# Branching Brownian motions in random environment

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# 1. Introduction.

## Continuous time Galton-Watson Processes.

▷ $T$: splitting time of a particle

$$\mathbb{P}(T > t) = e^{-ct}$$

▷ $\{p_n\}_{n=1}^{\infty}$, $0 \le p_n \le 1$, $p_1 \ne 1$, $\sum_{n=1}^{\infty} p_n = 1$: offspring distribution

▷ $m = \sum_{n=1}^{\infty} n p_n$: **expected offspring number**

▷ $\overline{N}_t$: **total population size at time** $t$

**Fact.** **(i)** $\mathbb{E}\left[\overline{N}_t\right] = e^{c(m-1)t}$

**(ii)** $\overline{M}_t := e^{-c(m-1)t}\overline{N}_t$ **is a positive martingale**

**Theorem ($L \log L$ condition [KS66-1, KS66-2], [AN72]).**

**If** $\sum_{n=1}^{\infty}(n \log n) p_n < \infty$, **then**

$$\lim_{t\to\infty} e^{-c(m-1)t}\overline{N}_t \in (0,\infty) \qquad \textbf{a.s.}$$

## Branching Brownian motions (BBMs).

▷ $N_t(A)$: population size on a set $A \subset \mathbb{R}^d$ at time $t$

▷ $\overline{N}_t := N_t(\mathbb{R}^d)$: total population size at time $t$

**Theorem (Diffusivity [S. Watanabe]).**

If $\sum_{n=1}^{\infty} n^2 p_n < \infty$, then

$$\lim_{t\to\infty} \frac{N_t(\sqrt{t}D)}{\overline{N}_t} = \frac{1}{(2\pi)^{d/2}} \int_D \exp\left(-\frac{|x|^2}{2}\right) \mathrm{d}x \quad \text{a.s.}$$

for any bounded domain $D \subset \mathbb{R}^d$

## BBMs in random environment (BBMsRE).

○ **(Time-space) random environment**

**Purpose.**

**(i) To introduce a model of BBMsRE**

**(ii) To study slow growth and localization property**

**Related models.**

◦ **Discrete time setting.**

**(i) [SW69], [AK71-1, AK71-2]: Branching processes in RE**

**(ii) [Y08], [HY09]: Branching random walks in RE**

◦ **Continuous time setting.**

**(iii) [K73]: Branching processes in RE**

**(iv) [E08]: Branching Brownian motions in RE**

## 2. Model.

▷ $\eta$: **Poisson random measure on** $\mathbb{R}_+ \times \mathbb{R}^d$ $(\mathbb{R}_+ := [0, \infty))$:

- $\eta(\mathrm{d}t\, \mathrm{d}x)$: $\mathbb{Z}_+$**-valued measure on** $\mathbb{R}_+ \times \mathbb{R}^d$

- $\eta(A_1), \eta(A_2), \cdots, \eta(A_n)$ **are independent for any disjoint and bounded sets** $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$

- $Q(\eta(A) = k) = \exp(-|A|) \dfrac{|A|^k}{k!}, \; k = 0, 1, 2, \cdots$

▷ $\mathrm{M} = \left(\{B_t\}_{t \geq 0}, P\right)$: **BM on $\mathbb{R}^d$ starting from the origin**

○ **The idea of the following formulation comes from [CY05]:**

▷ $U(x)$: **closed ball centered at $x \in \mathbb{R}^d$ with unit volume**

▷ $V_t := \left\{(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \,\middle|\, s \in (0, t],\ x \in U(B_s)\right\}$

⇝ $\eta(V_t)$: **the number of Poisson points "hit" by the Brownian particle**

▷ $\mathbb{P}^\eta$: **law of a BBM on** $\mathbb{R}^d$ **with branching rate** $\alpha\eta$ $(\alpha > 0)$

- **At time** $t = 0$, **a Brownian particle starts from the origin**
- **At time** $T$, **this particle splits into two Brownian particles, where**

$$\boxed{\mathbb{P}^\eta (T > t) = E\left[\exp\left(-\alpha\eta(V_t)\right)\right]}$$

- **These offspring reproduce independently in a similar way**

$$\boxed{\mathbb{P}(\mathrm{d}\omega\, \mathrm{d}\eta) := Q(\mathrm{d}\eta)\mathbb{P}^\eta(\mathrm{d}\omega)}$$

## 3. Results.

### 3.1. Expected total population size

▷ $N_t(A)$: population size on a set $A \subset \mathbb{R}^d$ at time $t$

▷ $\overline{N}_t := N_t(\mathbb{R}^d)$: total population size at time $t$

$$e^{\beta} := 2 - e^{-\alpha}, \quad \lambda := e^{\beta} - 1$$

**Lemma.**

$$\mathbb{E}^{\eta}\left[\overline{N}_t\right] = E\left[e^{\beta\eta(V_t)}\right], \quad \mathbb{E}\left[\overline{N}_t\right] = e^{\lambda t}$$

**Fact.** $\overline{M}_t := e^{-\lambda t}\overline{N}_t$ **is a $\mathbb{P}$-martingale and** $\mathbb{E}\left[\overline{M}_t\right] \equiv 1$

▷ $\overline{M}_\infty := \lim_{t\to\infty} \overline{M}_t \quad \mathbb{P}$**-a.s.**

## 3.2. Regular growth and diffusivity.

▷ $\rho_t(\mathrm{d}x) := \dfrac{N_t(\mathrm{d}x)}{\overline{N}_t}$**: population density at time** $t$

▷ $\rho(x) = \dfrac{1}{(2\pi)^{d/2}} \exp\left(-\dfrac{|x|^2}{2}\right)$

**Theorem 1 (Regular growth and diffusivity).**

Assume $d \geq 3$ and

$$E\left[\exp\left(\lambda^2 \int_0^\infty \left| U\left(B_t^1\right) \cap U\left(B_t^2\right) \right| \, \mathrm{d}t\right)\right] < \infty \quad (\star)$$

for independent BMs $\{B_t^1\}_{t\geq 0}$ and $\{B_t^2\}_{t\geq 0}$. Then

(i) $\mathbb{P}\left(\overline{M}_\infty \in (0,\infty)\right) = 1$

(ii) $\lim_{t\to\infty} \int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right) \rho_t(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\rho(x)\,\mathrm{d}x$

in $\mathbb{P}$-probability, $\forall f \in C_b(\mathbb{R}^d)$.

**Remark.** **(i)** $(\star)$ **is equivalent to one of the following:**

**(a)** $\sup_{t>0} \mathbb{E}\left[\overline{M}_t^2\right] < \infty$;

**(b)** $E\left[\exp\left(\frac{\lambda^2}{2}\int_0^\infty |U(0) \cap U(B_t)|\,\mathrm{d}t\right)\right] < \infty$;

**(c) (Gaugeability [C02], [T02])**

$$\inf\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x \,\middle|\, u \in C_0^\infty(\mathbb{R}^d),\right.$$
$$\left.\frac{\lambda^2}{2}\int_{\mathbb{R}^d} u(x)^2\,|U(0) \cap U(x)|\,\mathrm{d}x = 1\right\} > 1.$$

**(ii)** $(\star)$ **does not hold for** $d = 1$ **and** $2$.

## 3.2. Slow growth and localization.

**Theorem 2 (Slow growth).**

$\exists \beta(d) \geq 0$ s.t. $\mathbb{P}\left(\overline{M}_\infty = 0\right) = 1$ holds for any $\beta > \beta(d)$.

Moreover,

$$\boxed{\limsup_{t \to \infty} \frac{\log \overline{M}_t}{t} < -c(\beta) \quad \mathbb{P}\text{-a.s.}}$$

for some positive constant $c(\beta) > 0$.

**Note**: Regular growth $\Longrightarrow \lim_{t \to \infty} \frac{\log \overline{M}_t}{t} = 0$

**Remark.** (i) $\beta(d) > 0$ for any $d \geq 3$

(ii) $\beta(1) = \beta(2) = 0$ by [B08, B09]

▷ $\overline{\rho}_t := \sup_{x \in \mathbb{R}^d} \rho_t(U(x))$: density at the most populated site

**Theorem 3 (Localization).**

For any $\beta > \beta(d)$,

$$\limsup_{t \to \infty} \overline{\rho}_t > c_1(\beta) \quad \mathbb{P}\text{-a.s.}$$

for some non-random positive constant $c_1(\beta) \in (0, 1)$.

## 4. Replica overlap.

▷ $R_t := \int_{\mathbb{R}^d} \rho_t(U(x))^2 \, \mathrm{d}x$: replica overlap

$\Longrightarrow \exists c_2 = c_2(d) \in (0,1)$ s.t. $\boxed{c_2 \overline{\rho}_t^2 \leq R_t \leq \overline{\rho}_t}$

**Theorem 4.**

$$\{\overline{M}_\infty = 0\} = \left\{ \int_0^\infty R_t \, \mathrm{d}t = \infty \right\} \quad \mathbb{P}\text{-a.s.}$$

Furthermore, if $\mathbb{P}\left(\overline{M}_\infty = 0\right) = 1$, then

$$-c_3 \log \overline{M}_t \leq \int_0^t R_s \, \mathrm{d}s \leq -c_4 \log \overline{M}_t \quad \text{for all large } t$$

**Theorem 2 + Theorem 4 $\Longrightarrow$ Theorem 3**

**Proof of Theorem 3.**

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t R_s \,\mathrm{d}s \overset{\text{Theorem 4}}{\geq} -c_3 \limsup_{t\to\infty} \frac{\log \overline{M}_t}{t} \overset{\text{Theorem 2}}{>} c_1(\beta)$$

$$\Longrightarrow \quad \limsup_{t\to\infty} \overline{\rho}_t \geq \limsup_{t\to\infty} R_t > c_1(\beta) \qquad \square$$

## 5. Proof of Theorem 4.

▷ $\overline{M}_{t-} := \lim_{s \to t-0} \overline{M}_s$

▷ $\Delta \overline{M}_t := \overline{M}_t - \overline{M}_{t-}$

▷ $[\overline{M}]_t := \overline{M}_0^2 + \sum_{\substack{0<s\le t \\ \Delta \overline{M}_s \neq 0}} (\Delta \overline{M}_s)^2$: **quadratic variation**

**By Ito's formula applied to $-\log \overline{M}_t$,**

$$-\log \overline{M}_t \asymp -\int_0^t \frac{1}{\overline{M}_{s-}} \mathrm{d}\overline{M}_s + \int_0^t \frac{1}{\overline{M}_{s-}^2} \mathrm{d}[\overline{M}]_s$$

$$\boxed{\overline{M}_\infty > 0 \iff \int_0^\infty \frac{1}{\overline{M}_{t-}^2}\,\mathrm{d}[\overline{M}]_t < \infty}$$

▷ $\langle \overline{M} \rangle_t$: predictable quadratic variation

**Fact ([HWY 92]).**

(i) $$\int_0^\infty \frac{1}{\overline{M}_{t-}^2}\,\mathrm{d}[\overline{M}]_t < \infty \iff \int_0^\infty \frac{1}{\overline{M}_{t}^2}\,\mathrm{d}\langle \overline{M} \rangle_t < \infty$$

(ii) If $\displaystyle\int_0^\infty \frac{1}{\overline{M}_{t}^2}\,\mathrm{d}\langle \overline{M} \rangle_t = \infty$, then

$$\int_0^t \frac{1}{\overline{M}_{s-}^2}\,\mathrm{d}[\overline{M}]_s \sim \int_0^t \frac{1}{\overline{M}_{s}^2}\,\mathrm{d}\langle \overline{M} \rangle_s \quad \text{as } t \to \infty$$

**Proposition.**

(i) $$\int_0^t \frac{1}{\overline{\overline{M}}_s^2}\,\mathrm{d}\langle \overline{M}\rangle_s = \lambda^2 \int_0^t R_s\,\mathrm{d}s + (\lambda^2 - \lambda)\int_0^t \frac{1}{\overline{\overline{N}}_s}\,\mathrm{d}s$$

(ii) $$\int_0^\infty \frac{1}{\overline{\overline{N}}_t}\,\mathrm{d}t < \infty$$

- **Assume first** $\int_0^\infty R_t\,\mathrm{d}t < \infty$

$$\overset{\text{Prop.}}{\Longleftrightarrow} \int_0^\infty \frac{1}{\overline{\overline{M}}_t^2}\,\mathrm{d}\langle \overline{M}\rangle_t < \infty \overset{\text{Fact (i)}}{\Longleftrightarrow} \int_0^\infty \frac{1}{\overline{\overline{M}}_{t-}^2}\,\mathrm{d}[\overline{M}]_t < \infty$$

- **Assume next** $\int_0^\infty R_t \,\mathrm{d}t = \infty$

$$\lambda^2 \int_0^t R_s \,\mathrm{d}s \overset{\text{Prop.}}{\sim} \int_0^t \frac{1}{\overline{M}_s^2} \,\mathrm{d}\langle \overline{M} \rangle_s$$

$$\overset{\text{Fact (ii)}}{\sim} \int_0^t \frac{1}{\overline{M}_{s-}^2} \,\mathrm{d}[\overline{M}]_s$$

$$\Longrightarrow \; -\log \overline{M}_t \asymp \int_0^t \frac{1}{\overline{M}_{s-}^2} \,\mathrm{d}[\overline{M}]_s \asymp \int_0^t R_s \,\mathrm{d}s \qquad \square$$