Upper rate functions of Brownian motion type

for symmetric jump processes

塩沢 裕一 (大阪大学)

Jian Wang (Fujian Normal University)

日本数学会 2018 年度年会

東京大学

2018 年 3 月 18 日

1. Introduction

$*$ Range of symm. jump proc. with finite second moment

$\triangleright$ $X = (\{X_t\}_{t\geq 0}, P)$: symmetric Lévy process on $\mathbb{R}$,

$$X_0 = 0, \text{ a.s., } \underline{E[X_1^2] < \infty}$$

$$\boxed{\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{2t\log\log t}} = E[X_1^2]^{1/2} \quad \text{a.s.}}$$

[Gnedenko ('43), J.G.Wang ('93), Sato ('01)]

- Brownian motion
- rel. stable proc. $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ $(\alpha \in (0,2), m > 0)$

$$\boxed{\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{2t\log\log t}} = E[X_1^2]^{1/2} \quad \text{a.s.}}$$

▷ $R_\varepsilon(t) := \sqrt{(2+\varepsilon)E[X_1^2]t\log\log t} \quad (\varepsilon > 0)$

$\Rightarrow P\,(\exists T > 0$ s.t. $|X_t| \leq R_\varepsilon(t)$ for all $t \geq T) = 1 \ (\forall \varepsilon > 0)$

$R_\varepsilon(t)$: upper rate function/upper radius

- Kolmogorov test for BM [Itô-McKean ('74)]
- in terms of the distribution function [Sirao ('53)]

Q. How about non-Lévy processes?

2. Result

▷ $J(x,y)$: nonneg. symm. measurable funct. on $\mathbb{R}^d \times \mathbb{R}^d$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 J(x,y)\, \mathrm{d}x\mathrm{d}y$$

Assumption.

(i) $\exists \alpha_1, \alpha_2$ $(0 < \alpha_1 \le \alpha_2 < 2)$, $\exists \kappa_1, \kappa_2$ $(0 < \kappa_1 \le \kappa_2)$ **s.t.**

$$\frac{\kappa_1}{|x-y|^{d+\alpha_1}} \le J(x,y) \le \frac{\kappa_2}{|x-y|^{d+\alpha_2}} \quad (0 < |x-y| < 1)$$

(ii) $\exists \varepsilon > 0$, $\exists c > 0$ **s.t.**

$$J(x,y) \le \frac{c}{|x-y|^{d+2+\varepsilon}} \quad (|x-y| \ge 1)$$

Remark. Assumption $\Rightarrow$ $\boxed{\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^2 J(x,y)\,\mathrm{d}y<\infty}$

▷ $C_0^{\mathrm{lip}}(\mathbb{R}^d)$: **totality of Lip. conti. functions with cpt support**

$$\mathcal{E}(u,u)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))^2 J(x,y)\,\mathrm{d}x\mathrm{d}y$$

▷ $\mathcal{F}:=\overline{C_0^{\mathrm{lip}}(\mathbb{R}^d)}^{\sqrt{\mathcal{E}(\cdot,\cdot)+\|\cdot\|^2_{L^2(\mathbb{R}^d)}}}$

$\Rightarrow$ $(\mathcal{E},\mathcal{F})$: **regular Dirichlet form on** $L^2(\mathbb{R}^d)$

$\rightsquigarrow \mathbb{M}=\left(\{X_t\}_{t\geq 0},\{P_x\}_{x\in\mathbb{R}^d\setminus\mathcal{N}}\right)$: **symmetric Hunt process of pure jump type**

▷ $\mathcal{N}\subset\mathbb{R}^d$: **properly exceptional Borel set**

$\triangleright\ \psi(t) := \sqrt{t \log\log t}$

Theorem. **Under Assumption,**

(i) $\exists C_1 > 0$ **s.t.** $\forall x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$P_x\left(\exists T > 0 \text{ s.t. } |X_t - x| \le C_1 \psi(t) \text{ for all } t \ge T\right) = 1$$

(ii) $\exists C_2 > 0$ **s.t.** $\forall x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$P_x\left(\exists T > 0 \text{ s.t. } |X_t - x| \le C_2 \psi(t) \text{ for all } t \ge T\right) = 0$$

Prove (i): $\exists C > 0, \exists C_1 > 0, \exists \varepsilon > 0$ **s.t.** $\forall k \ge 1$: **large**

$$P_x(\exists s \in [2^k, 2^{k+1}] \text{ s.t. } |X_s - x| > C_1 \psi(s)) \le \frac{C}{k^{1+\varepsilon}}$$

- **Heat kernel:** $P_x(X_t \in A) = \int_A \exists p(t,x,y)\,\mathrm{d}y$

[Barlow-Bass-Chen-Kassmann ('09), Chen-Kim-Kumagai ('11)]

[Carlen-Kusuoka-Stroock ('87)]

Theorem. **Under Assumption,** $\exists c_i > 0, \exists \theta_0 > 0$ **s.t.** $\forall t$: **large,**

$$p(t,x,y) \le \begin{cases} \dfrac{c_1}{t^{d/2}}, & t \ge |x-y|^2 \\ \dfrac{c_2}{t^{d/2}} e^{-c_3|x-y|^2/t}, & \dfrac{\theta_0 |x-y|^2}{\log(1+|x-y|)} \le t \le |x-y|^2 \\ \dfrac{c_4 t}{|x-y|^{d+2+\varepsilon}}, & t \le \dfrac{\theta_0 |x-y|^2}{\log(1+|x-y|)} \end{cases}$$