Topology of complete Finsler manifolds with radial flag curvature bounded below[*†]

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Abstract

We recently established a Toponogov type triangle comparison theorem for a certain class of Finsler manifolds whose radial flag curvatures are bounded below by that of a von Mangoldt surface of revolution. In this article, as its applications, we prove the finiteness of topological type and a diffeomorphism theorem to Euclidean spaces.

1 Introduction

This article is a continuation of [KOT]. In [KOT], we have established a Toponogov type triangle comparison theorem for a certain class of Finsler manifolds whose radial flag curvatures are bounded below by that of a von Mangoldt surface of revolution (see Theorem 2.4 for the precise statement). In this article, we prove several applications of our Toponogov theorem on the relationship between the topology and the curvature of a Finsler manifold. We remark that, compared to the Riemannian case, there are only a small number of such kind of results, e.g., Rademacher's sphere theorem ([Ra]), Shen's finiteness theorem under lower Ricci and mean (or **S**-) curvature bounds ([Sh1]), and the second author's generalized splitting theorems under nonnegative weighted Ricci curvature ([Oh4], see also the suspension theorem in [Oh1, Section 5] which is available for (reversible) Finsler manifolds thanks to [Oh3]).

In order to state our results, let us introduce several notions in Finsler geometry as well as the geometry of radial curvature. Let (M, F, p) denote a pair of a forward complete, connected, n-dimensional C^∞-Finsler manifold (M, F) with a base point $p \in M$, and $d : M \times M \longrightarrow [0, \infty)$ denote the distance function induced from F. We remark that the *reversibility* $F(-v) = F(v)$ is not assumed in general, so that $d(x, y) \neq d(y, x)$ is allowed.

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For a local coordinate $(x^i)_{i=1}^n$ of an open subset $\mathcal{O} \subset M$, let $(x^i, v^j)_{i,j=1}^n$ be the coordinate of the tangent bundle $T\mathcal{O}$ over $\mathcal{O}$ such that

$$v := \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}\Big|_x, \qquad x \in \mathcal{O}.$$

For each $v \in T_xM \setminus \{0\}$, the positive-definite $n \times n$ matrix

$$\big(g_{ij}(v)\big)_{i,j=1}^n := \left(\frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i \partial v^j}(v)\right)_{i,j=1}^n$$

provides us the Riemannian structure g_v of T_xM by

$$g_v\left(\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_x, \sum_{j=1}^{n} b^j \frac{\partial}{\partial x^j}\Big|_x\right) := \sum_{i,j=1}^{n} g_{ij}(v) a^i b^j.$$

This is a Riemannian approximation (up to the second order) of F in the direction v. For two linearly independent vectors $v, w \in T_xM \setminus \{0\}$, the *flag curvature* is defined by

$$K_M(v,w) := \frac{g_v(R^v(w,v)v, w)}{g_v(v,v)g_v(w,w) - g_v(v,w)^2},$$

where R^v denotes the curvature tensor induced from the Chern connection (see [BCS, §3.9] for details). We remark that $K_M(v,w)$ depends not only on the *flag* $\{sv + tw \mid s, t \in \mathbb{R}\}$, but also on the *flag pole* $\{sv \mid s > 0\}$.

Given $v, w \in T_xM \setminus \{0\}$, define the *tangent curvature* by

$$\mathcal{T}_M(v,w) := g_X\big(D_Y^Y Y(x) - D_Y^X Y(x), X(x)\big),$$

where the vector fields X, Y are extensions of v, w, and $D_v^w X(x)$ denotes the covariant derivative of X by v with reference vector w. Independence of $\mathcal{T}_M(v,w)$ from the choices of X, Y is easily checked. Note that $\mathcal{T}_M \equiv 0$ if and only if M is of *Berwald type* (see [Sh2, Propositions 7.2.2, 10.1.1]). In Berwald spaces, for any $x, y \in M$, the tangent spaces $(T_xM, F|_{T_xM})$ and $(T_yM, F|_{T_yM})$ are mutually linearly isometric (cf. [BCS, Chapter 10]). In this sense, $\mathcal{T}_M$ measures the variety of tangent Minkowski normed spaces.

Let $\widetilde{M}$ be a complete 2-dimensional Riemannian manifold, which is homeomorphic to $\mathbb{R}^2$ if $\widetilde{M}$ is non-compact, or to $\mathbb{S}^2$ if $\widetilde{M}$ is compact. Fix a base point $\tilde{p} \in \widetilde{M}$. Then we call the pair $(\widetilde{M}, \tilde{p})$ a *model surface of revolution* if its Riemannian metric $d\tilde{s}^2$ is expressed in terms of the geodesic polar coordinate around $\tilde{p}$ as

$$d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2, \qquad (t, \theta) \in (0,a) \times \mathbb{S}^1_{\tilde{p}},$$

where $0 < a \leq \infty$, $f : (0,a) \longrightarrow \mathbb{R}$ is a positive smooth function which is extensible to a smooth odd function around 0, and $\mathbb{S}^1_{\tilde{p}} := \{v \in T_{\tilde{p}}\widetilde{M} \mid \|v\| = 1\}$. Define the *radial curvature function* $G : [0,a) \longrightarrow \mathbb{R}$ such that $G(t)$ is the Gaussian curvature at $\tilde{\gamma}(t)$, where $\tilde{\gamma} : [0,a) \longrightarrow \widetilde{M}$ is any (unit speed) meridian emanating from $\tilde{p}$. Note that f satisfies the differential equation $f'' + Gf = 0$ with initial conditions $f(0) = 0$ and $f'(0) = 1$. We call $(\widetilde{M}, \tilde{p})$ a *von Mangoldt surface* if G is non-increasing on $[0, a)$. A round sphere is the only compact, 'smooth' von Mangoldt surface, i.e., f satisfies $\lim_{t \to a} f'(t) = -1$. Paraboloids and 2-sheeted hyperboloids are typical examples of non-compact von Mangoldt surfaces. An atypical example of such a surface is the following.

Example 1.1 ([KT1, Example 1.2]) Set $f(t) := e^{-t^2} \tanh t$ on $[0, \infty)$. Then the non-compact surface of revolution $(\widetilde{M}, \tilde{p})$ with $d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2$ is of von Mangoldt type, and G changes the sign. Indeed, $\lim_{t \downarrow 0} G(t) = 8$ and $\lim_{t \to \infty} G(t) = -\infty$.

We say that a Finsler manifold (M, F, p) has the *radial flag curvature bounded below by that of a model surface of revolution* $(\widetilde{M}, \tilde{p})$ if, along every unit speed minimal geodesic $\gamma : [0, l) \longrightarrow M$ emanating from p, we have

$$K_M\big(\dot{\gamma}(t), w\big) \ge G(t)$$

for all $t \in [0, l)$ and $w \in T_{\gamma(t)}M$ linearly independent to $\dot{\gamma}(t)$. (Note that $\sup_{q \in M} d(p, q) \le \sup_{\tilde{q} \in \widetilde{M}} \tilde{d}(\tilde{p}, \tilde{q})$ is a priori assumed ($\tilde{d}$ is the distance function of $(\widetilde{M}, d\tilde{s}^2)$), however, the curvature bound in fact excludes $\sup_{q \in M} d(p, q) > \sup_{\tilde{q} \in \widetilde{M}} \tilde{d}(\tilde{p}, \tilde{q})$.)

Our first main result is a finiteness theorem of topological type.

Theorem A *Let (M, F, p) be a forward complete, non-compact, connected C^∞-Finsler manifold whose radial flag curvature is bounded below by that of a von Mangoldt surface $(\widetilde{M}, \tilde{p})$ satisfying $f'(\rho) = 0$ for unique $\rho \in (0, \infty)$. Assume that, for some $t_0 > \rho$,*

(1) $\mathrm{diam}(\partial B^+_t(p)) = O(t^\alpha)$ *for some* $\alpha \in (0, 1)$ *as* $t \to \infty$,

(2) $g_v(w, w) \ge F(w)^2$ *for all* $x \in M \setminus \overline{B^+_{t_0}(p)}$, $v \in \mathcal{G}_p(x)$ *and* $w \in T_xM$,

(3) $\mathcal{T}_M(v, w) = 0$ *for all* $x \in M \setminus \overline{B^+_{t_0}(p)}$, $v \in \mathcal{G}_p(x)$ *and* $w \in T_xM$,

(4) *the reverse curve* $\bar{c}(s) := c(l - s)$ *of any minimal geodesic segment* $c : [0, l] \longrightarrow M \setminus \overline{B^+_{t_0}(p)}$ *is geodesic.*

Then M has finite topological type, i.e., M is homeomorphic to the interior of a compact manifold with boundary.

We set

$$(1.1) \qquad \mathcal{G}_p(x) := \{\dot{\gamma}(l) \in T_xM \,|\, \gamma \text{ is a minimal geodesic segment from } p \text{ to } x\},$$

where $\gamma : [0, l] \longrightarrow M$ with $l = d(p, x)$, and

$$B^+_r(p) := \{x \in M \,|\, d(p, x) < r\}, \qquad \mathrm{diam}\big(\partial B^+_r(p)\big) := \sup_{q_1,\, q_2 \in \partial B^+_r(p)} d(q_1, q_2).$$

The condition (2) is the 2-*uniform convexity* with the sharp constant (see [Oh2]), but only for special points x and directions v. The sharpness means that $g_v(w, w) \ge F(w)^2$ holds for all $(x, v) \in TM \setminus \{0\}$ only if F is Riemannian. It is not difficult to construct non-Riemannian spaces satisfying (2) (see [KOT]). The conditions (3) and (4) are satisfied if (M, F) is of Berwald type (on $M \setminus \overline{B^+_{t_0}(p)}$). Of course (4) always holds true if F is reversible (on $M \setminus \overline{B^+_{t_0}(p)}$).

In the Riemannian case, the diameter growth bound (1) is in a sense a restrictive condition. Indeed, if we employ a non-negatively curved non-compact model surface of revolution $(\widetilde{M}, \tilde{p})$ having the diameter growth $o(t^{1/2})$, then M is isometric to the n-dimensional model space $\widetilde{M}^n$ (see [ST, Theorem 1.2], [KT2, Example 1.1]). Other results related to Theorem A include [KT2, Theorem 2.2] and [TK, Theorem 1.3], where we proved the finiteness of topological type of a complete non-compact Riemannian manifold with radial curvature bounded below by that of a non-compact model surface of revolution having a finite total curvature.

If F is reversible, then we can improve Theorem A as follows (by an entirely different technique).

Theorem B *Let (M, F, p), f, ρ and t_0 as in Theorem* A, *and in addition assume that F is reversible. Then p has no cut point. In particular, M is diffeomorphic to $\mathbb{R}^n$ and, for every unit speed minimal geodesic $\gamma : [0, \infty) \longrightarrow M$ emanating from p, we have $K_M(\dot{\gamma}(t), w) = G(t)$ for all $t > 0$.*

One of the related results is Shiohama and the third author's [ST, Theorem 1.2], where it is proved that a complete non-compact Riemannian manifold is isometric to the n-dimensional model space $\widetilde{M}^n$ if its radial curvature is bounded below by that of a non-compact model surface of revolution $\widetilde{M}$ satisfying $\int_1^\infty f(t)^{-2}\, dt = \infty$. Observe that our von Mangoldt surface always satisfies this integration assumption. In our Finsler situation, however, it is difficult (and in fact impossible in many cases) to obtain isometry to a model space. For instance, spaces of constant flag curvatures are not unique (all Minkowski normed spaces have the flat flag curvature, all Hilbert geometries satisfy $K_M \equiv -1$ (cf. [Sh3]), and Bryant [Br] constructed a family of (non-reversible) Finsler metrics on $\mathbb{S}^2$ with $K_M \equiv 1$). Another related result to Theorem B is the first and the third authors' [KT3, Theorem 1.1] on a complete non-compact connected Riemannian manifold with smooth convex boundary.

2 A Toponogov type triangle comparison theorem

We first recall the Toponogov type triangle comparison theorem established in [KOT, Theorem 1.2]. We refer to [BCS] and [Sh2] for the basics of Finsler geometry.

Let (M, F, p) be a forward complete, connected C^∞-Finsler manifold with a base point $p \in M$, and denote by d its distance function. The forward completeness guarantees that any two points in M can be joined by a minimal geodesic segment (by the Hopf-Rinow theorem, [BCS, Theorem 6.6.1]). Since $d(x, y) \neq d(y, x)$ in general, we also introduce

$$d_{\mathrm{m}}(x, y) := \max\{d(x, y), d(y, x)\}.$$

It is clear that d_{m} is a distance function of M. We can define the 'angles' with respect to d_{m} as follows.

Definition 2.1 (Angles) Let $c : [0,a] \longrightarrow M$ be a unit speed minimal geodesic segment (i.e., $F(\dot{c}) \equiv 1$) with $p \notin c([0,a])$. The *forward* and the *backward angles* $\overrightarrow{\angle}(pc(s)c(a))$, $\overleftarrow{\angle}(pc(s)c(0)) \in [0,\pi]$ at $c(s)$ are defined via

$$\cos \overrightarrow{\angle}\big(pc(s)c(a)\big) := -\lim_{h\downarrow 0} \frac{d(p,c(s+h)) - d(p,c(s))}{d_{\mathrm{m}}(c(s),c(s+h))} \quad \text{for } s \in [0,a),$$

$$\cos \overleftarrow{\angle}\big(pc(s)c(0)\big) := \lim_{h\downarrow 0} \frac{d(p,c(s)) - d(p,c(s-h))}{d_{\mathrm{m}}(c(s-h),c(s))} \quad \text{for } s \in (0,a].$$

(These limits indeed exist in $[-1,1]$ thanks to the definition of d_{m}, see [KOT, Lemma 2.2]).

Definition 2.2 (Forward triangles) For three distinct points $p,x,y \in M$,

$$\triangle(\overrightarrow{px},\overrightarrow{py}) := (p,x,y;\gamma,\sigma,c)$$

will denote the *forward triangle* consisting of unit speed minimal geodesic segments γ emanating from p to x, σ from p to y, and c from x to y. Then the corresponding *interior angles* $\overrightarrow{\angle}x$, $\overleftarrow{\angle}y$ at the vertices x, y are defined by

$$\overrightarrow{\angle}x := \overrightarrow{\angle}\big(pc(0)c(d(x,y))\big), \qquad \overleftarrow{\angle}y := \overleftarrow{\angle}\big(pc(d(x,y))c(0)\big).$$

Definition 2.3 (Comparison triangles) Fix a model surface of revolution $(\widetilde{M},\tilde{p})$. Given a forward triangle $\triangle(\overrightarrow{px},\overrightarrow{py}) = (p,x,y;\gamma,\sigma,c) \subset M$, a geodesic triangle $\triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \widetilde{M}$ is called its *comparison triangle* if

$$\tilde{d}(\tilde{p},\tilde{x}) = d(p,x), \qquad \tilde{d}(\tilde{p},\tilde{y}) = d(p,y), \qquad \tilde{d}(\tilde{x},\tilde{y}) = L_{\mathrm{m}}(c)$$

hold, where we set

$$L_{\mathrm{m}}(c) := \int_0^{d(x,y)} \max\{F(\dot{c}),F(-\dot{c})\}\,ds.$$

Now, the main result of [KOT] asserts the following.

Theorem 2.4 (TCT, [KOT]) *Assume that (M,F,p) is a forward complete, connected C^∞-Finsler manifold whose radial flag curvature is bounded below by that of a von Mangoldt surface $(\widetilde{M},\tilde{p})$ satisfying $f'(\rho) = 0$ for unique $\rho \in (0,\infty)$. Let $\triangle(\overrightarrow{px},\overrightarrow{py}) = (p,x,y;\gamma,\sigma,c) \subset M$ be a forward triangle satisfying that, for some open neighborhood $\mathcal{N}(c)$ of c,*

(1) $c([0,d(x,y)]) \subset M \setminus \overline{B_\rho^+(p)}$,

(2) $g_v(w,w) \geq F(w)^2$ *for all* $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ *and* $w \in T_zM$,

(3) $\mathcal{T}_M(v,w) = 0$ *for all* $z \in \mathcal{N}(c)$, $v \in \mathcal{G}_p(z)$ *and* $w \in T_zM$, *and the reverse curve* $\bar{c}(s) := c(d(x,y)-s)$ *of* c *is also geodesic.*

If such $\triangle(\overrightarrow{px},\overrightarrow{py})$ admits a comparison triangle $\triangle(\tilde{p}\tilde{x}\tilde{y})$ in $\widetilde{M}$, then we have $\overrightarrow{\angle}x \geq \angle\tilde{x}$ and $\overleftarrow{\angle}y \geq \angle\tilde{y}$.

3 Fundamental tools on model surfaces

We next introduce some fundamental tools in the geometry of model surfaces of revolution. We refer to [SST, Chapter 7] for more details. Let $(\widetilde{M}, \tilde{p})$ be a non-compact model surface of revolution with its metric $d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2$ on $(0,a) \times \mathbb{S}^1_{\tilde{p}}$. Given a unit speed geodesic $\tilde{c} : [0,a) \longrightarrow \widetilde{M}$ $(0 < a \leq \infty)$ expressed as $\tilde{c}(s) = (t(s), \theta(s))$, there exists a non-negative constant ν such that

$$\nu = f\big(t(s)\big)^2 |\theta'(s)| = f\big(t(s)\big) \sin \angle\big(\dot{\tilde{c}}(s), (\partial/\partial t)|_{\tilde{c}(s)}\big) \tag{3.1}$$

for all $s \in [0,a)$. The equation (3.1) is called the *Clairaut relation*, and ν is called the *Clairaut constant* of $\tilde{c}$. Note that $\nu = 0$ if and only if $\tilde{c}$ is (a part of) a meridian. Since $\tilde{c}$ has unit speed, we deduce from $|t'|^2 + |f(t)\theta'|^2 = 1$ that

$$|t'(s)| = \frac{\sqrt{f(t(s))^2 - \nu^2}}{f(t(s))}.$$

Thus we observe that $t'(s) = 0$ if and only if $f(t(s)) = \nu$. Moreover, if $a < \infty$, then the length $L(\tilde{c})$ of $\tilde{c}$ is not less than

$$t(a) - t(0) + \frac{\nu^2}{2} \int_{t(0)}^{t(a)} \frac{1}{f(t)\sqrt{f(t)^2 - \nu^2}}\, dt. \tag{3.2}$$

The proof of (3.2) can be found in (the proof of) [ST, Lemma 2.1].

4 Proof of Theorem A

Let (M, F, p), f and ρ be as in Theorem A. The following fact on the cut loci of a von Mangoldt surface is important.

Remark 4.1 The cut locus $\mathrm{Cut}(\tilde{x})$ of $\tilde{x} \neq \tilde{p}$ is either an empty set, or a ray properly contained in the meridian $\theta^{-1}(\theta(\tilde{x}) + \pi)$ opposite to $\tilde{x}$. Moreover, the endpoint of $\mathrm{Cut}(\tilde{x})$ is the first conjugate point to $\tilde{x}$ along the minimal geodesic from $\tilde{x}$ passing through $\tilde{p}$ ([Ta, Main Theorem]).

We first show an auxiliary lemma on the model surface.

Lemma 4.2 *If two distinct points $\tilde{x}, \tilde{y} \in \widetilde{M} \setminus \overline{B_\rho(\tilde{p})}$ satisfy $\tilde{d}(\tilde{p}, \tilde{x}) \leq \tilde{d}(\tilde{p}, \tilde{y})$, then*

$$\angle\big(\dot{\tilde{c}}(0), (\partial/\partial t)|_{\tilde{x}}\big) < \pi/2$$

holds for any unit speed minimal geodesic segment $\tilde{c}$ emanating from $\tilde{x}$ to $\tilde{y}$. In particular, we have $\tilde{c}([0, d(\tilde{x}, \tilde{y})]) \subset \widetilde{M} \setminus \overline{B_\rho(\tilde{p})}$.

Proof. Let us write $\tilde{c}(s) = (t(s), \theta(s))$. Suppose that $\angle(\dot{\tilde{c}}(0), (\partial/\partial t)|_{\tilde{x}}) \geq \pi/2$ which is equivalent to $t'(0) \leq 0$. Since $f' < 0$ on (ρ, ∞), it follows from [SST, (7.1.15)] that

$$t''(0) = f\big(t(0)\big) f'\big(t(0)\big) \theta'\big(t(0)\big)^2 < 0.$$

Hence $t(s)$ is decreasing on $[0,\delta]$ for some small $\delta > 0$. Since $t(d(\tilde{x},\tilde{y})) = \tilde{d}(\tilde{p},\tilde{y}) \geq \tilde{d}(\tilde{p},\tilde{x}) = t(0)$, there exists $s_0 \in (0, \tilde{d}(\tilde{p},\tilde{y}))$ such that $t'(s_0) = 0$ and $t(s_0) < t(0)$. By the Clairaut relation (3.1), for any $s \in [0, \tilde{d}(\tilde{p},\tilde{y})]$, we observe

$$f\big(t(s_0)\big) = f\big(t(s)\big) \sin \angle\big(\dot{\tilde{c}}(s), (\partial/\partial t)|_{\tilde{c}(s)}\big) \leq f\big(t(s)\big).$$

Since $f' < 0$ on (ρ,∞) and $t(s_0) < t(0)$, this shows $t(s_0) < \rho$. Thus $\tilde{c}$ intersects the parallel $t = \rho$ twice in $\theta^{-1}((\theta(\tilde{x}), \theta(\tilde{x}) + \pi))$, where we assume that $\theta(\tilde{x}) \leq \theta(\tilde{y})$. However, since $f'(\rho) = 0$, the parallel $t = \rho$ is geodesic. Therefore (by rotation) $\tilde{x}$ has a cut point in $\theta^{-1}((\theta(\tilde{x}), \theta(\tilde{x}) + \pi))$. This contradicts the structure of $\mathrm{Cut}(\tilde{x})$ (see Remark 4.1). $\square$

Lemma 4.3 *If two points $x, y \in M \setminus \overline{B_\rho^+(p)}$ satisfy $d(p,y) > d(p,x) \gg t_0$, then*

$$c\big([0, d(x,y)]\big) \cap \partial B_{t_0}^+(p) = \emptyset$$

holds for any minimal geodesic segment c emanating from x to y, where $t_0 > \rho$ is as in the assumption of Theorem A.

Proof. By the assumption (1) of Theorem A, there is a constant $C > 0$ such that

$$\frac{\mathrm{diam}(\partial B_t^+(p))}{t^\alpha} < C \tag{4.1}$$

for all $t \gg t_0$. Suppose that $c([0,d(x,y)]) \cap \partial B_{t_0}^+(p) \neq \emptyset$ for some minimal geodesic segment c emanating from x to y. Let S be the set of all $s \in (0, d(x,y))$ such that $c(s) \in \partial B_{t_0}^+(p)$, and set $s_0 := \sup S$. Since $d(p,y) > d(p,x)$, there exists $s_1 \in (s_0, d(x,y))$ such that $c(s_1) \in \partial B_{t_1}^+(p)$, where $t_1 := d(p,x)$. Observe from the triangle inequality that

$$s_1 - s_0 = d\big(c(s_0), c(s_1)\big) \geq d\big(p, c(s_1)\big) - d\big(p, c(s_0)\big) = t_1 - t_0.$$

Since $\mathrm{diam}(\partial B_{t_1}^+(p)) \geq s_1 > s_1 - s_0 \geq t_1 - t_0$, we obtain

$$\frac{\mathrm{diam}(\partial B_{t_1}^+(p))}{t_1^\alpha} > t_1^{1-\alpha} - \frac{t_0}{t_1^\alpha}.$$

This contradicts (4.1), because $t_1 \gg t_0$ and $\alpha < 1$. $\square$

Analogously to [GS], we define critical points of the distance function $d_p := d(p,\cdot)$ as follows. Recall (1.1) for the definition of $\mathcal{G}_p(x)$.

Definition 4.4 We say that a point $x \in M$ is a *forward critical point* for $p \in M$ if, for every $w \in T_xM \setminus \{0\}$, there exists $v \in \mathcal{G}_p(x)$ such that $g_v(v,w) \leq 0$.

An important consequence of the criticality is that, for any $y \in M$ and any forward triangle $\triangle(\overrightarrow{px}, \overrightarrow{py})$, we have $\overrightarrow{\angle} x \leq \pi/2$. We can prove Gromov's isotopy lemma [Gr] by a similar arguments to the Riemannian case.

Lemma 4.5 *Given $0 < r_1 < r_2 \leq \infty$, if $\overline{B_{r_2}^+(p)} \setminus B_{r_1}^+(p)$ has no critical point for $p \in M$, then $\overline{B_{r_2}^+(p)} \setminus B_{r_1}^+(p)$ is homeomorphic to $\partial B_{r_1}^+(p) \times [r_1, r_2]$.*

Now we are ready to prove Theorem A.

Proof of Theorem A. By virtue of Lemma 4.5, it is sufficient to prove that the set of forward critical points for p is bounded. Suppose that there is a divergent sequence $\{q_i\}_{i\in\mathbb{N}}$ of forward critical points for p. Then there exist $i_1, i_2 \in \mathbb{N}$ such that

$$d(p, q_{i_2}) > d(p, q_{i_1}) \gg t_0 > \rho.$$

Let $c : [0, a] \longrightarrow M$ be a minimal geodesic segment emanating from q_{i_1} to q_{i_2}. Note that $\overrightarrow{\angle}(pc(0)c(a)) \leq \pi/2$ by the criticality of q_{i_1}, and $c([0, a]) \cap \partial B^+_{t_0}(p) = \emptyset$ by Lemma 4.3.

We first consider the case where $d(p, q_{i_1}) = \min_{s\in[0,a]} d(p, c(s))$. For sufficiently small $s_1 \in (0, a)$, the forward triangle $\triangle(\overrightarrow{pq_{i_1}}, \overrightarrow{pc(s_1)})$ admits a comparison triangle $\triangle(\tilde{p}\widetilde{q_{i_1}}\widetilde{c(s_1)})$ in $\widetilde{M}$. Then, by Theorem 2.4, we observe that $\angle\widetilde{q_{i_1}} \leq \overrightarrow{\angle}(pc(0)c(a)) \leq \pi/2$. Since

$$\tilde{d}(\tilde{p}, \widetilde{q_{i_1}}) = d(p, q_{i_1}) \leq d\big(p, c(s_1)\big) = \tilde{d}\big(\tilde{p}, \widetilde{c(s_1)}\big),$$

this contradicts Lemma 4.2. If $\min_{s\in[0,a]} d(p, c(s)) < d(p, q_{i_1})$, then we fix $s_0 \in (0, a)$ such that $d(p, c(s_0)) = \min_{s\in[0,a]} d(p, c(s))$. By construction, it holds $\overrightarrow{\angle}(pc(s_0)c(a)) = \pi/2$ (note that $\overrightarrow{\angle}(pc(s_0)c(a)) > \pi/2$ can not happen by Theorem 2.4). Thus we derive a contradiction from the same argument as the first case. $\square$

5 Proof of Theorem B

Let (M, F, p), f, ρ and t_0 be as in Theorem B. Suppose that the cut locus $\mathrm{Cut}(p)$ of p is not empty. Then, since M is non-compact, $\mathrm{Cut}(p)$ is an unbounded set (consider a sequence in the open set $D_p := \{v \in U_pM \mid \gamma_v((0,\infty)) \cap \mathrm{Cut}(p) \neq \emptyset\}$ whose limit belongs to the complement $D^c_p = \{v \in U_pM \mid \gamma_v \text{ is a ray}\}$, where $U_pM := T_pM \cap F^{-1}(1)$ and $\gamma_v(t) := \exp_p(tv)$ for $t \geq 0$). Let $N(p)$ denote the set of all points $x \in M$ admitting at least two minimal geodesic segments emanating from p to x. Note that $N(p)$ is dense in $\mathrm{Cut}(p)$ (see [TS, Proposition 2.6]).

Take a divergent sequence $\{x_i\}_{i\in\mathbb{N}} \subset N(p)$, and fix $i_0 \in \mathbb{N}$ such that $d(p, x_{i_0}) > t_0$. Since M is non-compact and complete, there exists a unit speed ray $\sigma : [0, \infty) \longrightarrow M$ emanating from p. Take a divergent sequence $\{r_j\}_{j\in\mathbb{N}} \subset (d(p, x_{i_0}), \infty)$ and, for each j, let $c_j : [0, a_j] \longrightarrow M$ be a unit speed minimal geodesic segment emanating from x_{i_0} to $\sigma(r_j)$. By Lemma 4.3, $c_j([0, a_j]) \cap \partial B_{t_0}(p) = \emptyset$ holds for all $j \in \mathbb{N}$.

Take a subdivision $s_0 := 0 < s_1 < \cdots < s_{k-1} < s_k := a_j$ of $[0, a_j]$ such that $\triangle(\overrightarrow{pc_j(s_{l-1})}, \overrightarrow{pc_j(s_l)})$ admits a comparison triangle $\widetilde{\triangle}^l := \triangle(\tilde{p}\widetilde{c_j(s_{l-1})}\widetilde{c_j(s_l)}) \subset \widetilde{M}$ for each $l = 1, 2, \ldots, k$. Note that, by the reversibility of F,

$$\tilde{d}(\widetilde{c_j(s_{l-1})}, \widetilde{c_j(s_l)}) = L_{\mathrm{m}}(c_j|_{[s_{l-1}, s_l]}) = s_l - s_{l-1}. \tag{5.1}$$

It follows from Theorem 2.4 that

$$\overrightarrow{\angle} c_j(s_{l-1}) \geq \angle\big(\tilde{p}\widetilde{c_j(s_{l-1})}\widetilde{c_j(s_l)}\big), \qquad \overleftarrow{\angle} c_j(s_l) \geq \angle\big(\tilde{p}\widetilde{c_j(s_l)}\widetilde{c_j(s_{l-1})}\big) \tag{5.2}$$

for each $l = 1, 2, \ldots, k$. Starting from $\widetilde{\triangle}^1$, we inductively draw a geodesic triangle $\widetilde{\triangle}^{l+1} \subset \widetilde{M}$ which is adjacent to $\widetilde{\triangle}^l$ so as to have a common side $\tilde{p}\widetilde{c_j(s_l)}$, where $0 \le \theta(\widetilde{c_j(s_0)}) \le \theta(\widetilde{c_j(s_1)}) \le \cdots \le \theta(\widetilde{c_j(s_k)})$. We observe from the definition of the angles that $\overleftarrow{\angle} c_j(s_l) + \overrightarrow{\angle} c_j(s_l) \le \pi$ for each $l = 1, 2, \ldots, k-1$. Together with (5.2), we obtain

$$\angle\big(\tilde{p}\widetilde{c_j(s_l)}\widetilde{c_j(s_{l-1})}\big) + \angle\big(\tilde{p}\widetilde{c_j(s_l)}\widetilde{c_j(s_{l+1})}\big) \le \pi. \tag{5.3}$$

Let $\widehat{\xi}_j : [0, a_j] \longrightarrow \widetilde{M}$ denote the broken geodesic segment consisting of minimal geodesic segments from $\widetilde{c_j(s_{l-1})}$ to $\widetilde{c_j(s_l)}$, $l = 1, 2, \ldots, k$. We set $\widehat{\xi}_j(s) = (t(\widehat{\xi}_j(s)), \theta(\widehat{\xi}_j(s)))$. Then (5.3) gives us the unit speed (not necessarily minimal) geodesic $\widetilde{\eta}_j : [0, b_j] \longrightarrow \widetilde{M}$ emanating from $\widetilde{c_j(0)}$ to $\widetilde{c_j(a_j)}$ and passing under $\widehat{\xi}_j([0, a_j])$, i.e., $\theta(\widetilde{\eta}_j) \in [\theta(\widetilde{c_j(0)}), \theta(\widetilde{c_j(a_j)})]$ on $[0, b_j]$ and $t(\widehat{\xi}_j(s)) > t(\widetilde{\eta}_j(b))$ for all $(s, b) \in (0, a_j) \times (0, b_j)$ with $\theta(\widehat{\xi}_j(s)) = \theta(\widetilde{\eta}_j(b))$. On the one hand, by (5.1), we have

$$L(\widetilde{\eta}_j) \le L(\widehat{\xi}_j) = \sum_{l=1}^{k} \tilde{d}\big(\widetilde{c_j(s_{l-1})}, \widetilde{c_j(s_l)}\big) = s_k - s_0 = a_j,$$

where $L(\widetilde{\eta}_j)$ denotes the length of $\widetilde{\eta}_j$. Moreover, the reversibility of F and the triangle inequality show

$$L(\widetilde{\eta}_j) \le a_j = d\big(x_{i_0}, \sigma(r_j)\big) \le d(p, x_{i_0}) + r_j. \tag{5.4}$$

On the other hand, it follows from (3.2) that

$$\begin{aligned} L(\widetilde{\eta}_j) &\ge r_j - d(p, x_{i_0}) + \frac{\nu_j^2}{2} \int_{d(p, x_{i_0})}^{r_j} \frac{1}{f(t)\sqrt{f(t)^2 - \nu_j^2}}\, dt \\ &\ge r_j - d(p, x_{i_0}) + \frac{\nu_j^2}{2} \int_{d(p, x_{i_0})}^{r_j} f(t)^{-2}\, dt, \end{aligned}$$

where ν_j denotes the Clairaut constant of $\widetilde{\eta}_j$. Together with (5.4), we find

$$4d(p, x_{i_0}) \ge \nu_j^2 \int_{d(p, x_{i_0})}^{r_j} f(t)^{-2}\, dt.$$

Since f is decreasing on (ρ, ∞), this implies $\lim_{j\to\infty} \nu_j = 0$. Hence we have

$$\lim_{j\to\infty} \angle\big(\dot{\widetilde{\eta}}_j(0), (\partial/\partial t)|_{\widetilde{\eta}_j(0)}\big) = 0.$$

Combining this with $\angle(\dot{\widetilde{\eta}}_j(0), (\partial/\partial t)|_{\widetilde{\eta}_j(0)}) = \pi - \angle(\tilde{p}\widetilde{c_j(0)}\widetilde{c_j(s_1)})$ and (5.2), we obtain $\lim_{j\to\infty} \overrightarrow{\angle} c_j(0) = \pi$. This is a contradiction, since $c_j(0) = x_{i_0} \in N(p)$. Hence $\mathrm{Cut}(p) = \emptyset$, so that M is diffeomorphic to $\mathbb{R}^n$. The curvature equality follows from the same argument as [KT3, Theorem 4.8]. □

References

[BCS] D. Bao, S.-S. Chern, and Z. Shen, An introduction to Riemann-Finsler geometry, Springer, New York (2000).

[Br] R. L. Bryant, Projectively flat Finsler 2-spheres of constant curvature, Selecta Math. (N.S.) **3** (1997), 161–203.

[Gr] M. Gromov, Curvature, diameter and Betti numbers, Comment. Math. Helv. **56** (1981), 179–195.

[GS] K. Grove and K. Shiohama, A generalized sphere theorem, Ann. of Math. (2) **106** (1977), 201–211.

[KOT] K. Kondo, S. Ohta, and M. Tanaka, A Toponogov type triangle comparison theorem in Finsler geometry, Preprint (2012). Available at arXiv:1205.3913

[KT1] K. Kondo and M. Tanaka, Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. I, Math. Ann. **351** (2011), 251–266.

[KT2] K. Kondo and M. Tanaka, Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below. II, Trans. Amer. Math. Soc. **362** (2010), 6293–6324.

[KT3] K. Kondo and M. Tanaka, Applications of Toponogov's comparison theorems for open triangles, to appear in Osaka J. Math. Available at arXiv:1102.4156

[Oh1] S. Ohta, Products, cones, and suspensions of spaces with the measure contraction property, J. Lond. Math. Soc. (2) **76** (2007), 225–236.

[Oh2] S. Ohta, Uniform convexity and smoothness, and their applications in Finsler geometry, Math. Ann. **343** (2009), 669–699.

[Oh3] S. Ohta, Finsler interpolation inequalities, Calc. Var. Partial Differential Equations **36** (2009), 211–249.

[Oh4] S. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature, Preprint (2012). Available at arXiv:1203.0079

[Ra] H.-B. Rademacher, A sphere theorem for non-reversible Finsler metrics, Math. Ann. **328** (2004), 373–387.

[Sh1] Z. Shen, Volume comparison and its applications in Riemann-Finsler geometry, Adv. Math. **128** (1997), 306–328.

[Sh2] Z. Shen, Lectures on Finsler geometry, World scientific publishing co., Singapore, 2001.

[Sh3] Z. Shen, Differential geometry of spray and Finsler spaces, Kluwer Academic Publishers, Dordrecht, 2001.

[SST] K. Shiohama, T. Shioya, and M. Tanaka, The geometry of total curvature on complete open surfaces, Cambridge Tracks in Math. **159**, Cambridge University Press, Cambridge, 2003.

[ST] K. Shiohama and M. Tanaka, Compactification and maximal diameter theorem for noncompact manifolds with radial curvature bounded below, Math. Z. **241** (2002), 341–351.

[Ta] M. Tanaka, On the cut loci of a von Mangoldt's surface of revolution, J. Math. Soc. Japan **44** (1992), 631–641.

[TK] M. Tanaka and K. Kondo, The topology of an open manifold with radial curvature bounded from below by a model surface with finite total curvature and examples of model surfaces, to appear in Nagoya Math. J. Available at arXiv:1102.0852

[TS] M. Tanaka and S. V. Sabau, The cut locus and distance function from a closed subset of a Finsler manifold, Preprint (2012). Available at arXiv:1207.0918

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