Escape rate of the Brownian motions on hyperbolic spaces

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Workshop on Dirichlet Forms and Stochastic Analysis

Kansai University
November 12, 2016

1. Introduction

▷ $\mathbb{H}^d$: d-dim. hyperbolic space $(d \geq 2)$

$$\left(\mathrm{d}s^2 = \mathrm{d}r^2 + (\sinh r)^2\,\mathrm{d}\theta^2\right)$$

▷ $\mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{H}^d}\right)$: BM generated by $\Delta_{\mathbb{H}^d}/2$

| Purpose | To discuss the upper/lower rate functions for $\mathbb{M}$

- Upper rate function | how far the particle can go
- Lower rate function | how fast the particle goes to infinity

▷ $\rho(x) := d(o, x)$ ($o \in \mathbb{H}^d$: fixed)

<u>Definition.</u>

(i) $R(t)$ **is an** **upper rate function** **for** $\mathbb{M}$ $\iff$

$$P\left(\exists T > 0 \text{ s.t. } \rho(X_t) \leq R(t) \text{ for all } t \geq T\right) = 1$$

(ii) $r(t)$ **is a** **lower rate function** **for** $\mathbb{M}$ $\iff$

$$P\left(\exists T > 0 \text{ s.t. } \rho(X_t) \geq r(t) \text{ for all } t \geq T\right) = 1$$

▷ $(\{B_t\}_{t\geq 0}, P)$: **Brownian motion on** $\mathbb{R}^d$, $B_0 = 0$ **a.s.**

Kolmogorov's test **(e.g., see Itô-McKean)**

▷ $R(t) = \sqrt{t}g(t)$ **(**$g(t) \nearrow \infty$ **as** $t \to \infty$**)**

$$(\mathrm{U}) \qquad \int_{\cdot}^{\infty} g(t)^d \exp\left(-\frac{g(t)^2}{2}\right)\frac{\mathrm{d}t}{t} < \infty \ (\text{or} = \infty)$$

$$\Longrightarrow P\,(\exists T > 0 \text{ s.t. } |B_t| \leq R(t) \text{ for all } t \geq T) = 1 \ (\text{or } 0)$$

Example.

▷ $R(t) = \sqrt{(2+\varepsilon)t\log\log t} \ \left(\Rightarrow g(t) = \sqrt{(2+\varepsilon)\log\log t}\right)$

$$\boxed{(\mathrm{U}) \iff \varepsilon > 0}$$

Dvoretzky-Erdős' test ('51) $[d \geq 3]$

▷ $r(t) = \sqrt{t}h(t)$ $\quad (0 < h(t) \searrow 0$ as $t \to \infty)$

$$\text{(L)} \qquad \int_{\cdot}^{\infty} h(t)^{d-2}\frac{\mathrm{d}t}{t} < \infty \ (\text{or} = \infty)$$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |B_t| \geq r(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$

Example.

▷ $r(t) = \sqrt{t}/(\log t)^{\frac{1+\varepsilon}{d-2}}$ $\quad \left(\Longrightarrow h(t) = 1/(\log t)^{\frac{1+\varepsilon}{d-2}}\right)$

$$\boxed{\text{(L)} \iff \varepsilon > 0}$$

■ Upper rate functions for symmetric diffusion processes

∘ **Volume growth rate of the underlying measure**

∘ **Coefficient growth/degeneracy rate**

Takeda ('89), Grigor'yan ('99), Grigor'yan-Hsu ('08), Hsu-Qin ('10), Ouyang ('16)

▷ $(\mathcal{E}, \mathcal{F})$: **strongly local regular Dirichlet form on** $L^2(X; m)$

$\Rightarrow \mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in X}\right)$: m-**symm. diffusion proc.**

$$\mathcal{E}(u,u) = \frac{1}{2}\int_X \mathrm{d}\mu_{\langle u\rangle}^{(c)} \quad \left(\text{“}\mu_{\langle u\rangle}^{(c)}(\mathrm{d}x) = |\nabla u|^2\,\mathrm{d}x\text{”}\right)$$

Assumption. $\exists \rho : X \to [0, \infty)$ s.t.

(i) $\rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X)$ and $\rho(x) \to \infty$ as $x \to \Delta$

(ii) $B_\rho(r) := \{x \in X \mid \rho(x) \le r\}$: compact $(\forall r > 0)$

(iii) $\exists \Gamma(\rho) = \dfrac{\mathrm{d}\mu^{(c)}_{\langle \rho \rangle}}{\mathrm{d}m}$ ("$\Gamma(\rho) = |\nabla \rho|^2$")

▷ $\lambda_\rho(r) := \sup_{x \in B_\rho(r)} \Gamma(\rho)(x)$

▷ $\psi(R) := \displaystyle\int_6^R \frac{r}{\lambda_\rho(r)(\log m(B_\rho(r)) + \log\log r)} \mathrm{d}r$

Theorem.

If $\lim_{R\to\infty} \psi(R) = \infty$, then $\exists c > 0$ s.t. for m-a.e. $x \in X$,

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi^{-1}(ct) \text{ for all } t \ge T\right) = 1$$

$$(ct =)\ \psi(R) = \int_6^R \frac{r}{\lambda_\rho(r)(\log m(B_\rho(r)) + \log\log r)}\,\mathrm{d}r$$

Remark.

(i) Grigor'yan ('99) | $\psi(R) = \dfrac{R^2}{\log m(B(R))}$

(ii) Hsu-Qin ('10) add "$\log\log R$"

(iii) Intrinsic metric | Biroli-Mosco ('91), Sturm ('94)

Example.

▷ (X, d): complete, noncompact Riemannian manifold

▷ $\mathbb{M}$: Brownian motion on X

$\Longrightarrow \rho(x) = d(x, o)$ (**$o \in M$: fixed point**)

- $m(B(r)) \asymp r^{\alpha}$ $(\alpha > 0) \Longrightarrow \psi^{-1}(t) \asymp \sqrt{t \log t}$

- $m(B(r)) \asymp e^{cr^{\alpha}}$ $(0 < \alpha < 2) \Longrightarrow \psi^{-1}(t) \asymp t^{\frac{1}{2-\alpha}}$

- $m(B(r)) \asymp e^{c_1 r^2} \Longrightarrow \psi^{-1}(t) \asymp e^{c_2 t}$

■ Main interest in this talk

Under the exponential volume growth condition,

(1) to get estimates for lower rate functions [Grigor'yan ('99)]

(2) to find the 0-1 laws for rate functions

- **Grigor'yan-Hsu ('08):**

  Sharpness of the order for upper rate functions

$$V'(r) = V'(r_0) \exp\left(\int_{r_0}^{r} m(s)\,\mathrm{d}s \right), \quad t = \int_0^{R(t)} \frac{\mathrm{d}r}{m(r)}$$

$\Longrightarrow R((1+\varepsilon)t)$: **(not) upper rate function for** $\varepsilon > 0$ **(**$\varepsilon < 0$**)**

2. Result

▷ $\mathbb{H}^d$: d-dim. hyperbolic space $(d \geq 2)$

$(\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}\theta^2)$

▷ $\mathbb{M} = \left(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{H}^d}\right)$: BM generated by $\Delta_{\mathbb{H}^d}/2$

By Itô's formula applied to ρ [e.g., Hsu ('02)],

$$\begin{aligned}\rho(X_t) &= B_t + \frac{d-1}{2}\int_0^t \tanh \rho(X_s)\, \mathrm{d}s \\ &\geq B_t + \frac{d-1}{2} t\end{aligned}$$

$$\Longrightarrow \lim_{t\to\infty} \rho(X_t) = \infty \Longrightarrow \boxed{\lim_{t\to\infty} \frac{\rho(X_t)}{t} = \frac{d-1}{2}}$$

Theorem. Under some assumption on $g(t) : (0, \infty) \to (0, \infty)$,

(i) **(Upper rate functions)** $R(t) := (d-1)t/2 + \sqrt{t}g(t)$

$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le R(t) \text{ for all } t \ge T\right) = 1$ (or 0)

according as

$$\int^{\infty}_{\cdot} (1 \vee g(t)) \exp\left(-\frac{g(t)^2}{2t}\right) \frac{\mathrm{d}t}{t} < \infty \text{ (or } = \infty) \qquad (*)$$

(ii) **(Lower rate functions)** $r(t) := (d-1)t/2 - \sqrt{t}g(t)$

$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } \rho(X_t) \ge r(t) \text{ for all } t \ge T\right) = 1$ (or 0)

according as $(*)$ holds.

Remark.

(i) $(*)$: a generalized Kolmogorov's test [Keprta ('97, '98)]

(ii) [Anker-Setti ('92), cf. Babillot ('94)]

- M: complete, noncompact Riemannian manifold
- $m(B(R)) \asymp e^{2KR}$, $\lambda_0 := \inf \sigma(-\Delta/2) > 0$

If $\boxed{\lambda_0 = K^2/2}$ and $g(t) \nearrow \infty$ as $t \to \infty$, then

$$\lim_{t\to\infty} P\left(Kt - \sqrt{t}g(t) \le \rho(X_t) \le Kt + \sqrt{t}g(t)\right) = 1$$

$$\boxed{M = \mathbb{H}^d \Longrightarrow K = \frac{d-1}{2},\ \lambda_0 = \frac{(d-1)^2}{8}}$$

3. Proof

- **Proof for upper rate functions**

$$\rho(X_t) = B_t + \frac{d-1}{2}\int_0^t \tanh\rho(X_s)\,\mathrm{d}s$$
$$\geq B_t + \frac{d-1}{2}t = o(t) + \frac{d-1}{2}t$$

$\Longrightarrow \exists c > 0$ s.t.

$$P\left(\exists T > 0 \text{ s.t. } \rho(X_t) \geq ct \text{ for all } t \geq T\right) = 1$$

$$\rho(X_t) = B_t + \frac{d-1}{2}\int_0^t \tanh\rho(X_s)\,\mathrm{d}s$$
$$= B_t + \frac{d-1}{2}t + \frac{d-1}{2}\int_0^t (\tanh\rho(X_s) - 1)\;\mathrm{d}s$$

If $t \geq T$, then

$$\tanh \rho(X_t) - 1 = \frac{2}{e^{\rho(X_t)} - 1} \leq \frac{2}{e^{ct} - 1}$$

so that

$$\begin{aligned} &\int_0^t (\tanh \rho(X_s) - 1)\ \mathrm{d}s \\ &= \int_0^T (\tanh \rho(X_s) - 1)\ \mathrm{d}s + \int_T^t (\tanh \rho(X_s) - 1)\ \mathrm{d}s \\ &\leq \int_0^T (\tanh \rho(X_s) - 1)\ \mathrm{d}s + \int_T^\infty \frac{2}{e^{cs} - 1} \mathrm{d}s < \infty \end{aligned}$$

$\Longrightarrow$ $\exists\mathbb{N}$-valued r.v. N such that for all $t \geq T$,

$$\rho(X_t) = B_t + \frac{d-1}{2}\int_0^t (\tanh \rho(X_s) - 1)\ \mathrm{d}s + \frac{d-1}{2}t$$
$$\leq B_t + N + \frac{d-1}{2}t$$

Assume that

$$(*) \qquad \int_{\cdot}^{\infty} (1 \vee g(t)) \exp\left(-\frac{g(t)^2}{2t}\right) \frac{\mathrm{d}t}{t} < \infty.$$

Since $h_n(t) := g(t) - n/\sqrt{t}$ also satisfies $(*)$, a (generalized) Kolmogorov's test implies that for each $n \in \mathbb{N}$,

$$P\left(\exists T_n > 0 \text{ s.t. } B_t \leq h_n(t) \text{ for all } t \geq T_n\right) = 1$$

For each $n \in \mathbb{N}$,

$$P\left(\exists T_n > 0 \text{ s.t. } B_t \leq h_n(t) \text{ for all } t \geq T_n\right) = 1$$

$$\Rightarrow P\left(\forall n \in \mathbb{N}, \exists T_n > 0 \text{ s.t. } B_t \leq h_n(t) \text{ for all } t \geq T_n\right) = 1$$

Hence for all $t \geq T \vee T_N$,

$$\begin{aligned}\rho(X_t) &\leq B_t + N + \frac{d-1}{2}t \\ &\leq \underline{\sqrt{t}h_N(t) + N} + \frac{d-1}{2}t = \underline{\sqrt{t}g(t)} + \frac{d-1}{2}t \\ &\left(h_N(t) = g(t) - N/\sqrt{t}\right)\end{aligned}$$