Escape rate of symmetric jump-diffusion processes

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1. Introduction

▷ $(\{B_t\}_{t\geq 0}, P)$: d-dim. Brownian motion, $B_0 = 0$, a.s.

Khintchine's law of the iterated logarithm

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t \log\log t}} = 1 \quad P\text{-a.s.}$$

▷ $R_\varepsilon(t) = \sqrt{(2+\varepsilon)t \log\log t} \quad (\varepsilon > 0)$

$$\Longrightarrow \; P\left(\exists T > 0 \text{ s.t. } |B_t| \leq R_\varepsilon(t) \text{ for all } t \geq T\right) = 1$$

- R_ε: upper rate function (forefront of Brownian particles)

○ Brownian motions on Riemannian manifolds

Grigor'yan (99), Grigor'yan-Hsu (08), Hsu-Qin (10)

$$\psi(R) = \int_{\cdot}^{R} \frac{r}{\log m(B_x(r)) + \log\log r} \, \mathrm{d}r$$

$\lim_{R\to\infty} \psi(R) = \infty \Longrightarrow \exists c > 0$ **s.t.**

$$P_x\left(\exists T > 0 \text{ s.t. } d(x, X_t) \le \psi^{-1}(ct) \text{ for all } t \ge T\right) = 1$$

- $m(B(r)) \asymp r^{\alpha} \Longrightarrow \psi^{-1}(t) \asymp \sqrt{t \log t}$
- $m(B(r)) \asymp e^{cr^{\alpha}} \Longrightarrow \psi^{-1}(t) \asymp t^{\frac{1}{2-\alpha}}$ $(0 < \alpha < 2)$
- $m(B(r)) \asymp e^{c_0 r^2} \Longrightarrow \psi^{-1}(t) \asymp e^{c_1 t}$

∗ **Upper escape rate**

. . . **Quantitative version of conservativeness**

∘ **Symmetric diffusion processes**

Takeda (89), Ouyang (13)

∘ **Markov chains on weighted graphs**

Huang (14), Huang-S. (14)

▷ $a_{ij}(x)$: symm. measurable functions on $\mathbb{R}^d$ s.t.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \asymp (1+|x|)^{p}|\xi|^2, \quad \forall x, \forall \xi \in \mathbb{R}^d$$

$$\mathcal{E}(u,u) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x$$

$$P_x\,(\exists T > 0 \text{ s.t. } d(x, X_t) \le R(ct) \text{ for all } t \ge T) = 1$$

- $p = 0 \Longrightarrow R(t) \asymp \sqrt{t \log t}$
- $0 < p < 2 \Longrightarrow R(t) \asymp t^{\frac{1}{2-p}}$
- $p = 2 \Longrightarrow R(t) \asymp e^{ct}$

▷ $0 < \alpha < 2$

▷ $(\{X_t\}_{t \geq 0}, P)$: d-dim. symm. α-stable proc., $X_0 = 0$, a.s.

Khintchine (38)

▷ $R_\varepsilon(t) = t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}} \quad (\varepsilon > 0)$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |X_t| \leq R_\varepsilon(t) \text{ for all } t \geq T\right) = 1$$

Main interest in this talk.

Upper rate functions for symm. jump-diffusion processes

$*$ **Conservativeness of symm. jump-diffusion processes**

$\circ$ **Volume growth**

$$\boxed{m(B(r)) \asymp e^{cr\log r} \Longrightarrow \text{conservative}}$$

Masamune-Uemura (11)

Grigor'yan-Huang-Masamune (12)

Masamune-Uemura-J. Wang (12)

○ **Coefficient growth**

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \mathrm{d}y$$

▷ $0 < \alpha < 2, \; p \geq 0, \; q \geq 0$

$$c(x,y) \asymp \begin{cases} (1+|x|)^p + (1+|y|)^p & |x-y| < 1 \\ (1+|x|)^q + (1+|y|)^q & |x-y| \geq 1 \end{cases}$$

$p \in [0,2], \; q \in [0,\alpha) \Longrightarrow$ **conservative**

S.-Uemura (14), S. (14+)

2. Result

▷ (X, d): **locally compact separable metric space**

▷ m: **positive Radon measure on X with full support**

▷ $(\mathcal{E}, \mathcal{F})$: **regular Dirichlet form on $L^2(X; m)$ s.t.**

$$\mathcal{E}(u, u) = \iint_{X \times X \setminus d} (u(x) - u(y))^2 J(x, \mathrm{d}y) m(\mathrm{d}x)$$

$$J(x, \mathrm{d}y) m(\mathrm{d}x) = J(y, \mathrm{d}x) m(\mathrm{d}y).$$

▷ $\mathbb{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$: m-**symmetric Hunt process generated by** $(\mathcal{E}, \mathcal{F})$

- ▷ $\rho(x) = d(x, x_0)$ ($x_0 \in X$: fixed)

- ▷ $B(r) = \{y \in X \mid \rho(y) < r\}$

- ▷ $v(r)$ $(r > 0)$: increasing s.t.

$$m(B(r)) \leq v(r), \ \forall r > 0$$

- ▷ $F(r)$ $(r > 0)$: nondecreasing s.t.

$$F(r) \leq \frac{1}{32} \cdot \frac{r}{\log v(r) + \log \log r}, \ \forall r \geq 6$$

▷ $$M_1(R) = \sup_{x \in B(R)} \int_{d(x,y)<F(R)} (\rho(x) - \rho(y))^2 J(x, \mathrm{d}y)$$

▷ $$M_2(R) = \sup_{x \in X} \int_{d(x,y)\geq F(R)} J(x, \mathrm{d}y)$$

▷ $$\psi_\eta(R) = \frac{R^\eta}{M_1(R)(\log v(R) + \log\log R)} \quad (\eta \in (0, 2])$$

Assumption.

(i) $\mathbb{M}$ **is conservative;**

(ii) $\psi_\eta(R) \nearrow \infty$ **as** $R \to \infty$**;**

(iii) $\exists c > 0$, $\exists \nu > 1$ **s.t. for all large** $R > 0$,

$$\psi_\eta(R) M_2(R) \leq c/(\log R)^\nu.$$

Theorem.

Under Assumption, $\exists c > 0$ **s.t. for q.e.** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi_\eta^{-1}(ct) \text{ for all } t \ge T\right) = 1.$$

Corollary.

Under Assumption, $\exists c > 0$ **s.t. for q.e.** $x \in X$**,**

$$\limsup_{t\to\infty} \frac{\rho(X_t)}{\psi_\eta^{-1}(ct)} \le 1, \quad P_x\text{-a.s.}$$

▷ $\psi_\eta(R) = \dfrac{R^\eta}{M_1(R)(\log v(R) + \log\log R)}$ $(\eta \in [0,2])$

▷ $\psi_\eta(R) M_2(R) \le \dfrac{c}{(\log R)^\nu}$

Remark. $F(R)$ **and** ρ**: more general**

$\Longrightarrow$ **unbounded coefficient case**

Theorem.

Under Assumption, $\exists c > 0$ **s.t. for q.e.** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \leq \psi_\eta^{-1}(ct) \text{ for all } t \geq T\right) = 1.$$

Corollary.

Under Assumption, $\exists c > 0$ **s.t. for q.e.** $x \in X$**,**

$$\limsup_{t\to\infty} \frac{\rho(X_t)}{\psi_\eta^{-1}(ct)} \leq 1, \quad P_x\text{-a.s.}$$

3. Examples

∘ Coefficient growth

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \mathrm{d}y$$

▷ $0 < \alpha < 2, \; 0 \leq q < \alpha$

$$c(x,y) \asymp \begin{cases} (1+|x|)^2 + (1+|y|)^2 & |x-y| < 1 \\ (1+|x|)^q + (1+|y|)^q & |x-y| \geq 1 \end{cases}$$

- $\rho(x) = \log(2+|x|)$, $F_R(x,y) = c\{(R+|x|) \vee (R+|y|)\}$

(i) $0 \leq q < \alpha - 1$**:**

$\exists c > 0$ **s.t. for q.e.** $x \in \mathbb{R}^d$**,**

$$\limsup_{t\to\infty} \frac{|X_t - X_0|}{e^{ct}} \leq 1, \quad P_x\text{-a.s.}$$

(ii) $\alpha - 1 \leq q < \alpha$**:**

$\forall \varepsilon > 0, \exists c > 0$ **s.t. for q.e.** $x \in \mathbb{R}^d$**,**

$$\limsup_{t\to\infty} \frac{|X_t - X_0|}{\exp\left(ct^{\frac{1}{\alpha-q}}(\log t)^{\frac{1+\varepsilon}{\alpha-q}}\right)} \leq 1, \quad P_x\text{-a.s.}$$

Comparison with the diffusion case [Ouyang (13)]

▷ $a_{ij}(x)$: symm. measurable functions on $\mathbb{R}^d$ s.t.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \asymp (1+|x|)^2|\xi|^2, \quad \forall x, \forall \xi \in \mathbb{R}^d$$

$$\mathcal{E}(u,u) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x$$

$\exists c > 0$ s.t. for q.e. $x \in \mathbb{R}^d$,

$$\limsup_{t\to\infty} \frac{|X_t - X_0|}{e^{ct}} \le 1, \quad P_x\text{-a.s.}$$

◦ **Coefficient degeneracy**

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \mathrm{d}y$$

▷ $0 < \alpha < 2, \ p > 0$

$$c(x,y) \asymp \left(1 \wedge \frac{1}{|x|^p}\right)\left(1 \wedge \frac{1}{|y|^p}\right)$$

$\forall \varepsilon > 0, \ \exists c > 0$ **s.t. for q.e.** $x \in \mathbb{R}^d$,

$$\limsup_{t \to \infty} \frac{|X_t - X_0|}{t^{\frac{2+p}{2(\alpha+p)}} (\log t)^{\frac{(1+\varepsilon)(2+p)}{2(\alpha+p)}}} \le 1, \quad P_x\text{-a.s.}$$

- $\rho(x) = |x|$, $F(R) = R^{\frac{2}{2+p}}$

Comparison with symmetric stable processes

▷ $\left\{ t^{\frac{1}{\alpha}} (\log t)^{\frac{1+\varepsilon}{\alpha}} \right\}^{\frac{\alpha(2+p)}{2(\alpha+p)}}$: upper rate function for $\mathbb{M}$

▷ $t^{\frac{1}{\alpha}} (\log t)^{\frac{1+\varepsilon}{\alpha}}$: upper rate function for the α-stable case

$$\frac{\alpha(2+p)}{2(\alpha+p)} < 1, \quad \forall p > 0$$

$\Longrightarrow$ For large time, the sample path range of $\mathbb{M}$ is narrower than that for the α-stable case.

○ **Time changed processes**

▷ $\mathbb{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$: **symmetric α-stable process**

▷ $h(x) \asymp 1/(1+|x|)^p \quad (p \geq 0)$

▷ $A_t = \int_0^t h(X_s)\,\mathrm{d}s$

$$\check{X}_t = X_{\tau_t}, \quad \tau_t = \inf\{s > 0 \mid A_s > t\}$$

▷ $\check{\mathcal{L}} = -\frac{1}{h}(-\Delta)^{\frac{\alpha}{2}}$: **generator of** $(\{\check{X}_t\}_{t\geq 0})$

$$(\{\check{X}_t\}_{t\geq 0}) \text{ is conservative} \iff 0 \leq p \leq \alpha$$

S (14+), S-Uemura (14), J. Wang (11)

▷ $R(t) = t^{\frac{1}{\alpha}}(\log t)^{\frac{1+\varepsilon}{\alpha}}$: upper rate function for $\mathbb{M}$

$$A_t \asymp \int_0^t \frac{1}{(1+|X_s|)^p}\,\mathrm{d}s \geq c\int_0^t \frac{1}{(1+R(s))^p}\,\mathrm{d}s$$

$$= g(t)$$

$$d(x, \check{X}_t) = d(x, X_{\tau_t}) \leq R(\tau_t) \lesssim R(g^{-1}(t)) = \check{R}(t)$$

[Huang-S. (14)]

$$\boxed{0 < p < \alpha \Longrightarrow \check{R}(t) \asymp t^{\frac{1}{\alpha-p}}(\log t)^{\frac{1+\varepsilon}{\alpha-p}}}$$

Remark. $d > \alpha \Rightarrow \check{r}(t) = \dfrac{t^{\frac{1}{\alpha-p}}}{(\log t)^{\frac{\alpha}{\alpha-p}\cdot\frac{1+\varepsilon}{d-\alpha}}}$: lower rate funct.

4. Approach to Theorem

$$\boxed{P_x\left(\exists T > 0 \text{ s.t. } \rho(X_t) \le \psi_\eta^{-1}(ct) \text{ for all } t \ge T\right) = 1}$$

$\triangleright$ $A_n = \left\{\exists t \in (t_n, t_{n+1}] \text{ s.t. } \rho(X_t) \ge \psi_\eta^{-1}(ct)\right\}$

$$\boxed{\sum_{n=n_0}^{\infty} P_x(A_n) < \infty, \quad P_x\text{-a.s. } m\text{-a.e. } x \in X}$$

$\Longrightarrow$ **By Borel-Cantelli's lemma,**

$$\rho(X_t) \le \psi_\eta^{-1}(ct) \quad \forall t \gg 1, \ P_x\text{-a.s. } m\text{-a.e. } x \in X$$

- $\triangleright$ $\boldsymbol{A_n = \left\{\exists t \in (t_n, t_{n+1}] \text{ s.t. } \rho(X_t) \geq \psi_\eta^{-1}(ct)\right\}}$
- $\triangleright$ $\boldsymbol{R_n = \psi_\eta^{-1}(ct_n)}$

$$\boldsymbol{\Longrightarrow \psi_\eta^{-1}(ct) \geq \psi_\eta^{-1}(ct_n) = R_n, \ \forall t \in (t_n, t_{n+1}]}$$

$$\boldsymbol{\Longrightarrow A_n \subset \left\{\sup_{0<t\leq t_{n+1}} \rho(X_t) \geq R_n\right\} = \left\{\tau_{B(R_n)} \leq t_{n+1}\right\}}$$

- $\triangleright$ $\boldsymbol{\tau_{B(R)} = \inf\{t > 0 \mid X_t \notin B(R)\}}$

$$\boxed{\boldsymbol{P_x(A_n) \leq P_x\left(\tau_{B(R_n)} \leq t_{n+1}\right)}}$$

$$\mathcal{E}^R(u,u) = \iint_{d(x,y)<F(R)} (u(x)-u(y))^2 \, J(x,\mathrm{d}y)m(\mathrm{d}x)$$

- ▷ $\mathbb{M}^R = (\{X_t^R\}_{t\geq 0}, \{P_x\}_{x\in X})$: **associated with** $(\mathcal{E}^R, \mathcal{F})$
- ▷ $C_R = \dfrac{1}{32} \cdot \dfrac{R}{\log v(R) + \log\log R} \; (\geq F(R))$
- ▷ $\tau^R_{B(R-C_R)} = \inf\{t > 0 \mid X_t^R \notin B(R - C_R)\}$

$\Longrightarrow X^R_{\tau^R_{B(R-C_R)}} \in B(R) \quad (\because$ **jump range** $< F(R) \leq C_R)$

- ▷ $u_R(t,x) = P_x\left(\tau^R_{B(R-C_R)} \leq t\right)$

Proposition 1.

For m-a.e. $x \in X$,

$$P_x\left(\tau_{B(R-C_R)} \le t\right) \le \underbrace{u_R(t,x)}_{\text{small jumps}} + \underbrace{tM_2(R)}_{\text{big jumps}}.$$

▷ $u_R(t,x) = P_x\left(\tau^R_{B(R-C_R)} \le t\right)$

Ikeda-Nagasawa-Watanabe (66), Meyer (75)

Barlow-Bass-Chen-Kaßmann (09), Grigor'yan-Hu-Lau (14)

$$\begin{aligned} P_x(A_n) &\le P_x\left(\tau_{B(R_n-C_{R_n})} \le t_{n+1}\right) \\ &\le u_{R_n}(t_{n+1},x) + t_{n+1}M_2(R_n) \end{aligned}$$

$$I_R(t) = \int_{B(R-C_R)} e^{-\xi_R(t,x)} u_R(t,x)^2 \varphi_R(x)^2 \, m(\mathrm{d}x)$$

▷ $\xi_R(t,x) = c_1(R)t + 2c_2(R)\rho(x)$

▷ $\varphi_R(x) = \dfrac{\left(e^{c_2(R)(R-C_R)} - e^{c_2(R)\rho(x)}\right)_+}{e^{c_2(R)(R-C_R)} - 1}$

▷ $c_1(R) = 512 \cdot \dfrac{M_1(R)}{R^2} \cdot (\log v(R) + \log\log R)$

▷ $c_2(R) = \dfrac{8}{R} \cdot (\log v(R) + \log\log R) \left(= \dfrac{1}{4C_R}\right)$

Proposition 2.

$\exists c > 0$ **s.t.** $\forall R \gg 1$, $\forall t > 0$,

$$\int_{B((R-C_R)/2)} u_R(t,x)^2\, m(\mathrm{d}x) \leq \frac{c e^{c_1(R)t}}{v(R)^3 (\log R)^4}.$$

▷ $R_n = \theta^{n/2} \quad (1 < \theta < 2)$

▷ $t_n = \dfrac{1}{1024} \cdot \psi_\eta(R_n)$

$$\Longrightarrow c_1(R_n) t_{n+1} \lesssim \log v(R_n) + \log\log R_n$$

$$\sqrt{\int_{B((R_n - C_{R_n})/2)} u_{R_n}(t_{n+1},x)^2\, m(\mathrm{d}x)} \leq \frac{c(\theta)}{v(R_n) n^{3/2}}$$

$K \subset X$: **compact**

$$\int_K \left(\sum_{n=n_0}^{\infty} P_x(A_n) \right) m(\mathrm{d}x)$$

$$\leq \sum_{n=n_0}^{\infty} \int_K \left(u_{R_n}(t_{n+1}, x) + t_{n+1} M_2(R_n) \right) m(\mathrm{d}x) < \infty$$

$$\int_K u_{R_n}(t_{n+1}, x)\, m(\mathrm{d}x) \lesssim \sqrt{\int_K u_{R_n}(t_{n+1}, x)^2\, m(\mathrm{d}x)}$$

$$\lesssim n^{-3/2}$$

$$t_{n+1} M_2(R_n) \lesssim \psi_\eta(R_n) M_2(R_n) \lesssim n^{-\nu} \quad (\nu > 1)$$

Assumption.

(iii) $\exists c > 0$, $\underline{\exists \nu > 1}$ **s.t. for all large** $R > 0$,

$$\psi_\eta(R) M_2(R) \le c/(\log R)^{\nu}.$$