

Brownian motion in Minkowski normed spaces

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Abstract

A Minkowski normed space is the Euclidean space equipped with a (possibly asymmetric) uniformly convex and smooth norm, forming a particular class of Finsler manifolds. We construct a stochastic process with one-dimensional time marginal densities given by the fundamental solution to the nonlinear Finsler heat equation in Minkowski normed spaces. This process is constructed as a solution to a singular McKean–Vlasov stochastic differential equation and constitutes a nonlinear Markov process in the sense of McKean. Furthermore, we show that solutions to this stochastic differential equation are pathwise unique, and thus probabilistically strong solutions, though the equation has singular coefficients beyond the subcritical regime. Since our construction is a natural extension of the construction of standard Brownian motion from the standard heat kernel, we call this process *Brownian motion in Minkowski normed spaces*. To the best of our knowledge, this is the first construction of stochastic processes associated with nonlinear heat equation in Finslerian spaces.

Keywords: Minkowski normed space, Finsler manifold, nonlinear heat flow, McKean–Vlasov SDE, nonlinear Markov process, probabilistic representation.
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1 Introduction

1.1 Framework and goal

One of the cornerstones of stochastic analysis and linear PDE theory is the intimate relation between heat equation and Brownian motion, namely that the fundamental solutions to

$$\partial_t u = \frac{1}{2} \Delta u \tag{1.1}$$

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(i.e., the classical heat kernel $p(t, z, x)$) are the one-dimensional time marginal densities of Brownian motion; more precisely, the density of the distribution of $W_t^z := W_t + z$ is $p(t, z, \cdot)$, where W denotes standard Brownian motion. Moreover, the path laws $\{P^z\}_{z \in \mathbb{R}^d}$ of W^z form a Markov process, which is uniquely determined by its transition kernel $p(t, z, x)$. These links, allowing to pose analytic problems for (1.1) as probabilistic ones and vice versa, is fundamental, for instance in classical potential theory (see, among many other references, [8, 10, 11]), and inspired a powerful general theory of linear PDEs, stochastic processes and Markov processes. One central aspect of this theory is the construction of probabilistic counterparts for solutions u to large classes of PDEs of type

$$\partial_t u = \Delta(a(x)u) - \nabla \cdot (b(x)u)$$

for $a : \mathbb{R}^d \rightarrow [0, \infty)$ and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Typically, these counterparts are constructed as solutions X to the associated stochastic differential equation

$$dX_t = b(X_t) dt + \sqrt{2a(X_t)} dW_t$$

such that the distribution of X_t is $u(t, x) dx$ for all $t \geq 0$ (for the heat equation, one has $a = 1/2$ and $b = 0$).

Heat equation in Minkowski normed spaces. A *Minkowski normed space* $(\mathbb{R}^d, \|\cdot\|)$ is the Euclidean space $\mathbb{R}^d$ equipped with a uniformly convex, smooth and possibly asymmetric norm (i.e., $\|-x\| \neq \|x\|$ in general); see Section 2.1. Since such a norm does not necessarily arise from an inner product, the associated geometry is typically *Finslerian* rather than *Riemannian*, where the infinitesimal structures of the space (i.e., tangent spaces) are not equipped with a scalar product, but with a (possibly asymmetric) norm. In the context of Finsler manifolds, the natural *nonlinear Laplacian* Δ can be defined analogously to the Riemannian case. In the present Minkowski setting, it is explicitly given as

$$\Delta u = \sum_{i,j=1}^d \partial_i (\partial_{ij} Q^*(\nabla u) \partial_j u), \quad (1.2)$$

where Q^* is the Legendre dual of the Lagrangian $Q := \|\cdot\|^2/2$ and $\nabla u := (\partial_1 u, \dots, \partial_d u)$ ($\partial_i u$ denotes the standard partial derivative of u). When $\|\cdot\|$ is the standard Euclidean norm, (1.2) is the classical Laplacian. The notable feature is that the operator in (1.2) is generally *nonlinear* ($\Delta(u_1 + u_2) \neq \Delta u_1 + \Delta u_2$) and *inhomogeneous* ($\Delta(-u) \neq -\Delta u$) due to the non-constancy of $\partial_{ij} Q^*$ and the asymmetry of Q . Analogously to the classical heat equation, the *Finsler heat equation* is

$$\partial_t u = \frac{1}{2} \Delta u. \quad (1.3)$$

In the Minkowski case, the *Finsler heat kernel* centered at $z \in \mathbb{R}^d$ is given by

$$f_Q^z(t, x) := \frac{1}{(2t)^{d/2} C_Q} \exp\left(-\frac{Q(z-x)}{t}\right), \quad (1.4)$$

with the normalizing constant $C_Q > 0$ (see (2.5) for the precise definition of C_Q). This function solves (1.3) and is also called the *fundamental solution* to (1.3) (see [23, Example 4.3]).

Heat equation on Finslerian spaces has attracted great attention across non-linear PDEs, geometric analysis and optimal transport. From the PDE point of view, although Δ is nonlinear, it is uniformly elliptic and the existing PDE techniques apply to some extent; we refer to [12, 20, 21, 22, 23, 24, 25] for some results. However, the noncompact case has certain limitations: some basic geometric and analytic inequalities are known only under additional assumptions; see, e.g., [22]. From the optimal transport point of view, the heat equation (1.3) can be regarded as the gradient flow equation for the Dirichlet energy in the L^2 -space or for the relative entropy in the (reverse) L^2 -Wasserstein space (see [23, 26]).

A fundamental missing piece, however, lies on the probabilistic side: unlike the linear heat equation in Riemannian manifolds, where the heat equation is inseparably linked to Brownian motion and stochastic differential equations, the nonlinear Finsler heat equation has so far lacked a comparable stochastic counterpart.

Goal. A long-standing open question is whether there is a canonical notion of *Brownian motion* in this setting. More fundamentally, one may ask what the appropriate probabilistic counterpart of the nonlinear Finsler Laplacian should be.

A naïve analogy with the classical Riemannian case would suggest defining Brownian motion as the Markov process whose transition semigroup solves the Finsler heat equation. However, this analogy immediately breaks down at a fundamental level: the Finsler heat equation is nonlinear because the Finsler Laplacian itself is nonlinear while the transition semigroup of any Markov process is necessarily linear. Thus, a classical Markov process cannot have the nonlinear Finsler heat flow as its transition semigroup. This nonlinearity is one of the main obstructions preventing us from defining Brownian motion on Finsler manifolds by an analogy of the Riemannian case. Even if one leaves aside Markovianity, it is an open problem how to construct a stochastic process with one-dimensional time marginal densities given by the fundamental solution to the Finsler heat equation.

Our main goal of this paper is to provide an answer to this problem for Minkowski normed spaces by constructing a unique *nonlinear Markov process* whose one-dimensional time marginal densities are given by the fundamental solution to the Finsler heat equation (1.3), i.e., by (1.4). Our approach extends a classical construction of Brownian motion starting from the classical heat equation to the Minkowski setting, therefore, we call this process *Brownian motion in Minkowski normed spaces* (see Definition 1.2). This is, to the best of our knowledge, the first construction of Brownian motion in the Finslerian setting.

The notion of *nonlinear Markov process* used in this paper (see Definition

4.1) is based on ideas by McKean [19] and was recently introduced and studied in [27]. In contrast to classical Markov processes, the transition law may depend not only on the current state, but also on the current distribution of the process.

1.2 Main results

Setting. In this paper, we work with a Minkowski norm $\|\cdot\|$ on $\mathbb{R}^d$ with $d \geq 2$, being uniformly convex and uniformly smooth, but possibly asymmetric. More precisely, in terms of the Lagrangian $Q := \|\cdot\|^2/2 : \mathbb{R}^d \rightarrow [0, \infty)$, we impose the following conditions:

(H1) Q is positively 2-homogeneous:

$$Q(cx) = c^2 Q(x), \quad \forall c > 0, x \in \mathbb{R}^d;$$

(H2) $Q \in C^1(\mathbb{R}^d) \cap C^3(\mathbb{R}^d \setminus \{0\})$;

(H3) $\exists 0 < \lambda \leq \Lambda < \infty$ such that for every $x \in \mathbb{R}^d \setminus \{0\}$, as quadratic forms,

$$\lambda I \leq D^2 Q(x) \leq \Lambda I.$$

The McKean-Vlasov SDE and nonlinear Markov property. As our main result, we construct a nonlinear Markov process whose one-dimensional time-marginal densities are given by the Finsler heat kernel $\{f_Q^z(t, \cdot)\}_{z \in \mathbb{R}^d, t > 0}$ as in (1.4), i.e., by the fundamental solution to the Finsler heat equation (1.3). We also show that these solutions are probabilistically strong solutions. To this end, we first associate the Finsler heat equation to a distribution-dependent McKean–Vlasov SDE, and then prove the existence and uniqueness of probabilistically weak solutions to this SDE. Its coefficients are given as follows:

$$\begin{aligned} a_{ij}(x, \varphi) &:= \frac{1}{2} \partial_{ij} Q^*(\nabla \varphi(x)), \\ b_i(x, \varphi) &:= \partial_j a_{ij}(x, \varphi) = \frac{1}{2} \partial_j \left(\partial_{ij} Q^*(\nabla \varphi(x)) \right), \quad (x, \varphi) \in \mathbb{R}^d \times \mathcal{C} \end{aligned}$$

(see (2.6) for details, in particular for the domain $\mathcal{C}$ of a_{ij} and b_i). In the following theorem, we summarize the main results of this paper. We denote by $\mathcal{L}(X)$ the distribution of a random variable X ($\mathbb{R}^d$ - or path-valued; in the latter case, $\mathcal{L}(X)$ is the path law of a stochastic process X).

Theorem 1.1 (see Theorems 3.4, 4.5 and Proposition 6.1). *Let $d \geq 2$ and assume (H1)–(H3). For each $z \in \mathbb{R}^d$, the McKean–Vlasov SDE*

$$\begin{cases} dX_t = b(X_t, u(t)) dt + \sqrt{2a(X_t, u(t))} dW_t, \\ \mathcal{L}(X_t) = u(t, x) dx, \quad \forall t > 0, \quad X_0 = z \end{cases} \quad (1.5)$$

has a unique probabilistically weak solution X^z with the prescribed one-dimensional time marginals $\mathcal{L}(X_t^z) = f_Q^z(t, x) dx$ for all $t > 0$. Furthermore, the path laws $P^z := \mathcal{L}(X^z)$ of X^z form a nonlinear Markov process with one-dimensional time marginal densities $f_Q^z(t, \cdot)$, $(t, z) \in (0, \infty) \times \mathbb{R}^d$. Finally, the solutions X^z are probabilistically strong after every strictly positive time.

Outline of the proof. The proof splits into the existence and the uniqueness of solutions to the McKean–Vlasov SDE.

Existence. The proof proceeds in three steps: first, we recast the Finsler heat equation as the *nonlinear Fokker–Planck equation*

$$\partial_t u(t, x) = \partial_{ij} \left(a_{ij}(x, u(t)) u(t, x) \right) - \partial_i \left(b_i(x, u(t)) u(t, x) \right); \quad (1.6)$$

secondly, we prove that f_Q^z is a distributional solution to (1.6) (Lemma 2.7); thirdly, we construct a weak solution to (1.5) by aid of the Ambrosio–Figalli–Trevisan *superposition principle*, which allows to construct a solution to the McKean–Vlasov SDE (1.5) with one-dimensional time marginal densities given by solutions to (1.6) (see Proposition 3.2 below, [30] and also [5, 6]).

Uniqueness. To prove the uniqueness, it suffices to show the uniqueness for the *linearized Fokker–Planck equation*

$$\partial_t u(t, x) = \partial_{ij} \left(a_{ij}(x, f_Q^z(s+t)) u(t, x) \right) - \partial_i \left(b_i(x, f_Q^z(s+t)) u(t, x) \right) \quad (1.7)$$

for all $s > 0$ (see Proposition 4.3 and Lemma 4.4 for details).

To this end, we prove the strictly stronger result of pathwise uniqueness for the associated (non-distribution dependent) SDE with prescribed one-dimensional time marginals $f_Q^z(s+t, x) dx$, i.e.,

$$\begin{cases} dX_t = b(X_t, f_Q^z(s+t)) dt + \sqrt{2a(X_t, f_Q^z(s+t))} dW_t \\ \quad = \frac{1}{2} \operatorname{div} \left(D^2 Q^* (\nabla Q(z - X_t)) \right) dt + \sqrt{D^2 Q^* (\nabla Q(z - X_t))} dW_t, \\ \mathcal{L}(X_t) = f_Q^z(s+t, x) dx, \quad \forall t > 0 \end{cases} \quad (1.8)$$

(see Lemma 2.5 for the second equality).

We point out that the SDE (1.8) in $\mathbb{R}^d$ does not fall within the standard subcritical framework for the pathwise uniqueness of singular SDEs in Krylov–Röckner [17] nor within further developments [13, 14, 16, 18, 31]. Indeed, our drift part can be as singular as

$$\left| \operatorname{div} \left(D^2 Q^* (\nabla Q(z - x)) \right) \right| \sim \frac{1}{|z - x|} \quad \text{as } x \rightarrow z$$

(see Remark 2.2, Example 7.1). Thus the drift is locally in L^p only for $p < d$. In particular, it lies outside the usual subcritical regime

$$\frac{d}{p} + \frac{2}{q} < 1,$$

which requires $p > d$. Moreover, the diffusion coefficient is also singular: its gradient can be as singular as

$$\left| \nabla \sqrt{D^2 Q^* (\nabla Q(z - x))} \right| \sim \frac{1}{|z - x|} \quad \text{as } x \rightarrow z.$$

Our pathwise uniqueness proof consists of two main steps. In the first step, the equation is localized away from the singularity. On each compact annulus excluding the singular point, the coefficients are sufficiently regular, so that the standard Grönwall-type argument implies uniqueness up to the corresponding exit time. In the second step, we prove that the singular point is in fact almost surely unattainable. This is achieved by studying an appropriate radial process and deriving a Bessel-type estimate showing that the hitting probability of the singularity is zero. Since the solutions never reach the singularity, the local uniqueness argument extends to the entire time interval, and pathwise uniqueness follows.

We note that $s > 0$ is essential and that we cannot (and do not need to) extend the proof to $s = 0$ (the latter corresponds to the initial datum δ_z , but our pathwise uniqueness proof crucially depends on the absolute continuity of the initial condition; see Remark 5.5).

We emphasize that our pathwise uniqueness result yields uniqueness for the linearized PDE (1.7) in the a priori much larger class of *all* probability solutions. This is stronger than what is needed to apply Lemma 4.4 from [27], for which it suffices to prove uniqueness only within the class of solutions that are dominated by f_Q^z .

The fact that the corresponding unique solution path laws constitute a nonlinear Markov process then follows as a consequence of this uniqueness result.

Finally, using the Yamada–Watanabe theorem, we obtain that the solution X^z to the McKean–Vlasov SDE (1.5) is the unique probabilistically *strong* solution after any strictly positive time $s > 0$, i.e., $(X_t^z)_{t \geq s}$ is adapted to the canonical filtration of $W_{s+t} - W_s$, where W denotes the driving Brownian motion.

Conclusion. We construct a new family of stochastic processes X^z as a natural probabilistic counterpart to the Finsler heat kernel associated with a large class of smooth uniformly convex Minkowski norms $\|\cdot\| = \sqrt{2Q(\cdot)}$ on $\mathbb{R}^d$. These processes are constructed as the unique weak solutions to the McKean–Vlasov stochastic differential equation (1.5) with Finsler heat kernel as prescribed one-dimensional time marginal densities, form a uniquely determined nonlinear Markov process and are actually probabilistically strong solutions after any strictly positive time. Our construction generalizes the construction of standard Brownian motion as the unique Markov process with one-dimensional time marginal densities given by the classical heat kernel. Indeed, for the special case $Q(\cdot) = |\cdot|^2/2$, where $|\cdot|$ denotes the usual Euclidean norm, (1.5) becomes $dX_t = dW_t$, and X^z from Theorem 1.1 becomes $X^z = W + z$ for the standard Brownian motion W . Motivated by this close analogy, we propose the following definition.

Definition 1.2 (see Definition 4.7). We call X^z from Theorem 4.5 *Brownian motion starting at z in the Minkowski normed space $(\mathbb{R}^d, \|\cdot\|)$* .

Literature comparison. In a series of recent papers, nonlinear Markov processes with one-dimensional time marginal densities given by (fundamental) solutions to nonlinear PDEs have been constructed for the porous medium [27], 2D vorticity Euler [28] and Navier–Stokes [7], p -Laplace [4], and Leibenson equations [3]. In each case, the nonlinear Markov process consists of unique solution path laws to a McKean–Vlasov stochastic differential equation associated with the given nonlinear PDE. The techniques used in the present work are substantially different from those in these papers, where the construction of nonlinear Markov processes with prescribed one-dimensional time marginal densities relied on analytic proofs of uniqueness results for associated linear PDEs, instead of the stronger probabilistic pathwise uniqueness result beyond the subcritical regime developed in the present work.

Organization. In Section 2, we review some properties of Q , and show that f_Q^z solves the nonlinear Fokker–Planck equation. Section 3 introduces the associated McKean–Vlasov stochastic differential equation and our first main result (Theorem 3.4), while Section 4 recalls the notion of nonlinear Markov process and contains our second main result (Theorem 4.5) as well as our definition of Brownian motion in Minkowski normed spaces (Definition 4.7). In Section 5, we prove Theorem 4.5, which in particular amounts to proving the pathwise uniqueness result of Proposition 5.3. In Section 6, we prove that the weak solutions constructed in Theorem 3.4 are strong after any strictly positive time. Finally, in Section 7, we present some examples of Minkowski normed spaces to which our results apply.

Notation. We denote by $S^{d-1} \subset \mathbb{R}^d$ the unit sphere in $\mathbb{R}^d$, by I the $(d \times d)$ -identity matrix, and by A^T the transpose of a $d \times d$ -matrix A . For $x, y \in \mathbb{R}^d$, we denote by $|x|$ and $x \cdot y$ the Euclidean norm and inner product, respectively. For a differentiable function f on $\mathbb{R}^d$, we write $\nabla f := (\partial_1 f, \dots, \partial_d f)$ (instead of the Finsler notion of gradient vector).

We write δ_z for the Dirac measure at $z \in \mathbb{R}^d$ and $\mathcal{P}$ for the set of Borel probability measures on $\mathbb{R}^d$. As already used in the introduction, we denote by $\mathcal{L}(X)$ the distribution of an $\mathbb{R}^d$ -valued or path-valued random variable, i.e., in the latter case, $\mathcal{L}(X)$ denotes the path law of a stochastic process X with paths in $C([0, \infty); \mathbb{R}^d)$.

For an open set $U \subset \mathbb{R}^d$ and $k \in \mathbb{N} \cup \{\infty\}$, we denote by $C^k(U)$ ($C_c^k(U)$, respectively) the usual spaces of k -times continuously differentiable functions $\varphi : U \rightarrow \mathbb{R}$ (of compact support, respectively), and by $C^k(U; \mathbb{R}^d)$ the space of k -times continuously differentiable vector fields $\varphi : U \rightarrow \mathbb{R}^d$. We write $D^2\varphi$ for the Hessian of a twice differentiable function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$. For a measure space $(\Omega, \mathcal{F}, \mu)$ and $k \in \mathbb{N}$, we write $L^k(\Omega; \mu)$ for the usual space of measurable functions $g : \Omega \rightarrow \mathbb{R}$ with $\int_\Omega |g|^k d\mu < \infty$, and $L_{\text{loc}}^p(\Omega; \mu)$ for the corresponding spaces of *locally* integrable functions.

Throughout, we use Einstein summation convention.

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2 Finsler heat equation as a nonlinear Fokker–Planck equation

2.1 Minkowski normed spaces

Let $d \geq 2$ and assume throughout that $Q : \mathbb{R}^d \rightarrow [0, \infty)$ satisfies the following properties (the same as already listed in the introduction).

(H1) Q is positively 2-homogeneous:

$$Q(cx) = c^2 Q(x), \quad \forall c > 0, x \in \mathbb{R}^d;$$

(H2) $Q \in C^1(\mathbb{R}^d) \cap C^3(\mathbb{R}^d \setminus \{0\})$;

(H3) $\exists 0 < \lambda \leq \Lambda < \infty$ such that for every $x \in \mathbb{R}^d \setminus \{0\}$, as quadratic forms,

$$\lambda I \leq D^2 Q(x) \leq \Lambda I.$$

Under (H1)–(H3), $\|\cdot\| := \sqrt{2Q}$ provides a uniformly convex and smooth Minkowski norm (see, e.g., [21]). We call $(\mathbb{R}^d, \|\cdot\|)$ a *Minkowski normed space*. Regarding (H3), the existence of an upper constant Λ follows from the C^2 -regularity of Q ; thus the existence of a positive lower bound λ (i.e., the uniform convexity of Q) is the actual assumption in (H3). See Section 7 for some examples. Note that the positive definite matrix $D^2 Q$ gives the fundamental tensor in Finsler geometry (see [2, 21]) and coincides with the canonical inner product when $\|\cdot\|$ is the standard Euclidean norm. Since $D^2 Q$ is positively 0-homogeneous ($D^2 Q(cx) = D^2 Q(x)$ for $c > 0$) as seen in the next lemma, we can equivalently assume (H3) only for all $x \in S^{d-1}$.

Lemma 2.1 (Properties of Q). *Under (H1)–(H3), the following hold.*

(i) $D^2 Q$ is positively 0-homogeneous on $\mathbb{R}^d \setminus \{0\}$, and A defined as

$$A(x) := (D^2 Q(x))^{-1} \quad (x \neq 0), \quad A(0) := I,$$

is bounded, Borel measurable, symmetric and uniformly elliptic:

$$\frac{1}{\Lambda} I \leq A(x) \leq \frac{1}{\lambda} I \quad \forall x \in \mathbb{R}^d \setminus \{0\};$$

(ii) $x \cdot \nabla Q(x) = 2Q(x)$ for all $x \in \mathbb{R}^d$;

(iii) We have

$$A(x)\nabla Q(x) = x, \quad \forall x \in \mathbb{R}^d \setminus \{0\}; \quad (2.1)$$

(iv) For $B = (B_1, \dots, B_d)$ with $B_i(x) := \partial_j A_{ij}(x)$, $x \in \mathbb{R}^d \setminus \{0\}$, there exists $C > 0$ such that

$$|B(x)| \leq \frac{C}{|x|} \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (2.2)$$

Proof. The symmetry and measurability of A are obvious. Since Q is positively 2-homogeneous, D^2Q is positively 0-homogeneous on $\mathbb{R}^d \setminus \{0\}$. Combined with (H3), this gives the claimed uniform ellipticity.

By differentiating the 2-homogeneity $Q(cx) = c^2Q(x)$ in the radial parameter c at $c = 1$, we have

$$x \cdot \nabla Q(x) = 2Q(x), \quad x \neq 0,$$

and this identity extends to $x = 0$ by continuity. Differentiation then yields

$$D^2Q(x)x = \nabla Q(x), \quad x \neq 0,$$

which implies (2.1). Finally, since $A \in C^1(\mathbb{R}^d \setminus \{0\})$ (by $Q \in C^3(\mathbb{R}^d \setminus \{0\})$) and is positively 0-homogeneous on $\mathbb{R}^d \setminus \{0\}$, its first derivatives are positively (-1) -homogeneous on $\mathbb{R}^d \setminus \{0\}$. Hence, $|\nabla A(x)| \leq C|x|^{-1}$ for some $C > 0$, which gives (2.2). $\square$

Remark 2.2. In the case of the standard Euclidean norm, since $D^2Q(x)$ is independent of x , one has $B \equiv 0$. In general, however, B does not vanish (see Example 7.1(a) for an example) and then, by the (-1) -homogeneity, it is as singular as $|B(x)| \sim 1/|x|$.

Denote by Q^* the Legendre dual of Q , i.e.,

$$Q^*(\xi) := \sup_{x \in \mathbb{R}^d} \{\xi \cdot x - Q(x)\}, \quad \xi \in \mathbb{R}^d.$$

2.2 Finsler heat equation

With Q and $\|\cdot\|$ as above, the *Finsler heat equation* on $(\mathbb{R}^d, \|\cdot\|)$ is given as in (1.2)–(1.3):

$$\begin{aligned} \partial_t u &= \frac{1}{2} \Delta u \\ &= \frac{1}{2} \partial_i (\partial_i Q^*(\nabla u)) \\ &= \frac{1}{2} \partial_i (\partial_{ij} Q^*(\nabla u) \partial_j u). \end{aligned} \quad (2.3)$$

The final equality holds, since the 1-homogeneity of $\partial_i Q^*$ and (ii) of Lemma 2.1 imply

$$\partial_i Q^*(\xi) = \xi_j \partial_{ij} Q^*(\xi), \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}. \quad (2.4)$$

We refer to [21, 23] for the Finsler heat equation on general (measured) Finsler manifolds (without the coefficient $1/2$).

The fundamental solution to (2.3) (with initial point source $z \in \mathbb{R}^d$) is the *Finsler heat kernel*

$$f_Q^z(t, x) := \frac{1}{(2t)^{d/2} |B_Q| \Gamma(1 + \frac{d}{2})} \exp\left(-\frac{Q(z-x)}{t}\right), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad (2.5)$$

with $B_Q := \{x \in \mathbb{R}^d \mid Q(x) < 1/2\}$, which has finite Lebesgue measure $|B_Q| < \infty$ (see [23, Example 4.3]). We collect some properties of f_Q^z used in the sequel.

Lemma 2.3 (Properties of f_Q^z).

(i) $f_Q^z(t) := f_Q^z(t, \cdot)$ is a probability density for all $t > 0$, $t \mapsto f_Q^z(t, x) dx$ is weakly continuous on $(0, \infty)$ in the sense of measures, and $f_Q^z(t, x) dx \rightarrow \delta_z$ weakly as $t \rightarrow 0$.

(ii) $f_Q^z(t) \in C^1(\mathbb{R}^d)$ and $\nabla f_Q^z(t, x) = \frac{1}{t} f_Q^z(t, x) \nabla Q(z-x)$ for all $t > 0$.

(iii) Let $\varphi \in C_c^1(\mathbb{R}^d)$ and $t > 0$. Then

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) f_Q^z(t, x) dx - \varphi(z) \\ &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \nabla Q^*(\nabla f_Q^z(r, x)) \cdot \nabla \varphi(x) dx dr \\ &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_{ij} Q^*(\nabla f_Q^z(r, x)) \partial_j f_Q^z(r, x) \partial_i \varphi(x) dx dr. \end{aligned}$$

Proof. (i) $\int_{\mathbb{R}^d} f_Q^z(t, x) dx = 1$ for all $t > 0$ follows from a straightforward calculation, while the claimed weak continuity on $(0, \infty)$ follows for instance from the continuity of f_Q^z in (t, x) . The final claim about weak convergence as $t \rightarrow 0$ can be proven analogously to the proof that a standard normal distribution with covariance tI converges to δ_0 in the sense of distributions as $t \rightarrow 0$.

(ii) Both claims follow directly from the definition of f_Q^z .

(iii) The first identity follows from the fact that f_Q^z is a weak solution to (1.3), and the second one from (2.4). $\square$

2.3 Associated nonlinear Fokker–Planck equation

We introduce coefficients $a = (a_{ij})_{1 \leq i, j \leq d}$, $b = (b_1, \dots, b_d)$, defined on $\mathbb{R}^d \times \mathfrak{C}$ (the latter is defined below),

$$\begin{aligned} a_{ij}(x, \varphi) &:= \frac{1}{2} \partial_{ij} Q^*(\nabla \varphi(x)), \\ b_i(x, \varphi) &:= \partial_j a_{ij}(x, \varphi) = \frac{1}{2} \partial_j \left(\partial_{ij} Q^*(\nabla \varphi(x)) \right), \end{aligned} \quad (2.6)$$

with $a_{ij}(x, \varphi) := \frac{1}{2}\delta_{ij}$ and $b_i(x, \varphi) := 0$ for (x, φ) with $\nabla\varphi(x) = 0$ (this is necessary since $\partial_i Q^*$ is not differentiable at 0). Here,

$$\mathfrak{C} := \{\varphi \in C^1(\mathbb{R}^d) \mid \nabla\varphi \in C^1(\mathbb{R}^d \setminus O_{\nabla\varphi}; \mathbb{R}^d)\},$$

and we define the closed set $O_{\nabla\varphi} := \{x \in \mathbb{R}^d \mid \nabla\varphi(x) = 0\}$. Since $a(x, \varphi)$ is symmetric, we have $b(x, \varphi) = \operatorname{div} a(x, \varphi)$.

Remark 2.4. $f_Q^z(t) \in \mathfrak{C}$ for all $z \in \mathbb{R}^d$ and $t > 0$. This follows from Lemma 2.3(ii), $O_{\nabla Q} = \{0\}$, and $\nabla Q \in C^2(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$.

The next lemma shows that, for $\varphi = f_Q^z(t)$, the coefficients in (2.6) take a particular appealing form independent of $t > 0$.

Lemma 2.5 (Properties of a_{ij}, b_i). *Let $1 \leq i, j \leq d$, $z \in \mathbb{R}^d$ and $t > 0$. Then*

$$a_{ij}(x, f_Q^z(t)) = \frac{1}{2}\partial_{ij}Q^*(\nabla Q(z-x)), \quad \forall x \in \mathbb{R}^d \setminus \{z\},$$

and $a_{ij}(z, f_Q^z(t)) = \frac{1}{2}\delta_{ij}$, i.e.,

$$a_{ij}(x, f_Q^z(t)) = \frac{1}{2}A_{ij}(z-x), \quad b_i(x, f_Q^z(t)) = -\frac{1}{2}B_i(z-x), \quad (2.7)$$

where A and B are as in Lemma 2.1. Moreover, $(t, x) \mapsto a_{ij}(x, f_Q^z(t))$ is bounded on $(0, \infty) \times \mathbb{R}^d$.

Proof. Since $\partial_{ij}Q^*$ is 0-homogeneous, for $x \neq z$, we find

$$\begin{aligned} a_{ij}(x, f_Q^z(t)) &= \frac{1}{2}\partial_{ij}Q^*(\nabla f_Q^z(t, x)) = \frac{1}{2}\partial_{ij}Q^*\left(\frac{1}{t}f_Q^z(t, x)\nabla Q(z-x)\right) \\ &= \frac{1}{2}\partial_{ij}Q^*(\nabla Q(z-x)). \end{aligned}$$

For $x = z$, $\nabla f_Q^z(t, z) = 0$; thus, by definition, $a_{ij}(z, f_Q^z(t)) = \frac{1}{2}\delta_{ij}$ as claimed. Then (2.7) follows from the 2-homogeneity of Q (which implies $D^2Q^*(\nabla Q(y)) = (D^2Q(y))^{-1}$ for all $y \neq 0$), and Lemma 2.1(i) implies the boundedness of $a_{ij}(x, f_Q^z(t))$. $\square$

We now associate the Finsler heat equation (2.3) with the *nonlinear Fokker–Planck equation*

$$\partial_t u(t, x) = \partial_{ij}\left(a_{ij}(x, u(t))u(t, x)\right) - \partial_i\left(b_i(x, u(t))u(t, x)\right) \quad (2.8)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$, with a and b as above.

Definition 2.6 (Solutions to the nonlinear Fokker–Planck equation). Let $z \in \mathbb{R}^d$. We say that $u : (0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$ is a *probability solution* to (2.8) with initial datum z if $u(t) := u(t, \cdot) \in \mathfrak{C}$ for all $t > 0$ and

- (i) $u(t, x) dx \in \mathcal{P}$ for all $t > 0$, $t \mapsto u(t, x) dx$ is weakly continuous on $(0, \infty)$, and $u(t, x) dx \rightarrow \delta_z$ weakly as $t \rightarrow 0$;
- (ii) For each $T > 0$, the maps $(t, x) \mapsto a_{ij}(x, u(t))$ and $(t, x) \mapsto b_i(x, u(t))$ belong to $L^1([0, T] \times \mathbb{R}^d; u(t, x) dx dt)$;
- (iii) For all $\varphi \in C_c^2(\mathbb{R}^d)$ and $t > 0$,

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx - \varphi(z) \\ &= \int_0^t \int_{\mathbb{R}^d} \left(a_{ij}(x, u(r)) \partial_{ij} \varphi(x) + b_i(x, u(r)) \partial_i \varphi(x) \right) u(r, x) dx dr. \end{aligned}$$

This definition is consistent with the definition of solutions to general nonlinear Fokker–Planck equations, for instance as in [27].

Formally, the Finsler heat equation (2.3) and the nonlinear Fokker–Planck equation (2.8) are equivalent, as a straightforward integration by parts on the RHS of (2.3) with a test function φ shows. However, rigorously proving this equivalence is delicate. For our purpose, the following result is sufficient.

Lemma 2.7. *For any $z \in \mathbb{R}^d$, f_Q^z is a probability solution to (2.8) with initial datum z .*

Proof. Thanks to Lemma 2.3(i) and Remark 2.4, it remains to prove (ii) and (iii) of Definition 2.6 for $u = f_Q^z$. First, note that the Borel measurability of $a_{ij}(x, f_Q^z(t))$ and $b_i(x, f_Q^z(t))$ in (t, x) follows from Lemma 2.5. Secondly, since $(t, x) \mapsto a_{ij}(x, f_Q^z(t))$ is bounded on $(0, \infty) \times \mathbb{R}^d$ and $\int_0^T \int_{\mathbb{R}^d} f_Q^z(t, x) dx dt = T$ for each $T > 0$, $(t, x) \mapsto a_{ij}(x, f_Q^z(t))$ belongs to $L^1([0, T] \times \mathbb{R}^d; f_Q^z(t, x) dx dt)$. Moreover, Lemma 2.1(iv) yields $|b_i(x, f_Q^z(t))| \leq C|z - x|^{-1}$ for all $x \neq z$ and $t > 0$, and hence the map $(t, x) \mapsto b_i(x, f_Q^z(t))$ belongs to $L^1([0, T] \times \mathbb{R}^d; f_Q^z(t, x) dx dt)$ as well, since in dimension $d \geq 2$, we have

$$\int_{\mathbb{R}^d} |z - x|^{-1} f_Q^z(t, x) dx \leq C t^{-\frac{1}{2}},$$

where $C > 0$ depends on Q , but not on $t > 0$. Finally, let $\varphi \in C_c^2(\mathbb{R}^d)$ and $t > 0$. Then, using Lemma 2.3(iii) and integration by parts, we obtain by the definition of a_{ij} and b_i ,

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) f_Q^z(t, x) dx - \varphi(z) \\ &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_{ij} Q^*(\nabla f_Q^z(r, x)) \partial_j f_Q^z(r, x) \partial_i \varphi(x) dx dr \\ &= \int_0^t \int_{\mathbb{R}^d} \left(a_{ij}(x, f_Q^z(r)) \partial_{ij} \varphi(x) + b_i(x, f_Q^z(r)) \partial_i \varphi(x) \right) f_Q^z(r, x) dx dr. \end{aligned}$$

Thus, f_Q^z satisfies (iii) of Definition 2.6, which completes the proof. $\square$

2.4 Linearized equation

Fix $s \geq 0$ and consider the *linear* Fokker–Planck equation

$$\partial_t u(t, x) = \partial_{ij} \left(a_{ij}(x, f_Q^z(s+t)) u(t, x) \right) - \partial_i \left(b_i(x, f_Q^z(s+t)) u(t, x) \right) \quad (2.9)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$, obtained by a priori fixing $f_Q^z(s+t)$ in the second arguments of a_{ij} and b_i . The time shift by $s \geq 0$ will become relevant later on. Equation (2.9) is a linear Fokker–Planck equation with time-dependent coefficients

$$a_{ij}^{s,z}(t, x) := a_{ij}(x, f_Q^z(s+t)), \quad b_i^{s,z}(t, x) := b_i(x, f_Q^z(s+t)). \quad (2.10)$$

This kind of linearization has been studied on general Finsler manifolds in [24] to develop a nonlinear analog to the Γ -calculus technique (see also [20, 21, 22, 25] for further applications).

Definition 2.8 (Solutions to the linear Fokker–Planck equation). Let $z \in \mathbb{R}^d$ and $s \geq 0$. We say that $u : (0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$ is a *probability solution* to (2.9) with initial datum $\zeta \in \mathcal{P}$ if

- (i) $u(t, x) dx \in \mathcal{P}$ for all $t > 0$, $t \mapsto u(t, x) dx$ is weakly continuous on $(0, \infty)$, and $u(t, x) dx \rightarrow \zeta$ weakly as $t \rightarrow 0$;
- (ii) The maps $(t, x) \mapsto a_{ij}(x, f_Q^z(s+t))$ and $(t, x) \mapsto b_i(x, f_Q^z(s+t))$ belong to $L^1([0, T] \times \mathbb{R}^d; u(t, x) dx dt)$ for each $T > 0$;
- (iii) For all $\varphi \in C_c^2(\mathbb{R}^d)$ and $t > 0$,

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x) u(t, x) dx - \int_{\mathbb{R}^d} \varphi(x) d\zeta(x) \\ &= \int_0^t \int_{\mathbb{R}^d} \left(a_{ij}(x, f_Q^z(s+r)) \partial_{ij} \varphi(x) + b_i(x, f_Q^z(s+r)) \partial_i \varphi(x) \right) u(r, x) dx dr. \end{aligned}$$

We make the following immediate, yet important observation.

Lemma 2.9. *Let $z \in \mathbb{R}^d$ and $s \geq 0$. Then $u(t, x) = f_Q^z(s+t, x)$ is a probability solution to (2.9) with initial datum $\zeta = f_Q^z(s, x) dx$ (here, with slight abuse of notation, we write $f_Q^z(0, x) dx = \delta_z$ if $s = 0$).*

Proof. This follows directly from Lemma 2.7. $\square$

In general, (2.9) may have many solutions which are not related to f_Q^z at all. The previous lemma states that, in any case, one particular solution is given by $u(t, x) = f_Q^z(s+t, x)$.

3 The associated McKean–Vlasov equation

In this section, we study the following McKean–Vlasov SDE:

$$\begin{cases} dX_t = b(X_t, u(t)) dt + \sqrt{2a(X_t, u(t))} dW_t \\ \quad = \frac{1}{2} \operatorname{div} \left(D^2 Q^* (\nabla u(t, X_t)) \right) dt + \sqrt{D^2 Q^* (\nabla u(t, X_t))} dW_t, \\ \mathcal{L}(X_t) = u(t, x) dx, \quad \forall t > 0, \end{cases} \quad (3.1)$$

where a, b are as in (2.6), $W = (W_t)_{t \geq 0}$ is the standard d -dimensional Brownian motion, and $\mathcal{L}(X_t)$ denotes the distribution of X_t .

This SDE is the natural corresponding probabilistic model for the Finsler heat equation, more precisely for its associated nonlinear Fokker–Planck formulation (2.8). Indeed, it is readily seen from Itô’s formula that the curve of one-dimensional time marginals $\mu_t = \mathcal{L}(X_t)$ of any solution X to (3.1) solves (2.8). The goal of this section is the substantially more delicate reverse implication, i.e., that for a given solution u to (2.8) consisting of probability densities $u(t)$, there exists a solution to (3.1) with $\mathcal{L}(X_t) = u(t, x) dx$ for all $t \geq 0$. More precisely, we will construct such an SDE-solution with one-marginal densities $u(t) = f_Q^z(t)$.

Definition 3.1 (Solutions to the McKean–Vlasov SDE).

- (i) A *(probabilistically weak) solution* to (3.1) is a quadruple consisting of a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, an $(\mathcal{F}_t)$ -standard Brownian motion $W = (W_t)_{t \geq 0}$, a family of probability densities $u(t) \in \mathfrak{C}$, $t > 0$, and an $(\mathcal{F}_t)$ -adapted stochastic process $X = (X_t)_{t \geq 0}$ on Ω such that $\mathcal{L}(X_t) = u(t, x) dx$ for all $t > 0$,

$$\mathbb{E} \left[\int_0^T \left(|b(X_t, \mathcal{L}(X_t))| + |a(X_t, \mathcal{L}(X_t))| \right) dt \right] < \infty, \quad \forall T > 0$$

(where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$), and, $\mathbb{P}$ -a.s.,

$$X_t - X_0 = \int_0^t b(X_r, \mathcal{L}(X_r)) dr + \int_0^t \sqrt{2a(X_r, \mathcal{L}(X_r))} dW_r, \quad \forall t \geq 0.$$

Below, we usually simply refer to X instead of the entire quadruple as a solution.

- (ii) For each solution X , we call its path law $P = \mathcal{L}(X)$ (a probability measure on $C([0, \infty); \mathbb{R}^d)$) a *solution path law* to (3.1).
- (iii) We say that solutions to (3.1) with initial datum $\zeta \in \mathcal{P}$ are *weakly unique* if $\mathcal{L}(X) = \mathcal{L}(Y)$ for any two solutions X, Y with $\mathcal{L}(X_0) = \mathcal{L}(Y_0) = \zeta$.

If we prescribe u in (3.1) as $u(t, x) dx = f_Q^z(t, x) dx$, then by Lemma 2.5, (3.1) turns into

$$\begin{cases} dX_t = \frac{1}{2} \operatorname{div} \left(D^2 Q^* (\nabla Q(z - X_t)) \right) dt + \sqrt{D^2 Q^* (\nabla Q(z - X_t))} dW_t, \\ \mathcal{L}(X_t) = f_Q^z(t, x) dx, \quad \forall t > 0. \end{cases} \quad (3.2)$$

The notion of solution to (3.2) is that of probabilistically weak solution X to the SDE with coefficients $b(x) = \frac{1}{2} \operatorname{div}(D^2 Q^* (\nabla Q(z - x)))$ and $\sigma(x) = \sqrt{D^2 Q^* (\nabla Q(z - x))}$ in the usual sense, satisfying in addition the marginal condition $\mathcal{L}(X_t) = f_Q^z(t, x) dx$.

As our first main result (Theorem 3.4), we construct weak solutions to (3.1) with marginal densities $u(t, x) = f_Q^z(t, x)$. The proof relies on the following celebrated Ambrosio–Figalli–Trevisan *superposition principle* (see [30, Theorem 2.5]).

Proposition 3.2 (Superposition principle). *Let $a^0 = (a_{ij}^0)_{1 \leq i, j \leq d}$ and $b^0 = (b_1^0, \dots, b_d^0)$ consist of Borel measurable coefficients*

$$a_{ij}^0, b_i^0 : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that a^0 is pointwise symmetric and non-negative definite. Suppose that $u : (0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$ is a probability solution to the linear Fokker–Planck equation

$$\partial_t u(t, x) = \partial_{ij} (a_{ij}^0(t, x) u(t, x)) - \partial_i (b_i^0(t, x) u(t, x)) \quad (3.3)$$

with a probability measure-valued initial datum ζ , satisfying

$$a_{ij}^0, b_i^0 \in L^1([0, T] \times \mathbb{R}^d; u(t, x) dx dt), \quad \forall T > 0. \quad (3.4)$$

Then there exists a probabilistically weak solution $X = (X_t)_{t \geq 0}$ to the SDE

$$dX_t = b^0(t, X_t) dt + \sigma^0(t, X_t) dW_t,$$

with $\sigma^0 : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ such that $\sigma^0(\sigma^0)^T = 2a^0$, $\mathcal{L}(X_t) = u(t, x) dx$ for all $t > 0$, and $\mathcal{L}(X_0) = \zeta$.

Remark 3.3. We stress that the previous proposition only applies to solutions u to the linear Fokker–Planck equation (3.3) which satisfy the global in space integrability condition (3.4) (or a slightly more general, but still global condition; see [9]). To define solutions to (3.3) most generally, it suffices to require

$$a_{ij}^0, b_i^0 \in L_{\text{loc}}^1([0, \infty) \times \mathbb{R}^d; u(t, x) dx dt),$$

but for such merely locally integrable solutions, the previous result does not apply in general. Our definition of probability solutions to the linear Fokker–Planck equation (2.9) (see Definition 2.8) includes this global condition, and we explicitly verify it for $u = f_Q^z$ in the proof of Lemma 2.7. Hence, in the proof of Theorem 3.4 below, we may indeed apply this superposition principle to f_Q^z .

The following theorem is our first main result.

Theorem 3.4 (Existence). *Let $d \geq 2$, $z \in \mathbb{R}^d$. Then there exists a probabilistically weak solution $X^z = (X_t^z)_{t \geq 0}$ to the McKean–Vlasov equation (3.1) with $\mathcal{L}(X_t^z) = f_Q^z(t, x) dx$ for all $t > 0$ and $X_0^z = z$.*

Proof. By Lemma 2.9 with $s = 0$, f_Q^z is a probability solution to (2.9) with initial datum δ_z . By Proposition 3.2 applied to the linear Fokker–Planck equation with coefficients $a_{ij}^{0,z}, b_i^{0,z}$ from (2.10), there exists a weak solution $X^z = (X_t^z)_{t \geq 0}$ to (3.2). The integrability condition (3.4) has been already shown in the proof of Lemma 2.7. Hence X^z also solves (3.1) with $\mathcal{L}(X_t^z) = f_Q^z(t, x) dx$, as claimed. $\square$

4 Construction of Brownian motion in Minkowski normed spaces

We turn to our main result (Theorem 4.5), stating that the solutions X^z to the McKean–Vlasov equation (3.1) constructed in Theorem 3.4 are weakly unique conditioned on their one-dimensional time marginal densities being $f_Q^z(t)$, and that these solutions form a nonlinear Markov process.

4.1 Nonlinear Markov processes and cores

We recall some basic notions and key results from [1, 27]. We refer to these papers for more details on nonlinear Markov processes and cores.

Below we use the following canonical model-notation: $\mathcal{C} := C([0, \infty); \mathbb{R}^d)$ is the space of continuous paths from $[0, \infty)$ to $\mathbb{R}^d$ endowed with the topology of locally uniform convergence, $\pi_t : \mathcal{C} \rightarrow \mathbb{R}^d$ is the evaluation map $\pi_t(w) := w(t)$, $\Pi_t : \mathcal{C} \rightarrow \mathcal{C}$ is the shift $\Pi_t(w) := w(t + \cdot)$, and $\mathcal{F}_s := \sigma(\pi_r, 0 \leq r \leq s)$ denotes the corresponding canonical filtration. Then the Borel σ -algebra on $\mathcal{C}$ is given by $\mathcal{B}(\mathcal{C}) = \sigma(\pi_r, r \geq 0)$ and we write $\mathcal{P}(\mathcal{C})$ for the set of Borel probability measures on $\mathcal{C}$. For $P \in \mathcal{P}(\mathcal{C})$, we denote by $P_t \in \mathcal{P}$ the image measure (push-forward) of P under π_t , i.e., $P_t = P \circ \pi_t^{-1}$.

Definition 4.1 (Nonlinear Markov processes; see Definition 2.1 in [27]). Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A (*time-homogeneous*) *nonlinear Markov process* is a family $\{P^\zeta\}_{\zeta \in \mathcal{P}_0} \subseteq \mathcal{P}(\mathcal{C})$ such that, setting $\mu_t^\zeta := P_t^\zeta \in \mathcal{P}$ for $(t, \zeta) \in [0, \infty) \times \mathcal{P}_0$,

- (i) $\mu_0^\zeta = \zeta$ for all $\zeta \in \mathcal{P}_0$, and $\mu_t^\zeta \in \mathcal{P}_0$ for all $(t, \zeta) \in (0, \infty) \times \mathcal{P}_0$;
- (ii) the (*time-homogeneous*) *nonlinear Markov property* holds, i.e.,

$$P^\zeta(\pi_{t+s} \in A | \mathcal{F}_s)(\cdot) = P^{\mu_s^\zeta, \pi_s(\cdot)}(\pi_t \in A) \quad P^\zeta\text{-a.s.}, \quad \forall s, t \geq 0, \zeta \in \mathcal{P}_0$$

for every Borel set $A \subseteq \mathbb{R}^d$, where for $\zeta \in \mathcal{P}_0$, $\{P^{\zeta, y}\}_{y \in \mathbb{R}^d}$ is the disintegration family of P^ζ with respect to π_0 , i.e., the ζ -a.s. unique family $\{Q^y\}_{y \in \mathbb{R}^d}$ in $\mathcal{P}(\mathcal{C})$ such that $y \mapsto Q^y(W)$ is measurable and $P^\zeta(W) = \int_{\mathbb{R}^d} Q^y(W) d\zeta(y)$ for all $W \in \mathcal{B}(\mathcal{C})$.

By definition, the set of initial data $\mathcal{P}_0$ for a nonlinear Markov process is invariant under the flow of its one-dimensional time-marginals. This excludes, except for deterministic cases, the choice $\mathcal{P}_0 = \{\delta_z \mid z \in \mathbb{R}^d\}$ which, on the other hand, appears a natural choice if one wants to construct a nonlinear Markov process with one-dimensional time marginals given by the family of fundamental solutions of a nonlinear PDE. This led to the following notion of *nonlinear Markov core* from [1]. The idea is to construct the *minimal* nonlinear Markov process containing such one-dimensional time marginals.

Definition 4.2 (Nonlinear Markov cores). Let $\{\mu_t^z\}_{t \geq 0, z \in \mathbb{R}^d} \subseteq \mathcal{P}$ be such that $\mu_0^z = \delta_z$ for all $z \in \mathbb{R}^d$ and $(t, z) \mapsto \mu_t^z$ is injective from $[0, \infty) \times \mathbb{R}^d$ to $\mathcal{P}$. We call a family $\{P^z\}_{z \in \mathbb{R}^d} \subseteq \mathcal{P}(\mathcal{C})$ a (*time-homogeneous*) *nonlinear Markov core* for $\{\mu_t^z\}_{t \geq 0, z \in \mathbb{R}^d}$ if

- (i) $P_t^z = \mu_t^z$ for all $(t, z) \in [0, \infty) \times \mathbb{R}^d$;
- (ii) the family $\{P^\zeta\}_{\zeta \in \mathcal{P}_0} \subseteq \mathcal{P}(\mathcal{C})$, defined by $\mathcal{P}_0 := \{\mu_t^z\}_{t \geq 0, z \in \mathbb{R}^d}$ and

$$P^\zeta := P^z \circ \Pi_t^{-1} \quad \text{for } \zeta = \mu_t^z,$$

is a nonlinear Markov process in the sense of Definition 4.1.

We call $\{P^\zeta\}_{\zeta \in \mathcal{P}_0}$ from (ii) the nonlinear Markov process *induced by* $\{P^z\}_{z \in \mathbb{R}^d}$. If for each $(t, z) \in (0, \infty) \times \mathbb{R}^d$, μ_t^z is given by a density as $\mu_t^z = u^z(t, x) dx$, then we also call $\{P^z\}_{z \in \mathbb{R}^d}$ a nonlinear Markov core for $\{u^z(t)\}_{t > 0, z \in \mathbb{R}^d}$.

The previous definition implies that the nonlinear Markov process $\{P^\zeta\}_{\zeta \in \mathcal{P}_0}$ induced by a nonlinear Markov core $\{P^z\}_{z \in \mathbb{R}^d}$ is uniquely determined by the latter. Hence, even though a nonlinear Markov core does not satisfy Definition 4.1, it is equivalent to its induced nonlinear Markov process.

For the following results from [1] which we are going to use below, consider the following general nonlinear and linear Fokker–Planck equations and McKean–Vlasov SDE:

$$\partial_t u(t, x) = \partial_{ij} \left(a_{ij}^0(x, u(t)) u(t, x) \right) - \partial_i \left(b_i^0(x, u(t)) u(t, x) \right), \quad (4.1)$$

$$\partial_t u(t, x) = \partial_{ij} \left(a_{ij}^0(x, v(t)) u(t, x) \right) - \partial_i \left(b_i^0(x, v(t)) u(t, x) \right), \quad (4.2)$$

$$\begin{cases} dX_t = b^0(X_t, u(t)) dt + \sqrt{2a^0(X_t, u(t))} dW_t, \\ \mathcal{L}(X_t) = u(t, x) dx, \quad t > 0. \end{cases} \quad (4.3)$$

Here $a^0 = (a_{ij}^0)_{1 \leq i, j \leq d}$ takes values in the space of symmetric non-negative definite matrices, and $b^0 = (b_1^0, \dots, b_d^0)$. In (4.2), $t \mapsto v(t)$ is a curve of probability densities which is fixed a priori in the second arguments of a^0 and b^0 . Note that (2.8), (2.9) and (3.1) are special cases of (4.1), (4.2) and (4.3), respectively. For the standard definition of solutions to the above equations, see for instance [27, Definition 3.1].

The following general result from [1, Theorem 4.6] (see also [27, Theorem 3.8]) is one key ingredient for the proof of our main uniqueness result in Theorem 4.5 below. For $s \geq 0$, we write u_{s+}^z for the map $u_{s+}^z(t, x) := u^z(s + t, x)$, $(t, x) \in (0, \infty) \times \mathbb{R}^d$. Note that since (4.2) is a linear equation, its set of solutions with a prescribed initial datum is convex.

Proposition 4.3. *For each $z \in \mathbb{R}^d$, let u^z be a probability solution to the nonlinear Fokker–Planck equation (4.1) with initial datum z , such that*

- (i) $(t, z) \mapsto u^z(t, x) dx$ is injective from $[0, \infty) \times \mathbb{R}^d$ to $\mathcal{P}$ (here we write $u^z(0, x) dx := \delta_z$);
- (ii) for each $z \in \mathbb{R}^d$ and $s \in [0, \infty) \setminus \{0\}$, u_{s+}^z is an extreme point in the convex set of probability solutions to (4.2) with $u^z(s + t)$ replacing $v(t)$ and with initial datum $u^z(s, x) dx$.

Then, for each $z \in \mathbb{R}^d$, there exists a unique weak solution X^z to the McKean–Vlasov SDE (4.3) with the prescribed one-dimensional time marginals $\mathcal{L}(X_t^z) = u^z(t, x) dx$, $t > 0$, and $X_0^z = z$. The family of solution path laws $\{P^z\}_{z \in \mathbb{R}^d}$, $P^z = \mathcal{L}(X^z)$, is a nonlinear Markov core for $\{u^z(t)\}_{t > 0, z \in \mathbb{R}^d}$ and is uniquely determined by $\{u^z\}_{z \in \mathbb{R}^d}$ and (4.3).

For later use, we also recall a crucial lemma from [27, Lemma 3.5], which characterizes the extremality condition in Proposition 4.3(ii) by a restricted uniqueness condition for the associated linearized Fokker–Planck equation.

Lemma 4.4. *Let $(s, z) \in (0, \infty) \times \mathbb{R}^d$. In the setting of the previous proposition, u_{s+}^z is an extreme point in the set of probability solutions to (4.2) with initial datum $u^z(s, x) dx$ if and only if u_{s+}^z is a unique probability solution to (4.2) with initial datum $u^z(s, x) dx$ in the set*

$$\{[t \mapsto v(t, x) dx] \in C_w([0, \infty); \mathcal{P}) \mid \exists C \geq 1, v(t, x) \leq C u_{s+}^z(t, x) dx dt\text{-a.e.}\}, \quad (4.4)$$

where $C_w([0, \infty); \mathcal{P})$ denotes the set of weakly continuous paths $t \mapsto \mu_t \in \mathcal{P}$ on $[0, \infty)$.

4.2 Brownian motion in Minkowski normed spaces and nonlinear Markov property

The following theorem is our main result of this paper.

Theorem 4.5 (Uniqueness). *Let $d \geq 2$.*

- (i) *For each $z \in \mathbb{R}^d$, the solution X^z from Theorem 3.4 is the unique weak solution to (3.1) with the prescribed one-dimensional time marginals $\mathcal{L}(X_t^z) = f_Q^z(t, x) dx$, $t > 0$, and $X_0^z = z$;*
- (ii) *The family of path laws $\{P^z\}_{z \in \mathbb{R}^d} \subseteq \mathcal{P}(\mathcal{C})$, $P^z := \mathcal{L}(X^z)$, is a nonlinear Markov core for $\{f_Q^z(t)\}_{t > 0, z \in \mathbb{R}^d}$.*

Via this result, we construct a probabilistic counterpart of the Finsler heat kernel as the nonlinear Markov process consisting of the family of unique solutions to the McKean–Vlasov SDE (3.1) with Finsler heat kernel one-dimensional time marginal densities.

Remark 4.6 (Comparison with Euclidean case). In the special case $Q(x) = Q_2(x) := |x|^2/2$, where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$, this construction yields standard Brownian motion. Indeed, in this case $f_{Q_2}^z$ is the standard heat kernel

$$f_{Q_2}^z(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|z - x|^2}{2t}\right),$$

and the Finsler Laplacian becomes the standard Laplacian, i.e., (1.3) becomes the standard heat equation. Likewise, (3.1) reduces to $dX_t = dW_t$. Clearly, by imposing the initial condition $X_0 = z$, the unique solutions are simply $X_t^z := W_t + z$ for a standard Brownian motion W . Hence, in this case the nonlinear Markov core constructed in Theorem 4.5 is actually a classical Markov process, consisting of the shifted Wiener measures $\mathbb{W}^z = \mathcal{L}(W + z)$.

Having this analogy to the standard Brownian motion in the Euclidean case, we propose the following definition.

Definition 4.7 (Brownian motion in Minkowski normed spaces). Let $d \geq 2$. For $z \in \mathbb{R}^d$, we call the solution X^z to (3.1) from Theorem 4.5(i) (defined on any stochastic basis supporting a standard Brownian motion) *Brownian motion starting at z in the Minkowski normed space $(\mathbb{R}^d, \|\cdot\|)$* .

5 Proof of Theorem 4.5

All assertions of Theorem 4.5 will follow by applying Proposition 4.3 to $u^z = f_Q^z$, (2.8) and (3.1). Thus, below we verify its assumptions for this case. Injectivity of $(t, z) \mapsto f_Q^z(t, x) dx$ from $[0, \infty) \times \mathbb{R}^d$ to $\mathcal{P}$ is obvious (again, with slight abuse of notation, we write $f_Q^z(0, x) dx = \delta_z$). Hence, taking Lemma 4.4 into account, the following lemma verifies all the assumptions of Proposition 4.3, which in turn completes the proof of Theorem 4.5.

Lemma 5.1. *Let $d \geq 2, s > 0$ and $z \in \mathbb{R}^d$. Then the linear equation (2.9) has a probability solution with initial datum $\zeta = f_Q^z(s, x) dx$ in the sense of Definition 2.8, which is the unique solution in*

$$\begin{aligned} \mathcal{S}_{s,z} := \\ \{[t \mapsto u(t, x) dx] \in C_w([0, \infty); \mathcal{P}) \mid \exists C \geq 1, u(t, x) \leq C f_Q^z(s + t, x) dx dt\text{-a.e.}\}. \end{aligned} \quad (5.1)$$

By Lemma 2.9, this solution is identified as $u(t, x) = f_Q^z(s + t, x)$.

Remark 5.2. As the proof below shows (see in particular Proposition 5.3), we in fact prove a strictly stronger statement, namely uniqueness in the class of *all* probability solutions with initial datum $\zeta = f_Q^z(s, x) dx$.

Before proving the previous lemma, let us point out that while its assertion concerns uniqueness for a *linear* PDE, a topic with a long and well-developed history, its proof is not standard in the present setting because

- we claim uniqueness in the potentially large class of *distributional* solutions – and not (as would be much simpler), for instance, in the class of *weak* solutions which have weak first order spatial derivatives (cf. [21, 23]). In fact, while f_Q^z is regular, a priori this need not be true for elements in $\mathcal{S}_{s,z}$.
- our coefficients have critical/supercritical singularities as seen in Remark 2.2.

5.1 Proof of Lemma 5.1

Lemma 5.1 will follow as a consequence of the following strictly stronger statement.

Proposition 5.3 (Pathwise uniqueness). *Let $d \geq 2$, $z \in \mathbb{R}^d$ and $s > 0$. Probabilistically weak solutions to*

$$dX_t = \frac{1}{2} \operatorname{div} \left(D^2 Q^* (\nabla Q(z - X_t)) \right) dt + \sqrt{D^2 Q^* (\nabla Q(z - X_t))} dW_t \quad (5.2)$$

(that is, to the first equation in (3.2)) with initial datum $\mathcal{L}(X_0) = f_Q^z(s, x) dx$ are pathwise unique, i.e., $X_t = \tilde{X}_t$ for all $t \geq 0$ a.s. for any two probabilistically weak solutions X and $\tilde{X}$ to (5.2) with initial datum $f_Q^z(s, x) dx$ on the same stochastic basis with the same standard Brownian motion W such that $X_0 = \tilde{X}_0$ a.s.

From this result, one obtains Lemma 5.1 as follows.

Proof of Lemma 5.1. Suppose that u and $\tilde{u}$ are two probability solutions to (2.9) with initial condition $f_Q^z(s, x) dx$. Then, by the superposition principle (Proposition 3.2), there exist probabilistically weak solutions $X, \tilde{X}$ to (5.2) such that

$$\mathcal{L}(X_t) = u(t, x) dx, \quad \mathcal{L}(\tilde{X}_t) = \tilde{u}(t, x) dx, \quad \forall t \geq 0.$$

Since Proposition 5.3 in particular implies weak uniqueness for (5.2) with initial condition $f_Q^z(s, x) dx$, we obtain $\mathcal{L}(X_t) = \mathcal{L}(\tilde{X}_t)$ for all $t \geq 0$, hence $u(t, x) = \tilde{u}(t, x)$ dx -a.e. for all $t \geq 0$. $\square$

5.2 Proof of Proposition 5.3

Lemma 5.4. *Let $s > 0$ and X be a weak solution of (5.2) with initial datum $f_Q^z(s, x) dx$, defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Then, for all $T > 0$, we have*

$$\mathbb{P}(X_t \neq z \text{ for all } t \in [0, T]) = 1. \quad (5.3)$$

Proof. Fix $T > 0$ throughout the proof. We set

$$Y_t := 2Q(z - X_t), \quad t \geq 0,$$

and shall prove $Y_t > 0$ for all $t \in [0, T]$ a.s., which completes the proof.

Since $\mathcal{L}(X_0) = f_Q^z(s, x) dx$, one has $\mathbb{P}(X_0 = z) = 0$, hence $Y_0 > 0$ a.s. Fix $0 < \varepsilon < R < \infty$ and consider the $(\mathcal{F}_t)$ -stopping time

$$\tau_{\varepsilon, R} := \inf\{t \geq 0 \mid Y_t \leq \varepsilon \text{ or } Y_t \geq R\}.$$

Choose $\chi_{\varepsilon, R} \in C_c^\infty(\mathbb{R}^d \setminus \{z\})$ such that

$$\begin{aligned} 0 \leq \chi_{\varepsilon, R} \leq 1, \quad \chi_{\varepsilon, R} &\equiv 1 \text{ on } K_{\varepsilon, R} := \{x \in \mathbb{R}^d \mid \varepsilon \leq 2Q(z - x) \leq R\}, \\ \text{supp } \chi_{\varepsilon, R} &\subset \{x \in \mathbb{R}^d \mid \varepsilon/2 < 2Q(z - x) < 2R\}, \end{aligned}$$

and define $q_{\varepsilon, R}(x) := \chi_{\varepsilon, R}(x)Q(z - x) \in C_c^3(\mathbb{R}^d)$. Applying Itô's formula up to time $\tau_{\varepsilon, R}$, we may compute $q_{\varepsilon, R}(X_t)$ and $q_{\varepsilon, R}(X_t)^2$ on the set $K_{\varepsilon, R}$ on which $q_{\varepsilon, R} = Q(z - \cdot)$ and all derivatives of $q_{\varepsilon, R}$ agree with those of $Q(z - \cdot)$.

Observe that the linearized PDE (2.9), which corresponds to the SDE (5.2), can be written as

$$\partial_t u = \frac{1}{2}Lu, \quad Lu := \text{div}(A(z - \cdot)\nabla u),$$

for A given in Lemma 2.1 (for sufficiently regular u). Indeed, it follows from Lemma 2.5 that (2.9) becomes

$$\begin{aligned} \partial_t u &= \partial_{ij} \left(\frac{1}{2} A_{ij}(z - \cdot) u \right) + \partial_i \left(\frac{1}{2} \partial_j A_{ij}(z - \cdot) u \right) \\ &= \partial_i \left(\frac{1}{2} A_{ij}(z - \cdot) \partial_j u \right). \end{aligned}$$

To apply Itô's lemma, we compute $LQ(z - \cdot)$ and $L(Q(z - \cdot)^2)$. First, by (2.1) we have

$$LQ(z - \cdot) = -\text{div}(A(z - \cdot)\nabla Q(z - \cdot)) = -\text{div}(z - \cdot) = d. \quad (5.4)$$

Next,

$$\begin{aligned} L(Q(z - \cdot)^2) - 2Q(z - \cdot)LQ(z - \cdot) &= 2\nabla Q(z - \cdot) \cdot (A(z - \cdot)\nabla Q(z - \cdot)) \\ &= 2\nabla Q(z - \cdot) \cdot (z - \cdot) \\ &= 4Q(z - \cdot), \end{aligned} \quad (5.5)$$

where we used (2.1) again as well as Lemma 2.1(ii).

Claim 1. *The stopped process*

$$M_t^{\varepsilon, R} := Y_{t \wedge \tau_{\varepsilon, R}} - Y_0 - d(t \wedge \tau_{\varepsilon, R})$$

is a continuous local martingale with quadratic variation

$$[M^{\varepsilon, R}]_t = 8 \int_0^{t \wedge \tau_{\varepsilon, R}} Y_r dr.$$

Proof. Since $f_{\varepsilon,R} := 2q_{\varepsilon,R} \in C_c^\infty(\mathbb{R}^d)$, Itô's formula applied with the expression $\partial_t u = \frac{1}{2}Lu$ of the linearized PDE (2.9) shows that

$$f_{\varepsilon,R}(X_t) - f_{\varepsilon,R}(X_0) - \int_0^t \frac{1}{2} Lf_{\varepsilon,R}(X_r) dr$$

is a continuous local martingale, hence so is its stopped version at $\tau_{\varepsilon,R}$.

Now, on $\{r < \tau_{\varepsilon,R}\}$ we have $\varepsilon < Y_r < R$, so combined with (5.4), we observe

$$f_{\varepsilon,R}(X_r) = 2Q(z - X_r) = Y_r, \quad Lf_{\varepsilon,R}(X_r) = L(2Q(z - \cdot))(X_r) = 2d.$$

Thus, it follows that $M_t^{\varepsilon,R} = Y_{t \wedge \tau_{\varepsilon,R}} - Y_0 - d(t \wedge \tau_{\varepsilon,R})$ is a continuous local martingale.

The standard relation between the generator and the quadratic variation (see, e.g., [15, Proposition 4.7.11, eq (4.79)]) yields

$$[M^{\varepsilon,R}]_t = \int_0^{t \wedge \tau_{\varepsilon,R}} (L(f_{\varepsilon,R}^2) - 2f_{\varepsilon,R}Lf_{\varepsilon,R})(X_r) dr.$$

Since on $\{r < \tau_{\varepsilon,R}\}$ one has $f_{\varepsilon,R}(X_r) = Y_r$, using (5.5), we obtain

$$L(f_{\varepsilon,R}^2)(X_r) - 2f_{\varepsilon,R}(X_r)Lf_{\varepsilon,R}(X_r) = 8Y_r.$$

This completes the proof of Claim 1. $\square$

We now define the one-dimensional generator

$$(\mathcal{G}\phi)(y) := 2y\phi''(y) + d\phi'(y), \quad y > 0.$$

We set the following notation:

$$\tau_\varepsilon := \inf\{t \geq 0 \mid Y_t \leq \varepsilon\}, \quad \tau^R := \inf\{t \geq 0 \mid Y_t \geq R\},$$

so that $\tau_{\varepsilon,R} = \tau_\varepsilon \wedge \tau^R$.

Claim 2. *For every $R > 0$,*

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\tau_\varepsilon < T \wedge \tau^R) = 0.$$

Proof. We split the proof into the two cases $d > 2$ and $d = 2$.

First, when $d > 2$, we consider the function

$$s(y) := y^{1-\frac{d}{2}}, \quad y > 0.$$

Observe that $s \in C^\infty((0, \infty))$ and is strictly decreasing, and a direct computation gives $\mathcal{G}s = 0$ on $(0, \infty)$. Let

$$h_{\varepsilon,R}(y) := \frac{s(y) - s(R)}{s(\varepsilon) - s(R)}, \quad y > 0.$$

Clearly $0 \leq h_{\varepsilon,R} \leq 1$ on $[\varepsilon, R]$, $h_{\varepsilon,R}(\varepsilon) = 1$, $h_{\varepsilon,R}(R) = 0$, and $\mathcal{G}h_{\varepsilon,R} = 0$ on $(0, \infty)$. Since $h_{\varepsilon,R}$ is smooth, Itô's formula for the stopped semimartingale $Y_{t \wedge \tau_{\varepsilon,R}}$ gives

$$h_{\varepsilon,R}(Y_{t \wedge \tau_{\varepsilon,R}}) = h_{\varepsilon,R}(Y_0) + N_t + \int_0^{t \wedge \tau_{\varepsilon,R}} (\mathcal{G}h_{\varepsilon,R})(Y_r) dr = h_{\varepsilon,R}(Y_0) + N_t,$$

where N is a continuous local martingale. Hence, $h_{\varepsilon,R}(Y_{t \wedge \tau_{\varepsilon,R}})$ is a continuous local martingale. Set

$$B_{\varepsilon,R} := \{\varepsilon \leq Y_0 \leq R\} \in \mathcal{F}_0.$$

On $B_{\varepsilon,R}$, the path-continuity implies $Y_{t \wedge \tau_{\varepsilon,R}} \in [\varepsilon, R]$ for all $t \in [0, T]$, thus $0 \leq h_{\varepsilon,R}(Y_{t \wedge \tau_{\varepsilon,R}}) \leq 1$ on $B_{\varepsilon,R}$. Therefore, the process $\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_{t \wedge \tau_{\varepsilon,R}})$ is a bounded local martingale, thus a martingale. Evaluating at $t = T$ gives

$$\mathbb{E}[\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_{T \wedge \tau_{\varepsilon,R}})] = \mathbb{E}[\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_0)].$$

On the event $\{\tau_{\varepsilon} < T \wedge \tau^R\} \cap B_{\varepsilon,R}$ one has $Y_{T \wedge \tau_{\varepsilon,R}} = \varepsilon$, thus $h_{\varepsilon,R}(Y_{T \wedge \tau_{\varepsilon,R}}) = 1$. This implies

$$\mathbb{P}(\{\tau_{\varepsilon} < T \wedge \tau^R\} \cap B_{\varepsilon,R}) \leq \mathbb{E}[\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_0)].$$

Since $\{\tau_{\varepsilon} < T \wedge \tau^R\} \subset \{Y_0 < R\}$, we conclude

$$\begin{aligned} \mathbb{P}(\{\tau_{\varepsilon} < T \wedge \tau^R\}) &= \mathbb{P}(\{\tau_{\varepsilon} < T \wedge \tau^R\} \cap B_{\varepsilon,R}) + \mathbb{P}(\{\tau_{\varepsilon} < T \wedge \tau^R\} \cap B_{\varepsilon,R}^c) \\ &\leq \mathbb{E}[\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_0)] + \mathbb{P}(Y_0 < \varepsilon). \end{aligned}$$

Now, it follows from $Y_0 > 0$ a.s. that $\mathbb{P}(Y_0 < \varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$. Moreover, noting $\lim_{\varepsilon \downarrow 0} s(\varepsilon) = \infty$, we have $\mathbf{1}_{\{\varepsilon \leq y \leq R\}} h_{\varepsilon,R}(y) \rightarrow 0$ as $\varepsilon \downarrow 0$ for every $y > 0$. Combining this with $0 \leq \mathbf{1}_{\{\varepsilon \leq y \leq R\}} h_{\varepsilon,R}(y) \leq 1$, by the dominated convergence theorem, we obtain

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\mathbf{1}_{B_{\varepsilon,R}} h_{\varepsilon,R}(Y_0)] = 0.$$

This shows the claim.

Next, when $d = 2$, we set

$$s(y) := -\log y, \quad y > 0.$$

Again, $s \in C^\infty((0, \infty))$ and is strictly decreasing, $\mathcal{G}s = 0$ on $(0, \infty)$, and the function

$$h_{\varepsilon,R}(y) := \frac{s(y) - s(R)}{s(\varepsilon) - s(R)} = \frac{\log(y/R)}{\log(\varepsilon/R)}, \quad y > 0,$$

satisfies $0 \leq h_{\varepsilon,R} \leq 1$ on $[\varepsilon, R]$, $h_{\varepsilon,R}(\varepsilon) = 1$, $h_{\varepsilon,R}(R) = 0$, and $\mathcal{G}h_{\varepsilon,R} = 0$ on $(0, \infty)$. The rest of the proof is completely the same as in the case $d > 2$ above. $\square$

To conclude the proof of Lemma 5.4, for each $m \in \mathbb{N}$, define

$$\Omega_m := \left\{ \sup_{0 \leq t \leq T} Y_t < m \right\}.$$

Then $\Omega_m \uparrow \Omega$ as $m \rightarrow \infty$, because every continuous path is bounded on $[0, T]$. Note that, for each fixed $m \in \mathbb{N}$,

$$\{\tau_0 \leq T\} \cap \Omega_m \subset \bigcap_{n \in \mathbb{N}} \{\tau_{1/n} < T \wedge \tau^m\}.$$

Thus, Fatou's lemma and Claim 2 yield

$$\begin{aligned} \mathbb{P}(\{\tau_0 \leq T\} \cap \Omega_m) &\leq \mathbb{P}\left(\bigcap_{n \in \mathbb{N}} \{\tau_{1/n} < T \wedge \tau^m\}\right) \\ &\leq \liminf_{m \rightarrow \infty} \mathbb{P}(\tau_{1/n} < T \wedge \tau^m) = 0. \end{aligned}$$

Since $\Omega_m \uparrow \Omega$, we conclude

$$\mathbb{P}(\tau_0 \leq T) = \lim_{m \rightarrow \infty} \mathbb{P}(\{\tau_0 \leq T\} \cap \Omega_m) = 0.$$

Therefore, $Y_t > 0$ for all $t \in [0, T]$ a.s., as desired. $\square$

Now we are prepared to prove Proposition 5.3.

Proof of Proposition 5.3. Let X and $\tilde{X}$ be two weak solutions of (5.2) on the same stochastic basis, driven by the same Brownian motion such that $\mathcal{L}(X_0) = \mathcal{L}(\tilde{X}_0) = f_Q^z(s, x) dx$ and $X_0 = \tilde{X}_0$ a.s. By Lemma 5.4,

$$\mathbb{P}(X_t \neq z \text{ and } \tilde{X}_t \neq z \text{ for all } t \in [0, T]) = 1.$$

Fix $0 < \varepsilon < R < \infty$ and define

$$\tau_{\varepsilon, R} := \inf\{t \geq 0 \mid Y_t \leq \varepsilon, \tilde{Y}_t \leq \varepsilon, Y_t \geq R, \text{ or } \tilde{Y}_t \geq R\},$$

where $Y_t := 2Q(z - X_t)$ and $\tilde{Y}_t := 2Q(z - \tilde{X}_t)$. On the compact annulus

$$K_{\varepsilon, R} := \{x \in \mathbb{R}^d \mid \varepsilon \leq 2Q(z - x) \leq R\},$$

the coefficients of the SDE (5.2) are smooth, hence Lipschitz continuous. Applying Itô's formula to $|X_{t \wedge \tau_{\varepsilon, R}} - \tilde{X}_{t \wedge \tau_{\varepsilon, R}}|^2$, taking expectation, and using this Lipschitz continuity gives

$$\mathbb{E}|X_{t \wedge \tau_{\varepsilon, R}} - \tilde{X}_{t \wedge \tau_{\varepsilon, R}}|^2 \leq C_{\varepsilon, R} \int_0^t \mathbb{E}|X_{r \wedge \tau_{\varepsilon, R}} - \tilde{X}_{r \wedge \tau_{\varepsilon, R}}|^2 dr$$

for some $C_{\varepsilon, R} > 0$. Since $X_0 = \tilde{X}_0$ a.s., Grönwall's lemma implies

$$\mathbb{E}|X_{t \wedge \tau_{\varepsilon, R}} - \tilde{X}_{t \wedge \tau_{\varepsilon, R}}|^2 = 0, \quad \forall 0 \leq t \leq T,$$

so that

$$X_{t \wedge \tau_{\varepsilon, R}} = \tilde{X}_{t \wedge \tau_{\varepsilon, R}} \quad \text{a.s. for every } t \in [0, T].$$

Because both Y and $\tilde{Y}$ are strictly positive on $[0, T]$ and have continuous paths, we have $T \wedge \tau_{\varepsilon, R} \uparrow T$ a.s. as $\varepsilon \downarrow 0$ and $R \uparrow \infty$. This yields $X_t = \tilde{X}_t$ a.s. for every $t \in [0, T]$, therefore, the path continuity of X and $\tilde{X}$ shows $X = \tilde{X}$ on $[0, T]$ a.s. $\square$

Remark 5.5. We stress that the above proof of Lemma 5.4 crucially relies on $s > 0$, more precisely, on the initial condition $f_Q^z(s, x) dx$ being absolutely continuous with respect to the Lebesgue measure. This shows that, in the condition (ii) of Proposition 4.3, it is essential that one need not consider $s = 0$. Notably, nevertheless, in Theorem 4.5 we obtain weak uniqueness of solutions to (3.1) with Dirac initial conditions.

6 Brownian motion in Minkowski normed spaces as probabilistically strong solutions

Let us point out an additional consequence of Proposition 5.3. Given $(s, z) \in (0, \infty) \times \mathbb{R}^d$, Proposition 5.3 yields pathwise uniqueness for (5.2) with initial condition $f_Q^z(s, x) dx$. The existence of a probabilistically weak solution to (5.2) with this initial condition follows from Theorem 3.4 (indeed, consider $(X_{s+t}^z)_{t \geq 0}$, where X^z denotes the solution obtained in Theorem 3.4). Thus, the Yamada–Watanabe theorem implies that Brownian motion on a Minkowski normed space is adapted to the filtration of its driving Brownian motion from any strictly positive time $s > 0$. More precisely, we have the following result.

Proposition 6.1. *Let $d \geq 2$, $z \in \mathbb{R}^d$ and $s > 0$. Solutions to (3.1) with $u(t, x)$ replaced by $f_Q^z(s+t, x)$ and with initial datum $f_Q^z(s, x) dx$ are probabilistically strong and pathwise unique.*

In particular, if $(X_t^z)_{t \geq 0}$ is a Brownian motion in the Minkowski normed space under investigation from z as in Definition 4.7 on a stochastic basis, solving (3.1) with a standard Brownian motion W , then $(X_{s+t}^z)_{t \geq 0}$ is adapted to the canonical filtration of $(W_{s+t} - W_s)_{t \geq 0}$.

7 Examples of Minkowski normed spaces

Finally, we give several examples of Minkowski norms that are not induced from inner products. Recall that $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$. We first discuss asymmetric norms arising from fundamental examples of irreversible Finsler metrics.

Example 7.1 (Asymmetric norms).

- (a) *Randers metrics* provide an important class of irreversible Finsler structures (see [2, 21]). In our setting, it is given as, for fixed $a \in \mathbb{R}^d \setminus \{0\}$ with $|a| < 1$,

$$\|x\| := |x| + a \cdot x.$$

The unit sphere $\{x \in \mathbb{R}^d \mid \|x\| = 1\}$ is an ellipsoid whose center is not the origin, and $\|-a\| \neq \|a\|$. In this case, one can explicitly calculate A and B as in Lemma 2.1 as follows (see [21, Lemma 6.14]):

$$\begin{aligned} A_{ij}(x) &= \frac{|x|}{\|x\|} \delta_{ij} + \frac{a \cdot x + |a|^2 |x|}{\|x\|^3} x_i x_j - \frac{|x|}{\|x\|^2} (a_i x_j + a_j x_i), \\ B_i(x) &= \frac{d+1}{\|x\|^3} ((a \cdot x) x_i + |a|^2 |x| x_i - \|x\| |x| a_i), \end{aligned}$$

for $x \in \mathbb{R}^d \setminus \{0\}$. Thus, for i with $a_i \neq 0$, we have $B_i(x) \neq 0$ when $x_i = 0$.

- (b) More generally, for $a \in \mathbb{R}^d \setminus \{0\}$ with $|a| < 1$ and a positive C^∞ -function $\phi : (-1, 1) \rightarrow (0, \infty)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$$

for some $b \in (0, 1)$ and all $s \in [-b, b]$, the associated (α, β) -metric is defined by

$$\|x\| := |x| \cdot \phi\left(\frac{a \cdot x}{|x|}\right) \quad (x \neq 0), \quad \|0\| := 0$$

(see, e.g., [29]). Note that choosing $\phi(s) = 1 + s$ recovers the Randers metric for a .

Next, we consider symmetric norms.

Example 7.2 (Symmetric norms).

- (a) For $d \geq 2$ and $\varepsilon \in \mathbb{R}$, define

$$\|x\| := \left(|x|^2 + \varepsilon \frac{x_1^4 + \cdots + x_d^4}{|x|^2} \right)^{1/2} \quad (x \neq 0), \quad \|0\| := 0.$$

When $|\varepsilon| > 0$ is sufficiently small, $Q_\varepsilon(x) := \|x\|^2/2$ satisfies our hypotheses (H1)–(H3). Indeed, since $Q_\varepsilon(x) = O(|x|^2)$ and $\nabla Q_\varepsilon(x) = O(|x|)$ near 0, $\nabla Q_\varepsilon(0) = 0$ and $Q_\varepsilon \in C^1(\mathbb{R}^d)$. Moreover, by $D^2 Q_0 = I$ and the continuity of $(\varepsilon, \theta) \mapsto D^2 Q_\varepsilon(\theta)$ for $\theta \in S^{d-1}$, when $|\varepsilon|$ is sufficiently small, we find $\lambda, \Lambda > 0$ such that

$$\lambda I \leq D^2 Q_\varepsilon(\theta) \leq \Lambda I \quad \forall \theta \in S^{d-1}.$$

To see that $\|\cdot\|$ does not come from an inner product, let e_1, e_2 be the first two basis vectors. Then $\|e_1\|^2 = \|e_2\|^2 = 1 + \varepsilon$ and $\|e_1 + e_2\|^2 = \|e_1 - e_2\|^2 = 2 + \varepsilon$. Therefore, the parallelogram identity fails (and hence $\|\cdot\|$ is not induced by an inner product) unless $\varepsilon = 0$:

$$2\|e_1\|^2 + 2\|e_2\|^2 - \|e_1 + e_2\|^2 - \|e_1 - e_2\|^2 = 2\varepsilon \neq 0.$$

- (b) A norm $\|\cdot\|$ is determined by its unit sphere $S := \{x \in \mathbb{R}^d \mid \|x\| = 1\}$; then conditions (H1)–(H3) are met when S is C^3 and positively curved. For example, in the polar coordinates (r, θ) of $\mathbb{R}^2$, given $\varepsilon \in (0, 1/3)$, set

$$S := \{(r, \theta) \mid r = \rho(\theta)\}, \quad \rho(\theta) := \sqrt{1 + \varepsilon \cos 2\theta}.$$

Then, the associated norm is

$$\|(r, \theta)\| = \frac{r}{\rho(\theta)} \quad (r > 0), \quad \|0\| = 0.$$

To show that this norm is uniformly convex, observe that for a function of the form $Q(r, \theta) = r^2 q(\theta)$ with $q > 0$, the Hessian in the orthonormal polar frame (e_r, e_θ) is

$$D^2 Q = \begin{pmatrix} 2q & rq' \\ rq' & r^2(q'' + 2q) \end{pmatrix}.$$

Hence, $D^2 Q$ is positive definite on $\mathbb{R}^2 \setminus \{0\}$ if and only if

$$2q(q'' + 2q) - (q')^2 > 0.$$

For $q(\theta) = 1/(1 + \varepsilon \cos 2\theta)$, a direct computation gives

$$2q(q'' + 2q) - (q')^2 = \frac{4(1 + 3\varepsilon^2 + 4\varepsilon \cos 2\theta)}{(1 + \varepsilon \cos 2\theta)^4} \geq \frac{4(1 + 3\varepsilon^2 - 4\varepsilon)}{(1 + \varepsilon \cos 2\theta)^4},$$

which is positive provided $\varepsilon \in (0, 1/3)$. The failure of the parallelogram identity can be seen in the same way as (a):

$$\begin{aligned} 2\|e_1\|^2 + 2\|e_2\|^2 - \|e_1 + e_2\|^2 - \|e_1 - e_2\|^2 &= \frac{2}{1 + \varepsilon} + \frac{2}{1 - \varepsilon} - 4 \\ &= \frac{4\varepsilon^2}{1 - \varepsilon^2} \neq 0. \end{aligned}$$

- (c) One can generalize the construction in (b) to

$$\|x\| = |x| \left(1 + \varepsilon \psi \left(\frac{x}{|x|} \right) \right) \quad (x \neq 0), \quad \|0\| = 0,$$

where $\psi \in C^\infty(S^{d-1})$ is even and nonconstant. When $|\varepsilon|$ is sufficiently small, this norm satisfies (H1)–(H3).

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