

# Weighted Ricci curvature estimates for Hilbert and Funk geometries

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## Abstract

We consider Hilbert and Funk geometries on a strongly convex domain in the Euclidean space. We show that, with respect to the Lebesgue measure on the domain, Hilbert (resp. Funk) metric has the bounded (resp. constant negative) weighted Ricci curvature. As one of corollaries, these metric measure spaces satisfy the curvature-dimension condition in the sense of Lott, Sturm and Villani.

## 1 Introduction

Hilbert [Hi] introduced the distance function  $d_{\mathcal{H}}$  on a bounded convex domain  $D \subset \mathbb{R}^n$ , related to his fourth problem. Given distinct points  $x, y \in D$ , denoting by  $x' = x + s(y - x)$  and  $y' = x + t(y - x)$  the intersections of the boundary  $\partial D$  and the line passing through  $x$  and  $y$  with  $s < 0 < t$  (see Figure), Hilbert's distance  $d_{\mathcal{H}}$  is given by

$$d_{\mathcal{H}}(x, y) = \frac{1}{2} \log \left( \frac{|x' - y| \cdot |x - y'|}{|x' - x| \cdot |y - y'|} \right),$$

where  $|\cdot|$  stands for the Euclidean norm. This is indeed a distance function on  $D$ , and satisfies the interesting property that line segments between any points are minimizing. In the particular case where  $D$  is the unit ball,  $(D, d_{\mathcal{H}})$  coincides with the Klein model of the hyperbolic space. The structure of  $(D, d_{\mathcal{H}})$  has been investigated from geometric and dynamical aspects (see, for example, [Eg], [Be], [CV]). For instance,  $(D, d_{\mathcal{H}})$  is known to be Gromov hyperbolic under mild smoothness and convexity assumptions on  $D$ .

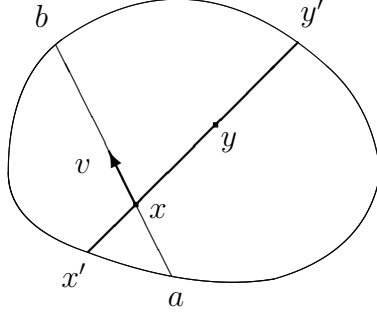
Funk [Fu] introduced a non-symmetrization of  $d_{\mathcal{H}}$ , namely

$$d_{\mathcal{F}}(x, y) = \log \left( \frac{|x - y'|}{|y - y'|} \right).$$

Note that  $d_{\mathcal{F}}(x, y) \neq d_{\mathcal{F}}(y, x)$ , while the triangle inequality  $d_{\mathcal{F}}(x, z) \leq d_{\mathcal{F}}(x, y) + d_{\mathcal{F}}(y, z)$  still holds. Clearly we have  $2d_{\mathcal{H}}(x, y) = d_{\mathcal{F}}(x, y) + d_{\mathcal{F}}(y, x)$ , and line segments are minimizing also with respect to Funk's distance.

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Figure

If  $\partial D$  is smooth and  $D$  is strongly convex (in other words,  $\partial D$  is positively curved; see Definition 2.1), then  $d_{\mathcal{H}}$  and  $d_{\mathcal{F}}$  are realized by the smooth Finsler structures

$$F_{\mathcal{H}}(x, v) = \frac{|v|}{2} \left\{ \frac{1}{|x-a|} + \frac{1}{|x-b|} \right\}, \quad F_{\mathcal{F}}(x, v) = \frac{|v|}{|x-b|} \quad \text{for } v \in T_x D = \mathbb{R}^n, \quad (1.1)$$

respectively (cf. [Sh2, §2.3]), where  $a = x + sv$  and  $b = x + tv$  denote the intersections of  $\partial D$  and the line passing through  $x$  in the direction  $v$  with  $s < 0 < t$  (see Figure). Note that  $2F_{\mathcal{H}}(x, v) = F_{\mathcal{F}}(x, v) + F_{\mathcal{F}}(x, -v)$ . A remarkable feature of these metrics is that they have the constant negative flag curvature  $-1$  and  $-1/4$ , respectively (cf. [Ok, Theorem 1], [Sh2, Theorem 12.2.11]), provided that  $n \geq 2$  as a matter of course. The flag curvature is a generalization of the sectional curvature in Riemannian geometry, so that it is natural that  $(D, d_{\mathcal{H}})$  and  $(D, d_{\mathcal{F}})$  enjoy properties of negatively curved spaces.

Recently, the theory of the *weighted Ricci curvature* (see Definition 2.2) for Finsler manifolds equipped with arbitrary measures has been developed in connection with optimal transport theory. It turned out that the weighted Ricci curvature is a natural quantity and quite useful in the study of geometry and analysis on Finsler manifolds (see [Oh2], [Oh4], [OS1], [OS2]). The aim of this article is to show that the weighted Ricci curvature for Hilbert and Funk geometries admits uniform bounds with respect to the Lebesgue measure  $m_{\mathbf{L}}$  restricted on  $D$ .

**Theorem 1.1 (Funk case)** *Let  $D \subset \mathbb{R}^n$  with  $n \geq 2$  be a strongly convex domain such that  $\partial D$  is smooth. Then  $(D, F_{\mathcal{F}}, m_{\mathbf{L}})$  has the constant negative weighted Ricci curvature as, for any unit vector  $v \in TD$ ,*

$$\text{Ric}_{\infty}(v) = -\frac{n-1}{4}, \quad \text{Ric}_N(v) = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)} \quad \text{for } N \in (n, \infty).$$

**Theorem 1.2 (Hilbert case)** *Let  $D \subset \mathbb{R}^n$  with  $n \geq 2$  be a strongly convex domain such that  $\partial D$  is smooth. Then the weighted Ricci curvature of  $(D, F_{\mathcal{H}}, m_{\mathbf{L}})$  is bounded as, for any unit vector  $v \in TD$ ,*

$$\text{Ric}_{\infty}(v) \in (-(n-1), 2], \quad \text{Ric}_N(v) \in \left( -(n-1) - \frac{(n+1)^2}{N-n}, 2 \right] \quad \text{for } N \in (n, \infty).$$

We stress that our estimates are independent of the choice of the domain  $D$ . There are several applications (Corollaries 5.1, 5.2) via the theory of the weighted Ricci curvature.

The article is organized as follows. After preliminaries for Finsler geometry and the weighted Ricci curvature, we prove Theorems 1.1, 1.2 in Sections 3, 4, respectively. We finally discuss applications and remarks in Section 5.

## 2 Preliminaries

We very briefly review the necessary notions in Finsler geometry, we refer to [BCS], [Sh1] and [Sh2] for further reading. Let  $M$  be a connected,  $n$ -dimensional  $\mathcal{C}^\infty$ -manifold without boundary such that  $n \geq 2$ . Given a local coordinate  $(x^i)_{i=1}^n$  on an open set  $\Omega \subset M$ , we always use the coordinate  $(x^i, v^j)_{i,j=1}^n$  of  $T\Omega$  such that

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j} \Big|_x \in T_x M \quad \text{for } x \in \Omega.$$

**Definition 2.1 (Finsler structures)** A nonnegative function  $F : TM \longrightarrow [0, \infty)$  is called a  $\mathcal{C}^\infty$ -Finsler structure of  $M$  if the following three conditions hold.

- (1) (Regularity)  $F$  is  $\mathcal{C}^\infty$  on  $TM \setminus 0$ , where  $0$  stands for the zero section.
- (2) (Positive 1-homogeneity) It holds  $F(cv) = cF(v)$  for all  $v \in TM$  and  $c > 0$ .
- (3) (Strong convexity) The  $n \times n$  matrix

$$(g_{ij}(v))_{i,j=1}^n := \left( \frac{1}{2} \frac{\partial^2 (F^2)}{\partial v^i \partial v^j} (v) \right)_{i,j=1}^n \quad (2.1)$$

is positive-definite for all  $v \in TM \setminus 0$ .

For  $x, y \in M$ , we can define the *distance* from  $x$  to  $y$  in a natural way by

$$d(x, y) := \inf_{\eta} \int_0^1 F(\dot{\eta}(t)) dt,$$

where the infimum is taken over all  $\mathcal{C}^1$ -curves  $\eta : [0, 1] \longrightarrow M$  with  $\eta(0) = x$  and  $\eta(1) = y$ . We remark that this distance can be *nonsymmetric* (namely  $d(y, x) \neq d(x, y)$ ), since  $F$  is only positively homogeneous. A  $\mathcal{C}^\infty$ -curve  $\eta$  on  $M$  is called a *geodesic* if it is locally minimizing and has a constant speed (i.e.,  $F(\dot{\eta})$  is constant).

Given  $v \in T_x M$ , if there is a geodesic  $\eta : [0, 1] \longrightarrow M$  with  $\dot{\eta}(0) = v$ , then we define the *exponential map* by  $\exp_x(v) := \eta(1)$ . We say that  $(M, F)$  is *forward complete* if the exponential map is defined on whole  $TM$ . If the *reverse* Finsler manifold  $(M, \overleftarrow{F})$  with  $\overleftarrow{F}(v) := F(-v)$  is forward complete, then  $(M, F)$  is said to be *backward complete*. We remark that  $(D, F_{\mathcal{H}})$  is both forward and backward complete (they are indeed equivalent since  $\overleftarrow{F}_{\mathcal{H}} = F_{\mathcal{H}}$ ), while  $(D, F_{\mathcal{F}})$  is only forward complete.

For each  $v \in T_x M \setminus 0$ , the positive-definite matrix  $(g_{ij}(v))_{i,j=1}^n$  in (2.1) induces the Riemannian structure  $g_v$  of  $T_x M$  as

$$g_v \left( \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_x, \sum_{j=1}^n b_j \frac{\partial}{\partial x^j} \Big|_x \right) := \sum_{i,j=1}^n a_i b_j g_{ij}(v). \quad (2.2)$$

Note that  $g_{cv} = g_v$  for  $c > 0$ . This inner product is regarded as the best Riemannian approximation of  $F|_{T_x M}$  in the direction  $v$ , in the sense that the unit sphere of  $g_v$  is tangent to that of  $F|_{T_x M}$  at  $v/F(v)$  up to the second order. In particular, we have  $g_v(v, v) = F(v)^2$ .

The *Ricci curvature* (as the trace of the *flag curvature*) for a Finsler manifold is defined by using the Chern connection. Instead of giving the precise definition in coordinates, we explain a useful interpretation due to Shen (see [Sh1, §6.2], [Sh3, Lemma 2.4]). Given a unit vector  $v \in T_x M \cap F^{-1}(1)$ , we extend it to a non-vanishing  $\mathcal{C}^\infty$ -vector field  $V$  on a neighborhood of  $x$  in such a way that every integral curve of  $V$  is geodesic, and consider the Riemannian structure  $g_V$  induced from (2.2). Then the Ricci curvature  $\text{Ric}(v)$  of  $v$  with respect to  $F$  coincides with the Ricci curvature of  $v$  with respect to  $g_V$  (in particular, it is independent of the choice of  $V$ ).

Let us fix a positive  $\mathcal{C}^\infty$ -measure  $m$  on  $M$ . Inspired by the above interpretation of the Finsler Ricci curvature and the theory of weighted Riemannian manifolds, the weighted Ricci curvature for the triple  $(M, F, m)$  was introduced in [Oh2] as follows.

**Definition 2.2 (Weighted Ricci curvature)** Given a unit vector  $v \in T_x M \cap F^{-1}(1)$ , let  $\eta : (-\varepsilon, \varepsilon) \rightarrow M$  be the geodesic such that  $\dot{\eta}(0) = v$ . We decompose  $m$  along  $\eta$  using the Riemannian volume measure  $\text{vol}_\eta$  of  $g_\eta$  as  $m = e^{-\Psi} \text{vol}_\eta$ , where  $\Psi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ . Then we define the *weighted Ricci curvature* involving a parameter  $N \in [n, \infty]$  by

- (1)  $\text{Ric}_n(v) := \begin{cases} \text{Ric}(v) + \Psi''(0) & \text{if } \Psi'(0) = 0, \\ -\infty & \text{if } \Psi'(0) \neq 0, \end{cases}$
- (2)  $\text{Ric}_N(v) := \text{Ric}(v) + \Psi''(0) - \frac{\Psi'(0)^2}{N - n}$  for  $N \in (n, \infty)$ ,
- (3)  $\text{Ric}_\infty(v) := \text{Ric}(v) + \Psi''(0)$ .

We also set  $\text{Ric}_N(cv) := c^2 \text{Ric}_N(v)$  for  $c \geq 0$ .

We will say that  $\text{Ric}_N \geq K$  holds for some  $K \in \mathbb{R}$  if  $\text{Ric}_N(v) \geq KF(v)^2$  for all  $v \in TM$ . Observe that  $\text{Ric}_N(v) \leq \text{Ric}_{N'}(v)$  for  $N < N'$ , and that for the scaled space  $M' = (M, F, am)$  with  $a > 0$  we have  $\text{Ric}_N^{M'}(v) = \text{Ric}_N^M(v)$ . It was shown in [Oh2, Theorem 1.2] that  $\text{Ric}_N \geq K$  is equivalent to Lott, Sturm and Villani's *curvature-dimension condition*  $\text{CD}(K, N)$ . (Roughly speaking, the curvature-dimension condition is a convexity condition of an entropy functional on the space of probability measures; we refer to [St1], [St2], [LV1], [LV2] and [Vi, Part III] for details and further theories.) This equivalence extends the corresponding result on (weighted) Riemannian manifolds, and has many analytic and geometric applications (see [Oh2]).

### 3 Proof of Theorem 1.1 (Funk case)

Let us first treat the Funk case. In this section, we will denote the Funk metric simply by  $F$  for brevity, and consider the standard coordinate of  $D \subset \mathbb{R}^n$ . The following lemma enables us to translate all the vertical derivatives ( $\partial/\partial v^i$ ) to the horizontal derivatives ( $\partial/\partial x^i$ ).

**Lemma 3.1** ([Ok, Proposition 1], [Sh2, Lemma 2.3.1]) *For any  $v \in TD \setminus 0$  and  $i = 1, 2, \dots, n$ , we have*

$$\frac{\partial F}{\partial x^i}(v) = F(v) \frac{\partial F}{\partial v^i}(v).$$

Observe that, on  $TD \setminus 0$ ,

$$\frac{1}{2} \frac{\partial^2(F^2)}{\partial v^i \partial v^j} = \frac{\partial}{\partial v^i} \left[ \frac{\partial F}{\partial x^j} \right] = \frac{\partial}{\partial x^j} \left[ \frac{1}{F} \frac{\partial F}{\partial x^i} \right] = \frac{1}{F} \frac{\partial^2 F}{\partial x^i \partial x^j} - \frac{1}{F^2} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j}. \quad (3.1)$$

Now, we fix a unit vector  $v \in T_x D \cap F^{-1}(1)$  and choose an appropriate coordinate that  $x$  is the origin,  $v = \partial/\partial x^n$  and that  $g_{in}(v) = 0$  for all  $i = 1, 2, \dots, n-1$ . We remark that such a coordinate exchange multiplies the Lebesgue measure merely by a positive constant, so that the weighted Ricci curvature does not change. Put  $V := \partial/\partial x^n$  on  $D$  and recall that the all integral curves of  $V$  are minimizing (and hence re-parametrizations of geodesics). Therefore it suffices to calculate the weighted Ricci curvature of  $(D, g_V, m_{\mathbf{L}})$ .

We can represent  $\partial D \cap \{x \in \mathbb{R}^n \mid x^n > 0\}$  as the graph of the  $\mathcal{C}^\infty$ -function  $h : U \rightarrow (0, \infty)$  for a sufficiently small neighborhood  $U \subset \mathbb{R}^{n-1}$  of 0, namely

$$\partial D \cap \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z \in U, t > 0\} = \{(z, h(z)) \mid z \in U\}. \quad (3.2)$$

Then (1.1) yields

$$F(V(z, t)) = \frac{1}{h(z) - t} \quad \text{for } (z, t) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Putting  $\partial_i := \partial/\partial x^i$  for simplicity, we deduce from (3.1) that

$$\begin{aligned} g_{ij}(V) &= (h-t) \partial_i \partial_j \left( \frac{1}{h-t} \right) - (h-t)^2 \partial_i \left( \frac{1}{h-t} \right) \partial_j \left( \frac{1}{h-t} \right) \\ &= (h-t) \left\{ -\frac{\partial_i \partial_j (h-t)}{(h-t)^2} + \frac{2\partial_i (h-t) \partial_j (h-t)}{(h-t)^3} \right\} - \frac{\partial_i (h-t) \partial_j (h-t)}{(h-t)^2} \\ &= -\frac{\partial_i \partial_j (h-t)}{h-t} + \frac{\partial_i (h-t) \partial_j (h-t)}{(h-t)^2}, \end{aligned}$$

where the evaluations at  $(z, t) \in D$  were omitted. We remark that, for  $i, j \neq n$ ,

$$g_{ij}(V) = -\frac{\partial_i \partial_j h}{h-t} + \frac{\partial_i h \partial_j h}{(h-t)^2}, \quad g_{in}(V) = -\frac{\partial_i h}{(h-t)^2}, \quad g_{nn}(V) = \frac{1}{(h-t)^2}.$$

Hence, when differentiating  $g_{ij}(V(z, t))$  by  $t$ , we need to take only the denominators into account. Thus we find

$$\frac{\partial[g_{ij}(V)]}{\partial t} = -\frac{\partial_i \partial_j (h-t)}{(h-t)^2} + \frac{2\partial_i (h-t) \partial_j (h-t)}{(h-t)^3} = \frac{1}{h-t} \left\{ g_{ij}(V) + \frac{\partial_i (h-t) \partial_j (h-t)}{(h-t)^2} \right\}.$$

Decomposing  $m_{\mathbf{L}}$  as  $m_{\mathbf{L}} = e^{-\Psi} \sqrt{\det(g_{ij}(V))} dx^1 dx^2 \cdots dx^n$  along the curve  $\eta(t) = (0, t) \in D$ , we observe

$$\Psi(t) = \frac{1}{2} \log \left( \det(g_{ij}(t)) \right), \quad \Psi'(t) = \frac{1}{2} \text{trace} \left[ (g^{ij}(t)) \cdot (g'_{ij}(t)) \right],$$

where we abbreviated as  $g_{ij}(t) := g_{ij}(V(0, t))$  and  $(g^{ij}(t))$  stands for the inverse matrix of  $(g_{ij}(t))$ . Dividing  $\Psi'(t)$  by the speed  $F(\dot{\eta}(t)) = F(V(0, t)) = (h(0) - t)^{-1}$ , we obtain

$$(h(0) - t)\Psi'(t) = \frac{1}{2} \text{trace} \left[ (g^{ij}(t)) \cdot \left( g_{ij}(t) + \frac{\partial_i(h(0) - t)\partial_j(h(0) - t)}{(h(0) - t)^2} \right) \right] \equiv \frac{n+1}{2},$$

where the second equality follows from the fact that  $g_{in}(t) = -\partial_i h(0)/(h(0) - t)^2 = 0$  for  $i \neq n$  guaranteed by  $g_{in}(v) = 0$ . Therefore we conclude, as  $(D, F)$  has the constant flag curvature  $-1/4$ ,

$$\text{Ric}_{\infty}(v) = -\frac{n-1}{4}, \quad \text{Ric}_N(v) = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)}.$$

□

## 4 Proof of Theorem 1.2 (Hilbert case)

We next consider the Hilbert case, where the calculation is similar but more involved. We will denote the Hilbert metric of  $D$  by  $F$  in this section.

Given a unit vector  $v \in T_x D \cap F^{-1}(1)$ , similarly to the previous section, we choose a coordinate such that  $x$  is the origin,  $v = \partial/\partial x^n$  and that  $g_{in}(v) = 0$  for all  $i = 1, 2, \dots, n-1$ . Put  $V := \partial/\partial x^n$  again. In addition to  $h : U \rightarrow (0, \infty)$  as in (3.2), we introduce the function  $b : U \rightarrow (-\infty, 0)$  such that

$$\partial D \cap \{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid z \in U, t < 0\} = \{(z, b(z)) \mid z \in U\}.$$

Using the Funk metric  $F_+$  of  $D$  and its reverse  $F_-(v) := F_+(-v)$ , we can write  $F(V)$  as (recall (1.1))

$$F(V(z, t)) = \frac{F_+(V(z, t)) + F_-(V(z, t))}{2} = \frac{1}{2} \left\{ \frac{1}{h(z) - t} + \frac{1}{t - b(z)} \right\}.$$

It follows from Lemma 3.1 and  $F_-(v) = F_+(-v)$  that

$$\frac{\partial F_-}{\partial x^i} = -F_- \frac{\partial F_-}{\partial v^i}.$$

This yields that, by putting  $\partial_i := \partial/\partial x^i$ ,

$$\begin{aligned} 2 \frac{\partial^2(F^2)}{\partial v^i \partial v^j} &= \frac{1}{2} \frac{\partial^2}{\partial v^i \partial v^j} (F_+^2 + 2F_+F_- + F_-^2) \\ &= \frac{1}{2} \frac{\partial^2(F_+^2)}{\partial v^i \partial v^j} + \frac{1}{2} \frac{\partial^2(F_-^2)}{\partial v^i \partial v^j} - \frac{\partial_i F_+ \partial_j F_-}{F_+ F_-} - \frac{\partial_j F_+ \partial_i F_-}{F_+ F_-} \\ &\quad + \left( \frac{\partial_i \partial_j F_+}{F_+^2} - \frac{2\partial_i F_+ \partial_j F_+}{F_+^3} \right) F_- + \left( \frac{\partial_i \partial_j F_-}{F_-^2} - \frac{2\partial_i F_- \partial_j F_-}{F_-^3} \right) F_+. \end{aligned}$$

By (3.1) we have, omitting the evaluations at  $(z, t) \in D$ ,

$$\begin{aligned}
4g_{ij}(V) &= -\frac{\partial_i \partial_j (h-t)}{h-t} + \frac{\partial_i (h-t) \partial_j (h-t)}{(h-t)^2} - \frac{\partial_i \partial_j (t-b)}{t-b} + \frac{\partial_i (t-b) \partial_j (t-b)}{(t-b)^2} \\
&\quad - \left\{ \frac{\partial_i (h-t)}{h-t} \frac{\partial_j (t-b)}{t-b} + \frac{\partial_j (h-t)}{h-t} \frac{\partial_i (t-b)}{t-b} \right\} - \frac{\partial_i \partial_j (h-t)}{t-b} - \frac{\partial_i \partial_j (t-b)}{h-t} \\
&= -\{\partial_i \partial_j (h-t) + \partial_i \partial_j (t-b)\} \left( \frac{1}{h-t} + \frac{1}{t-b} \right) \\
&\quad + \left\{ \frac{\partial_i (h-t)}{h-t} - \frac{\partial_i (t-b)}{t-b} \right\} \left\{ \frac{\partial_j (h-t)}{h-t} - \frac{\partial_j (t-b)}{t-b} \right\}.
\end{aligned}$$

Note that the assumption  $g_{in}(v) = 0$  implies

$$\frac{\partial_i h(0)}{h(0)} - \frac{\partial_i b(0)}{b(0)} = 0 \quad \text{for } i = 1, 2, \dots, n-1. \quad (4.1)$$

We also observe for later convenience that, for  $i, j \neq n$ ,

$$4g_{ij}(v) = -\{\partial_i \partial_j h(0) - \partial_i \partial_j b(0)\} \left( \frac{1}{h(0)} - \frac{1}{b(0)} \right), \quad 4g_{nn}(v) = \left( \frac{1}{h(0)} - \frac{1}{b(0)} \right)^2.$$

By the same reasoning as the Funk case, the numerators can be neglected when one differentiates  $g_{ij}(V)$  by  $t$ . Thus we find

$$\begin{aligned}
4 \frac{\partial [g_{ij}(V)]}{\partial t} &= -\{\partial_i \partial_j (h-t) + \partial_i \partial_j (t-b)\} \left\{ \frac{1}{(h-t)^2} - \frac{1}{(t-b)^2} \right\} \\
&\quad + \left\{ \frac{\partial_i (h-t)}{(h-t)^2} + \frac{\partial_i (t-b)}{(t-b)^2} \right\} \left\{ \frac{\partial_j (h-t)}{h-t} - \frac{\partial_j (t-b)}{t-b} \right\} \\
&\quad + \left\{ \frac{\partial_i (h-t)}{h-t} - \frac{\partial_i (t-b)}{t-b} \right\} \left\{ \frac{\partial_j (h-t)}{(h-t)^2} + \frac{\partial_j (t-b)}{(t-b)^2} \right\}.
\end{aligned}$$

We further calculate

$$\begin{aligned}
4 \frac{\partial^2 [g_{ij}(V)]}{\partial t^2} &= -\{\partial_i \partial_j (h-t) + \partial_i \partial_j (t-b)\} \left\{ \frac{2}{(h-t)^3} + \frac{2}{(t-b)^3} \right\} \\
&\quad + \left\{ \frac{2\partial_i (h-t)}{(h-t)^3} - \frac{2\partial_i (t-b)}{(t-b)^3} \right\} \left\{ \frac{\partial_j (h-t)}{h-t} - \frac{\partial_j (t-b)}{t-b} \right\} \\
&\quad + \left\{ \frac{\partial_i (h-t)}{h-t} - \frac{\partial_i (t-b)}{t-b} \right\} \left\{ \frac{2\partial_j (h-t)}{(h-t)^3} - \frac{2\partial_j (t-b)}{(t-b)^3} \right\} \\
&\quad + 2 \left\{ \frac{\partial_i (h-t)}{(h-t)^2} + \frac{\partial_i (t-b)}{(t-b)^2} \right\} \left\{ \frac{\partial_j (h-t)}{(h-t)^2} + \frac{\partial_j (t-b)}{(t-b)^2} \right\}.
\end{aligned}$$

We abbreviate as  $g_{ij}(t) := g_{ij}(V(0, t))$  and deduce from (4.1) that, for  $i, j \neq n$ ,

$$\begin{aligned}
4g'_{ij}(0) &= 4g_{ij}(0) \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right), \quad 4g'_{in}(0) = -\left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{1}{h(0)} - \frac{1}{b(0)} \right), \\
4g'_{nn}(0) &= 8g_{nn}(0) \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right).
\end{aligned}$$

We also obtain

$$\begin{aligned}
4g''_{ij}(0) &= 8g_{ij}(0) \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) \\
&\quad + 2 \left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2} \right), \\
4g''_{nn}(0) &= 8g_{nn}(0) \left\{ 2 \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) + \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 \right\}.
\end{aligned}$$

Put  $\Psi(t) = 2^{-1} \log(\det(g_{ij}(t)))$  and observe

$$\begin{aligned}
\Psi'(t) &= \frac{1}{2} \text{trace} \left[ (g^{ij}(t)) \cdot (g'_{ij}(t)) \right], \\
\Psi''(t) &= \frac{1}{2} \text{trace} \left[ (g^{ij}(t)) \cdot (g''_{ij}(t)) - \{(g^{ij}(t)) \cdot (g'_{ij}(t))\}^2 \right].
\end{aligned}$$

Comparing  $g_{ij}(0)$  and  $g'_{ij}(0)$ , we have

$$\Psi'(0) = \frac{1}{2} \left\{ (n-1) \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right) + 2 \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right) \right\} = \frac{n+1}{2} \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right).$$

It similarly holds

$$\begin{aligned}
\frac{1}{2} \text{trace} \left[ (g^{ij}(0)) \cdot (g''_{ij}(0)) \right] &= (n-1) \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n-1} g^{ij}(0) \left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2} \right) \\
&\quad + 2 \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) + \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 \\
&= (n+1) \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) + \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n-1} g^{ij}(0) \left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2} \right).
\end{aligned}$$

Combining this with

$$\begin{aligned}
&\text{trace} \left[ \{(g^{ij}(0)) \cdot (g'_{ij}(0))\}^2 \right] \\
&= (n-1) \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 + 4 \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 \\
&\quad + \frac{g^{nn}(0)}{8} \sum_{i,j=1}^{n-1} g^{ij}(0) \left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2} \right) \left( \frac{1}{h(0)} - \frac{1}{b(0)} \right)^2 \\
&= (n+3) \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 + \frac{1}{2} \sum_{i,j=1}^{n-1} g^{ij}(0) \left( \frac{\partial_i h(0)}{h(0)^2} - \frac{\partial_i b(0)}{b(0)^2} \right) \left( \frac{\partial_j h(0)}{h(0)^2} - \frac{\partial_j b(0)}{b(0)^2} \right),
\end{aligned}$$



we obtain

$$\begin{aligned}\Psi''(0) &= (n+1) \left( \frac{1}{h(0)^2} + \frac{1}{h(0)b(0)} + \frac{1}{b(0)^2} \right) - \frac{n+1}{2} \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 \\ &= \frac{n+1}{2} \left( \frac{1}{h(0)^2} + \frac{1}{b(0)^2} \right).\end{aligned}$$

Therefore we have, as  $F(v) = (h(0)^{-1} - b(0)^{-1})/2 = 1$ ,

$$\frac{d}{dt} \left[ \frac{\Psi'(t)}{F(V(0,t))} \right]_{t=0} = \Psi''(0) - \frac{\Psi'(0)}{2} \left( \frac{1}{h(0)^2} - \frac{1}{b(0)^2} \right) = -\frac{n+1}{h(0)b(0)}.$$

Since

$$0 < -\frac{1}{h(0)b(0)} \leq \frac{1}{4} \left( \frac{1}{h(0)} - \frac{1}{b(0)} \right)^2 = 1,$$

this yields  $\text{Ric}_\infty(v) \in (-(n-1), 2]$ . Moreover,

$$\Psi'(0)^2 = \frac{(n+1)^2}{4} \left( \frac{1}{h(0)} + \frac{1}{b(0)} \right)^2 = (n+1)^2 \left( 1 + \frac{1}{h(0)b(0)} \right) \in [0, (n+1)^2)$$

shows

$$\text{Ric}_N(v) \in \left( -(n-1) - \frac{(n+1)^2}{N-n}, 2 \right].$$

□

## 5 Applications and remarks

As mentioned in Section 2,  $\text{Ric}_N \geq K$  is equivalent to the curvature-dimension condition  $\text{CD}(K, N)$ . Spaces satisfying  $\text{CD}(K, N)$  enjoy a number of properties similar to Riemannian manifolds of  $\text{Ric} \geq K$  and  $\dim \leq N$ . Since  $\text{CD}(K, N)$  (between compactly supported measures) is preserved under the *pointed measured Gromov-Hausdorff convergence* of locally compact, complete metric measure spaces ([Vi, Theorem 29.25]), we can deal with merely bounded, convex domains  $D$ .

**Corollary 5.1** *Let  $D \subset \mathbb{R}^n$  be a bounded convex domain with  $n \geq 2$ . Then the metric measure spaces  $(D, d_{\mathcal{F}}, m_{\mathbf{L}})$  and  $(D, d_{\mathcal{H}}, m_{\mathbf{L}})$  satisfy  $\text{CD}(K, N)$  for  $N \in (n, \infty]$  with*

$$K = -\frac{n-1}{4} - \frac{(n+1)^2}{4(N-n)}, \quad K = -(n-1) - \frac{(n+1)^2}{N-n},$$

respectively, where we read  $K = -(n-1)/4$  and  $K = -(n-1)$  when  $N = \infty$ . In particular, they satisfy

- the Brunn-Minkowski inequality by  $\text{CD}(K, N)$  with  $N \in (n, \infty]$ ;
- the Bishop-Gromov volume comparison theorem by  $\text{CD}(K, N)$  with  $N \in (n, \infty)$ .

See [St2, Proposition 2.1, Theorem 2.3] (and [Vi, Theorem 30.7], [Oh3, Theorem 6.1] as well for the case of  $N = \infty$ ) for the precise statements of the Brunn-Minkowski inequality and the Bishop-Gromov volume comparison. Beyond the general theory of the curvature-dimension condition, the weighted Ricci curvature bound implies the following.

**Corollary 5.2** *Let  $D \subset \mathbb{R}^n$  with  $n \geq 2$  be a strongly convex domain such that  $\partial D$  is smooth. For  $K$  as in Corollary 5.1,  $(D, F_{\mathcal{F}}, m_{\mathbf{L}})$  and  $(D, F_{\mathcal{H}}, m_{\mathbf{L}})$  satisfy*

- *the Laplacian comparison theorem for  $N \in (n, \infty)$ ;*
- *the Bochner-Weitzenböck inequality for  $N \in (n, \infty)$ .*

See [OS1, Theorem 5.2] for the Laplacian comparison, and [OS2, Theorems 3.3, 3.6] for the Bochner-Weitzenböck formula (by the Bochner-Weitzenböck inequality we meant the inequality given by plugging the weighted Ricci curvature bound into the Bochner-Weitzenböck formula).

We conclude the article with remarks on possible improvements of the estimates in Theorems 1.1, 1.2. Our estimates on  $\text{Ric}_N$  with respect to  $m_{\mathbf{L}}$  are independent of the shape of  $D$ . In particular, Theorem 1.2 provides the same (far from optimal) estimates even for the Klein model of the hyperbolic spaces. Thus there would be a better choice of a measure depending on the shape of  $D$ . Then, as an arbitrary measure is represented by  $e^{-\psi} m_{\mathbf{L}}$ , its weighted Ricci curvature is calculated by combining Theorems 1.1, 1.2 and the convexity of  $\psi$ . One may think of the squared distance function from some point as a candidate of  $\psi$ , however, in order to estimate its convexity along geodesics, we need to bound not only the flag curvature but also the *uniform convexity* as well as the *tangent curvature* (see [Oh1, Theorem 5.1]). The uniform convexity is measured by the constant

$$\mathbf{C} = \sup_{x \in M} \sup_{v, w \in T_x M \setminus \{0\}} \frac{F(w)}{g_v(w, w)^{1/2}},$$

and it is infinite for Funk metrics. As for Hilbert geometry, one could bound  $\mathbf{C}$  by the convexity of  $\partial D$  (whereas it seems unclear; see [Eg, Remark 2.1]). The author has no idea about the tangent curvature, which measures how the tangent spaces are distorted as one moves in  $M$ .

There are several natural constructive measures  $m$  on  $D$ , and it is interesting to consider the corresponding weighted Ricci curvature  $\text{Ric}_N^m(V)$ . Then, however, it seems not easy (at least more difficult than  $m_{\mathbf{L}}$ ) to calculate  $\text{Ric}_N^m(V)$  because  $m$  should depend on the shape of whole  $\partial D$ , while  $g_V$  is induced only from the behavior of  $F_{\mathcal{F}}$  or  $F_{\mathcal{H}}$  near the direction  $V$ .

We also remark that, in Hilbert geometry (which is both forward and backward complete),  $\text{Ric}_N$  with  $N < \infty$  can not be nonnegative for any measure. Otherwise,  $g_V$  splits isometrically that is a contradiction ([Oh4, Proposition 4.3]). Due to the same reasoning,  $\text{Ric}_{\infty}$  can be nonnegative only when  $\sup \Psi = \infty$ .

## References

- [BCS] D. Bao, S.-S. Chern and Z. Shen, An introduction to Riemann-Finsler geometry, Springer-Verlag, New York, 2000.

- [Be] Y. Benoist, Convexes hyperboliques et fonctions quasisymétriques (French), *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 181–237.
- [CV] B. Colbois and P. Verovic, Hilbert geometry for strictly convex domains, *Geom. Dedicata* **105** (2004), 29–42.
- [Eg] D. Egloff, Uniform Finsler Hadamard manifolds, *Ann. Inst. H. Poincaré Phys. Théor.* **66** (1997), 323–357.
- [Fu] P. Funk, Über Geometrien, bei denen die Geraden die Kürzesten sind (German), *Math. Ann.* **101** (1929), 226–237.
- [Hi] D. Hilbert, Über die gerade Linie als kürzeste Verbindung zweier Punkte (German), *Math. Ann.* **46** (1895), 91–96.
- [LV1] J. Lott and C. Villani, Weak curvature conditions and functional inequalities, *J. Funct. Anal.* **245** (2007), 311–333.
- [LV2] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math. (2)* **169** (2009), 903–991.
- [Oh1] S. Ohta, Uniform convexity and smoothness, and their applications in Finsler geometry, *Math. Ann.* **343** (2009), 669–699.
- [Oh2] S. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations* **36** (2009), 211–249.
- [Oh3] S. Ohta, Ricci curvature, entropy and optimal transport, to appear in *Séminaires et Congrès*.
- [Oh4] S. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature, Preprint (2012). Available at [arXiv:1203.0079](https://arxiv.org/abs/1203.0079)
- [OS1] S. Ohta and K.-T. Sturm, Heat flow on Finsler manifolds, *Comm. Pure Appl. Math.* **62** (2009), 1386–1433.
- [OS2] S. Ohta and K.-T. Sturm, Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds, Preprint (2011). Available at [arXiv:1104.5276](https://arxiv.org/abs/1104.5276)
- [Ok] T. Okada, On models of projectively flat Finsler spaces of constant negative curvature, *Tensor (N.S.)* **40** (1983), 117–124.
- [Sh1] Z. Shen, *Lectures on Finsler geometry*, World Scientific Publishing Co., Singapore, 2001.
- [Sh2] Z. Shen, *Differential geometry of spray and Finsler spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
- [Sh3] Z. Shen, *Curvature, distance and volume in Finsler geometry*, Preprint. Available at <http://www.math.iupui.edu/~zshen/>
- [St1] K.-T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* **196** (2006), 65–131.
- [St2] K.-T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* **196** (2006), 133–177.

[Vi] C. Villani, Optimal transport, old and new, Springer-Verlag, Berlin, 2009.