Convexities of metric spaces[*][†]

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Abstract

We introduce two kinds of the notion of convexity of a metric space, called k-convexity and L-convexity, as generalizations of the CAT(0)-property and of the nonpositively curved property in the sense of Busemann, respectively. 2-uniformly convex Banach spaces as well as CAT(1)-spaces with small diameters satisfy both these convexities. Among several geometric and analytic results, we prove the solvability of the Dirichlet problem for maps into a wide class of metric spaces.

1 Introduction

CAT(0)-spaces or, more generally, nonpositively curved metric spaces in the sense of Busemann (NPC spaces for short) are one of the most important objects in both of the geometry and the analysis on metric spaces (see [Ba], [BH], [J], [KS], and references therein). On one hand, the CAT(0)-property of a geodesic metric space is defined as a generalization of the nonpositivity of the sectional curvature on a Riemannian manifold. On the other hand, the definition can also be regarded as the convexity of (the square of) the distance function.

In this article, we introduce two generalized notions of the convexity, the k-convexity and the L-convexity, of geodesic metric spaces (Definitions 2.1 and 2.6). Actually the k-convexity and the L-convexity include the CAT(0)-property and the NPC property as special cases, respectively. Moreover, 2-uniformly convex Banach spaces (e.g., l^p-spaces with $1 < p \le 2$) as well as CAT(1)-spaces with diameters less than $\pi/2$ are both k-convex and L-convex (Section 3). It is interesting and is an advantage of this context to be able to treat these different kinds of spaces simultaneously.

For a k-convex and/or L-convex metric space (X, d_X), it is natural to ask whether the known facts on CAT(0)-spaces and NPC spaces are still true or not. We first consider

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some geometric properties. Since X may be a Banach space, the canonical discussion using the 'angle' is needed to be modified in this setting. For instance, we can not regard spaces of directions and the boundary at infinity as metric spaces. Nevertheless, it is possible to define directly the distances on tangent cones and the cone at infinity, and then the convexity of X descends to them (Propositions 4.2 and 4.4). We also prove the first variation formula for arclength (Theorem 5.2), which generalizes that on Alexandrov spaces ([OS], [OT]) and plays a key role in Section 8.

In the latter half of the article, we study the Cheeger-type Sobolev spaces for maps from an arbitrary metric measure space into a k-convex and/or L-convex metric space, and extend some results obtained in [O1]. In this setting, the energy forms are not convex in the usual sense, and this is the most critical difficulty compared with the case where target spaces are NPC. We intend to use the L-convexity as a substitution for the NPC property, and prove the existence of a minimizer of the energy form. The essential idea is seen in the proof of the unique existence of a minimal generalized upper gradient (Theorem 7.3). As one of our main results, we solve the Dirichlet problem (Theorem 9.4) in the case where the target space is proper and L-convex. This extends the results in [EF], [F], [KS], and [O1] to maps into a much wider class of metric spaces, for target spaces admit not only positive cuvatures, but also Finsler structures. We mention that the regularity of harmonic maps, with respect to the Korevaar-Schoen-type energy form ([KS]), from a Riemannian manifold into a CAT(1)-space is studied by Serbinowski in the appendix of [EF] (see also [F]).

The article is organized as follows: We define the k-convexity and the L-convexity in Section 2, and give examples in Section 3. In Section 4, we consider some geometric properties. We prove the first variation formula for arclength in Section 5. Sections 6–9 are devoted to the study of the Cheeger-type Sobolev spaces for maps into a k-convex and/or L-convex metric space. After recalling the definition of the Cheeger-type Sobolev space in Section 6, we prove the unique existence of a minimal generalized upper gradient and the minimality of $\mathrm{Lip}\, u$ for a Lipschitz map u in Sections 7 and 8, respectively. In the last section, we treat the Dirichlet problem.

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2 Definitions of convexities

Throughout this article, let (X, d_X) be a geodesic metric space. A rectifiable curve $\gamma : [0, l] \longrightarrow X$ is called a *geodesic* if it is locally minimizing and has a constant speed (i.e., parametrized proportionally to the arclength). A metric space (X, d_X) is said to be *geodesic* if any two points $x, y \in X$ can be connected by a rectifiable curve $\gamma : [0, l] \longrightarrow X$ satisfying $\gamma(0) = x$, $\gamma(l) = y$, and $\mathrm{length}(\gamma) = d_X(x, y)$. Clearly such γ is a geodesic if it has a constant speed, so that we call such a curve a *minimal* geodesic between x and y. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball with center x and radius r. For

$a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Henceforce, we denote $d_X(x, y)$ by $|x - y|_X$ for brevity.

2.1 k-convexity

Definition 2.1 Let $k \in (0, 2]$.

(1) An open set U in a geodesic metric space (X, d_X) is called a *C_k-domain* for k if, for any three points $x, y, z \in U$, any minimal geodesic $\gamma : [0, 1] \longrightarrow X$ between y and z, and for all $t \in [0, 1]$, we have

$$|x - \gamma(t)|_X^2 \le (1 - t)|x - y|_X^2 + t|x - z|_X^2 - \frac{k}{2}(1 - t)t|y - z|_X^2. \tag{2.1}$$

(2) A geodesic metric space (X, d_X) is said to be *k-convex* for k if X itself is a C_k-domain.

(3) A geodesic metric space (X, d_X) is said to be *locally k-convex* for k if every point in X is contained in a C_k-domain.

(4) A geodesic metric space (X, d_X) is said to be *locally (k)-convex* if every point $x \in X$ is contained in a C_k-domain for some $k = k(x) \in (0, 2]$.

The inequality (2.1) means that $(d^2/dt^2)|x - \gamma(t)|_X^2 \ge k|\gamma(0) - \gamma(1)|_X^2$ in the weak sense. We remark that the k-convexity for $k = 2$ coincides with the CAT(0)-property. As for the CAT(0)-property, the inequality (2.1) for $t = 1/2$ implies that for all $t \in [0, 1]$.

Lemma 2.2 *If an open ball $B(x, r) \subset X$ is a C_k-domain, then, for any two points in $B(x, r)$, a minimal geodesic between them is unique. In particulr, any two points in a k-convex metric space are connected by a unique minimal geodesic.*

Proof. Fix two points $y, z \in B(x, r)$ and let $\gamma, \xi : [0, 1] \longrightarrow X$ be two minimal geodesics between y and z. Note that γ and ξ are contained in $B(x, r)$ by the k-convexity. For $t \in (0, 1)$, take a minimal geodesic $\eta : [0, 1] \longrightarrow X$ from $\gamma(t)$ to $\xi(t)$. It follows from the k-convexity that

$$\begin{aligned} |y - \eta(1/2)|_X^2 &\le \frac{1}{2}|y - \gamma(t)|_X^2 + \frac{1}{2}|y - \xi(t)|_X^2 - \frac{k}{8}|\gamma(t) - \xi(t)|_X^2 \\ &= t^2|y - z|_X^2 - \frac{k}{8}|\gamma(t) - \xi(t)|_X^2, \end{aligned}$$

and that

$$|z - \eta(1/2)|_X^2 \le (1 - t)^2|y - z|_X^2 - \frac{k}{8}|\gamma(t) - \xi(t)|_X^2.$$

Therefore we have

$$\begin{aligned} |y - z|_X \le &\sqrt{t^2|y - z|_X^2 - \frac{k}{8}|\gamma(t) - \xi(t)|_X^2} \\ &+ \sqrt{(1 - t)^2|y - z|_X^2 - \frac{k}{8}|\gamma(t) - \xi(t)|_X^2}. \end{aligned}$$

It implies $\gamma(t) = \xi(t)$. □

For $y, z \in B(x, r)$, we denote the unique minimal geodesic from y to z by $\gamma_{yz} : [0, 1] \longrightarrow B(x, r)$, and $\gamma_{yz}(t)$ by $(1-t)y + tz$.

Lemma 2.3 *Assume that $B(x, r)$ is a C_k-domain. Then, for any $y, z \in B(x, r)$ and all $t \in [0, 1]$, we have*

$$|x - \gamma_{yz}(t)|_X^2 \le \frac{2}{k}\{(1-t)|x-y|_X^2 + t|x-z|_X^2 - (1-t)t|y-z|_X^2\}. \tag{2.2}$$

Proof. Put $\gamma = \gamma_{yz}$, fix $t \in (0, 1)$, and let $\xi : [0, 1] \longrightarrow X$ be a minimal geodesic from x to $\gamma(t)$. By the k-convexity, for any $s \in (0, 1)$, we see

$$\begin{aligned} |y - \xi(s)|_X^2 &\le (1-s)|y-x|_X^2 + s|y-\gamma(t)|_X^2 - \frac{k}{2}(1-s)s|x-\gamma(t)|_X^2 \\ &= (1-s)|x-y|_X^2 + st^2|y-z|_X^2 - \frac{k}{2}(1-s)s|x-\gamma(t)|_X^2. \end{aligned}$$

Similarly, we have

$$|z - \xi(s)|_X^2 \le (1-s)|x-z|_X^2 + s(1-t)^2|y-z|_X^2 - \frac{k}{2}(1-s)s|x-\gamma(t)|_X^2.$$

These yield

$$\begin{aligned} &|y-z|_X^2 \le (|y-\xi(s)|_X + |\xi(s)-z|_X)^2 \\ &\le \frac{1}{t}|y-\xi(s)|_X^2 + \frac{1}{1-t}|z-\xi(s)|_X^2 \\ &\le \frac{1}{t}\left\{(1-s)|x-y|_X^2 + st^2|y-z|_X^2 - \frac{k}{2}(1-s)s|x-\gamma(t)|_X^2\right\} \\ &\quad + \frac{1}{1-t}\left\{(1-s)|x-z|_X^2 + s(1-t)^2|y-z|_X^2 - \frac{k}{2}(1-s)s|x-\gamma(t)|_X^2\right\} \\ &= (1-s)\left(\frac{1}{t}|x-y|_X^2 + \frac{1}{1-t}|x-z|_X^2\right) + s|y-z|_X^2 \\ &\quad - \frac{k}{2(1-t)t}(1-s)s|x-\gamma(t)|_X^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |x-\gamma(t)|_X^2 &\le \frac{2(1-t)t}{ks}\left(\frac{1}{t}|x-y|_X^2 + \frac{1}{1-t}|x-z|_X^2 - |y-z|_X^2\right) \\ &= \frac{2}{ks}\{(1-t)|x-y|_X^2 + t|x-z|_X^2 - (1-t)t|y-z|_X^2\}. \end{aligned}$$

Letting s tend to 1, we complete the proof. □

The inequality (2.2) can be regarded as the '$(2/k) \times$ CAT(0)-inequality'.

Corollary 2.4 *Every k-convex metric space is contractible.*

Proof. Fix a point $x \in X$. For $t \in [0,1]$, we define a map $\Phi_t : X \longrightarrow X$ by $\Phi_t(y) := \gamma_{xy}(t)$. Then $\Phi_0(y) = x$ and $\Phi_1(y) = y$ for all $y \in X$. It suffices to show that $\Phi_t(y)$ is continuous in y. Lemma 2.3 yields that, for any $y, z \in X$,

$$\begin{aligned}
&|\Phi_t(y) - \Phi_t(z)|_X^2 \\
&\le \frac{2}{k}\big\{(1-t)|\Phi_t(y) - x|_X^2 + t|\Phi_t(y) - z|_X^2 - (1-t)t|x-z|_X^2\big\} \\
&\le \frac{2}{k}\big\{(1-t)t^2|x-y|_X^2 + t(|\Phi_t(y) - y|_X + |y-z|_X)^2 \\
&\quad - (1-t)t(|x-y|_X - |y-z|_X)^2\big\} \\
&= \frac{2}{k}\big\{(1-t)t^2|x-y|_X^2 + t(1-t)^2|x-y|_X^2 + 2t(1-t)|x-y|_X|y-z|_X \\
&\quad + t|y-z|_X^2 - (1-t)t|x-y|_X^2 + 2(1-t)t|x-y|_X|y-z|_X \\
&\quad - (1-t)t|y-z|_X^2\big\} \\
&= \frac{2}{k}\big\{t^2|y-z|_X^2 + 4(1-t)t|x-y|_X|y-z|_X\big\}.
\end{aligned}$$

Hence $|\Phi_t(y) - \Phi_t(z)|_X$ tends to zero as $|y-z|_X$ goes to zero. □

Remark 2.5 In [O3], we consider the converse inequality of (2.1) in a Banach space $(V, |\cdot|)$, more precisely,

$$2|v|^2 + 2|w|^2 - |v+w|^2 \le K|v-w|^2$$

for $K \ge 1$ and $v, w \in V$. We have observed that this condition geometrically means that the (one-dimensional) curvature of the unit sphere of V is not greater than K (see [O3, Section 2]). In this context, the k-convexity on a Banach space can be interpreted as a kind of lower curvature bound on its unit sphere. Compare this with Subsection 3.2 in this article.

2.2 L-convexity

Definition 2.6 Let $L_1, L_2 \ge 0$. An open set U in a geodesic metric space (X, d_X) is called a *C_L-domain* for (L_1, L_2) if, for any three points $x, y, z \in U$, any minimal geodesics $\gamma, \xi : [0,1] \longrightarrow X$ with $\gamma(0) = \xi(0) = x$, $\gamma(1) = y$, and $\xi(1) = z$, and for all $t \in [0,1]$, we have

$$|\gamma(t) - \xi(t)|_X \le \left(1 + L_1 \frac{(|x-y|_X + |x-z|_X) \wedge 2L_2}{2}\right) t|y-z|_X.$$

The *L-convexity*, the *local L-convexity*, and the *local (L)-convexity* of a geodesic metric space are defined in the same manner as for the k-convexity (see Definition 2.1).

The following are straightforward by the definition.

Lemma 2.7 *Any two points in a C_L-domain are connected by a unique minimal geodesic.*

Lemma 2.8 *Every L-convex metric space is contractible.*

Lemma 2.9 *Let $U \subset X$ be a C_L-domain, $x, y, z \in U$, and let $\gamma : [0,1] \longrightarrow U$ be a (unique) minimal geodesic between y and z. Then, for all $t \in [0,1]$, we have*

$$|\gamma_{xy}(t) - \gamma_{xz}(t)|_X \le \left(1 + L_1 \int_0^1 (|x - \gamma(s)|_X \wedge L_2)\, ds\right) t|y - z|_X.$$

The L-convexity for $L_1 = 0$ amounts to the nonpositively curved property in the sense of Busemann (see [J] for the definition). Therefore the k-convextiy for $k = 2$ (i.e., the CAT(0)-property) implies the L-convexity for $L_1 = 0$. However, it is not clear whether the analogue holds for general k, that is, whether the k-convexity for $k \in (0,2)$ implies the L-convexity for some $(L_1, L_2) = (L_1(k), L_2(k))$ or not. We know only that we can not take $L_1(k) = 0$ for $k \in (0,2)$ (see Proposition 3.1 below). On the other hand, since every strictly convex Banach space is clearly L-convex for $L_1 = 0$, the k-convexity does not follow from the L-convexity even for $L_1 = 0$ (see Proposition 3.3 below).

3 Examples

3.1 CAT(1)-spaces

In this subsection, we will show the following:

Proposition 3.1 (i) *A* CAT(1)*-space (X, d_X) with* $\operatorname{diam} X \le \pi/2 - \varepsilon$, $\varepsilon \in (0, \pi/2)$, *is k-convex for $k = (\pi - 2\varepsilon)\sin\varepsilon / \cos\varepsilon$.*

(ii) *Let (X, d_X) be a* CAT(1)*-space in which no triangle has a perimeter greater than 2π. If, in addition,* $\operatorname{diam} X \le \pi - \varepsilon$ *with $\varepsilon \in (0, \pi)$, then (X, d_X) is L-convex for*

$$(L_1, L_2) = \left(\frac{2\{(\pi - \varepsilon) - \sin\varepsilon\}}{(\pi - \varepsilon)\sin\varepsilon}, \pi - \varepsilon\right).$$

Proof. (i) We first show the k-convexity. To do this, it is sufficient to consider the two-dimensional sphere $\mathbb{S}^2$. Take three points $x, y, z \in \mathbb{S}^2$ with $|x-y|_{\mathbb{S}^2} \vee |x-z|_{\mathbb{S}^2} \vee |y-z|_{\mathbb{S}^2} \le \pi/2 - \varepsilon$ and set $a := |x-y|_{\mathbb{S}^2}$, $b := |x-z|_{\mathbb{S}^2}$, $c := |y-z|_{\mathbb{S}^2}/2$, and $d := |x - \gamma_{yz}(1/2)|_{\mathbb{S}^2}$. Then we need to estimate

$$f(a,b,c) := \frac{2}{c^2}\left(\frac{1}{2}a^2 + \frac{1}{2}b^2 - d^2\right)$$

from below. The proof consists of three steps.

Step 1 We may assume $a = b$.

Fix b and c, assume $a \le b$, and put $f(a) := f(a,b,c)$. Then d is a function of a. By using the spherical cosine formula, we have

$$\cos d(a) = \frac{\cos a + \cos b}{2\cos c}, \qquad d'(a) = -\frac{(\cos d)'}{\sin d} = \frac{\sin a}{2\sin d\cos c}.$$

Thus we find

$$f'(a) = \frac{2}{c^2}\{a - 2dd'(a)\} = \frac{2}{c^2}\left\{a - \frac{d\sin a}{\sin d\cos c}\right\}.$$

Note that $f'(0) = 0$ and, for $0 < a \leq d$,

$$f'(a) \leq \frac{2}{c^2}\left\{a - \frac{d\sin a}{\sin d}\right\} \leq 0.$$

Moreover, for $d \leq a \leq b$, it follows from $\cos d \leq \cos a/\cos c$ that

$$f'(a) \leq \frac{2}{c^2}\left\{a - \frac{d\sin a\cos d}{\sin d\cos a}\right\} \leq 0.$$

Hence we obtain $f(a) \geq f(b)$.

Step 2 We can suppose $a = \pi/2 - \varepsilon$.

Fix c, assume $a = b$, and put $f(a) := f(a, a, c)$. Then d is a function of a and we have

$$f'(a) = \frac{2}{c^2}\left\{a - \frac{d\sin a\cos d}{\sin d\cos a}\right\} \leq 0$$

since $a \geq d$. Thus we obtain $f(a) \geq f(\pi/2 - \varepsilon)$.

Step 3 It is sufficient to take a limit as $c \to 0$.

We assume $a = b = \pi/2 - \varepsilon$ and put $f(c) := f(\pi/2 - \varepsilon, \pi/2 - \varepsilon, c)$. By the spherical cosine formula, we know

$$\cos d = \frac{\cos(\pi/2 - \varepsilon)}{\cos c} = \frac{\sin\varepsilon}{\cos c}.$$

If we consider d as a function of c, then we observe

$$d'(c) = -\frac{(\cos d)'}{\sin d} = -\frac{1}{\sin d}\frac{\sin\varepsilon\sin c}{\cos^2 c} = -\frac{\sin c\cos d}{\cos c\sin d}.$$

On the other hand, we calculate

$$f(c) = \frac{2}{c^2}\left\{\left(\frac{\pi}{2} - \varepsilon\right)^2 - d^2\right\},\ f'(c) = \frac{2}{c^4}\left[-2dd'c^2 - 2c\left\{\left(\frac{\pi}{2} - \varepsilon\right)^2 - d^2\right\}\right].$$

Put $g(c) := -d'dc - (\pi/2 - \varepsilon)^2 + d^2$. Then $g(0) = 0$ and

$$\begin{aligned} g'(c) = &\frac{\cos d}{\cos^2 c\sin^3 d}\{cd(\sin^2 d + \sin^2 c) - d\cos c\sin c\sin^2 d \\ &- c\sin^2 c\cos d\sin d\} \\ &= \frac{\cos d}{\cos^2 c\sin^3 d}\{c\sin^2 c(d - \cos d\sin d) + d\sin^2 d(c - \cos c\sin c)\} \\ &\geq 0. \end{aligned}$$

Thus we see $f'(c) \geq 0$ and it follows from $\lim_{c\to 0} d(c) = (\pi/2 - \varepsilon)$ and L'Hospital's rule that

$$\lim_{c\to 0} f(c) = \lim_{c\to 0} \frac{-4dd'}{2c} = \lim_{c\to 0} \frac{2d\sin c\cos d}{c\cos c\sin d} = (\pi - 2\varepsilon)\frac{\sin\varepsilon}{\cos\varepsilon}.$$

This completes the proof of the k-convexity.

(ii) We next consider the L-convexity. Take three points $x, y, z \in \mathbb{S}^2$ with $|x-y|_{\mathbb{S}^2} \vee |x-z|_{\mathbb{S}^2} \vee |y-z|_{\mathbb{S}^2} \leq \pi - \varepsilon$, and set $a := |x-y|_{\mathbb{S}^2}$, $b := |x-z|_{\mathbb{S}^2}$, $c(t) := |\gamma_{xy}(t) - \gamma_{xz}(t)|_{\mathbb{S}^2}$, and $d := c(1) = |y-z|_{\mathbb{S}^2}$. We have, for any $t \in (0,1)$,

$$\frac{c(t)}{t} \leq \lim_{t\to 0} \frac{c(t)}{t} = \{a^2 + b^2 - 2ab\cos\angle yxz\}^{1/2},$$

and so we estimate $\{a^2 + b^2 - 2ab\cos\angle yxz\}/d^2$ from above. It follows from the spherical cosine formula together with

$$(a-b)^2 \sin a \sin b \leq 2ab\big(1 - \cos(a-b)\big)$$

that

$$\begin{aligned}
\frac{a^2 + b^2 - 2ab\cos\angle yxz}{d^2} &\leq \frac{a^2 + b^2 - 2ab\cos\angle yxz}{2(1-\cos d)} \\
&= \frac{(a-b)^2 + 2ab(1-\cos\angle yxz)}{2(1-\cos(a-b)) + 2\sin a\sin b(1-\cos\angle yxz)} \\
&\leq \frac{ab}{\sin a \sin b} \leq \left(\frac{a\vee b}{\sin(a\vee b)}\right)^2.
\end{aligned}$$

Now we assume $a \geq b$ and calculate

$$\begin{aligned}
\frac{a}{\sin a} &= 1 + \frac{a - \sin a}{a\sin a} a \leq 1 + \frac{(\pi-\varepsilon) - \sin(\pi-\varepsilon)}{(\pi-\varepsilon)\sin(\pi-\varepsilon)} a \\
&\leq 1 + \frac{2\{(\pi-\varepsilon) - \sin\varepsilon\}}{(\pi-\varepsilon)\sin\varepsilon}\frac{a+b}{2}.
\end{aligned}$$

Here the second inequality follows from

$$\frac{d}{da}\left(\frac{a-\sin a}{a\sin a}\right) = \frac{\sin^2 a - a^2\cos a}{a^2\sin^2 a},$$

$$\begin{aligned}
\frac{d}{da}(\sin^2 a - a^2\cos a) &= 2\sin a\cos a - 2a\cos a + a^2\sin a \\
&\geq 2\sin a\cos a - 2\sin a + a^2 \sin a \\
&= \sin a(2\cos a - 2 + a^2) \geq 0.
\end{aligned}$$

This completes the proof. □

Note that

$$\lim_{\varepsilon\to\pi/2} (\pi - 2\varepsilon)\frac{\sin\varepsilon}{\cos\varepsilon} = 2, \qquad \lim_{\varepsilon\to\pi} \frac{2\{(\pi-\varepsilon) - \sin\varepsilon\}}{(\pi-\varepsilon)\sin\varepsilon} = 0.$$

We also remark that, by definition, if (X, d_X) is k-convex or L-convex for some k or (L_1, L_2), then the scaled metric space $(X, a\cdot d_X)$ with $a > 0$ is k-convex or L-convex for k or $(L_1/a, aL_2)$, respectively.

Corollary 3.2 *An Alexandrov space with a local upper curvature bound is both locally k-convex and locally L-convex for any $k \in (0,2)$ and $L_1, L_2 > 0$.*

By Proposition 3.1, a k-convex metric space for $k \in (0,2)$ is not necessarily L-convex for $L_1 = 0$, in other words, not necessarily nonpositively curved in the sense of Busemann. Moreover, we also observe that the local k-convexity (or the L-convexity) of a simply-connected metric space does not imply the global one. In particular, the Cartan-Hadamard-type theorem does not hold for both of the k-convexity and the L-convexity. See [BH, Chapter II.4] for the cases of the CAT(0)-property and the nonpositively curved property in the sense of Busemann.

3.2 k-convex Banach spaces

Let $(V, |\cdot|)$ be a Banach space and define the *modulus of convexity* of V as

$$\delta(\varepsilon) := \inf \left\{ 1 - \left| \frac{x+y}{2} \right| \,\middle|\, x, y \in V, |x| = 1, |y| = 1, |x-y| = \varepsilon \right\}$$

for $\varepsilon \in [0,2]$ (cf. [LT, II, Section 1.e]). We say that V is *uniformly convex with the modulus of convexity of power type* 2 if there is a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^2$ holds for all $\varepsilon \in [0,2]$. It is shown in [BCL] that the k-convexity is equivalent to the uniform convexity with the modulus of convexity of power type 2.

Proposition 3.3 ([BCL, Proposition 7]) *If a Banach space $(V, |\cdot|)$ is k-convex as a metric space, then it satisfies $\delta(\varepsilon) \geq k\varepsilon^2/16$. Conversely, if $\delta(\varepsilon) \geq c\varepsilon^2$ holds for all $\varepsilon \in [0,2]$, then $(V, |\cdot|)$ is k-convex for some $k \in (0,2]$ depending only on c.*

Proof. We recall only the easier direction. Assume that $(V, |\cdot|)$ is k-convex. For $x, y \in V$ with $|x| = 1$, $|y| = 1$, and with $|x-y| = \varepsilon$, we see

$$\left| \frac{x+y}{2} \right|^2 \leq \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{k}{8}|x-y|^2 = 1 - \frac{k}{8}\varepsilon^2,$$

and hence

$$1 - \left| \frac{x+y}{2} \right| \geq \frac{1}{2}\left(1 - \left| \frac{x+y}{2} \right|^2 \right) \geq \frac{k}{16}\varepsilon^2.$$

□

Example 3.4 ([BCL, Theorem 1]) For a measure space (Z, μ) and $p \in (1,2]$, the Banach space $L^p(Z)$ is k-convex for $k = 2(p-1)$.

3.3 Riemannian polyhedra without focal points

Example 3.5 ([Bo, Theorem 1.4]) A universal covering of a compact Riemannian polyhedron without focal points is k-convex for some $k > 0$.

4 Fundamental geometric properties

In this section, we consider some geometric properties of k-convex and/or L-convex metric spaces. See [Ba] and [BH] for the case of CAT(0)-spaces. The results in this section are not necessary in the following sections, but they seem to be of independent interest.

4.1 Spaces of directions and tangent cones

Let (X, d_X) be an L-convex metric space (possibly for $L_2 = \infty$). For $x \in X$, we define Σ'_x as the set of unit speed minimal geodesics emanating from x. For three points $x, y, z \in X$, we set $\tilde{\angle} xyz := \angle \tilde{x}\tilde{y}\tilde{z}$, where $\triangle \tilde{x}\tilde{y}\tilde{z}$ denotes a comparison triangle in $\mathbb{R}^2$, i.e., a (geodesic) triangle in $\mathbb{R}^2$ with $|\tilde{x} - \tilde{y}|_{\mathbb{R}^2} = |x - y|_X$, $|\tilde{y} - \tilde{z}|_{\mathbb{R}^2} = |y - z|_X$, and $|\tilde{z} - \tilde{x}|_{\mathbb{R}^2} = |z - x|_X$.

Lemma 4.1 *For any $\gamma, \xi \in \Sigma'_x$ and $s, t \ge 0$, the limit*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X$$

exists. In particular, $\lim_{\varepsilon \to 0} \tilde{\angle} \gamma(\varepsilon) x \xi(\varepsilon)$ exists.

Proof. For all $\varepsilon \in (0, 1)$, it follows from the L-convexity that

$$\varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X \le \left(1 + L_1 \frac{s+t}{2}\right) |\gamma(s) - \xi(t)|_X.$$

Hence there exists a monotone decreasing sequence $\{\varepsilon_i\}_{i=1}^\infty$ tending to zero such that the limit $\lim_{i\to\infty} \varepsilon_i^{-1} |\gamma(s\varepsilon_i) - \xi(t\varepsilon_i)|_X$ exists. Denote the limit by $a \ge 0$. Then, for $\varepsilon \in [\varepsilon_{i+1}, \varepsilon_i]$, we have

$$\begin{aligned} \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X &= \varepsilon_i^{-1} \frac{\varepsilon_i}{\varepsilon} \left| \gamma\left(s\varepsilon_i \frac{\varepsilon}{\varepsilon_i}\right) - \xi\left(t\varepsilon_i \frac{\varepsilon}{\varepsilon_i}\right) \right|_X \\ &\le \varepsilon_i^{-1} \left(1 + L_1 \frac{s\varepsilon_i + t\varepsilon_i}{2}\right) |\gamma(s\varepsilon_i) - \xi(t\varepsilon_i)|_X \end{aligned}$$

and, similarly,

$$\varepsilon_{i+1}^{-1} |\gamma(s\varepsilon_{i+1}) - \xi(t\varepsilon_{i+1})|_X \le \left(1 + L_1 \varepsilon \frac{s+t}{2}\right) \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X.$$

Thus we see $\lim_{\varepsilon\to 0} \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X = a$. $\square$

Different from CAT(0)-spaces, the 'angle' $\lim_{\varepsilon\to 0} \tilde{\angle} \gamma(\varepsilon) x \xi(\varepsilon)$ depends on the choices of the parametrizations of γ and ξ. Define the *space of directions* Σ_x at $x \in X$ by $\Sigma_x := \Sigma'_x / \sim$, where $\gamma \sim \xi$ holds if $\lim_{\varepsilon\to 0} \tilde{\angle} \gamma(\varepsilon) x \xi(\varepsilon) = 0$ (or, equivalently, $\lim_{\varepsilon\to 0} \varepsilon^{-1} |\gamma(\varepsilon) - \xi(\varepsilon)|_X = 0$). Put

$$K'_x := \Sigma_x \times [0, \infty) / \sim,$$

where $(\gamma, s) \sim (\xi, t)$ holds if $\lim_{\varepsilon\to 0} \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X = 0$. Then

$$|(\gamma, s) - (\xi, t)|_{K'_x} := \lim_{\varepsilon \to 0} \varepsilon^{-1} |\gamma(s\varepsilon) - \xi(t\varepsilon)|_X$$

gives a distance function on K'_x. Define the *tangent cone* $(K_x, |\cdot|_{K_x})$ at $x \in X$ as the completion of $(K'_x, |\cdot|_{K'_x})$.

Proposition 4.2 *For an L-convex and k-convex metric space (X, d_X) and $x \in X$, the tangent cone $(K_x, |\cdot|_{K_x})$ is a geodesic space. Moreover, it is L-convex for $L_1 = 0$ and k-convex for the same k as (X, d_X).*

Proof. The proof is similar to the case of CAT(0)-spaces (see [BH, Chapter II.3, Theorem 3.19]). We give the outline of the proof for completeness.

We first show that $(K_x, |\cdot|_{K_x})$ is geodesic. Fix two points $[\gamma, s], [\xi, t] \in K_x'$, where we denote by $[\gamma, s]$ the equivalent class containing $(\gamma, s) \in \Sigma_x \times [0, \infty)$. For $\varepsilon > 0$, set $y_\varepsilon := (1/2)\gamma(s\varepsilon) + (1/2)\xi(t\varepsilon) \in X$ and $v_\varepsilon := [\gamma_{xy_\varepsilon}, \varepsilon^{-1}] \in K_x'$. Recall that y_ε is the unique midpoint of $\gamma(s\varepsilon)$ and $\xi(t\varepsilon)$ (see the sentence following Lemma 2.2). By the L-convexity, we have

$$\begin{aligned}
|[\gamma, s] - v_\varepsilon|_{K_x'} &= \lim_{\delta \to 0} \frac{1}{\varepsilon\delta} |\gamma(s\varepsilon\delta) - \gamma_{xy_\varepsilon}(\delta)|_X \\
&\le \frac{1}{\varepsilon}\left(1 + L_1 \frac{|x - \gamma(s\varepsilon)|_X + |x - y_\varepsilon|_X}{2}\right) |\gamma(s\varepsilon) - y_\varepsilon|_X \\
&= \left(1 + L_1 \frac{|x - \gamma(s\varepsilon)|_X + |x - y_\varepsilon|_X}{2}\right) \frac{|\gamma(s\varepsilon) - \xi(t\varepsilon)|_X}{2\varepsilon} \\
&\to \frac{1}{2} |[\gamma, s] - [\xi, t]|_{K_x'}
\end{aligned}$$

as ε tends to zero. Thus v_ε's are approximate midpoints between $[\gamma, s]$ and $[\xi, t]$.

The L-convexity also yields that, for $\varepsilon > \varepsilon' > 0$,

$$|v_\varepsilon - v_{\varepsilon'}|_{K_x'} \le \frac{1}{\varepsilon'}\left(1 + L_1 \frac{\varepsilon^{-1}\varepsilon' |x - y_\varepsilon|_X + |x - y_{\varepsilon'}|_X}{2}\right) |\gamma_{xy_\varepsilon}(\varepsilon^{-1}\varepsilon') - y_{\varepsilon'}|_X.$$

By Lemma 2.3, we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} (\varepsilon')^{-2} |\gamma_{xy_\varepsilon}(\varepsilon^{-1}\varepsilon') - y_{\varepsilon'}|_X^2 \\
&\le \lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} \frac{2}{k(\varepsilon')^2} \left\{ \frac{1}{2} |\gamma_{xy_\varepsilon}(\varepsilon^{-1}\varepsilon') - \gamma(s\varepsilon')|_X^2 + \frac{1}{2} |\gamma_{xy_\varepsilon}(\varepsilon^{-1}\varepsilon') - \xi(t\varepsilon')|_X^2 \right. \\
&\qquad \left. - \frac{1}{4} |\gamma(s\varepsilon') - \xi(t\varepsilon')|_X^2 \right\} \\
&= \lim_{\varepsilon \to 0} \frac{2}{k} \left\{ \frac{1}{2} |v_\varepsilon - [\gamma, s]|_{K_x}^2 + \frac{1}{2} |v_\varepsilon - [\xi, t]|_{K_x}^2 - \frac{1}{4} |[\gamma, s] - [\xi, t]|_{K_x}^2 \right\} \\
&= 0.
\end{aligned}$$

Therefore the sequence $\{v_\varepsilon\}_{\varepsilon > 0}$ is a Cauchy sequence and tends a midpoint between $[\gamma, s]$ and $[\xi, t]$. For general two points $\sigma, \tau \in K_x$, approximate them by points in K_x' and observe that their midpoints form a Cauchy sequence converging to a midpoint of σ and τ. Hence K_x is geodesic. A similar discussion shows that minimal geodesics are unique. The L-convexity of $(K_x, |\cdot|_{K_x})$ for $L_1 = 0$ follows from the L-convexity of (X, d_X) by taking a scaling limit. It is similar to the k-convexity. $\square$

4.2 The cone at infinity

Let (X, d_X) be a complete, L-convex, and k-convex metric space. A nonconstant geodesic $\gamma : [0, \infty) \longrightarrow X$ is called a *geodesic ray* if it is globally minimizing, i.e., $\mathrm{length}(\gamma|_{[0,l]}) = |\gamma(0) - \gamma(l)|_X$ holds for all $l > 0$. Note that, if $L_1 = 0$, then every geodesic is globally minimizing. Two geodesic rays γ and ξ are said to be *asymptotic* if $|\gamma(t) - \xi(t)|_X$ is bounded from above uniformly in $t \in [0, \infty)$. Define

$$X(\infty) := \{\text{unit speed geodesic rays}\}/\sim,$$

where $\gamma \sim \xi$ holds if they are asymptotic.

Lemma 4.3 *Fix a point $x_0 \in X$. For any $\sigma \in X(\infty)$, there exists a unique unit speed geodesic ray γ satisfying $\gamma(0) = x_0$ and $\gamma \in \sigma$.*

Proof. We first show the uniqueness. Let $\gamma, \xi : [0, \infty) \longrightarrow X$ be two mutually asymptotic, unit speed geodesic rays with $\gamma(0) = \xi(0) = x_0$. For any $t \in (0, \infty)$ and $T > t$, the L-convexity implies that

$$|\gamma(t) - \xi(t)|_X \le (1 + L_1 L_2)\frac{t}{T}|\gamma(T) - \xi(T)|_X.$$

Since γ and ξ are asymptotic, by letting T diverge to the infinity, we have $\gamma(t) = \xi(t)$.

We next consider the existence. Fix $\xi \in \sigma$, put $l_T := |x_0 - \xi(T)|_X$ for each $T > 0$, and let $\gamma_T : [0, l_T] \longrightarrow X$ be a unit speed geodesic from x_0 to $\xi(T)$. For $T \ge S > 0$ and $t \in (0, l_-)$, where we put $l_- := l_S \wedge l_T$ and $l_+ := l_S \vee l_T$, it follows from the L-convexity and the k-convexity that

$$\begin{aligned}
&|\gamma_S(t) - \gamma_T(t)|_X^2 \le (1 + L_1 L_2)^2 \frac{t^2}{l_-^2}|\gamma_S(l_-) - \gamma_T(l_-)|_X^2 \\
&\le (1 + L_1 L_2)^2 \frac{2t^2}{k l_-^2}\left\{\left(1 - \frac{l_-}{l_+}\right)l_-^2 + \frac{l_-}{l_+}|\xi(S) - \xi(T)|_X^2 - \left(1 - \frac{l_-}{l_+}\right)\frac{l_-}{l_+}l_+^2\right\} \\
&= (1 + L_1 L_2)^2 \frac{2t^2}{k l_- l_+}\{l_-(l_+ - l_-) + (T - S)^2 - (l_+ - l_-)l_+\} \\
&= (1 + L_1 L_2)^2 \frac{2t^2}{k l_S l_T}\{(T - S)^2 - (l_+ - l_-)^2\} \\
&= (1 + L_1 L_2)^2 \frac{2t^2}{k l_S l_T}\{(T - l_T) - (S - l_S)\}\{(T + l_T) - (S + l_S)\} \\
&\le (1 + L_1 L_2)^2 \frac{2t^2}{k} 2|x_0 - \xi(0)|_X \left|\frac{T + l_T}{l_T l_S} - \frac{S + l_S}{l_T l_S}\right|.
\end{aligned}$$

Letting S and T go to the infinity, we see that γ_T converges to a geodesic ray γ with

$\gamma(0) = x_0$. Again by the L-convexity, it holds that, for any $T > 0$ and $t \in (0, l_T \wedge T)$,

$$\begin{aligned}
|\gamma_T(t) - \xi(t)|_X &\le \left|\gamma_T\left(\frac{t}{l_T} l_T\right) - \xi\left(\frac{t}{l_T} T\right)\right|_X + t\left|1 - \frac{T}{l_T}\right| \\
&\le (1 + L_1 L_2)\left(1 - \frac{t}{l_T}\right)|\gamma_T(0) - \xi(0)|_X + \frac{t}{l_T}|l_T - T| \\
&\le (1 + L_1 L_2)\left(1 - \frac{t}{l_T}\right)|x_0 - \xi(0)|_X + \frac{t}{l_T}|x_0 - \xi(0)|_X \\
&\to (1 + L_1 L_2)|x_0 - \xi(0)|_X
\end{aligned}$$

as T diverges to the infinity. Therefore γ is asymptotic to ξ, and it completes the proof. □

Let γ and ξ be two unit speed geodesic rays emanating from a common point and $s, t \ge 0$. Then, for any $T > 1$, we have

$$(1 + L_1 L_2)^{-1}|\gamma(s) - \xi(t)|_X \le T^{-1}|\gamma(sT) - \xi(tT)|_X \le s + t.$$

It implies $\limsup_{T\to\infty} T^{-1}|\gamma(sT) - \xi(tT)|_X < \infty$ and is positive unless $\gamma(\lambda s) = \xi(\lambda t)$ for all $\lambda \ge 0$. Moreover, it clearly holds that

$$\limsup_{T\to\infty} T^{-1}|\gamma'(sT) - \xi'(tT)|_X = \limsup_{T\to\infty} T^{-1}|\gamma(sT) - \xi(tT)|_X$$

for any unit speed geodesic rays γ' and ξ' which are asymptotic to γ and ξ, respectively. We remark that, if $L_1 = 0$, then $T^{-1}|\gamma(sT) - \xi(tT)|_X$ is monotone non-decreasing in T, and hence $\lim_{T\to\infty} T^{-1}|\gamma(sT) - \xi(tT)|_X$ exists.

We define the *cone at infinity* of X as the set

$$C_\infty X := X(\infty) \times [0, \infty)/\sim,$$

where $(\sigma, 0) \sim (\tau, 0)$ for all $\sigma, \tau \in X(\infty)$, equipped with the distance

$$|(\sigma, s) - (\tau, t)|_{C_\infty X} := \limsup_{T\to\infty} T^{-1}|\gamma(sT) - \xi(tT)|_X,$$

where $\gamma \in \sigma$ and $\xi \in \tau$. We already observed that $|\cdot|_{C_\infty X}$ is well-defined and non-degenerate.

Proposition 4.4 *Let (X, d_X) be a complete, L-convex, and k-convex metric space. Then the cone at infinity $(C_\infty X, |\cdot|_{C_\infty X})$ is complete. If, in addition, $L_1 = 0$, then $(C_\infty X, |\cdot|_{C_\infty X})$ is L-convex for $L_1 = 0$ and k-convex for the same k as (X, d_X).*

Proof. In this proof, by virtue of Lemma 4.3, we fix a point $x_0 \in X$ and identify $X(\infty)$ with the set of unit speed geodesic rays starting at x_0. Then the completeness is easily deduced from the fact that, for any $\gamma, \xi \in X(\infty)$ and $s, t, \lambda \ge 0$, we have

$$|\gamma(\lambda s) - \xi(\lambda t)|_X \le (1 + L_1 L_2)\lambda \limsup_{T\to\infty} T^{-1}|\gamma(sT) - \xi(tT)|_X.$$

Assume $L_1 = 0$ in the following. We first verify that $(C_\infty X, |\cdot|_{C_\infty X})$ is geodesic. To do this, we fix $\gamma, \xi \in X(\infty)$ and $s, t \ge 0$, and will find a midpoint of $[\gamma, s]$ and $[\xi, t]$, where we denote by $[\gamma, s] \in C_\infty X$ the equivalent class containing $(\gamma, s) \in X(\infty) \times [0, \infty)$. Without loss of generality, we may assume $s, t > 0$ and $0 < |[\gamma, s] - [\xi, t]|_{C_\infty X} < s + t$. For each $T > 0$, put $x_T := (1/2)\gamma(sT) + (1/2)\xi(tT)$ and $l_T := |x_0 - x_T|_X$. The triangle inequality yields

$$\frac{1}{2}\{s + t - T^{-1}|\gamma(sT) - \xi(tT)|_X\} \le \frac{l_T}{T} \le \frac{1}{2}\{s + t + T^{-1}|\gamma(sT) - \xi(tT)|_X\},$$

so that there exists a sequence $\{T_i\}_{i=1}^\infty$ going to the infinity for which $\lim_{i\to\infty} l_{T_i}/T_i$ exists. Denote the limit by $a \in (0, s+t)$. For $j \gg i \ge 1$, it follows from the L-convexity for $L_1 = 0$ that

$$\begin{aligned}
&\frac{1}{T_i}\left|\left\{\left(1 - \frac{l_{T_i}}{l_{T_j}}\right)x_0 + \frac{l_{T_i}}{l_{T_j}}x_{T_j}\right\} - \gamma(sT_i)\right|_X \\
&\le \frac{1}{T_i}\left|\left\{\left(1 - \frac{T_i}{T_j}\right)x_0 + \frac{T_i}{T_j}x_{T_j}\right\} - \gamma(sT_i)\right|_X + \frac{1}{T_i}\left|\frac{l_{T_i}}{l_{T_j}} - \frac{T_i}{T_j}\right| l_{T_j} \\
&\le \frac{1}{T_j}|x_{T_j} - \gamma(sT_j)|_X + \left|\frac{l_{T_i}}{T_i} - \frac{l_{T_j}}{T_j}\right| \\
&\to \frac{1}{2}|[\gamma, s] - [\xi, t]|_{C_\infty X}
\end{aligned}$$

as i and j diverge to the infinity. Hence we have, by Lemma 2.3,

$$\begin{aligned}
&\frac{1}{T_i^2}\left|x_{T_i} - \left\{\left(1 - \frac{l_{T_i}}{l_{T_j}}\right)x_0 + \frac{l_{T_i}}{l_{T_j}}x_{T_j}\right\}\right|_X^2 \\
&\le \frac{2}{kT_i^2}\left[\frac{1}{2}\left|\gamma(sT_i) - \left\{\left(1 - \frac{l_{T_i}}{l_{T_j}}\right)x_0 + \frac{l_{T_i}}{l_{T_j}}x_{T_j}\right\}\right|_X^2\right. \\
&\quad \left. + \frac{1}{2}\left|\xi(tT_i) - \left\{\left(1 - \frac{l_{T_i}}{l_{T_j}}\right)x_0 + \frac{l_{T_i}}{l_{T_j}}x_{T_j}\right\}\right|_X^2 - \frac{1}{4}|\gamma(sT_i) - \xi(tT_i)|_X^2\right] \\
&\to 0
\end{aligned}$$

as i and j go to the infinity. It implies that $\{\gamma_{x_0 x_{T_i}}\}_{i=1}^\infty$ converges to a geodesic ray which is clearly a midpoint of $[\gamma, s]$ and $[\xi, t]$ by construction.

The uniqueness of a minimal geodesic follows from the same discussion. Consequently, we obtain the L-convexity for $L_1 = 0$ and the k-convexity of $(C_\infty X, |\cdot|_{C_\infty X})$ by taking a scaling limit of d_X. □

4.3 Foot-points

Let (X, d_X) be a complete, k-convex metric space. A subset $A \subset X$ is said to be *geodesically convex* if, for every two points $x, y \in A$, we have $\gamma_{xy} \subset A$. For a closed, geodesically convex subset $A \subset X$ and a point $x \in X$, the *foot-point* $F_A(x)$ of x to A is defined as a point in A which satisfies $|x - F_A(x)|_X = \mathrm{dist}(x, A) := \inf\{|x - y|_X \mid y \in A\}$. It follows from the k-convexity that such a point exists uniquely.

Proposition 4.5 *Let (X, d_X) be a complete, k-convex metric space and $A \subset X$ be a closed, geodesically convex subset. Then, for any $x, y \in X$, we have*

$$|F_A(x) - F_A(y)|_X^2 \le \frac{8}{k}\big\{|x-y|_X + \mathrm{dist}(\{x,y\},A)\big\}|x-y|_X.$$

In particular, the map F_A is $(1/2)$-Hölder continuous on each bounded set.

Proof. Fix $x, y \in X$ and put $x' := F_A(x)$, $y' := F_A(y)$, and $z_t := (1-t)x' + ty'$ for $t \in [0,1]$. We may assume $\mathrm{dist}(\{x,y\},A) = \mathrm{dist}(x,A)$. By the k-convexity, we have

$$|x-x'|_X^2 \le |x-z_t|_X^2 \le (1-t)|x-x'|_X^2 + t|x-y'|_X^2 - \frac{k}{2}(1-t)t|x'-y'|_X^2.$$

By dividing both sides by t and letting t tend to 0, it implies

$$\begin{aligned}
(k/2)|x'-y'|_X^2 &\le |x-y'|_X^2 - |x-x'|_X^2 \\
&\le (|x-y|_X + |y-y'|_X)^2 - (|x-y|_X - |y-x'|_X)^2 \\
&\le (|x-y|_X + |y-x'|_X)^2 - (|x-y|_X - |y-x'|_X)^2 \\
&= 4|x-y|_X|y-x'|_X \\
&\le 4|x-y|_X\{|x-y|_X + \mathrm{dist}(x,A)\}.
\end{aligned}$$

This completes the proof. □

5 First variation formula

In this section, we prove the first variation formula which plays a key role in Section 8. We start with a variation of Lemma 4.1.

Lemma 5.1 *Let an open ball $B \subset X$ be a C_k-domain. Then, for any three distinct points $x, y, z \in B$, the function*

$$\cos \tilde{\angle} xy\gamma_{yz}(t) - \frac{1-(k/2)}{2}\frac{t|y-z|_X}{|x-y|_X}$$

is monotone non-increasing in $t \in (0,1]$. In particular, $\lim_{t\to 0}\tilde{\angle} xy\gamma_{yz}(t)$ exists.

Proof. Put $\gamma = \gamma_{yz}$ and fix $t \in (0,1]$ and $\lambda \in (0,1)$. It follows from the k-convexity that

$$|x-\gamma(\lambda t)|_X^2 \le (1-\lambda)|x-y|_X^2 + \lambda|x-\gamma(t)|_X^2 - \frac{k}{2}(1-\lambda)\lambda(t|y-z|_X)^2.$$

By the cosine formula, we obtain

$$
\begin{aligned}
\cos \tilde{\angle} xy\gamma(\lambda t) = &\frac{|x-y|_X^2 + (\lambda t|y-z|_X)^2 - |x-\gamma(\lambda t)|_X^2}{2\lambda t|x-y|_X|y-z|_X} \\
\geq &\frac{1}{2\lambda t|x-y|_X|y-z|_X} \\
&\times \left[\lambda|x-y|_X^2 - \lambda|x-\gamma(t)|_X^2 + \left\{\frac{k}{2}(1-\lambda)\lambda t^2 + (\lambda t)^2\right\}|y-z|_X^2\right] \\
= &\frac{|x-y|_X^2 + (t|y-z|_X)^2 - |x-\gamma(t)|_X^2}{2t|x-y|_X|y-z|_X} \\
&+ \frac{(\lambda-1)t^2 + (k/2)(1-\lambda)t^2}{2t|x-y|_X}|y-z|_X \\
= &\cos \tilde{\angle} xy\gamma(t) - (1-\lambda)\frac{1-(k/2)}{2|x-y|_X}t|y-z|_X.
\end{aligned}
$$

□

Different from CAT(κ)-spaces, two limits $\lim_{t\to 0} \tilde{\angle} xy\gamma_{yz}(t)$ and $\lim_{t\to 0} \tilde{\angle} \gamma_{yx}(t)yz$ may be different. A geodesic metric space (X, d_X) is said to be *locally geodesics extendable* if, for each $x \in X$, there exists $\delta = \delta(x) > 0$ for which every unit speed minimal geodesic $\gamma : [0,\varepsilon] \longrightarrow X$ with $\gamma(0) = x$ can be extended to a minimal geodesic $\bar{\gamma} : [0,\delta] \longrightarrow X$ satisfying $\bar{\gamma} = \gamma$ on $[0,\varepsilon]$. For $x \in X$ and $r > 0$, we define $S(x,r) := \{y \in X \,|\, |x-y|_X = r\}$. The symbols $\theta_{\alpha,\beta}(\varepsilon)$ and $O_{\alpha,\beta}(\varepsilon)$ denote functions depending only on α and β with $\lim_{\varepsilon\to 0} \theta_{\alpha,\beta}(\varepsilon) = 0$ and $\limsup_{\varepsilon\to 0} |O_{\alpha,\beta}(\varepsilon)|/\varepsilon < \infty$, respectively. The following first variation formula for arclength is an analogue of that in [OS, Theorem 3.5] (see also [OT] and [O2, Theorem 2.2.3]). We give a precise proof for the thoroughness.

Theorem 5.2 (First variation formula) *Let (X, d_X) be a locally compact, locally geodesics extendable, and geodesic metric space. We suppose that an open ball $B \subset X$ is a C_k- and C_L-domain, and take two distinct points $x, y \in B$. Then, for any $z \in B$, we have*

$$|x-y|_X - |x-z|_X = |y-z|_X \cos\left(\lim_{t\to 0} \tilde{\angle} xy\gamma_{yz}(t)\right) + O_{x,y}(|y-z|_X^2).$$

Proof. Fix a small $\varepsilon \in (0, \delta(y))$ so that $S(y,\varepsilon)$ is compact, and choose a finite set $\{z_i\}_{i=1}^N \subset S(y,\varepsilon)$ for which $\{B(z_i,\varepsilon^2)\}_{i=1}^N$ covers $S(y,\varepsilon)$. By Lemma 5.1, we can find $t_\varepsilon \in (0,\varepsilon]$ such that

$$\left|\cos \tilde{\angle} xy\gamma_{yz_i}(s) - \cos\left(\lim_{t\to 0} \tilde{\angle} xy\gamma_{yz_i}(t)\right)\right| \leq \varepsilon$$

holds for all i and $s \in (0, t_\varepsilon/\varepsilon]$.

Note that we may assume $z \in B(y, t_\varepsilon)$ and $\varepsilon = \theta_{x,y}(|y-z|_X)$. For $z \in B(y,t_\varepsilon) \setminus \{y\}$, take $\bar{z} \in S(y,\varepsilon)$ and i satisfying that $z = \gamma_{y\bar{z}}(s)$ for some $s \in (0, t_\varepsilon/\varepsilon)$ and that $|\bar{z} - z_i|_X \leq \varepsilon^2$. We put $z' := \gamma_{y\bar{z}}(t_\varepsilon/\varepsilon)$ and $z_i' := \gamma_{yz_i}(t_\varepsilon/\varepsilon)$. Then it follows from Lemma 5.1 that

$$\cos\left(\lim_{t\to 0} \tilde{\angle} xy\gamma_{yz}(t)\right) \geq \cos \tilde{\angle} xyz - \frac{1-(k/2)}{2|x-y|_X}s\varepsilon \geq \cos \tilde{\angle} xyz' - \frac{1-(k/2)}{2|x-y|_X}t_\varepsilon.$$

Moreover, by the L-convexity, we see

$$
\begin{aligned}
&|\cos\tilde{\angle}xyz' - \cos\tilde{\angle}xyz'_i| \\
&= \left| \frac{|x-y|_X^2 + |y-z'|_X^2 - |x-z'|_X^2}{2|x-y|_X|y-z'|_X} - \frac{|x-y|_X^2 + |y-z'_i|_X^2 - |x-z'_i|_X^2}{2|x-y|_X|y-z'_i|_X} \right| \\
&= \frac{1}{2t_\varepsilon|x-y|_X}\big||x-z'_i|_X^2 - |x-z'|_X^2\big| \\
&\le \frac{1}{t_\varepsilon|x-y|_X}(|x-y|_X + t_\varepsilon)|z'-z'_i|_X \\
&\le \frac{1}{t_\varepsilon|x-y|_X}(|x-y|_X + t_\varepsilon)(1+L_1\varepsilon)\frac{t_\varepsilon}{\varepsilon}|\bar{z}-z_i|_X \\
&\le \frac{(1+L_1\varepsilon)\varepsilon}{|x-y|_X}(|x-y|_X + t_\varepsilon).
\end{aligned}
$$

Thus we have $|\cos\tilde{\angle}xyz' - \cos\tilde{\angle}xyz'_i| = O_{x,y}(\varepsilon)$ and, by a similar discussion,

$$
|\cos\tilde{\angle}xy\gamma_{y\bar{z}}(t) - \cos\tilde{\angle}xy\gamma_{yz_i}(t)| = O_{x,y}(\varepsilon)
$$

for small $t > 0$. Therefore we obtain

$$
\begin{aligned}
\cos\left(\lim_{t\to 0}\tilde{\angle}xy\gamma_{yz}(t)\right) &\ge \cos\tilde{\angle}xyz + O_{x,y}(\varepsilon) \ge \cos\tilde{\angle}xyz' + O_{x,y}(\varepsilon) \\
&\ge \cos\tilde{\angle}xyz'_i + O_{x,y}(\varepsilon) \\
&= \cos\left(\lim_{t\to 0}\tilde{\angle}xy\gamma_{yz_i}(t)\right) + O_{x,y}(\varepsilon) \\
&= \cos\left(\lim_{t\to 0}\tilde{\angle}xy\gamma_{yz}(t)\right) + O_{x,y}(\varepsilon),
\end{aligned}
$$

and hence

$$
\left|\cos\tilde{\angle}xyz - \cos\left(\lim_{t\to 0}\tilde{\angle}xy\gamma_{yz}(t)\right)\right| = O_{x,y}(\varepsilon) = O_{x,y}(|y-z|_X).
$$

Combining this with

$$
\begin{aligned}
\left|\cos\tilde{\angle}xyz - \frac{|x-y|_X - |x-z|_X}{|y-z|_X}\right| &= \frac{|y-z|_X^2 - (|x-y|_X - |x-z|_X)^2}{2|x-y|_X|y-z|_X} \\
&\le \frac{|y-z|_X}{2|x-y|_X},
\end{aligned}
$$

we consequently obtain

$$
\begin{aligned}
|x-y|_X - |x-z|_X &= |y-z|_X\cos\tilde{\angle}xyz + O_{x,y}(|y-z|_X^2) \\
&= |y-z|_X\cos\left(\lim_{t\to 0}\tilde{\angle}xy\gamma_{yz}(t)\right) + O_{x,y}(|y-z|_X^2).
\end{aligned}
$$

□

6 Sobolev spaces

In this section, we recall two definitions of Sobolev spaces for maps from a metric measure space into a metric space, the Cheeger-type Sobolev space $H^{1,p}(U;X)$ and the Newtonian space $N^{1,p}(U;X)$, which will be turned out to give the same thing provided that $p>1$ and (X,d_X) is complete. Throughout the remainder of this article, without otherwise indicated, let (Z,d_Z) and (X,d_X) be metric spaces, $U\subset Z$ be an open set, and let μ be a Borel regular measure on Z such that any ball with finite positive radius is of finite positive measure.

Take a point $x_0\in X$ and fix it as a base point, and let $p\in[1,\infty)$. For two measurable maps $u,v:U\longrightarrow X$, we define $|u-v|_{L^p}:=\big(\int_U |u-v|_X^p\,d\mu\big)^{1/p}$ and

$$L^p(U;X):=\{u:U\longrightarrow X\,|\,\text{measurable}, |u-x_0|_{L^p}<\infty\}/\sim,$$

where x_0 denotes the constant map to x_0 and $u_1\sim u_2$ holds if $u_1=u_2$ a.e. on U. The function $|\cdot|_{L^p}$ defines a distance on $L^p(U;X)$.

First, we recall the Cheeger-type Sobolev space (see [C] and [O1]). Following the idea of Heinonen and Koskela [HK], a Borel function $g:U\longrightarrow[0,\infty]$ is called an *upper gradient* for a map $u:U\longrightarrow X$ if, for any unit speed curve $\gamma:[0,l]\longrightarrow U$, we have

$$\big|u\big(\gamma(0)\big)-u\big(\gamma(l)\big)\big|_X\le\int_0^l g\big(\gamma(s)\big)\,ds.$$

A function $g\in L^p(U)$ is called a *p-generalized upper gradient* for $u\in L^p(U;X)$ if there exists a sequence $\{(u_i,g_i)\}_{i=1}^\infty$ such that g_i is an upper gradient for u_i, and $u_i\to u$ in $L^p(U;X)$ and $g_i\to g$ in $L^p(U)$, respectively, as $i\to\infty$.

Definition 6.1 For $u\in L^p(U;X)$, we define

$$E_p(u):=\inf\{|g|^p_{L^p(U)}\,|\,g \text{ is a } p\text{-generalized upper gradient for } u\}.$$

Define the *Cheeger-type* $(1,p)$*-Sobolev space* by

$$H^{1,p}(U;X):=\{u\in L^p(U;X)\,|\,E_p(u)<\infty\}.$$

Note that it follows from the definition that $|g|^p_{L^p}\ge E_p(u)$ for any p-generalized upper gradient g for u. A p-generalized upper gradient $g\in L^p(U)$ for a map $u\in H^{1,p}(U;X)$ is said to be *minimal* if it satisfies $|g|^p_{L^p}=E_p(u)$.

Remark 6.2 The above definition of E_p is slightly modified from those in [C] and [O1] at the point that we require the convergence of $\{g_i\}_{i=1}^\infty$ in $L^p(U)$. We adopt it for the technical reason on proving Theorem 7.3. Accordingly, we do not know whether the lower semi-continuity of E_p in $L^p(U;X)$ holds or not. However, they coincide in the case where $p>1$ and (X,d_X) is L-convex for $L_1=0$ (thanks to the existence of minimal p-generalized upper gradients, [O1, Theorem 3.2]) or (X,d_X) is complete (due to N. Shanmugalingam, a personal communication).

We next recall the Newtonian space (see [S] and [HKST]). A Borel function $\rho : U \longrightarrow [0,\infty]$ is called a *p-weak upper gradient* for a map $u : U \longrightarrow X$ if

$$\big|u\big(\gamma(0)\big) - u\big(\gamma(l)\big)\big|_X \le \int_0^l \rho\big(\gamma(s)\big)\,ds \tag{6.1}$$

holds for p-a.e. curve $\gamma : [0,l] \longrightarrow U$. In other words, a family Γ of rectifiable curves for which (6.1) does not hold is *p-exceptional* in the sense that there exists a nonnegative Borel function $f \in L^p(U)$ satisfying $\int_\gamma f = \infty$ for all $\gamma \in \Gamma$.

Definition 6.3 For a measurable map u in $L^p(U;X)$, we define $E'_p(u) := \inf_\rho |\rho|^p_{L^p}$, where the infimum is taken over all p-weak upper gradient ρ in $L^p(U)$ for u. The *Newtonian space* is defined by

$$N^{1,p}(U;X) := \{u : U \longrightarrow X \mid u \in L^p(U;X), E'_p(u) < \infty\}/\sim,$$

where $u \sim v$ holds if $u = v$ a.e. on U.

We remark that, if u and v satisfy $E'_p(u), E'_p(v) < \infty$ and if $u = v$ holds a.e. on U, then we have $E'_p(u) = E'_p(v)$ by an analogue of [S, Corollary 3.3] through an isometric embedding of X into a Banach space $L^\infty(X)$. Moreover, if ρ in $L^p(U)$ is a p-weak upper gradient for u above, then ρ is also a p-weak upper gradient for v.

A p-weak upper gradient ρ in $L^p(U)$ for a map $u \in N^{1,p}(U;X)$ is said to be *minimal* if it satisfies $|\rho|^p_{L^p} = E'_p(u)$. If $p \in (1,\infty)$, then the uniform convexity of $L^p(U)$ implies that, for every map $u \in N^{1,p}(U;X)$, there exists a minimal p-weak upper gradient ρ_u in $L^p(U)$ for u which does not depend on the representative of u (in $N^{1,p}(U;X)$) and is unique upto a modification on a null measure set.

Proposition 6.4 *Let $p \in (1,\infty)$.*

(i) *If a representative $\bar{u}$ of $u \in L^p(U;X)$ belongs to $N^{1,p}(U;X)$, then $u \in H^{1,p}(U;X)$ and $E_p(u) \le E'_p(\bar{u})$. Conversely, if (X,d_X) is complete and $u \in H^{1,p}(U;X)$, then some representative $\bar{u}$ of u belongs to $N^{1,p}(U;X)$ and we have $E_p(u) = E'_p(\bar{u})$.*

(ii) *Assume that (X,d_X) is complete. For any $u \in H^{1,p}(U;X)$, there exists a unique minimal generalized upper gradient $g_u \in L^p(U)$ for u. Furthermore, it holds that $g_u = \rho_{\bar{u}}$ for $\bar{u}$ as in* (i).

Proof. We give only a sketch of the proof. See [S] and [HKST] for detail.

(i) It suffices to show that $g \in L^p(U)$ is a p-generalized upper gradient for $u \in L^p(U;X)$ if (and only if, in the case where (X,d_X) is complete) a representative $\bar{g}$ of g is a p-weak upper gradient for a representative $\bar{u}$ of u. If $\bar{g}$ is a p-weak upper gradient for $\bar{u}$, then, for $i \in \mathbb{N}$, we put $u_i := u$ and $g_i := \bar{g} + i^{-1}f$ for $f \in L^p(U)$ as in the definition of the p-weak upper gradient. As g_i is an upper gradient for u_i and g_i converges to g in $L^p(U)$, we see that g is a p-generalized upper gradient. The converse is a modification of [S, Lemma 4.11] provided that (X,d_X) is complete (due to N. Shanmugalingam, a personal communication).

(ii) It is clear by the proof of (i) of this lemma. □

7 Convexity of energy forms

In this section, let (X, d_X) be a (not necessarily complete) L-convex metric space, and consider an analogue of Proposition 6.4(ii). In this situation, the energy form E_p is not convex, but we can estimate how the convexity is violated. For two maps $u_1, u_2 : U \longrightarrow X$ and $t \in [0,1]$, denote by $(1-t)u_1 + tu_2$ the map $U \ni z \longmapsto (1-t)u_1(z) + tu_2(z) \in X$.

Lemma 7.1 *Let $u_1, u_2 : U \longrightarrow X$ be maps and $t \in [0,1]$. For any upper gradient g_1 and g_2 for u_1 and u_2, respectively, and for any function $\Phi : U \longrightarrow (0,\infty)$ with $\inf_{B(z,r)} \Phi > 0$ for every $r > 0$ and $z \in U$, the function*

$$g := \{(1-t)g_1 + tg_2\}\{1 + L_1(|u_1 - u_2|_X \wedge L_2) + \Phi\}$$

is an upper gradient for the map $v := (1-t)u_1 + tu_2$.

In particular, if $u_1, u_2 \in H^{1,p}(U;X)$ with $1 \le p < \infty$, then we have $v \in H^{1,p}(U;X)$ and

$$E_p(v)^{1/p} \le (1 + L_1 L_2)\{(1-t)E_p(u_1)^{1/p} + tE_p(u_2)^{1/p}\}.$$

Proof. Fix a unit speed curve $\gamma : [0,l] \longrightarrow U$. We may assume $\int_0^l g_i \circ \gamma\, ds < \infty$ for $i = 1,2$, and then $u_i \circ \gamma$ is uniformly continuous for $i = 1,2$. Take a sufficiently large $n \ge 1$ for which

$$\max_{i=1,2} \big|u_i\big(\gamma(s)\big) - u_i\big(\gamma(s')\big)\big|_X \le \inf_{\gamma([0,l])} \Phi/(2L_1)$$

holds if $|s - s'| \le l/n$. Set $l_j := (j/n)l$ and $z_j := \gamma(l_j)$ for $0 \le j \le n$. By the L-convexity, we have

$$\begin{aligned}
&\big|v\big(\gamma(0)\big) - v\big(\gamma(l)\big)\big|_X \le \sum_{j=1}^n |v(z_{j-1}) - v(z_j)|_X \\
&\le \sum_{j=1}^n \{|v(z_{j-1}) - \gamma_{u_1(z_j)u_2(z_{j-1})}(t)|_X + |\gamma_{u_1(z_j)u_2(z_{j-1})}(t) - v(z_j)|_X\} \\
&\le \sum_{j=1}^n \Bigg[\Bigg\{1 + L_1 \frac{(|u_2(z_{j-1}) - u_1(z_{j-1})|_X + |u_2(z_{j-1}) - u_1(z_j)|_X) \wedge 2L_2}{2}\Bigg\} \\
&\quad \times (1-t)|u_1(z_{j-1}) - u_1(z_j)|_X \\
&\quad + \Bigg\{1 + L_1 \frac{(|u_1(z_j) - u_2(z_{j-1})|_X + |u_1(z_j) - u_2(z_j)|_X) \wedge 2L_2}{2}\Bigg\} \\
&\quad \times t|u_2(z_{j-1}) - u_2(z_j)|_X\Bigg] \\
&\le \sum_{j=1}^n \int_{l_{j-1}}^{l_j} \big[\{1 + L_1(|u_1 - u_2|_X \wedge L_2) + \Phi\}\{(1-t)g_1 + tg_2\}\big] \circ \gamma\, ds \\
&= \int_0^l g \circ \gamma\, ds.
\end{aligned}$$

This completes the proof. □

We prove a simple lemma for later use.

Lemma 7.2 *Let $\{g_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty} \subset L^p(U)$ and $g \in L^p(U)$. If $g_i \to g$ and $f_i \to 0$ in $L^p(U)$ as $i \to \infty$, and if $|f_i| \le L < \infty$ holds uniformly in $i \ge 1$, then we have $g_i f_i \to 0$ in $L^p(U)$ as $i \to \infty$.*

Proof. It follows from $|f_i| \le L$ that

$$\left| \left(\int_U |g_i f_i|^p \, d\mu \right)^{1/p} - \left(\int_U |g f_i|^p \, d\mu \right)^{1/p} \right| \le L \left(\int_U |g_i - g|^p \, d\mu \right)^{1/p} \to 0$$

as $i \to \infty$. By the Lebesgue dominated convergence theorem, we have

$$\limsup_{i \to \infty} \int_U |g_i f_i|^p \, d\mu = \limsup_{i \to \infty} \int_U |g f_i|^p \, d\mu = 0.$$

□

Theorem 7.3 *For any $u \in H^{1,p}(U; X)$ with $1 < p < \infty$, there exists a unique minimal p-generalized upper gradient $g_u \in L^p(U)$ for u.*

Proof. The proof is parallel to that of [O1, Theorem 3.2]. For $n \ge 1$, take a sequence $\{(u_{n,i}, g_{n,i})\}_{i=1}^{\infty}$ and a function $g_n \in L^p(U)$ satisfying that $u_{n,i} \to u$ in $L^p(U; X)$ and $g_{n,i} \to g_n$ in $L^p(U)$ as $i \to \infty$, $g_{n,i}$ is an upper gradient for $u_{n,i}$, and that $|g_n|_{L^p} \le E_p(u)^{1/p} + n^{-1}$. If $E_p(u) = 0$, then clearly $g_n \to 0$ in $L^p(U)$ as $n \to \infty$, and hence the constant function 0 is a unique minimal p-generalized upper gradient.

We assume $E_p(u) > 0$ and fix $m > n \ge 1$. The triangle inequality yields that $(1/2)u_{m,i} + (1/2)u_{n,i} \to u$ in $L^p(U; X)$. By Lemma 7.1, the function

$$\left(\frac{1}{2} g_{m,i} + \frac{1}{2} g_{n,i} \right) \{ 1 + L_1(|u_{m,i} - u_{n,i}|_X \wedge L_2) + i^{-1} \}$$

is an upper gradient for $(1/2)u_{m,i} + (1/2)u_{n,i}$ and, by Lemma 7.2, which converges to $(1/2)g_m + (1/2)g_n$ in $L^p(U)$ as $i \to \infty$. Therefore we have

$$\frac{1}{2}|g_m|_{L^p} + \frac{1}{2}|g_n|_{L^p} \le E_p(u)^{1/p} + 2n^{-1} \le \left| \frac{1}{2} g_m + \frac{1}{2} g_n \right|_{L^p} + 2n^{-1}.$$

For each $n \ge 1$, take $i(n)$ large enough to satisfy $|g_{n,i(n)} - g_n|_{L^p} \le n^{-1}$. Then we see

$$\begin{aligned} \frac{1}{2}|g_{m,i(m)}|_{L^p} + \frac{1}{2}|g_{n,i(n)}|_{L^p} &\le \frac{1}{2}|g_m|_{L^p} + \frac{1}{2}|g_n|_{L^p} + n^{-1} \\ &\le \left| \frac{1}{2} g_m + \frac{1}{2} g_n \right|_{L^p} + 3n^{-1} \\ &\le \left| \frac{1}{2} g_{m,i(m)} + \frac{1}{2} g_{n,i(n)} \right|_{L^p} + 4n^{-1}. \end{aligned}$$

Since $L^p(U)$ is uniformly convex and $|g_n|_{L^p}^p \ge E_p(u) > 0$ for all $n \ge 1$, it implies that $\{g_{n,i(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(U)$, and hence it converges to a minimal p-generalized upper gradient for u. The uniqueness also follows from the uniform convexity of $L^p(U)$. □

For a continuous function $f : U \longrightarrow \mathbb{R}$ and a point $z \in U$, we define

$$\operatorname{Lip} f(z) := \lim_{r \to 0} \sup_{w \in B(z,r) \setminus \{z\}} \frac{|f(z) - f(w)|}{|z - w|_Z},$$

and we put $\operatorname{Lip} f(z) := 0$ if z is an isolated point. See Section 8 for more on this function. Along the discussions in [O1, Section 3], we obtain the following.

Lemma 7.4 *Let $u_1, u_2 : U \longrightarrow X$ be maps and $\phi : U \longrightarrow [0,1]$ be a function. For any upper gradients g_1, g_2, and g_3 for u_1, u_2, and ϕ, respectively, and for any Φ as in Lemma 7.1, the function*

$$\begin{aligned} g &:= g_3 \cdot (|u_1 - u_2|_X + \Phi) \\ &\quad + \{(1 - \phi + \Phi) g_1 + (\phi + \Phi) g_2\}\{1 + L_1(|u_1 - u_2|_X \wedge L_2) + \Phi\} \end{aligned}$$

is an upper gradient for the map $v := (1 - \phi) u_1 + \phi\, u_2$.

Let, in addition, ϕ be Lipschitz continuous and $1 \le p < \infty$. Then, for any p-generalized upper gradients $g_1, g_2 \in L^p(U)$ for $u_1, u_2 \in H^{1,p}(U; X)$, respectively, the function

$$g' := (\operatorname{Lip} \phi)|u_1 - u_2|_X + \{(1 - \phi) g_1 + \phi g_2\}\{1 + L_1(|u_1 - u_2|_X \wedge L_2)\}$$

is a p-generalized upper gradient for v.

Proof. The proof is same as that of [O1, Lemma 3.3] by using Lemma 7.1 instead of [O1, Lemma 3.1]. □

Proposition 7.5 *Let $1 \le p < \infty$, $W \subset U$ be an open set, and $u \in H^{1,p}(U; X)$. If g_U and g_W are p-generalized upper gradients for $u|_U$ and $u|_W$, respectively, then the function g defined by $g := g_U$ on $U \setminus W$ and $g := g_W$ on W is a p-generalized upper gradient for u. In particular, if $1 < p < \infty$, then we have $g_u = g_{(u|_W)}$ a.e. on W.*

Proof. See [O1, Proposition 3.4]. □

Corollary 7.6 *Let $1 < p < \infty$.*

(i) *If $g \in L^p(U)$ is a p-generalized upper gradient for $u \in H^{1,p}(U; X)$, then $g_u \le g$ holds a.e. on U.*

(ii) *For $u, v \in H^{1,p}(U; X)$, if $u = v$ a.e. on an open set $W \subset U$, then we have $g_u = g_v$ a.e. on W.*

Proof. See [O1, Corollaries 3.5, 3.6]. □

8 Minimality of $\operatorname{Lip} u$

For a continuous map $u : U \longrightarrow X$ and a point $z \in U$, we define

$$\operatorname{Lip} u(z) := \lim_{r\to 0} \sup_{w\in B(z,r)\setminus\{z\}} \frac{|u(z)-u(w)|_X}{|z-w|_Z},$$

and we put $\operatorname{Lip} u(z) := 0$ if z is an isolated point. Note that $\operatorname{Lip} u$ is Borel measurable and, if u is Lipschitz continuous, then it does not exceed the Lipschitz constant of u. It is easy to show that, for a locally Lipschitz map u, $\operatorname{Lip} u$ is an upper gradient for u ([O1, Proposition 5.2]). The first variation formula on a C_k- and C_L-domain (Theorem 5.2) allows us to obtain the minimality of $\operatorname{Lip} u$ for maps into a locally (k)-convex and locally (L)-convex metric space, and it generalizes [O1, Theorem 5.9].

Lemma 8.1 *Let (X, d_X) be a locally compact, locally geodesics extendable, locally (k)-convex, and locally (L)-convex metric space, and let $u : U \longrightarrow X$ be a locally Lipschitz map. Then, for every $z \in U$ and $\varepsilon > 0$, there exists a point $x \in X \setminus \{u(z)\}$ in a C_k- and C_L-domain containing $u(z)$ such that*

$$\operatorname{Lip} |u - x|_X(z) \ge \operatorname{Lip} u(z) - \varepsilon.$$

Moreover, for such x and each $y \in X$ near x, we have

$$\operatorname{Lip} |u - y|_X(z) \ge \operatorname{Lip} u(z) - \varepsilon + \theta_{x,u(z)}(|x - y|_X).$$

Proof. We may assume $\operatorname{Lip} u(z) > 0$. Take a sequence $\{z_i\}_{i=1}^{\infty} \subset U \setminus \{z\}$ which tends to z and satisfies

$$\lim_{i\to\infty} \frac{|u(z)-u(z_i)|_X}{|z-z_i|_Z} = \operatorname{Lip} u(z).$$

For a sufficiently small $\delta > 0$, by the local geodesics extendability, we find a point

$$x_i \in S(u(z), \delta^2) = \{w \in X \mid |u(z) - w|_X = \delta^2\}$$

satisfying $u(z_i) = \gamma_{u(z)x_i}(|u(z) - u(z_i)|_X/\delta^2)$ for each large i. As $S(u(z), \delta^2)$ is compact, we can extract a subsequence $\{x_j\}$ of $\{x_i\}$ which tends to a point $x' \in S(u(z), \delta^2)$, and we take $x \in S(u(z), \delta)$ with $x' = \gamma_{u(z)x}(\delta)$. By Theorem 5.2, we have

$$\begin{aligned} |u(z) - x|_X - |u(z_j) - x|_X &= |u(z) - u(z_j)|_X \cos\left(\lim_{t\to 0} \tilde{\angle} x u(z) \gamma_{u(z)x_j}(t)\right) \\ &\quad + O_{x,u(z)}(|u(z) - u(z_j)|_X^2). \end{aligned}$$

It follows from the k-convexity that

$$\begin{aligned} \cos \tilde{\angle} x u(z) \gamma_{u(z)x_j}(t) &= \frac{\delta^2 + t^2\delta^4 - |x - \gamma_{u(z)x_j}(t)|_X^2}{2t\delta^3} \\ &\ge \frac{1}{2t\delta^3}\left\{\delta^2 + t^2\delta^4 - (1-t)\delta^2 - t|x - x_j|_X^2 + \frac{k}{2}(1-t)t\delta^4\right\} \\ &= \frac{1}{2\delta^3}(\delta^2 + \delta^4 - |x - x_j|_X^2) - \left(1 - \frac{k}{2}\right)\frac{(1-t)\delta}{2} \\ &= \cos \tilde{\angle} x u(z) x_j - \left(1 - \frac{k}{2}\right)\frac{(1-t)\delta}{2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\frac{|u(z)-x|_X-|u(z_j)-x|_X}{|z-z_j|_Z}\\
&=\frac{|u(z)-u(z_j)|_X}{|z-z_j|_Z}\cos\left(\lim_{t\to 0}\tilde{\angle} xu(z)\gamma_{u(z)x_j}(t)\right)+O_{x,u(z)}(|u(z)-u(z_j)|_X)\\
&\geq\frac{|u(z)-u(z_j)|_X}{|z-z_j|_Z}\left\{\cos\tilde{\angle} xu(z)x_j-\left(1-\frac{k}{2}\right)\frac{\delta}{2}\right\}+O_{x,u(z)}(|z-z_j|_Z)\\
&\to\left\{1-\left(1-\frac{k}{2}\right)\frac{\delta}{2}\right\}\mathrm{Lip}\,u(z)
\end{aligned}$$

as j diverges to the infinity. This completes the proof of the first part.

Recall that $\delta=|x-u(z)|_X$. For $y\in B(x,\delta)$, $j\geq 1$, and for $t\in(0,1)$, Lemma 5.1 yields

$$\begin{aligned}
\cos\tilde{\angle} yu(z)\gamma_{u(z)x_j}(t)&\geq\cos\tilde{\angle} yu(z)x_j-\left(1-\frac{k}{2}\right)\frac{(1-t)\delta^2}{2|y-u(z)|_X}\\
&\geq\cos\tilde{\angle} yu(z)x_j-\left(1-\frac{k}{2}\right)\frac{\delta}{2}+\theta_\delta(|x-y|_X).
\end{aligned}$$

We remark that the term $\theta_\delta(|x-y|_X)$ does not depend on j. Therefore we have

$$\begin{aligned}
&\liminf_{j\to\infty}\frac{|u(z)-y|_X-|u(z_j)-y|_X}{|z-z_j|_Z}\\
&=\liminf_{j\to\infty}\left\{\frac{|u(z)-u(z_j)|_X}{|z-z_j|_Z}\cos\left(\lim_{t\to 0}\tilde{\angle} yu(z)\gamma_{u(z)x_j}(t)\right)\right.\\
&\qquad\left.+O_{y,u(z)}(|u(z)-u(z_j)|_X)\right\}\\
&\geq\left\{\cos\tilde{\angle} xu(z)y-\left(1-\frac{k}{2}\right)\frac{\delta}{2}\right\}\mathrm{Lip}\,u(z)+\theta_\delta(|x-y|_X)\\
&=\left\{1-\left(1-\frac{k}{2}\right)\frac{\delta}{2}\right\}\mathrm{Lip}\,u(z)+\theta_{x,u(z)}(|x-y|_X).
\end{aligned}$$

□

Now we can prove the minimality of $\mathrm{Lip}\,u$ just as in the proof of [O3, Theorem 5.9]. Before stating the theorem, we need to recall two important notions.

Definition 8.2 A metric measure space (Z,d_Z,μ) is said to satisfy the *doubling condition* if there exist constants $R_D>0$ and $C_D\geq 1$ such that $\mu(B(z,r))\leq C_D\,\mu(B(z,r/2))$ holds for all $z\in Z$ and $r\in(0,R_D]$.

It follows from this condition that any ball with radius R_D, say $B(z,R_D)$, is totally bounded. Hence, if (Z,d_Z) is complete, then any closed ball $\overline{B}(z,r)$ with $r\in(0,R_D)$ is compact, so that μ is a Radon measure on $\overline{B}(z,r)$.

Definition 8.3 A metric measure space (Z, d_Z, μ) is said to satisfy the *weak Poincaré inequality of type* $(1,p)$ if there exist constants $R_P > 0$, $C_P \geq 1$, and $\Lambda \geq 1$ such that we have

$$\fint_{B(z,r)} \left| f - \fint_{B(z,r)} f\, d\mu \right| d\mu \leq C_P\, r \left(\fint_{B(z,\Lambda r)} g^p\, d\mu \right)^{1/p}$$

for all $z \in Z$, $r \in (0, R_P]$, $f \in L^p(B(z,\Lambda r))$, and for all upper gradient $g : B(z,\Lambda r) \longrightarrow [0,\infty]$ for f. Here, as usual, we define $\fint_{B(z,r)} f\, d\mu := \mu(B(z,r))^{-1} \int_{B(z,r)} f\, d\mu$.

Theorem 8.4 *Let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1,p)$ for some $1 < p < \infty$, and let (X, d_X) be a locally compact, locally geodesics extendable, locally (k)-convex, and locally (L)-convex metric space. Then, for any locally Lipschitz map $u \in H^{1,p}(U;X)$, the function* $\operatorname{Lip} u$ *is a minimal p-generalized upper gradient for u, i.e.,*

$$E_p(u) = \int_U (\operatorname{Lip} u)^p\, d\mu.$$

If, in addition, (X, d_X) is complete or L-convex, then $g_u = \operatorname{Lip} u$ holds a.e. on U.

Proof. The proof is similar to that of [O1, Theorem 5.9] by virtue of Lemma 8.1. □

The above theorem contains [O1, Theorem 5.9] by Corollary 3.2. A similar discussion yields the following (which is actually equivalent to Theorem 8.4 if (X, d_X) is complete by Proposition 6.4).

Theorem 8.5 *Let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1,p)$ for some $1 < p < \infty$, and let (X, d_X) be a locally compact, locally geodesics extendable, locally (k)-convex, and locally (L)-convex metric space. Then, for any locally Lipschitz map $u \in N^{1,p}(U;X)$, the function* $\operatorname{Lip} u$ *is a minimal p-weak upper gradient for u.*

9 Dirichlet problem

This section is devoted to the study of the Dirichlet problem, i.e., the existence problem of an energy minimizer. The target space (X, d_X) is supposed to be complete throughout the section, so that all discussions are also applicable to the Newtonian space by Proposition 6.4.

9.1 Proper case

In this subsection, let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1,p)$ for some $p \in (1,\infty)$, and let (X, d_X) be a proper, L-convex metric space. Then $L^p(U;X)$ is complete. In addition, we suppose that there exists a constant $C > 0$ such that, for any $f \in H_0^{1,p}(U)$, it holds that

$$\left(\int_U |f|^p\, d\mu \right)^{1/p} \leq C \left(\int_U |g_f|^p\, d\mu \right)^{1/p}. \tag{9.1}$$

Remark 9.1 In the case where $p = 2$ and $\operatorname{diam} U < (\operatorname{diam} Z)/3$, the inequality (9.1) follows from the doubling condition and the weak Poincaré inequality of type $(1, 2)$ by combining [HaK, Theorem 1], [SC, Theorem 2.1], and [Bj, Proposition 3.1].

Define the distance $d_{H^{1,p}}$ on $H^{1,p}(U; X)$ by, for $u, v \in H^{1,p}(U; X)$,

$$d_{H^{1,p}}(u, v) := |u - v|_{L^p} + |g_u - g_v|_{L^p}.$$

(We do not use the notation $|u - v|_{H^{1,p}}$ in order to avoid the confusion with the Sobolev norm of the function $|u - v|_X$.) For $v \in H^{1,p}(U; X)$, we define

$$\mathring{H}^{1,p}_v(U; X) := \{u \in H^{1,p}(U; X) \mid \operatorname{supp} |u - v|_X \subset U\}$$

and denote by $H^{1,p}_v(U; X)$ its $d_{H^{1,p}}$-closure. Note that $\mathring{H}^{1,p}_v(U; X)$ is a convex subset in $H^{1,p}(U; X)$ and that

$$\inf\{E_p(u) \mid u \in H^{1,p}(U; X), \operatorname{supp} |u - v|_X \subset U\} = \inf_{u \in H^{1,p}_v(U;X)} E_p(u).$$

Definition 9.2 A map $v \in H^{1,p}(U; X)$ is said to be *p-harmonic* if it satisfies

$$E_p(v) = \inf_{u \in H^{1,p}_v(U;X)} E_p(u).$$

If $\partial U = \emptyset$, then $H^{1,p}_v(U; X) = H^{1,p}(U; X)$ and hence the constant map to the base point x_0 is p-harmonic. Therefore, henceforce, we assume $\partial U \neq \emptyset$ and fix a map $v \in H^{1,p}(U; X)$.

We first recall that the canonical embedding $H^{1,p}(U; X) \hookrightarrow L^p(U; X)$ is compact in the sense that every sequence $\{u_i\}_{i=1}^\infty$ in $H^{1,p}(U; X)$ such that $\{|u_i - x_0|_{L^p} + E_p(u_i)\}_{i=1}^\infty$ is uniformly bounded has a subsequence which is convergent in $L^p(U; X)$.

Lemma 9.3 *The embedding $H^{1,p}(U; X) \hookrightarrow L^p(U; X)$ is compact.*

Proof. It is well-known that the weak Poincaré inequality of type $(1, p)$ together with the doubling condition implies the compactness of the embedding $H^{1,p}(U) \hookrightarrow L^p(U)$. Then the lemma follows from the proof of [KS, Theorem 1.13] since X is assumed to be proper. □

We next consider the Dirichlet problem. The properness of X allows us to simplify the discussions in [J, Section 3.1] and [O1, Section 4].

Theorem 9.4 *Let (Z, d_Z, μ) be a complete metric measure space satisfying the doubling condition and the weak Poincaré inequality of type $(1, p)$ for some $p \in (1, \infty)$, let (X, d_X) be a proper, L-convex metric space, and suppose that the inequality (9.1) holds for every $f \in H^{1,p}_0(U)$. Then, for any $v \in H^{1,p}(U; X)$, there exists a p-harmonic map in $H^{1,p}_v(U; X)$.*

Proof. Take a sequence $\{u_i\}_{i=1}^\infty \subset \mathring{H}^{1,p}_v(U; X)$ satisfying $\lim_{i\to\infty} E_p(u_i) = \inf_{H^{1,p}_v(U;X)} E_p$. Since $g_{u_i} + g_v$ is a p-generalized upper gradient for the function $|u_i - v|_X$, by (9.1), we have

$$\begin{aligned} |u_i - v|_{L^p} &\le C|g_{|u_i - v|_X}|_{L^p} \le C|g_{u_i} + g_v|_{L^p} \le C(|g_{u_i}|_{L^p} + |g_v|_{L^p}) \\ &= C(E_p(u_i)^{1/p} + E_p(v)^{1/p}). \end{aligned}$$

Hence $\{|u_i - v|_{L^p}\}_{i=1}^{\infty}$ is uniformly bounded, so that it follows from Lemma 9.3 that a subsequence of $\{u_i\}_{i=1}^{\infty}$ converges to a map $u \in L^p(U;X)$ in $L^p(U;X)$. We again denote this subsequence by $\{u_i\}_{i=1}^{\infty}$.

As in the proof of Theorem 7.3, by

$$\lim_{i,j\to\infty}\left(\frac{1}{2}|g_{u_i}|_{L^p}+\frac{1}{2}|g_{u_j}|_{L^p}\right)=\inf_{H_v^{1,p}(U;X)}E_p^{1/p}\le\liminf_{i,j\to\infty}\left|\frac{1}{2}g_{u_i}+\frac{1}{2}g_{u_j}\right|_{L^p}$$

together with the uniform convexity of $L^p(U)$, we obtain that $\{g_{u_i}\}_{i=1}^{\infty}$ is a Cauchy sequence, so that it converges to some $g \in L^p(U)$. In particular, g is a p-generalized upper gradient for u, and hence $u \in H^{1,p}(U;X)$.

By a similar discussion to the latter half of the proof of [O1, Lemma 4.2] using Lemma 7.4 instead of [O1, Lemma 3.3], we see that $g = g_u$. Therefore u_i tends to u with respect to $d_{H^{1,p}}$ as $i \to \infty$, so that $u \in H_v^{1,p}(U;X)$. □

Remark 9.5 It is easily observed that a p-harmonic map in $H_v^{1,p}(U;X)$ is not necessarily unique. In fact, since $E_p(u)$ cares only the most stretching direction of u (see Theorem 8.4), we can deform u in a less stretching direction without changing the energy.

9.2 Improper case

Even in the case where (X,d_X) is not proper, if (X,d_X) is k-convex and L-convex for $L_1 = 0$ and if $p = 2$, then we can apply all discussions in [O1, Section 4] verbatim.

Theorem 9.6 *Let (Z,d_Z,μ) be a metric measure space, $U \subset Z$ be an open set, (X,d_X) be a complete, k-convex, and L-convex metric space for $L_1 = 0$, and suppose that the inequality* (9.1) *holds for every $f \in H_0^{1,2}(U)$. Then, for any $v \in H^{1,2}(U;X)$, there exists a 2-harmonic map in $H_v^{1,2}(U;X)$.*

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