$, we have $\Psi(\cdot, x) \equiv 0$ on Ω.

(2) For every $y \in Y \setminus X$, the function $\Psi(\cdot, y) : \Omega \longrightarrow [0, \infty)$ is ω-measurable and satisfies

$$\int_\Omega \Psi(a, y)\, d\omega(a) = 1.$$

(3) There is a Borel measurable map $\gamma : \Omega \longrightarrow X$ such that

$$\int_\Omega d_Y\big(\gamma(a), x\big) |\Psi(a, x) - \Psi(a, y)|\, d\omega(a) \leq K \cdot d_Y(x, y) \tag{4.1}$$

holds for all $x, y \in Y$.

Definition 4.2 ((K, L)-gentle partitions of unity) Let (Y, d_Y) be a metric space, $X \subset Y$ be its closed subset, and let (Ω, ω) be a measure space. Given $K, L \geq 1$, a function $\Psi : \Omega \times Y \longrightarrow [0, \infty)$ is called a *(K, L)-gentle partition of unity with respect to X* if the following hold:

(1) For every $x \in X$, we have $\Psi(\cdot, x) \equiv 0$ on Ω.

(2) For every $y \in Y \setminus X$, the function $\Psi(\cdot, y) : \Omega \longrightarrow [0, \infty)$ is ω-measurable and satisfies

$$\int_\Omega \Psi(a, y)\, d\omega(a) = 1.$$

(3) There is a Borel measurable map $\gamma : \Omega \longrightarrow X$ such that

$$\begin{aligned} &\operatorname{diam}\big(\{x, y\} \cup \{\gamma(a) \mid \Psi(a, x) + \Psi(a, y) > 0\}\big) \\ &\leq K \cdot \big[d_Y(x, y) + \max\{d_Y(x, X), d_Y(y, X)\}\big] \end{aligned} \tag{4.2}$$

holds for all $x, y \in Y$.

(4) For every distinct points $x, y \in Y$, we have

$$\int_\Omega |\Psi(a, x) - \Psi(a, y)|\, d\omega(a) \leq \frac{L \cdot d_Y(x, y)}{d_Y(x, y) + \max\{d_Y(x, X), d_Y(y, X)\}}. \tag{4.3}$$

We observe that K- and (K,L)-gentle partitions of unity generate δ_1^W- and δ_∞^V-stochastic Lipschitz retractions, respectively. In order to do this, we introduce another quantity δ_1' for simplicity. Given a metric space (X, d_X) and $\mu, \nu \in \mathcal{P}_1(X)$, we define

$$\delta_1'(\mu,\nu) := \frac{2}{|\mu-\nu|(X)} \int_{X\times X} d_X(x,y)\, d[\mu-\nu]_+(x)\, d[\nu-\mu]_+(y) \tag{4.4}$$

if $\mu \neq \nu$, and $\delta_1'(\mu,\mu) := 0$. Note that $\delta_1'(\mu,\nu) \leq \delta_\infty^V(\mu,\nu)$. In addition, it holds that $\delta_1^W(\mu,\nu) \leq \delta_1'(\mu,\nu)$ because

$$q := \varphi_*(\mu - [\mu-\nu]_+) + \frac{2}{|\mu-\nu|(X)} \cdot \big([\mu-\nu]_+ \times [\nu-\mu]_+\big)$$

is a coupling of μ and ν, where $\varphi : X \longrightarrow X \times X$ denotes the diagonal map $\varphi(x) = (x,x)$.

Lemma 4.3 *Let (Y, d_Y) be a metric space and $X \subset Y$ be a closed subset. If there is a K-gentle partition of unity with respect to X, say $\Psi : \Omega \times Y \longrightarrow [0,\infty)$, then X is a δ_1^W-stochastic K-Lipschitz retract of Y.*

Proof. Define a map $\rho : Y \longrightarrow \mathcal{P}(X)$ as follows:

(a) $\rho(x) := \delta_x$ for $x \in X$.

(b) $\rho(y) := \gamma_*[\Psi(\cdot,y)\cdot\omega]$ for $y \in Y \setminus X$.

Note that, for $x \in X$ and $y \in Y \setminus X$, we deduce from $\Psi(\cdot,x) \equiv 0$ and (4.1) that

$$\int_X d_X(x,z)\, d[\rho(y)](z) = \int_\Omega d_X\big(x,\gamma(a)\big)\Psi(a,y)\, d\omega(a) \leq K \cdot d_Y(x,y),$$

and hence $\rho(y) \in \mathcal{P}_1(X)$ for all $y \in Y$. We shall show that $\delta_1'(\rho(x),\rho(y)) \leq K \cdot d_Y(x,y)$ holds for all $x, y \in Y$. The case of $x, y \in X$ is clear by definition. If $x, y \in Y \setminus X$ and $\rho(x) \neq \rho(y)$, then it follows from (4.1) that

$$\begin{aligned}
&\delta_1'\big(\rho(x),\rho(y)\big)\\
&= \frac{2}{|\rho(x)-\rho(y)|(X)} \int_{X\times X} d_X(u,v)\, d[\rho(x)-\rho(y)]_+(u)\, d[\rho(y)-\rho(x)]_+(v)\\
&\leq \frac{2}{|\rho(x)-\rho(y)|(X)}\\
&\qquad \times \int_{X\times X} \{d_Y(u,x)+d_Y(x,v)\}\, d[\rho(x)-\rho(y)]_+(u)\, d[\rho(y)-\rho(x)]_+(v)\\
&= \int_X d_Y(u,x)\, d[\rho(x)-\rho(y)]_+(u) + \int_X d_Y(v,x)\, d[\rho(y)-\rho(x)]_+(v)\\
&\leq \int_\Omega d_Y\big(\gamma(a),x\big)[\Psi(a,x)-\Psi(a,y)]_+\, d\omega(a)\\
&\qquad + \int_\Omega d_Y\big(\gamma(a),x\big)[\Psi(a,y)-\Psi(a,x)]_+\, d\omega(a)\\
&= \int_\Omega d_Y\big(\gamma(a),x\big)|\Psi(a,x)-\Psi(a,y)|\, d\omega(a)\\
&\leq K \cdot d_Y(x,y).
\end{aligned}$$

Here, as usual, we set $[\Psi(a,x)-\Psi(a,y)]_+ := \max\{\Psi(a,x)-\Psi(a,y),0\}$. If $x \in X$ and $y \in Y \setminus X$, then we find

$$\begin{aligned}\delta_1'\big(\rho(x),\rho(y)\big) &= \delta_1'\big(\delta_x,\rho(y)\big) = \int_X d_Y(x,v)\,d[\rho(y)-\delta_x]_+(v)\\ &\le \int_X d_Y(x,v)\,d[\rho(y)](v) = \int_\Omega d_Y\big(x,\gamma(a)\big)\Psi(a,y)\,d\omega(a)\\ &\le K\cdot d_Y(x,y).\end{aligned}$$

As $\delta_1^W(\rho(x),\rho(y)) \le \delta_1'(\rho(x),\rho(y))$, we complete the proof. □

Lemma 4.4 *Let (Y,d_Y) be a metric space and $X \subset Y$ be a closed subset. If there is a (K,L)-gentle partition of unity with respect to X, say $\Psi : \Omega \times Y \longrightarrow [0,\infty)$, then X is a δ_∞^V-stochastic KL-Lipschitz retract of Y.*

Proof. As in the proof of Lemma 4.3, we define a map $\rho : Y \longrightarrow \mathcal{P}(X)$ by the following:

(a) $\rho(x) := \delta_x$ for $x \in X$.

(b) $\rho(y) := \gamma_*[\Psi(\cdot,y)\cdot\omega]$ for $y \in Y \setminus X$.

The condition (4.2) for $x = y \in Y$ implies that $\rho(x) \in \mathcal{P}_\infty(X)$ for all $x \in Y$. We will see that $\delta_\infty^V(\rho(x),\rho(y)) \le KL\cdot d_Y(x,y)$ holds for all $x,y \in Y$. The case of $x,y \in X$ is immediate by definition. For every $x \in Y$, we observe that

$$\operatorname{supp}\rho(x) \subset \big[\{x\}\cup\{\gamma(a)\,|\,\Psi(a,x)>0\}\big]^-,$$

and hence (4.2) says that, for all $x,y \in Y$,

$$\operatorname{diam}\Big(\operatorname{supp}\big(\rho(x)+\rho(y)\big)\Big) \le K\cdot\big[d_Y(x,y)+\max\{d_Y(x,X),d_Y(y,X)\}\big].$$

Combining this with (4.3), we obtain, for any distinct $x,y \in Y \setminus X$,

$$\begin{aligned}&\delta_\infty^V\big(\rho(x),\rho(y)\big) = \frac{1}{2}\operatorname{diam}\Big(\operatorname{supp}\big(\rho(x)+\rho(y)\big)\Big)\cdot|\rho(x)-\rho(y)|(X)\\ &\le \frac{K}{2}\cdot\big[d_Y(x,y)+\max\{d_Y(x,X),d_Y(y,X)\}\big]\cdot\int_\Omega|\Psi(a,x)-\Psi(a,y)|\,d\omega(a)\\ &\le \frac{KL}{2}\cdot d_Y(x,y).\end{aligned}$$

If $x \in X$ and $y \in Y \setminus X$, then a similar discussion yields that

$$\begin{aligned}&\delta_\infty^V\big(\rho(x),\rho(y)\big) = \delta_\infty^V\big(\delta_x,\rho(y)\big)\\ &= \operatorname{diam}\big(\{x\}\cup\operatorname{supp}\rho(y)\big)\cdot[\rho(y)-\delta_x]_+(X)\\ &\le K\cdot\{d_Y(x,y)+d_Y(y,X)\}\cdot\int_\Omega\Psi(a,y)\,d\omega(a)\\ &\le KL\cdot d_Y(x,y).\end{aligned}$$

□

We proceed to concrete examples obtained by combining Lemmas 4.3 and 4.4 with results in [LN]. We first enumerate absolute stochastic Lipschitz retracts. We remark that every X is separable.

Example 4.5 (Doubling metric spaces, cf. [LN, Corollary 3.12]) Let us take a *doubling* metric space (X, d_X), namely there is a constant $\lambda \in \mathbb{N}$ such that every (open or closed) ball in X can be covered by at most λ balls of half the radius. Then there exists a universal constant $C > 0$ (independent of X and λ) for which X is a δ_∞^V-absolute stochastic σ_r-Lipschitz retract with a uniform bound $\sigma_r \le C \log \lambda$.

For example, every subset of a complete, n-dimensional Riemannian manifold of non-negative Ricci curvature is a δ_∞^V-absolute stochastic σ_r-Lipschitz retract with $\sigma_r \le C'n$ for a universal constant $C' > 0$.

Example 4.6 (Graphs excluding minors, cf. [LN, Lemma 3.14]) For a countable graph $G = (V, E)$ with edge lengths (weights) in $[0, \infty]$, we denote by (Σ, d_Σ) the associated one-dimensional simplicial complex with the length metric (which possibly takes values 0 and ∞). Then there exists a universal constant $C > 0$ such that, if G does not contain the complete graph on k (≥ 3) vertices as a minor (see [LN] for the definition), then every metric space (X, d_X) isometrically embedded in (Σ, d_Σ) is a δ_∞^V-absolute stochastic σ_r-Lipschitz retract with a uniform bound $\sigma_r \le Ck^2$. In particular, trees ($k = 3$) and planar graphs ($k = 5$) are δ_∞^V-absolute stochastic σ_r-Lipschitz retracts with uniform bounds on σ_r.

Example 4.7 (Surfaces of bounded genus, cf. [LN, Corollary 3.15]) Let M^2 be a two-dimensional Riemannian manifold of genus g. Then every subset $X \subset M$ is a δ_∞^V-absolute stochastic σ_r-Lipschitz retract with a uniform bound $\sigma_r \le C(g+1)$ for a universal constant $C > 0$.

Example 4.8 (Finite metric spaces, cf. [LN, Theorem 4.3]) There exists a universal constant $C > 0$ such that every metric space (X, d_X) consisting of m points is a δ_1^W-absolute stochastic σ_r-Lipschitz retract with a uniform bound $\sigma_r \le C \cdot \max\{1, \log m/(\log\log m)\}$.

We finish the list with an example of a δ_∞^V-stochastic Lipschitz retract.

Example 4.9 (Euclidean spaces, cf. [LN, Lemma 3.16]) Let us consider an n-dimensional Euclidean space $(\mathbb{R}^n, d_{\mathbb{R}^n})$ with the standard Euclidean distance. Then every subset $X \subset \mathbb{R}^n$ is a δ_∞^V-stochastic σ_r-Lipschitz retract of $\mathbb{R}^n$ with $\sigma_r \le C\sqrt{n}$ for a universal constant $C > 0$. Note that it sharpens the estimate obtained in more general Example 4.5.

4.2 $1 < p < \infty$

We start with a simple negative example which reveals a difference from the case where $p = 1, \infty$.

Example 4.10 Let (Y, d_Y) be an interval $[0, 1]$ with the standard Euclidean distance. Then the subset $X := \{0, 1\} \subset Y$ is not a δ_p^W-stochastic Lipschitz retract of Y for any $p \in (1, \infty)$, while it is a δ_∞^V-stochastic 1-Lipschitz retract of Y. In particular, X is not a Lipschitz retract of Y in the usual sense.

Given a continuous map $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_p(X), \delta_p^W)$ with $\rho(0) = \delta_0$ and $\rho(1) = \delta_1$, we set $\varphi(t) := [\rho(t)](\{1\})$ for $t \in [0,1] = Y$. If ρ is σ-Lipschitz for some $\sigma \geq 1$, then it holds that

$$|\varphi(s) - \varphi(t)| = \delta_p^W\big(\rho(s), \rho(t)\big)^p \leq \sigma^p \cdot |s-t|^p$$

for every $s, t \in [0,1]$. However, it implies that φ is constant and contradicts to $\varphi(0) = 0$ and $\varphi(1) = 1$. Therefore X is not a δ_p^W-stochastic Lipschitz retract of Y. On the other hand, a δ_∞^V-stochastic 1-Lipschitz retraction $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_\infty(X), \delta_\infty^V)$ is given by $[\rho(t)](\{0\}) := 1 - t$ (as well as $[\rho(t)](\{1\}) := t$).

Example 4.11 (Wasserstein spaces) Let (X, d_X) be a metric space and put $(Y, d_Y) = (\mathcal{P}_p(X), \delta_p^W)$. We identify X with a subset of Y through a map $X \ni x \longmapsto \delta_x \in Y$. Then X is a δ_p^W-stochastic 1-Lipschitz retract of Y.

Indeed, define a map $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_p(X), \delta_p^W) = (Y, d_Y)$ as the identity map. Then clearly ρ is 1-Lipschitz and $\rho(x) = \delta_x$ for $x \in X$.

Example 4.12 (L_p-spaces) Given a metric space (X, d_X) and a probablity space (Ω, ω), we define $L_p(\Omega; X)$ as the set of all Borel measurable maps $\varphi : \Omega \longrightarrow X$ satisfying

$$\int_\Omega d_X\big(\varphi(a), x\big)^p \, d\omega(a) < \infty$$

for some (and hence all) point $x \in X$, equipped with an equivalence relation such that $\varphi \sim \psi$ holds if we have $\varphi(a) = \psi(a)$ for ω-a.e. $a \in \Omega$. For two maps $\varphi, \psi \in L_p(\Omega; X)$, we set

$$d_p^L(\varphi, \psi) := \bigg(\int_\Omega d_X\big(\varphi(a), \psi(a)\big)^p \, d\omega(a) \bigg)^{1/p}.$$

Then d_p^L provides a distance function on $L_p(\Omega; X)$ and we put $(Y, d_Y) = (L_p(\Omega; X), d_p^L)$. We can regard X as a subset of Y by associating $x \in X$ with a constant map to x. Then X is a δ_p^W-stochastic 1-Lipschitz retract of Y.

Define a map $\rho : (Y, d_Y) \longrightarrow (\mathcal{P}_p(X), \delta_p^W)$ by $\rho(\varphi) := \varphi_*\omega$ and note that $\rho(x) = \delta_x$ for $x \in X$. For every $\varphi, \psi \in L_p(\Omega; X)$, as $(\varphi \times \psi)_*\omega \in \mathcal{P}(X \times X)$ is a coupling of $\rho(\varphi) = \varphi_*\omega$ and $\rho(\psi) = \psi_*\omega$, we obtain

$$\delta_p^W\big(\rho(\varphi), \rho(\psi)\big) \leq \bigg(\int_\Omega d_X\big(\varphi(a), \psi(a)\big)^p \, d\omega(a) \bigg)^{1/p} = d_p^L(\varphi, \psi).$$

5 Examples: Target spaces

This section is devoted to examples of barycentric metric spaces. The linear case is easy, and the nonlinear case has a connection with upper curvature bounds.

5.1 Banach spaces

Example 5.1 Every separable Banach space $(Z, \|\cdot\|)$ is δ_1^W-barycentric with $\beta_c = 1$.

In order to see this, we set $c(\mu) := \int_Z z\, d\mu(z)$ for $\mu \in \mathcal{P}_1(Z)$. Then clearly $c(\delta_z) = z$ for each $z \in Z$. Moreover, for any $\mu, \nu \in \mathcal{P}_1(Z)$ and any coupling $q \in \mathcal{P}(Z \times Z)$ of μ and ν, we observe

$$\begin{aligned}\|c(\mu) - c(\nu)\| &= \left\| \int_Z z\, d\mu(z) - \int_Z w\, d\nu(w) \right\| = \left\| \int_{Z\times Z} (z-w)\, dq(z,w) \right\| \\ &\le \int_{Z\times Z} \|z - w\|\, dq(z,w).\end{aligned}$$

Taking the infimum over all couplings q, we obtain $\|c(\mu) - c(\nu)\| \le \delta_1^W(\mu, \nu)$.

The separability assumption on Z is used to guarantee that the identity map on Z is *Bochner integrable* with respect to the measures μ and ν (cf., for example, [BL, Chapter 5]). In view of Lemma 3.5, it is sufficient to suppose only the separability of the support of $f_*[\rho(y)]$ for all $y \in Y$, e.g., X is separable and f is continuous.

5.2 CAT(0)-spaces

We review some standard terminologies in metric geometry. Let (Z, d_Z) be a metric space. A rectifiable curve $\eta : [0, l] \longrightarrow Z$ is called a *geodesic* if it is locally minimizing and has a constant speed, i.e., parametrized proportionally to the arclength. A geodesic $\eta : [0, l] \longrightarrow Z$ is said to be *minimal* if it satisfies $\mathrm{length}(\eta) = d_Z(\eta(0), \eta(l))$. We say that (Z, d_Z) is *geodesic* if every two points in Z can be joined by a minimal geodesic between them.

For $\kappa \in \mathbb{R}$, we denote by $\mathbb{M}^2(\kappa)$ a complete, simply-connected, two-dimensional Riemannian manifold of constant sectional curvature κ. That is to say, $\mathbb{M}^2(\kappa)$ is a two-sphere ($\kappa > 0$) or a Euclidean plane ($\kappa = 0$) or a hyperbolic plane ($\kappa < 0$). Given a point $z \in Z$ and a minimal geodesic $\eta : [0,1] \longrightarrow Z$ (provided that $d_Z(z, \eta(0)) + d_Z(z, \eta(1)) + d_Z(\eta(0), \eta(1)) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$), we can take a corresponding point $\tilde{z} \in \mathbb{M}^2(\kappa)$ and a geodesic $\tilde{\eta} : [0,1] \longrightarrow \mathbb{M}^2(\kappa)$ (which are unique up to an isometry) such that

$$\begin{aligned}&d_{\mathbb{M}^2(\kappa)}\big(\tilde{z}, \tilde{\eta}(0)\big) = d_Z\big(z, \eta(0)\big), \qquad d_{\mathbb{M}^2(\kappa)}\big(\tilde{z}, \tilde{\eta}(1)\big) = d_Z\big(z, \eta(1)\big), \\ &d_{\mathbb{M}^2(\kappa)}\big(\tilde{\eta}(0), \tilde{\eta}(1)\big) = d_Z\big(\eta(0), \eta(1)\big).\end{aligned}$$

Definition 5.2 (CAT(κ)-spaces) Let (Z, d_Z) be a geodesic metric space and $\kappa \in \mathbb{R}$. We say that (Z, d_Z) is a *CAT(κ)-space* if, for any point $z \in Z$, any minimal geodesic $\eta : [0,1] \longrightarrow Z$ (provided that $d_Z(z, \eta(0)) + d_Z(z, \eta(1)) + d_Z(\eta(0), \eta(1)) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and for any $\lambda \in [0,1]$, we have

$$d_Z\big(z, \eta(\lambda)\big) \le d_{\mathbb{M}^2(\kappa)}\big(\tilde{z}, \tilde{\eta}(\lambda)\big). \tag{5.1}$$

In a particular case where $\kappa = 0$, the inequality (5.1) is rewritten as

$$d_Z\big(z, \eta(\lambda)\big)^2 \le (1-\lambda) d_Z\big(z, \eta(0)\big)^2 + \lambda d_Z\big(z, \eta(1)\big)^2 - (1-\lambda)\lambda d_Z\big(\eta(0), \eta(1)\big)^2. \tag{5.2}$$

Fundamental examples of CAT(0)-spaces are

(a) complete, simply-connected Riemannian manifolds of nonpositive sectional curvature,

(b) Hilbert spaces,

(c) Euclidean buildings,

(d) trees,

as well as their ℓ_2-products and spaces of L_2-maps into them. See [St] and references therein for more on CAT(0)-spaces. Now, [St, Theorem 6.3] asserts the following.

Example 5.3 Every complete CAT(0)-space (Z, d_Z) is δ_1^W-barycentric with $\beta_c = 1$.

Here we briefly review the discussion in [St]. For $\mu \in \mathcal{P}_1(Z)$ and $v \in Z$, the function

$$Z \ni z \longmapsto \int_Z \{d_Z(z,w)^2 - d_Z(v,w)^2\}\, d\mu(w)$$

possesses a unique minimizer $c(\mu) \in Z$ which is independent of the choice of $v \in Z$ (see Lemma 5.5 below). Note that $c(\delta_z) = z$ for $z \in Z$. As (Z, d_Z) is a CAT(0)-space, the set

$$A := \{(z,w,t) \in Z \times Z \times \mathbb{R} \,|\, d_Z(z,w) \le t\}$$

is a closed convex subset of $Z \times Z \times \mathbb{R}$. Define a map $\varphi : Z \times Z \longrightarrow Z \times Z \times \mathbb{R}$ by $\varphi(z,w) := (z, w, d_Z(z,w))$. Since the ℓ_2-product $Z \times Z \times \mathbb{R}$ is again a CAT(0)-space, for every $\mu, \nu \in \mathcal{P}_1(Z)$ and every coupling $q \in \mathcal{P}(Z \times Z)$ of μ and ν, we obtain

$$c(\varphi_* q) = \left(c(\mu), c(\nu), \int_{Z \times Z} d_Z(z,w)\, dq(z,w) \right) \in A.$$

By taking the definition of the set A into account, we find that $d_Z(c(\mu), c(\nu)) \le \delta_1^W(\mu, \nu)$.

5.3 2-uniformly convex metric spaces

Definition 5.4 (2-uniformly convex metric spaces) We say that a geodesic metric space (Z, d_Z) is 2-*uniformly convex* if there is a constant $C \ge 1$ such that, for any $z \in Z$, minimal geodesic $\eta : [0,1] \longrightarrow Z$ and for any $\lambda \in [0,1]$, we have

$$d_Z\big(z, \eta(\lambda)\big)^2 \le (1-\lambda) d_Z\big(z, \eta(0)\big)^2 + \lambda d_Z\big(z, \eta(1)\big)^2 - C^{-2}(1-\lambda)\lambda d_Z\big(\eta(0), \eta(1)\big)^2.$$

We denote the infimum of such constants $C \ge 1$ by C_Z.

The term '2-uniform convexity' (or, equivalently, the *modulus of convexity of power type* 2) has its root in the theory of Banach spaces (see [BCL]), and it is also regarded as a generalization of the CAT(0)-property which amounts to the case where $C = 1$ (see (5.2)). We refer to [O1] for some geometric and analytic properties of 2-uniformly convex metric spaces (which are called *k-convex* spaces there). Examples of 2-uniformly convex metric spaces besides CAT(0)-spaces are

(a) ℓ_p-spaces with $1 < p \le 2$, where $C_Z = 1/\sqrt{p-1}$,

(b) ([O1]) CAT(1)-spaces (e.g., convex sets in a unit sphere) whose diameters, say D, are less than $\pi/2$, where $C_Z \le \sqrt{D^{-1}\tan D}$.

Given $\mu \in \mathcal{P}_1(Z)$ and $v \in Z$, as in CAT(0)-spaces, we define a function $h_{v,\mu} : Z \longrightarrow [0,\infty)$ by

$$h_{v,\mu}(z) := \int_Z \{d_Z(z,w)^2 - d_Z(v,w)^2\}\, d\mu(w).$$

Note that

$$h_{v,\mu}(z) \le d_Z(z,v) \int_Z \{d_Z(z,w) + d_Z(v,w)\}\, d\mu(w) < \infty.$$

Lemma 5.5 *Let (Z, d_Z) be a complete, 2-uniformly convex metric space. For every $\mu \in \mathcal{P}_1(Z)$ and $v \in Z$, the infimum of the function $h_{v,\mu}$ is attained at a unique point $z_0 \in Z$ and it is independent of the choice of $v \in Z$.*

Proof. Set $h_0 := \inf_{z\in Z} h_{v,\mu}(z)$ and take a minimizing sequence $\{z_i\}_{i\in\mathbb{N}} \subset Z$ of $h_{v,\mu}$. Given $i, j \in \mathbb{N}$, let $\eta : [0,1] \longrightarrow Z$ be a minimal geodesic between them. Then the 2-uniform convexity of Z shows that

$$\begin{aligned}
h_0 &\le h_{v,\mu}\big(\eta(1/2)\big)\\
&\le \int_Z \left\{ \frac{1}{2} d_Z(z_i,w)^2 + \frac{1}{2} d_Z(z_j,w)^2 - \frac{C_Z^{-2}}{4} d_Z(z_i,z_j)^2 - d_Z(v,w)^2 \right\} d\mu(w)\\
&= \frac{1}{2} h_{v,\mu}(z_i) + \frac{1}{2} h_{v,\mu}(z_j) - \frac{C_Z^{-2}}{4} d_Z(z_i,z_j)^2.
\end{aligned}$$

Thus we have $\lim_{i,j\to\infty} d_Z(z_i,z_j) = 0$, that is, $\{z_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence. Therefore $\{z_i\}_{i\in\mathbb{N}}$ converges to a point $z_0 \in Z$ as i diverges to the infinity, and z_0 is a unique minimizer of $h_{v,\mu}$. The independence from the choice of $v \in Z$ is deduced from the fact that, for $v, v' \in Z$,

$$h_{v,\mu}(z) - h_{v',\mu}(z) = \int_Z \{d_Z(v',w)^2 - d_Z(v,w)^2\}\, d\mu(w)$$

does not depend on $z \in Z$. $\square$

We define $c(\mu) := z_0$ (given in Lemma 5.5) for $\mu \in \mathcal{P}_1(Z)$. For $z \in Z$ and $r > 0$, we denote by $B(z,r)$ and $\overline{B}(z,r)$ open and closed balls of center z and radius r, respectively.

Lemma 5.6 *Let (Z, d_Z) be a complete, 2-uniformly convex metric space. Given $\mu \in \mathcal{P}_\infty(Z)$, $z \in Z$ and $r > 0$, if $\operatorname{supp}\mu \subset \overline{B}(z,r)$, then we have $c(\mu) \in B(z,2r)$.*

Proof. Put $l := d_Z(z, c(\mu))$ and let $\eta : [0,l] \longrightarrow Z$ be a minimal geodesic from z to $c(\mu)$. We suppose $l \ge 2r$ and will derive a contradiction. For every $w \in \operatorname{supp}\mu \subset \overline{B}(z,r)$, it follows from the 2-uniform convexity of Z that

$$d_Z\big(w,\eta(r)\big)^2 \le \left(1 - \frac{r}{l}\right) d_Z(w,z)^2 + \frac{r}{l} d_Z\big(w,c(\mu)\big)^2 - C_Z^{-2}\left(1 - \frac{r}{l}\right)\frac{r}{l} l^2,$$

and hence

$$\begin{aligned} & d_Z\big(w, \eta(r)\big)^2 - d_Z\big(w, c(\mu)\big)^2 \\ & \le \left(1 - \frac{r}{l}\right)\big\{d_Z(w,z)^2 - d_Z\big(w, c(\mu)\big)^2 - C_Z^{-2} rl\big\}. \end{aligned}$$

However, since

$$d_Z\big(w, c(\mu)\big) \ge d_Z\big(z, c(\mu)\big) - d_Z(z,w) \ge 2r - r = r \ge d_Z(w,z),$$

it implies

$$d_Z\big(w, \eta(r)\big)^2 - d_Z\big(w, c(\mu)\big)^2 \le -C_Z^{-2}\left(1 - \frac{r}{l}\right) rl.$$

Integrating this inequality by μ, we find, for an arbitrarily fixed $v \in Z$,

$$h_{v,\mu}\big(\eta(r)\big) \le h_{v,\mu}\big(c(\mu)\big) - C_Z^{-2}\left(1 - \frac{r}{l}\right) rl.$$

It contradicts to the minimality of $c(\mu)$. Thus we obtain $c(\mu) \in B(z, 2r)$. $\square$

Proposition 5.7 *Every complete and 2-uniformly convex metric space (Z, d_Z) is δ_∞^V-barycentric with $\beta_c = 4C_Z^2$.*

Proof. We demonstrate along the line of [LPS, Lemma 4.3]. Fix $\mu, \nu \in \mathcal{P}_\infty(Z)$ and $v \in Z$. We first show that, for all $z \in Z$,

$$h_{v,\mu}\big(c(\mu)\big) \le h_{v,\mu}(z) - C_Z^{-2} \cdot d_Z\big(c(\mu), z\big). \tag{5.3}$$

Let $\eta : [0,1] \longrightarrow Z$ be a minimal geodesic from z to $c(\mu)$. Then the 2-uniform convexity of Z yields that, for each $\lambda \in (0,1)$,

$$\begin{aligned} & h_{v,\mu}\big(c(\mu)\big) \le h_{v,\mu}\big(\eta(\lambda)\big) \\ & \le (1-\lambda) \int_Z \{d_Z(z,w)^2 - d_Z(v,w)^2\}\, d\mu(w) \\ & \qquad + \lambda \int_Z \big\{d_Z\big(c(\mu), w\big)^2 - d_Z(v,w)^2\big\}\, d\mu(w) - C_Z^{-2}(1-\lambda)\lambda \cdot d_Z\big(z, c(\mu)\big)^2 \\ & = (1-\lambda) h_{v,\mu}(z) + \lambda h_{v,\mu}\big(c(\mu)\big) - C_Z^{-2}(1-\lambda)\lambda \cdot d_Z\big(z, c(\mu)\big)^2. \end{aligned}$$

Thus we have

$$h_{v,\mu}\big(c(\mu)\big) \le h_{v,\mu}(z) - C_Z^{-2}\lambda \cdot d_Z\big(z, c(\mu)\big)^2.$$

By letting λ tend to 1, we deduce (5.3).

We apply (5.3) to $(\mu, c(\nu))$ and $(\nu, c(\mu))$ and see

$$\begin{aligned}
&2C_Z^{-2} \cdot d_Z\big(c(\mu), c(\nu)\big)^2 \\
&\le h_{v,\mu}\big(c(\nu)\big) - h_{v,\mu}\big(c(\mu)\big) + h_{v,\nu}\big(c(\mu)\big) - h_{v,\nu}\big(c(\nu)\big) \\
&= \int_Z \big\{d_Z\big(c(\nu), w\big)^2 - d_Z\big(c(\mu), w\big)^2\big\}\, d\mu(w) + \int_Z \big\{d_Z\big(c(\mu), w\big)^2 - d_Z\big(c(\nu), w\big)^2\big\}\, d\nu(w) \\
&\le \int_Z \big|d_Z\big(c(\nu), w\big)^2 - d_Z\big(c(\mu), w\big)^2\big|\, d|\mu - \nu|(w).
\end{aligned}$$

Note that it follows from Lemma 5.6 that, for any $w \in \operatorname{supp}(\mu + \nu)$,

$$\begin{aligned}
&\big|d_Z\big(c(\nu), w\big)^2 - d_Z\big(c(\mu), w\big)^2\big| \\
&\le \big\{d_Z\big(c(\nu), w\big) + d_Z\big(c(\mu), w\big)\big\} \cdot d_Z\big(c(\mu), c(\nu)\big) \\
&\le 2\big[\operatorname{diam}(\operatorname{supp}\nu \cup \{w\}) + \operatorname{diam}(\operatorname{supp}\mu \cup \{w\})\big] \cdot d_Z\big(c(\mu), c(\nu)\big) \\
&\le 4 \operatorname{diam}\big(\operatorname{supp}(\mu + \nu)\big) \cdot d_Z\big(c(\mu), c(\nu)\big).
\end{aligned}$$

Thus we obtain

$$d_Z\big(c(\mu), c(\nu)\big) \le 2C_Z^2 \operatorname{diam}\big(\operatorname{supp}(\mu + \nu)\big) \cdot |\mu - \nu|(Z) = 4C_Z^2 \cdot \delta_\infty^V(\mu, \nu).$$

This completes the proof. □

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