Quantitative estimates for the Bakry–Ledoux isoperimetric inequality. II

Cong Hung MAI* Shin-ichi OHTA*,†

July 27, 2022

Abstract

Concerning quantitative isoperimetry for a weighted Riemannian manifold satisfying $\mathrm{Ric}_\infty \ge 1$, we give an L^1-estimate exhibiting that the push-forward of the reference measure by the guiding function (arising from the needle decomposition) is close to the Gaussian measure. We also show L^p- and W_2-estimates in the 1-dimensional case.

1 Introduction

This short article is devoted to several further applications of the detailed estimates in [MO] to quantitative isoperimetry. In [MO], on a weighted Riemannian manifold $(M, g, \mathfrak{m})$ (with $\mathfrak{m} = \mathrm{e}^{-\Psi} \mathrm{vol}_g$) satisfying $\mathfrak{m}(M) = 1$ and $\mathrm{Ric}_\infty \ge 1$, we investigated the stability of the *Bakry–Ledoux isoperimetric inequality* [BL]:

$$\mathsf{P}(A) \ge \mathcal{I}_{(\mathbb{R},\boldsymbol{\gamma})}\big(\mathfrak{m}(A)\big) \tag{1.1}$$

for any Borel set $A \subset M$, where $\mathsf{P}(A)$ is the perimeter of A, $\boldsymbol{\gamma}(dx) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}\,dx$ is the Gaussian measure on $\mathbb{R}$, and $\mathcal{I}_{(\mathbb{R},\boldsymbol{\gamma})}$ is its *isoperimetric profile* written as

$$\mathcal{I}_{(\mathbb{R},\boldsymbol{\gamma})}(\theta) = \frac{\mathrm{e}^{-a_\theta^2/2}}{\sqrt{2\pi}}, \qquad \theta = \boldsymbol{\gamma}\big((-\infty, a_\theta]\big). \tag{1.2}$$

It is known by [Mo, Theorem 18.7] (see also [Ma, §3]) that equality holds in (1.1) for some A with $\theta = \mathfrak{m}(A) \in (0,1)$ if and only if $(M, g, \mathfrak{m})$ is isometric to the product of $(\mathbb{R}, |\cdot|, \boldsymbol{\gamma})$ and a weighted Riemannian manifold $(\Sigma, g_\Sigma, \mathfrak{m}_\Sigma)$ of $\mathrm{Ric}_\infty \ge 1$. Moreover, A is necessarily of the form $(-\infty, a_\theta] \times \Sigma$ or $[-a_\theta, \infty) \times \Sigma$ (so-called a *half-space*). Then, the stability result [MO, Theorem 7.5] asserts that, if equality in (1.1) nearly holds, then A is close to a kind of half-space in the sense that the symmetric difference between them has a small volume.

*Department of Mathematics, Osaka University, Osaka 560-0043, Japan (hungmcuet@gmail.com, s.ohta@math.sci.osaka-u.ac.jp)

†RIKEN Center for Advanced Intelligence Project (AIP), 1-4-1 Nihonbashi, Tokyo 103-0027, Japan

The proof as well as the formulation of [MO, Theorem 7.5] are based on the *needle decomposition* paradigm (also called the *localization*), which was established by Klartag [Kl] for Riemannian manifolds and has provided a significant contribution specifically in the study of isoperimetric inequalities (we refer to [CM] for a generalization to metric measure spaces satisfying the curvature-dimension condition, and to [CMM] for a stability result). The half-space we mentioned above is in fact a sub-level or super-level set of the *guiding function* arising in the needle decomposition (see Section 3 and [MO] for more details). The needle decomposition enables us to decompose a global inequality on M into the corresponding 1-dimensional inequalities on minimal geodesics in M (called *needles* or *transport rays*). Therefore, a more detailed 1-dimensional analysis on needles will furnish a better estimate on M.

The 1-dimensional analysis in [MO] is concentrated in Proposition 3.2 in it (restated in Proposition 2.1 below), which gives a very detailed estimate on the difference from the Gaussian measure $\boldsymbol{\gamma}$. In this article, as an application of the analysis developed in [MO], we show an L^1-bound between $\boldsymbol{\gamma}$ and the push-forward measure $u_*\mathfrak{m}$ of $\mathfrak{m}$ by the guiding function u:

$$\|\rho \cdot \mathrm{e}^{\psi_{\mathrm{g}}} - 1\|_{L^1(\boldsymbol{\gamma})} \le C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)},$$

where $u_*\mathfrak{m} = \rho\, dx$ and $\boldsymbol{\gamma} = \mathrm{e}^{-\psi_{\mathrm{g}}}\, dx$ (see Theorem 3.1 for the precise statement). In the 1-dimensional case (on intervals), we also prove an L^p-bound with the improved (and sharp) order $\delta^{1/p}$ (Proposition 2.2; see Example 2.3 for the sharpness) and an estimate of the L^2-Wasserstein distance W_2 (Proposition 2.4). The use of L^p and W_2 (instead of the volume of the symmetric difference) is inspired by stability results for the Poincaré and log-Sobolev inequalities (e.g., [BF, BGRS, CF, IK, IM]). We refer to Remark 3.2 for some further related works and open problems.

Acknowledgements. We are grateful to Emanuel Indrei, whose question on the L^p-estimate led us to write this paper. CHM was supported by Grant-in-Aid for JSPS Fellows 20J11328. SO was supported in part by JSPS Grant-in-Aid for Scientific Research (KAKENHI) 19H01786.

2 Quantitative estimates on intervals

We first consider the 1-dimensional case (on intervals) and establish quantitative stability estimates in terms of the L^p-norm and the W_2-distance. The L^1-bound will be instrumental to study the Riemannian case in the next section.

2.1 An L^p-estimate

Throughout this section, let $I \subset \mathbb{R}$ be an open interval equipped with a probability measure $\mathfrak{m} = \mathrm{e}^{-\psi}\, dx$ such that ψ is 1-*convex* in the sense that

$$\psi\big((1-t)x + ty\big) \le (1-t)\psi(x) + t\psi(y) - \frac{1}{2}(1-t)t|x-y|^2$$

for all $x, y \in I$ and $t \in (0,1)$. This means that $(I, |\cdot|, \mathfrak{m})$ satisfies $\mathrm{Ric}_\infty \ge 1$ (or the curvature-dimension condition $\mathrm{CD}(1,\infty)$), and (1.1) holds. The 1-dimensional isoperimetric inequality is well investigated in convex analysis. An important fact due to Bobkov

[Bo, Proposition 2.1] is that an isoperimetric minimizer can be always taken as a half-space of the form $(-\infty, a] \cap I$ or $[b, \infty) \cap I$. Now we restate [MO, Proposition 3.2], which is the source of all the estimates. Recall that $\boldsymbol{\gamma} = \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}}\,dx$ is the Gaussian measure.

Proposition 2.1 ([MO]) *Fix $\theta \in (0,1)$ and suppose that*

$$\mathfrak{m}\big((-\infty, a_\theta] \cap I\big) = \theta \tag{2.1}$$

and

$$\mathrm{e}^{-\psi(a_\theta)} \le \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(a_\theta)} + \delta \tag{2.2}$$

hold for sufficiently small $\delta > 0$ (relative to θ). Then we have

$$\psi(x) - \boldsymbol{\psi}_{\mathrm{g}}(x) \ge \big(\psi'_+(a_\theta) - a_\theta\big)(x - a_\theta) - C(\theta)\delta \tag{2.3}$$

for every $x \in I$, and

$$\psi(x) - \boldsymbol{\psi}_{\mathrm{g}}(x) \le \big(\psi'_+(a_\theta) - a_\theta\big)(x - a_\theta) + C(\theta)\sqrt{\delta} \tag{2.4}$$

for every $x \in [S, T] \subset I$ such that $\lim_{\delta \to 0} S = -\infty$ and $\lim_{\delta \to 0} T = \infty$, where ψ'_+ denotes the right derivative of ψ and $C(\theta)$ is a positive constant depending only on θ.

The first condition (2.1) means that I is "centered" in comparison with $\boldsymbol{\gamma}$ which satisfies $\boldsymbol{\gamma}((-\infty, a_\theta]) = \theta$ (as in (1.2)). Note also that $\mathrm{e}^{-\psi(a_\theta)} \ge \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(a_\theta)}$ holds by the isoperimetric inequality (1.1) (since $\mathsf{P}((-\infty, a_\theta] \cap I) = \mathrm{e}^{-\psi(a_\theta)}$), and then (2.2) tells that the *deficit* of $(-\infty, a_\theta] \cap I$ in the isoperimetric inequality is less than or equal to δ.

Besides the above proposition, we also need the following estimate in its proof (see [MO, (3.9)]):

$$\limsup_{\delta \to 0} \frac{|\psi'_+(a_\theta) - a_\theta|}{\delta} \le C(\theta). \tag{2.5}$$

The lower bound (2.3) enables us to obtain the following L^p-estimate between $\boldsymbol{\gamma} = \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}}\,dx$ and $\mathfrak{m} = \mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi}\,\boldsymbol{\gamma}|_I$. (We remark that the upper bound (2.4) will not be used.)

Proposition 2.2 (An L^p-estimate on I) *Assume (2.1) and (2.2). Then we have*

$$\|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\|_{L^p(\boldsymbol{\gamma})} \le C(p, \theta)\delta^{1/p}$$

for all $p \in [1, \infty)$ and sufficiently small $\delta > 0$ (relative to θ and p), where we set $\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} := 0$ on $\mathbb{R} \setminus I$.

Proof. In this proof, we denote by C a positive constant depending on θ, and put $a := a_\theta$ for brevity. Since $\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1 \ge -1$ and $\mathfrak{m}(I) = \boldsymbol{\gamma}(\mathbb{R}) = 1$, we find

$$\begin{aligned}
\|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\|^p_{L^p(\boldsymbol{\gamma})} &= \int_I \big[\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\big]_+^p \,d\boldsymbol{\gamma} + \int_{-\infty}^{\infty} \big[1 - \mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi}\big]_+^p \,d\boldsymbol{\gamma} \\
&\le \int_I \big[\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\big]_+^p \,d\boldsymbol{\gamma} + \int_{-\infty}^{\infty} \big[1 - \mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi}\big]_+ \,d\boldsymbol{\gamma} \\
&= \int_I \big[\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\big]_+^p \,d\boldsymbol{\gamma} + \int_I \big[\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \psi} - 1\big]_+ \,d\boldsymbol{\gamma},
\end{aligned}$$

where $[r]_+ := \max\{r, 0\}$. Thus, we need to estimate only $[\mathrm{e}^{\psi_{\mathrm{g}}-\psi} - 1]_+$. Observe that

$$[\mathrm{e}^{(\psi_{\mathrm{g}}-\psi)(x)} - 1]_+^p \le \left(\mathrm{e}^{C\delta|x-a|+C\delta} - 1\right)^p \le \mathrm{e}^{p(C\delta|x-a|+C\delta)} - 1$$

from (2.3) and (2.5), and hence

$$\begin{aligned}\int_I [\mathrm{e}^{\psi_{\mathrm{g}}-\psi} - 1]_+^p \, d\gamma &\le \int_{-\infty}^{\infty} \left(\mathrm{e}^{p(C\delta|x-a|+C\delta)} - 1\right) \gamma(dx) \\ &= \frac{\mathrm{e}^{pC\delta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} + pC\delta|x-a|\right) dx - 1.\end{aligned}$$

Dividing the integral into $(-\infty, a]$ and $[a, \infty)$, we continue the calculation as

$$\begin{aligned}&\int_{-\infty}^{a} \exp\left(-\frac{x^2}{2} - pC\delta(x-a)\right) dx + \int_{a}^{\infty} \exp\left(-\frac{x^2}{2} + pC\delta(x-a)\right) dx \\ &= \int_{-\infty}^{a} \exp\left(-\frac{(x+pC\delta)^2}{2} + \frac{(pC\delta)^2}{2} + pCa\delta\right) dx \\ &\quad + \int_{a}^{\infty} \exp\left(-\frac{(x-pC\delta)^2}{2} + \frac{(pC\delta)^2}{2} - pCa\delta\right) dx \\ &\le \exp\left(\frac{(pC\delta)^2}{2} + pCa\delta\right) \left\{\int_{-\infty}^{a} \mathrm{e}^{-x^2/2} \, dx + pC\delta\right\} \\ &\quad + \exp\left(\frac{(pC\delta)^2}{2} - pCa\delta\right) \left\{\int_{a}^{\infty} \mathrm{e}^{-x^2/2} \, dx + pC\delta\right\} \\ &\le \exp\left(\frac{(pC\delta)^2}{2} + pC|a|\delta\right) \left(\sqrt{2\pi} + 2pC\delta\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\int_I [\mathrm{e}^{\psi_{\mathrm{g}}-\psi} - 1]_+^p \, d\gamma &\le \exp\left(pC\delta + pC|a|\delta + \frac{(pC\delta)^2}{2}\right) \left(1 + \frac{2pC\delta}{\sqrt{2\pi}}\right) - 1 \\ &\le C(p, \theta)\delta.\end{aligned}$$

This completes the proof. $\square$

We remark that, since

$$\left\{\exp\left(pC\delta + \frac{(pC\delta)^2}{2}\right) - 1\right\}^{1/p} \ge \exp\left(C\delta + \frac{p(C\delta)^2}{2}\right) - 1,$$

the constant $C(p, \theta)$ given by the above proof necessarily depends on p. The order $\delta^{1/p}$ in Proposition 2.2 may be compared with L^p-estimates in [IK] for the log-Sobolev inequality on Gaussian spaces. One can see that the order $\delta^{1/p}$ is optimal from the following example.

Example 2.3 Let $I = (-D, D)$ and $\mathfrak{m} = (1+\delta) \cdot \gamma|_I$, where $\delta > 0$ is given by $\gamma(I) = (1+\delta)^{-1}$. Then, at $\theta = 1/2$, we have $a_{1/2} = 0$, $\mathfrak{m}((-\infty, 0] \cap I) = 1/2$,

$$\mathrm{e}^{-\psi(0)} - \mathrm{e}^{-\psi_{\mathrm{g}}(0)} = \frac{\delta}{\sqrt{2\pi}},$$

and

$$\|\mathrm{e}^{\psi_{\mathrm{g}}-\psi} - 1\|_{L^p(\gamma)} = \left(\frac{\delta^p}{1+\delta} + \frac{\delta}{1+\delta}\right)^{1/p} = \left(\frac{1+\delta^{p-1}}{1+\delta}\right)^{1/p} \delta^{1/p}.$$

2.2 A W_2-estimate

From Proposition 2.1, one can also derive an upper bound of the L^2-*Wasserstein distance* between $\mathfrak{m}$ and $\boldsymbol{\gamma}$. We refer to [Vi] for the basics of optimal transport theory. What we need is only the following *Talagrand inequality* with $\boldsymbol{\gamma}$ as the base measure (see [Ta], [Vi, Theorem 22.14]):

$$W_2^2(\mathfrak{m},\boldsymbol{\gamma}) \le 2\,\mathrm{Ent}_{\boldsymbol{\gamma}}(\mathfrak{m}) = 2\int_I (\boldsymbol{\psi}_{\mathrm{g}}-\psi)\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}}-\psi}\,d\boldsymbol{\gamma}, \tag{2.6}$$

where $\mathrm{Ent}_{\boldsymbol{\gamma}}(\mathfrak{m})$ is the *relative entropy* of $\mathfrak{m}$ with respect to $\boldsymbol{\gamma}$. We remark that both $\boldsymbol{\gamma}$ and $\mathfrak{m}$ have finite second moment (by the 1-convexity of ψ).

Proposition 2.4 (A W_2-estimate on I) *Assume* (2.1) *and* (2.2)*. Then we have*

$$W_2(\mathfrak{m},\boldsymbol{\gamma}) \le C(\theta)\sqrt{\delta}$$

for sufficiently small $\delta>0$ (relative to θ).

Proof. We again denote a_θ by a, and C will be a positive constant depending only on θ. Similarly to the proof of Proposition 2.2, we observe from (2.3) and (2.5) that

$$\begin{aligned}\int_I (\boldsymbol{\psi}_{\mathrm{g}}-\psi)\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}}-\psi}\,d\boldsymbol{\gamma} &\le \int_{-\infty}^{\infty}(C\delta|x-a|+C\delta)\mathrm{e}^{C\delta|x-a|+C\delta}\,\boldsymbol{\gamma}(dx)\\ &= \frac{C\delta}{\sqrt{2\pi}}\mathrm{e}^{C\delta}\int_{-\infty}^{\infty}(|x-a|+1)\exp\left(-\frac{x^2}{2}+C\delta|x-a|\right)dx\\ &\le C\delta\left\{\int_{-\infty}^{\infty}|x-a|\exp\left(-\frac{x^2}{2}+C\delta|x-a|\right)dx + C\right\},\end{aligned}$$

where we used

$$\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2}+C\delta|x-a|\right)dx \le C$$

from the proof of Proposition 2.2. Then we have

$$\begin{aligned}&\int_{-\infty}^{a}(a-x)\exp\left(-\frac{x^2}{2}-C\delta(x-a)\right)dx\\ &= \exp\left(Ca\delta+\frac{(C\delta)^2}{2}\right)\int_{-\infty}^{a}(a-x)\exp\left(-\frac{(x+C\delta)^2}{2}\right)dx\\ &\le (1+C\delta)\left\{(a+C\delta)\int_{-\infty}^{a}\exp\left(-\frac{(x+C\delta)^2}{2}\right)dx + \left[\exp\left(-\frac{(x+C\delta)^2}{2}\right)\right]_{-\infty}^{a}\right\}\\ &\le (1+C\delta)\left\{a\int_{-\infty}^{a}\mathrm{e}^{-x^2/2}\,dx + C\delta + \exp\left(-\frac{(a+C\delta)^2}{2}\right)\right\}\\ &\le a\int_{-\infty}^{a}\mathrm{e}^{-x^2/2}\,dx + \mathrm{e}^{-a^2/2} + C\delta.\end{aligned}$$

We similarly find

$$\begin{aligned}
&\int_a^\infty (x-a)\exp\left(-\frac{x^2}{2}+C\delta(x-a)\right)\,dx\\
&=\exp\left(-Ca\delta+\frac{(C\delta)^2}{2}\right)\int_a^\infty (x-a)\exp\left(-\frac{(x-C\delta)^2}{2}\right)\,dx\\
&\le (1+C\delta)\left\{(-a+C\delta)\int_a^\infty \exp\left(-\frac{(x-C\delta)^2}{2}\right)\,dx-\left[\exp\left(-\frac{(x-C\delta)^2}{2}\right)\right]_a^\infty\right\}\\
&\le (1+C\delta)\left\{-a\int_a^\infty \mathrm{e}^{-x^2/2}\,dx+C\delta+\exp\left(-\frac{(a-C\delta)^2}{2}\right)\right\}\\
&\le -a\int_a^\infty \mathrm{e}^{-x^2/2}\,dx+\mathrm{e}^{-a^2/2}+C\delta.
\end{aligned}$$

Therefore, together with the Talagrand inequality (2.6), we obtain the desired estimate $W_2^2(\mathfrak{m},\boldsymbol{\gamma})\le C\delta$. □

We do not know whether the order $\sqrt{\delta}$ in Proposition 2.4 is optimal. Since $W_p(\mathfrak{m},\boldsymbol{\gamma})\le W_2(\mathfrak{m},\boldsymbol{\gamma})$ for any $p\in[1,2)$ by the Hölder inequality, we have, in particular, a bound of the L^1-Wasserstein distance:

$$W_1(\mathfrak{m},\boldsymbol{\gamma})\le C(\theta)\sqrt{\delta}.$$

One can alternatively infer this estimate from the Kantorovich–Rubinstein duality (see [Vi]); in fact,

$$W_1(\mathfrak{m},\boldsymbol{\gamma})\le\int_{-\infty}^{\infty}|x-a|\cdot|\mathrm{e}^{(\psi_{\mathrm{g}}-\psi)(x)}-1|\,\boldsymbol{\gamma}(dx)\le C(\theta)\sqrt{\delta}.$$

We also remark that, when we take a detour via the reverse Poincaré inequality in [MO, Proposition 5.1] and the stability result [CF, Theorem 1.2], we arrive at a weaker estimate

$$W_1(\mathfrak{m},\boldsymbol{\gamma})\le C(\theta,\varepsilon)\delta^{(1-\varepsilon)/4}.$$

We refer to [CMS, FGS] for stability results for the Poincaré inequality (equivalently, the spectral gap) on $\mathrm{CD}(N-1,N)$-spaces and $\mathrm{RCD}(N-1,N)$-spaces with $N\in(1,\infty)$.

3 An L^1-estimate on weighted Riemannian manifolds

Next, we consider a weighted Riemannian manifold, namely a connected, complete $\mathcal{C}^\infty$-Riemannian manifold (M,g) of dimension $n\ge 2$ equipped with a probability measure $\mathfrak{m}=\mathrm{e}^{-\Psi}\,\mathrm{vol}_g$, where $\Psi\in\mathcal{C}^\infty(M)$ and vol_g is the Riemannian volume measure. Assuming $\mathrm{Ric}_\infty\ge 1$, we have the Bakry–Ledoux isoperimetric inequality (1.1).

We begin with an outline of the proof of (1.1) via the *needle decomposition* (see [Kl]). Given a Borel set $A\subset M$ with $\theta=\mathfrak{m}(A)\in(0,1)$, we employ the function $f:=\chi_A-\theta$ (χ_A denotes the characteristic function of A) and an associated 1-Lipschitz function $u: M\longrightarrow\mathbb{R}$ attaining the maximum of $\int_M f\phi\,d\mathfrak{m}$ among all 1-Lipschitz functions ϕ. Then, analyzing the behavior of u, one can build a partition $\{X_q\}_{q\in Q}$ of M consisting of (the image of) minimal geodesics (called *needles*), and Q is endowed with a probability measure ν. For ν-almost every $q \in Q$, $u|_{X_q}$ has slope 1 ($|u(x) - u(y)| = d(x, y)$ for all $x, y \in X_q$) and X_q is equipped with a probability measure $\mathfrak{m}_q$ such that $\mathfrak{m}_q(A \cap X_q) = \theta$ and $(X_q, |\cdot|, \mathfrak{m}_q)$ satisfies $\mathrm{Ric}_\infty \geq 1$. Moreover, we have

$$\int_M h \, d\mathfrak{m} = \int_Q \left(\int_{X_q} h \, d\mathfrak{m}_q \right) \nu(dq) \tag{3.1}$$

for all $h \in L^1(\mathfrak{m})$. Then, (1.1) for A is obtained by integrating its 1-dimensional counterparts for $A \cap X_q$ with respect to ν.

The 1-Lipschitz function u is called the *guiding function*. We can assume $\int_M u \, d\mathfrak{m} = 0$ without loss of generality, and X_q will be identified with an interval via u (in other words, X_q is parametrized by u). Denote $\mathfrak{m}_q = \mathrm{e}^{-\sigma_q} \, dx$ and $\mu := u_* \mathfrak{m} = \rho \, dx$. Note that $\operatorname{supp} \mu$ is an interval and may not be the whole $\mathbb{R}$. Through the parametrization of X_q by u, we deduce from (3.1) that

$$\rho(x) = \int_Q \mathrm{e}^{-\sigma_q(x)} \, \nu(dq), \tag{3.2}$$

where we set $\mathrm{e}^{-\sigma_q(x)} := 0$ if $x \notin X_q$.

Theorem 3.1 (An L^1-estimate on M) *Assume* $\mathrm{Ric}_\infty \geq 1$ *and fix* $\varepsilon \in (0, 1)$. *If* $\mathsf{P}(A) \leq \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) + \delta$ *holds for some Borel set* $A \subset M$ *with* $\theta = \mathfrak{m}(A) \in (0, 1)$ *and sufficiently small* δ *(relative to* θ *and* ε*), then* $u_* \mathfrak{m} = \rho \, dx$ *satisfies*

$$\|\rho \cdot \mathrm{e}^{\psi_{\mathrm{g}}} - 1\|_{L^1(\gamma)} \leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)},$$

where u *is the guiding function associated with* A *such that* $\int_M u \, d\mathfrak{m} = 0$.

Proof. First of all, by (3.2) and Fubini's theorem, we have

$$\|\rho \cdot \mathrm{e}^{\psi_{\mathrm{g}}} - 1\|_{L^1(\gamma)} = \int_{-\infty}^{\infty} \left| \int_Q (\mathrm{e}^{\psi_{\mathrm{g}} - \sigma_q} - 1) \, \nu(dq) \right| d\gamma \leq \int_Q \|\mathrm{e}^{\psi_{\mathrm{g}} - \sigma_q} - 1\|_{L^1(\gamma)} \, \nu(dq).$$

We shall estimate $\|\mathrm{e}^{\psi_{\mathrm{g}} - \sigma_q} - 1\|_{L^1(\gamma)}$ by dividing into "good" needles and "bad" needles. Note that $\nu(Q_\ell) \geq 1 - \sqrt{\delta}$ holds for

$$Q_\ell := \left\{ q \in Q \,\middle|\, \mathfrak{m}_q(A \cap X_q) = \theta, \, \mathsf{P}(A \cap X_q) < \mathcal{I}_{(\mathbb{R},\gamma)}(\theta) + \sqrt{\delta} \right\}$$

by [MO, Lemma 7.1], where $\mathsf{P}(A \cap X_q)$ denotes the perimeter of $A \cap X_q$ in $(X_q, |\cdot|, \mathfrak{m}_q)$. Moreover, it follows from [MO, Proposition 7.3] that there exists a measurable set $Q_c \subset Q$ such that $\nu(Q_c) \geq 1 - \delta^{(1-\varepsilon)/(9-3\varepsilon)}$ and

$$\max \left\{ |a_\theta - r_q^-|, |a_{1-\theta} - r_q^+| \right\} \leq C(\theta, \varepsilon) \delta^{(1-\varepsilon)/(9-3\varepsilon)}$$

for all $q \in Q_c \cap Q_\ell$, where $\mathfrak{m}_q((-\infty, r_q^-] \cap X_q) = \mathfrak{m}_q([r_q^+, \infty) \cap X_q) = \theta$ (recall that $\gamma((-\infty, a_\theta]) = \gamma([a_{1-\theta}, \infty)) = \theta$).

On the one hand, for $q \in Q_c \cap Q_\ell$, note that either $\mathsf{P}(A \cap X_q) \ge \mathrm{e}^{-\sigma_q(r_q^-)}$ or $\mathsf{P}(A \cap X_q) \ge \mathrm{e}^{-\sigma_q(r_q^+)}$ holds by [Bo, Proposition 2.1] (recall Subsection 2.1). When $\mathsf{P}(A \cap X_q) \ge \mathrm{e}^{-\sigma_q(r_q^-)}$, we put

$$\boldsymbol{\gamma}_q(dx) = \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g},q}(x)}\,dx := \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}(x+a_\theta - r_q^-)}\,dx,$$

which is a translation of $\boldsymbol{\gamma}$ satisfying $\boldsymbol{\gamma}_q((-\infty, r_q^-]) = \theta$. Then, it follows from Proposition 2.2 (with $\mathrm{e}^{-\sigma_q(r_q^-)} \le \mathsf{P}(A \cap X_q) \le \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g},q}(r_q^-)} + \sqrt{\delta}$) and Cavalieri's principle that

$$\begin{aligned}
\|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \sigma_q} - 1\|_{L^1(\boldsymbol{\gamma})} &\le \|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g},q} - \sigma_q} - 1\|_{L^1(\boldsymbol{\gamma}_q)} + \|\mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g},q}} - \mathrm{e}^{-\boldsymbol{\psi}_{\mathrm{g}}}\|_{L^1(dx)} \\
&\le C(\theta)\sqrt{\delta} + 2\frac{|a_\theta - r_q^-|}{\sqrt{2\pi}} \\
&\le C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}.
\end{aligned}$$

We have the same bound also in the case where $\mathsf{P}(A \cap X_q) \ge \mathrm{e}^{-\sigma_q(r_q^+)}$ by reversing I in Proposition 2.2.

On the other hand, for $q \in Q \setminus (Q_c \cap Q_\ell)$, we have the trivial bound

$$\|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \sigma_q} - 1\|_{L^1(\boldsymbol{\gamma})} \le \|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \sigma_q}\|_{L^1(\boldsymbol{\gamma})} + \|1\|_{L^1(\boldsymbol{\gamma})} = 2.$$

Therefore, we obtain

$$\|\rho \cdot \mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}}} - 1\|_{L^1(\boldsymbol{\gamma})} \le C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)} + 2\big(1 - \nu(Q_c \cap Q_\ell)\big) \le C(\theta, \varepsilon)\delta^{(1-\varepsilon)/(9-3\varepsilon)}.$$

□

Note that $q \in Q_c \cap Q_\ell$ is well-behaved and can be handled by the 1-dimensional analysis, whereas one has a priori no information of $q \in Q \setminus (Q_c \cap Q_\ell)$. This could be a common problem for stability estimates via the needle decomposition (see, e.g., [MO, Theorem 6.2] showing a reverse Poincaré inequality on a manifold from a sharper estimate on intervals). In particular, it may be difficult to achieve the same order δ as in the 1-dimensional case (Proposition 2.2) by the needle decomposition. In the L^p-case, it is unclear (to the authors) with what we can replace the trivial bound $\|\mathrm{e}^{\boldsymbol{\psi}_{\mathrm{g}} - \sigma_q} - 1\|_{L^1(\boldsymbol{\gamma})} \le 2$. For the Wasserstein distance W_2 or W_1, we have the same problem on the control of $q \in Q \setminus (Q_c \cap Q_\ell)$.

Remark 3.2 (Further related works and open problems) (a) Theorem 3.1 holds true also for reversible Finsler manifolds by the same proof (see [MO, Remark 7.6(c)] and [Oh1, Oh2]).

(b) As we mentioned in the introduction, our L^p- and W_2-estimates are inspired by the quantitative stability for functional inequalities. We refer to [BGRS, FIL, IK, IM] for the study of the *log-Sobolev inequality* on the Gaussian space:

$$\mathrm{Ent}_{\boldsymbol{\gamma}}(f\boldsymbol{\gamma}) \le \frac{1}{2}\mathrm{I}_{\boldsymbol{\gamma}}(f\boldsymbol{\gamma}) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f}\,d\boldsymbol{\gamma},$$

where $\mathrm{I}_{\boldsymbol{\gamma}}(f\boldsymbol{\gamma})$ is the *Fisher information* of a probability measure $f\boldsymbol{\gamma}$ with respect to $\boldsymbol{\gamma}$. They investigated the difference between $\boldsymbol{\gamma}$ and $f\boldsymbol{\gamma}$, in terms of the additive deficit

$\delta(f) = \mathrm{I}_\gamma(f\gamma)/2 - \mathrm{Ent}_\gamma(f\gamma)$. For instance, W_2-bounds (under certain convexity and concavity conditions on f) were given in [BGRS, IM], and L^1- and L^p-bounds can be found in [IK]. In the setting of weighted Riemannian manifolds satisfying $\mathrm{Ric}_\infty \geq 1$ (as in Theorem 3.1), we have only the rigidity (see [OT]) and the stability is an open problem.

(c) We have seen in [MO, §6] that the reverse forms of the Poincaré and log-Sobolev inequalities can be derived from the isoperimetric deficit. The reverse Poincaré inequality then implies a W_1-estimate for the push-forward by an eigenfunction thanks to [BF, Theorem 1.3] (see also [FGS]). We also expect a direct W_1- or W_2-estimate for the push-forward by the guiding function, which remains an open question (see [MO, Remark 7.6(g)]).

(d) Another direction of research is a generalization to negative effective dimension, i.e., $\mathrm{Ric}_N \geq K > 0$ with $N < -1$. We have established rigidity in the isoperimetric inequality in [Ma], thereby it is natural to consider quantitative isoperimetry, though it seems to require longer calculations.

References

[BL] D. Bakry and M. Ledoux, Lévy–Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. Invent. Math. **123** (1996), 259–281.

[BF] J. Bertrand and M. Fathi, Stability of eigenvalues and observable diameter in $\mathrm{RCD}(1,\infty)$ spaces. Preprint (2021). Available at arXiv:2107.05324

[Bo] S. Bobkov, Extremal properties of half-spaces for log-concave distributions. Ann. Probab. **24** (1996), 35–48.

[BGRS] S. G. Bobkov, N. Gozlan, C. Roberto and P.-M. Samson, Bounds on the deficit in the logarithmic Sobolev inequality. J. Funct. Anal. **267** (2014), 4110–4138.

[CMM] F. Cavalletti, F. Maggi and A. Mondino, Quantitative isoperimetry à la Levy–Gromov. Comm. Pure Appl. Math. **72** (2019), 1631–1677.

[CM] F. Cavalletti and A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. **208** (2017), 803–849.

[CMS] F. Cavalletti, A. Mondino and D. Semola, Quantitative Obata's theorem. Anal. PDE (to appear). Available at arXiv:1910.06637

[CF] T. A. Courtade and M. Fathi, Stability of the Bakry–Émery theorem on $\mathbb{R}^n$. J. Funct. Anal. **279** (2020), 108523, 28 pp.

[FGS] M. Fathi, I. Gentil and J. Serres, Stability estimates for the sharp spectral gap bound under a curvature-dimension condition. Preprint (2022). Available at arXiv:2202.03769

[FIL] M. Fathi, E. Indrei and M. Ledoux, Quantitative logarithmic Sobolev inequalities and stability estimates. Discrete Contin. Dyn. Syst. **36** (2016), 6835–6853.

[IK] E. Indrei and D. Kim, Deficit estimates for the logarithmic Sobolev inequality. Differential Integral Equations **34** (2021), 437–466.

[IM] E. Indrei and D. Marcon, A quantitative log-Sobolev inequality for a two parameter family of functions. Int. Math. Res. Not. IMRN 2014, 5563–5580.

[Kl] B. Klartag, Needle decompositions in Riemannian geometry. Mem. Amer. Math. Soc. **249** (2017), no. 1180.

[Ma] C. H. Mai, Rigidity for the isoperimetric inequality of negative effective dimension on weighted Riemannian manifolds. Geom. Dedicata **202** (2019), 213–232.

[MO] C. H. Mai and S. Ohta, Quantitative estimates for the Bakry–Ledoux isoperimetric inequality. Comment. Math. Helv. **96** (2021), 693–739.

[Mo] F. Morgan, Geometric measure theory. A beginner's guide. Fifth edition. Elsevier/Academic Press, Amsterdam, 2016.

[Oh1] S. Ohta, Needle decompositions and isoperimetric inequalities in Finsler geometry. J. Math. Soc. Japan **70** (2018), 651–693.

[Oh2] S. Ohta, Comparison Finsler geometry. Springer Monographs in Mathematics. Springer, Cham, 2021.

[OT] S. Ohta and A. Takatsu, Equality in the logarithmic Sobolev inequality. Manuscripta Math. **162** (2020), 271–282.

[Ta] M. Talagrand, Transportation cost for Gaussian and other product measures. Geom. Funct. Anal. **6** (1996), 587–600.

[Vi] C. Villani, Optimal transport, old and new. Springer-Verlag, Berlin, 2009.