Corrections to "$\mathrm{CD}(K,N) \Rightarrow \mathrm{Ric}_N \geq K$"

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Abstract

We correct the proof of the implication $\mathrm{CD}(K,N) \Rightarrow \mathrm{Ric}_N \geq K$ in [7] ($N \in [n,\infty]$) and [8, 9] ($N < 0$, $N = 0$). The same correction applies to a review in [10, Chapter 18].

In [7] (see the proof of Theorem 1.2(i) in §8.2) and the following papers [8, 9], to show the implication from the curvature-dimension condition $\mathrm{CD}(K,N)$ to the lower weighted Ricci curvature bound $\mathrm{Ric}_N \geq K$, we used the Brunn–Minkowski inequality (derived from $\mathrm{CD}(K,N)$) and an asymptotic formula of the volume of the form

$$\frac{\mathsf{m}(Z_{1/2}(A_-, A_+))}{c_n \varepsilon^n} = \mathrm{e}^{-\mathcal{V}(v)} \left(1 + \frac{\mathrm{Ric}(v)}{2} r^2 \right) + O(r^3). \tag{1}$$

However, its rigorous proof was not given, and we found that (1) seems (reasonable but) technically involved. Thus, here we explain an alternative, direct proof along the lines of (5) $\Rightarrow$ (1) of [4, Theorem 7.3]. We refer to [5] for the derivation from the Brunn–Minkowski inequality to the lower weighted Ricci curvature bound (and hence to the curvature-dimension condition) on weighted Riemannian manifolds.

Let (M,F) be a (connected, smooth) Finsler manifold of dimension $n \geq 2$ and m be a smooth positive measure on M. Given $v \in T_xM \setminus \{0\}$, let $\eta\colon (-\varepsilon,\varepsilon) \longrightarrow M$ be the geodesic with $\dot\eta(0) = v$ and V be a smooth vector field on a neighborhood U of x such that $V(\eta(t)) = \dot\eta(t)$ and that every integral curve of V is geodesic. On U, we decompose m as $\mathsf{m} = \mathrm{e}^{-\psi} \mathrm{vol}_{g_V}$, where vol_{g_V} is the volume measure induced from the Riemannian metric g_V on U. Put $\psi_\eta := \psi \circ \eta$ for simplicity. Then, the weighted Ricci curvature is defined by

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + \psi''_\eta(0) - \frac{\psi'_\eta(0)^2}{N-n}$$

for $N \in (-\infty, 0] \cup (n,\infty)$, and $\mathrm{Ric}_n(v)$ and $\mathrm{Ric}_\infty(v)$ are defined as the limits. Note that, by definition, for $N \in (n,\infty)$ and $N' \in (-\infty,0)$,

$$\mathrm{Ric}_n(v) \leq \mathrm{Ric}_N(v) \leq \mathrm{Ric}_\infty(v) \leq \mathrm{Ric}_{N'}(v) \leq \mathrm{Ric}_0(v).$$

We prove that $\mathrm{CD}(K,N)$ implies $\mathrm{Ric}_N \geq K$ (i.e., $\mathrm{Ric}_N(v) \geq KF^2(v)$ for all $v \in TM$). We refer to [7] for $\mathrm{CD}(K,N)$ with $N \in [n,\infty]$, [8] for $N < 0$, and [9] for $N = 0$. A review can be found in [10, Chapter 18] as well.

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Proof. Assume on the contrary that $\mathrm{Ric}_N(v) < K - 2\varepsilon$ for some $\varepsilon > 0$ and $v \in T_x M$ with $F(v) = 1$. We divide the proof into five cases depending on N.

(i) We first consider $N \in (n, \infty)$. Take a g_v-orthonormal basis $(e_i)_{i=1}^n$ of $T_x M$ with $e_1 = v$, and a smooth function ϕ on U such that

$$\nabla\phi(x) = v, \qquad g_{\nabla\phi}\big(D_{e_i}^{\nabla\phi}[\nabla\phi], e_j\big) = -\frac{\psi_\eta'(0)}{N-n}\delta_{ij}.$$

For sufficiently small $\delta, s > 0$, we consider a probability measure

$$\mu_0 = \rho_0 \mathsf{m} := \mathsf{m}\big(B^+(x,\delta)\big)^{-1} \cdot \mathsf{m}|_{B^+(x,\delta)}$$

and a map

$$\mathbf{T}_t(z) := \exp_z\big(st\nabla\phi(z)\big), \quad t \in [0,1].$$

Then, $\mu_t := (\mathbf{T}_t)_\sharp \mu_0$ is the unique minimal geodesic from μ_0 to μ_1 with respect to the L^2-Kantorovich–Wasserstein distance (by [11, Theorem 13.5] with the help of [6, Theorem 5.1]). Setting

$$\Phi(t) := \log\Big(\det\big[\mathrm{d}\mathbf{T}_t(x)\big]\Big)$$

as in [8], we have $\Phi'(0) = s\,\mathrm{trace}(B)$, where

$$B_{ij} := g_{\nabla\phi}\big(D_{e_i}^{\nabla\phi}[\nabla\phi], e_j\big) = -\frac{\psi_\eta'(0)}{N-n}\delta_{ij}.$$

Moreover, the Riccati equation yields

$$\Phi''(0) = -\mathrm{Ric}(sv) - s^2\,\mathrm{trace}(B^2).$$

Now, we consider the functions

$$\begin{aligned} c_1(t) &:= \exp\left(\frac{\psi_\eta(0) - \psi_\eta(st)}{N-n}\right), \qquad c_2(t) := \mathrm{e}^{\Phi(t)/n}, \\ c(t) &:= c_1(t)^{(N-n)/N} \cdot c_2(t)^{n/N} = \Big(\mathrm{e}^{\psi_\eta(0)-\psi_\eta(st)} \cdot \det\big[\mathrm{d}\mathbf{T}_t(x)\big]\Big)^{1/N} \\ &= \Big(\det{}_{\mathsf{m}}\big[\mathrm{d}\mathbf{T}_t(x)\big]\Big)^{1/N}. \end{aligned}$$

It follows from

$$\frac{c_1'(0)}{c_1(0)} = -\frac{s\psi_\eta'(0)}{N-n} = \frac{s}{n}\mathrm{trace}(B) = \frac{c_2'(0)}{c_2(0)}$$

and the Riccati equation above that

$$\begin{aligned} N\frac{c''(0)}{c(0)} &= (N-n)\frac{c_1''(0)}{c_1(0)} + n\frac{c_2''(0)}{c_2(0)} - \frac{n(N-n)}{N}\left(\frac{c_1'(0)}{c_1(0)} - \frac{c_2'(0)}{c_2(0)}\right)^2 \\ &= -\mathrm{Ric}_N(sv) - s^2\,\mathrm{trace}(B^2) + \frac{(s\,\mathrm{trace}(B))^2}{n}. \end{aligned}$$

Recalling the choice of B, we find

$$\mathrm{trace}(B^2) = \frac{(\mathrm{trace}(B))^2}{n} = \frac{n\psi'_\eta(0)^2}{(N-n)^2}.$$

Therefore, we obtain

$$N\frac{c''(0)}{c(0)} = -s^2 \mathrm{Ric}_N(v) > -(K-2\varepsilon)s^2 F^2(v).$$

Let us denote c above by c_x, and set $d_z := F(s\nabla\phi(z)) = d(z, \mathbf{T}_1(z))$. Then, by continuity,

$$\frac{c_z''(t)}{c_z(t)} > -\frac{K-\varepsilon}{N} d_z^2$$

holds for every $z \in B^+(x,\delta)$ and $t \in [0,1]$ (by letting $\delta, s > 0$ smaller if necessary). This implies

$$c_z(t) \le \frac{\mathbf{s}_{(K-\varepsilon)/N}((1-t)d_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} c_z(0) + \frac{\mathbf{s}_{(K-\varepsilon)/N}(td_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} c_z(1)$$

by [3, Lemma 2.2] (or [8, Lemma 2.1]), and hence

$$\rho_t\big(\mathbf{T}_t(z)\big)^{-1/N} \le \frac{\mathbf{s}_{(K-\varepsilon)/N}((1-t)d_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_0(z)^{-1/N} + \frac{\mathbf{s}_{(K-\varepsilon)/N}(td_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_1\big(\mathbf{T}_1(z)\big)^{-1/N}$$

by the Monge–Ampère equation, where

$$\mathbf{s}_k(r) := \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}r) & k > 0, \\ r & k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}r) & k < 0. \end{cases}$$

This means that the reduced curvature-dimension condition $\mathrm{CD}^*(K,N)$ fails, and hence $\mathrm{CD}(K,N)$ also fails (see [1, Propositions 2.5, 2.8]).

(ii) $N = n$: If $\mathrm{Ric}_n(v) < K - 2\varepsilon$, then we have $\mathrm{Ric}_N(v) < K - \varepsilon$ for some $N \in (n, \infty)$. Thus, $\mathrm{CD}(K,N)$ does not hold by (i), and hence $\mathrm{CD}(K,n)$ also fails.

(iii) $N < 0$: In this case, in the same manner as (i), we have

$$\frac{c_z''(t)}{c_z(t)} < -\frac{K-\varepsilon}{N} d_z^2,$$

which yields

$$c_z(t) \ge \frac{\mathbf{s}_{(K-\varepsilon)/N}((1-t)d_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} c_z(0) + \frac{\mathbf{s}_{(K-\varepsilon)/N}(td_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} c_z(1)$$

and

$$\rho_t\big(\mathbf{T}_t(z)\big)^{-1/N} \ge \frac{\mathbf{s}_{(K-\varepsilon)/N}((1-t)d_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_0(z)^{-1/N} + \frac{\mathbf{s}_{(K-\varepsilon)/N}(td_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_1\big(\mathbf{T}_1(z)\big)^{-1/N}.$$

This means that $\mathrm{CD}^*(K,N)$ fails, and hence $\mathrm{CD}(K,N)$ also fails (see [8, Proposition 4.7] and the proof of [8, Theorem 4.10]).

(iv) $N = \infty$: If $\mathrm{Ric}_\infty(v) < K - 2\varepsilon$, then we have $\mathrm{Ric}_N(v) < K - \varepsilon$ for some $N < 0$. Thus, $\mathrm{CD}(K, N)$ does not hold by (iii), and hence $\mathrm{CD}(K, \infty)$ also fails.

(v) $N = 0$: Since $\mathrm{Ric}_N(v) \le \mathrm{Ric}_0(v)$ for all $N < 0$, we find

$$\rho_t\big(\mathbf{T}_t(z)\big) \ge \left(\frac{\mathbf{s}_{(K-\varepsilon)/N}\big((1-t)d_z\big)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_0(z)^{-1/N} + \frac{\mathbf{s}_{(K-\varepsilon)/N}(td_z)}{\mathbf{s}_{(K-\varepsilon)/N}(d_z)} \rho_1\big(\mathbf{T}_1(z)\big)^{-1/N} \right)^{-N}$$

by (iii), which implies

$$\rho_t\big(\mathbf{T}_t(z)\big) \ge \Big(\tau^{(1-t)}_{K-\varepsilon,N}(d_z)\rho_0(z)^{-1/N} + \tau^{(t)}_{K-\varepsilon,N}(d_z)\rho_1\big(\mathbf{T}_1(z)\big)^{-1/N} \Big)^{-N}$$

by [8, Proposition 4.7], where

$$\tau^{(t)}_{K,N}(r) := t^{1/N} \left(\frac{\mathbf{s}_{K/(N-1)}(tr)}{\mathbf{s}_{K/(N-1)}(r)} \right)^{(N-1)/N}.$$

Letting $N \uparrow 0$ (then $-1/N \uparrow \infty$), we obtain

$$\rho_t\big(\mathbf{T}_t(z)\big) \ge \max\left\{ \frac{\mathbf{s}_{-K+\varepsilon}\big((1-t)d_z\big)}{(1-t)\mathbf{s}_{-K+\varepsilon}(d_z)} \rho_0(z), \frac{\mathbf{s}_{-K+\varepsilon}(td_z)}{t\mathbf{s}_{-K+\varepsilon}(d_z)} \rho_1\big(\mathbf{T}_1(z)\big) \right\}.$$

Hence, $\mathrm{CD}(K, 0)$ does not hold. □

The above argument closely followed the proof of [2, Theorem 6.1], which established the equivalence between the timelike curvature-dimension condition $\mathrm{TCD}_q(K, N)$ (with an arbitrary exponent $q \in (0, 1)$) and the lower weighted Ricci curvature bound $\mathrm{Ric}_N \ge K$ in timelike directions for measured Finsler spacetimes (covering the same range $N \in (-\infty, 0] \cup [n, \infty]$ as above).

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