TOPICS IN ALGEBRA

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Abstract. This is a note for the course 'Topics in Algebra', which has been given during the winter semester of 2016-2017 at Osaka university.

Contents

0. Plan of the lecture

The aim of this series of lectures is very roughly twofold:

- introduce the so-called *noncommutative algebraic geometry* in the sense of M. Artin, Schelter, Tate, Van den Bergh,.....
- explain modern points of view on the subject; more precisely, abelian and derived categorical points of view.

The plan of this series of lectures can be found in Contents above. Some information of references can be found in Section 15.

1. Introduction and Overview

Noncommutative algebraic geometry was initiated as an attempt to investigate "non-commutative" analogue of commutative algebraic varieties from the point of view of graded algebras. In the late 80s, *Artin-Schelter regular algebras* was introduced and

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studied purely algebraically in [AS87]. They are, by definition, certain *noncommutative*[1] $\mathbb{Z}$-graded algebras which share some of the nice properties of commutative polynomial rings. By the theorem of Serre, polynomial ring $\mathbf{k}[x_0, \dots, x_d]$ with the grading $\deg x_i = 1$ give rise to the projective space $\mathbb{P}^d_{\mathbf{k}}$ in the sense that there exists a standard equivalence of $\mathbf{k}$-linear abelian categories

$$\operatorname{coh} \mathbb{P}^d_{\mathbf{k}} \simeq \operatorname{qgr} \mathbf{k}[x_0, \dots, x_d] = \operatorname{grmod} \mathbf{k}[x_0, \dots, x_d] / \operatorname{tors} \mathbf{k}[x_0, \dots, x_d], \tag{1.1}$$

where grmod is the $\mathbf{k}$-linear category of finitely generated (right) graded modules, tors is the full subcategory of finite dimensional modules, and qgr is the Serre quotient. Therefore one expects that $\operatorname{qgr} S$ should be something which resembles projective spaces, where S is an AS-regular algebra. This expectation is true modulo some modifications, and one of the goals of this lecture series is to explain it.

The classification of AS-regular algebras with $d = 2$ was not completed in [AS87], but was done in [ATVdB90]. An ingenious discovery of [ATVdB90] is that the classification is reduced to that of (possibly singular) curves of genus one + extra geometric data on it. Moreover, it turns out any $\operatorname{qgr} S$ for such a graded algebra S "contains" such a curve as a "divisor".

A smooth projective variety over a field $\mathbf{k}$ is

- the common zero
$$X = V(f_1, \dots, f_r) \subset \mathbb{P}^N \tag{1.2}$$
of homogeneous polynomials $f_1, \dots, f_r \in \mathbf{k}[x_0, \dots, x_N]$ such that
$$\operatorname{rank} \left(\frac{\partial f_j}{\partial x_i} (a_0, \dots, a_N) \right) = N - \dim X \tag{1.3}$$
for all $a = (a_0 : \cdots : a_N) \in X$, or
- a smooth and projective scheme X over $\operatorname{Spec} \mathbf{k}$.

Example 1.1. Consider the following surface, which is traditionally called the *Hesse pencil.*

$$S(3) = V\left(t_0\left(x^3 + y^3 + z^3\right) + t_1 xyz\right) \subset \mathbb{P}^2_{x:y:z} \times \mathbb{P}^1_{t_0:t_1} \tag{1.4}$$

Let $\pi \colon S(3) \to X(3) \coloneqq \mathbb{P}^1_{t_0:t_1}$ be the restriction of the second projection $\mathrm{pr}_2 \colon \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ to $X(3)$. Then π is a flat and projective morphism of relative dimension 1. The fiber over a point $t = (t_0 : t_1) \in X(3)$ is naturally identified with the plane cubic curve defined by the equation

$$t_0\left(x^3 + y^3 + z^3\right) + t_1 xyz = 0. \tag{1.5}$$

As the notations suggest, $X(3)$ is the smooth compactification of the fine moduli space $X'(3)$ of elliptic curves with level 3 structures. $\pi \colon S(3) \to X(3)$ is the compactified universal family which was first studied by Shioda [Shi72].

The 9 sections of π which represent the universal 3-torsion points are given by the common inflection points. All of them are (-1)-curves, and after contracting all of them we obtain $\mathbb{P}^2_{x:y:z}$.

See [AD09] for more details.

Exercise 1.2. Show that the morphism π in Example 1.1 is in fact flat (hint: use [Har77, Chapter II, Proposition 9.7]).

Exercise 1.3. Show that the morphism π in Example 1.1 is smooth over the open subset

$$X'(3) = \left\{(0:1), (1:-3), (1:-3\omega), (1:-3\omega^2)\right\}^c \subset X(3). \tag{1.6}$$

[1] noncommutative = not necessarily commutative

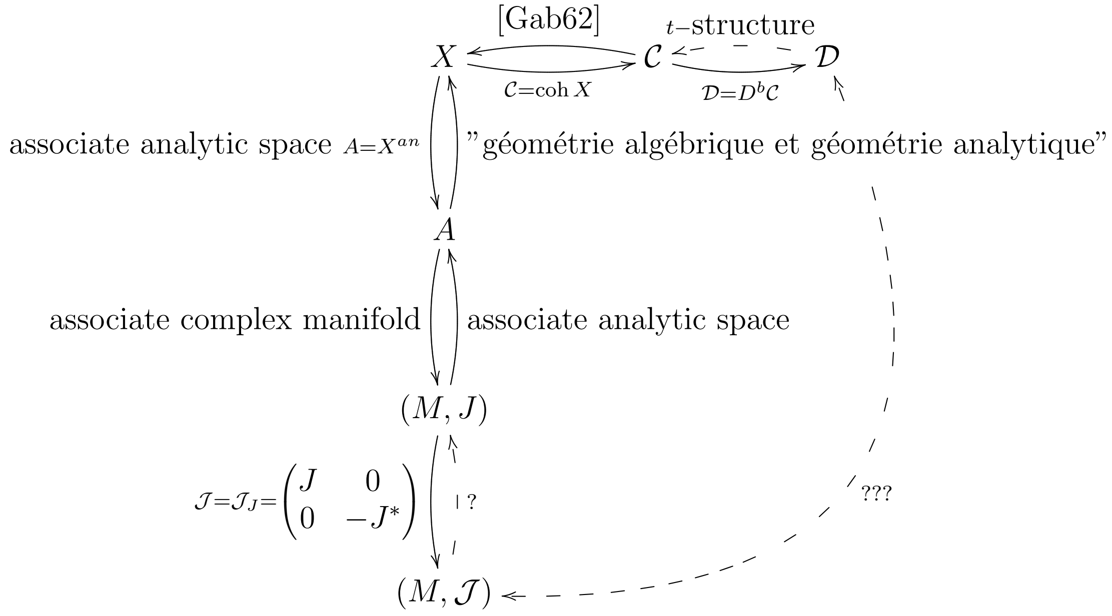

FIGURE 1.1. algebraic variety and related structures

There are various reinterpretations of X, as described in Figure 1.1. In particular, flat deformations of X as algebraic varieties induce flat deformation of abelian categories in the sense of [LVdB06]. Recall that a deformation of X over a pointed scheme $(Y, 0)$, where $0 \in Y(\mathbf{k})$, is a flat morphism of schemes $f \colon \mathcal{X} \to Y$ and an isomorphism $f^{-1}(0) = \mathcal{X} \times_Y \operatorname{Spec} \mathbf{k}(0) \xrightarrow{\sim} X$. Then it follows that $\operatorname{coh} \mathcal{X}$ is a flat deformation of $\operatorname{coh} X$ over $\mathcal{O}_Y$.

On the other hand, not all flat deformations of $\operatorname{coh} X$ come from those of X as algebraic varieties. For example, although $\mathbb{P}^2$ is rigid as an algebraic variety, the moduli of deformations of $\operatorname{coh} \mathbb{P}^2$ is of dimension 2. It follows that *any* noncommutative deformation of $\mathbb{P}^2$ over a complete local Noetherian ring comes from a AS-regular algebra of dimension 3; more precisely, a 3-dimensional quadratic AS-regular algebra. It is remarkable that AS-regular algebras are recovered naturally from such categorical consideration, although AS-regular algebras are defined as noncommutative algebras which share some nice homological properties with commutative polynomial rings.

2. Category and functor

Definition 2.1. [2] A *category* $\mathcal{C}$ consists of the following data:

- collection of *objects*, which will be denoted by $\operatorname{ob} \mathcal{C}$.
- collection of *morphisms*, which will be denoted by $\hom \mathcal{C}$.

They should satisfy the following properties.

(1) Every morphism $f \in \hom(\mathcal{C})$ has its source object $s(f) \in \operatorname{ob} \mathcal{C}$ and its target object $t(f) \in \operatorname{ob} \mathcal{C}$. Alternatively we also say 'f is a morphism from the object $s(f)$ to $t(f)$'.

[2] As is well known, we should in fact care about foundational issues concerning set theory. For example, we should assume ZFC and the universe axiom and argue based on a choice of universe. In this note, we will omit this point.

For objects $x, y \in \operatorname{ob} \mathcal{C}$, the collection of morphisms from x to y will be denoted by
$$\operatorname{Hom}_{\mathcal{C}}(x, y) \tag{2.1}$$
or simply
$$\mathcal{C}(x, y). \tag{2.2}$$
Unless otherwise stated, we will assume that $\mathcal{C}(x, y)$ is a set.

(2) If two morphisms $f, g \in \hom \mathcal{C}$ satisfy
$$s(f) = t(g), \tag{2.3}$$
then the *composition*
$$f \circ g \in \mathcal{C}(s(g), t(f)) \tag{2.4}$$
should be defined.

(3) If three morphisms f, g and h are composable (i.e. if $s(f) = t(g)$ and $s(g) = t(h)$), then the *associativity*
$$(f \circ g) \circ h = f \circ (g \circ h) \tag{2.5}$$
should hold.

(4) For each object $x \in \operatorname{ob} \mathcal{C}$, there should be a morphism $\mathrm{id}_x \in \mathcal{C}(x, x)$ satisfying
$$f \circ \mathrm{id}_x = f, \quad \mathrm{id}_x \circ g = g \tag{2.6}$$
for any $f, g \in \hom \mathcal{C}$ such that $s(f) = x$ and $t(g) = x$.

Notations 2.2. (1) To mean $x \in \operatorname{ob} \mathcal{C}$, we also write $x \in \mathcal{C}$.

(2) For an object $x \in \mathcal{C}$, the hom space $\mathcal{C}(x, x)$ will be also denoted by $\operatorname{End}_{\mathcal{C}}(x)$.

Exercise 2.3. Show that for each $x \in \operatorname{ob} \mathcal{C}$ there exists only one morphism id_x satisfying the conditions (2.6).

Definition 2.4. A morphism $f \in \mathcal{C}(x, y)$ is said to be *invertible* if there exists a morphism $g \in \mathcal{C}(y, x)$ such that
$$f \circ g = \mathrm{id}_y, \quad g \circ f = \mathrm{id}_x. \tag{2.7}$$
In fact, for each f there exists at most one such g. It will be called the *inverse* of f and denoted by f^{-1}.

Exercise 2.5. Show the uniqueness of the inverse, which is mentioned above.

Exercise 2.6. Show that a group is nothing but a category which has only one object and any morphism is invertible.

Definition 2.7. A *groupoid* is a category in which any morphism is invertible.

Example 2.8. The category $\mathcal{S}et$ has sets as objects and maps between sets as morphisms. Composition is nothing but the composition of maps in the usual sense, and for each set $X \in \operatorname{ob} \mathcal{S}et$ the identity morphism is nothing but the identity map itself.

Exercise 2.9. The category $\mathcal{C}\mathrm{orr}$ has sets as objects and maps between sets are correspondences. Namely, for sets X, Y, a map from X to Y is a subset $S \subset X \times Y$. Compositions of correspondences are defined as
$$T \circ S = \mathrm{pr}_{X \times Z}\left(\mathrm{pr}_{Y \times Z}^{-1}(T) \cap \mathrm{pr}_{X \times Y}^{-1}(S)\right) \tag{2.8}$$
for $S \subset X \times Y$ and $T \subset Y \times Z$, where $\mathrm{pr}_{X \times Y} \colon X \times Y \times Z \to X \times Y$ and so on are the projection morphisms.

Show that $\mathcal{C}\mathrm{orr}$ is in fact a category.

Example 2.10. The category $\mathcal{T}$op is defined as follows:

- objects are topological spaces.
- morphisms are continuous maps.

Example 2.11. The category $\mathcal{H}ot$ is defined as follows:

- objects are topological spaces.
- morphisms are homotopy classes of continuous maps.

Let us recall the relevant notions. Consider two continuous maps $f, g\colon X \rightrightarrows Y$ between topological spaces. A *homotopy* H from f to g is a continuous map

$$H\colon X \times [0,1] \to Y \tag{2.9}$$

such that

$$H(x,0) = f(x), \tag{2.10}$$

$$H(x,1) = g(x), \tag{2.11}$$

where $[0,1]$ is the closed interval with the standard (=Euclidean) topology and $X \times [0,1]$ is the product topological space. This is denoted by $f \sim_H g$. A continuous map f is said to be homotopic to g if there exists a homotopy from f to g. Thus we obtained a relation on the set of continuous maps from X to Y $(= (X,Y))$, and actually it is an equivalence relation.

Exercise 2.12. Show this!

Therefore the *homotopy class* makes sense.

Exercise 2.13. Show that $\mathcal{H}ot$ is well-defined as a category. In particular, define compositions and show its well-definedness.

Exercise 2.14. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ is defined as follows: objects of $\mathcal{C}^{op}$ are the same as those of $\mathcal{C}$. For $x, y, z \in \operatorname{ob}\mathcal{C}^{op} = \operatorname{ob}\mathcal{C}$, set $\mathcal{C}^{op}(x,y) = \mathcal{C}(y,x)$ and

$$g \circ_{\mathcal{C}^{op}} f = f \circ g \tag{2.12}$$

for $f \in \mathcal{C}^{op}(x,y) = \mathcal{C}(y,x)$ and $g \in \mathcal{C}^{op}(y,z) = \mathcal{C}(z,y)$.

Show that $\mathcal{C}^{op}$ is a category.

Example 2.15. The category $\mathfrak{Ab}$ is defined as follows:

- objects are abelian groups.
- morphisms are homomorphisms of groups.

Example 2.16. Let R be an associative ring with the unit. The category of right modules $\operatorname{Mod} R$ is defined as follows:

- objects are right R-modules.
- morphism are homomorphisms of right R-modules.

The category of left R-modules is similarly defined and will be denoted by $R \operatorname{Mod}$.

Definition 2.17. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- an object $F(x) \in \mathcal{D}$ for each object $x \in \mathcal{C}$.
- A map
$$F(x,y) : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y)) \tag{2.13}$$
for each pair of objects $x, y \in \mathcal{C}$.

They should satisfy the following properties.

$$\begin{tikzcd}
\mathcal{C}(y,z)\times\mathcal{C}(x,y) \arrow[rr] \arrow[d, "F(y,z)\times F(x,y)"'] & & \mathcal{C}(x,z) \arrow[d, "F(x,z)"] \\
\mathcal{D}(F(y),F(z))\times\mathcal{D}(F(x),F(y)) \arrow[rr] & & \mathcal{D}(F(x),F(z))
\end{tikzcd}$$

FIGURE 2.1. functoriality

$$\begin{tikzcd}
F(x) \arrow[r, "\theta(x)"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\theta(y)"'] & G(y)
\end{tikzcd}$$

FIGURE 2.2. natural transformation

(1) For each object $x \in \mathcal{C}$, $F(x,x)$ should send id_x to $\mathrm{id}_{F(x)}$.
(2) F should respect the compositions. Namely, the diagram Figure 2.1 should commute for any objects $x, y, z \in \mathcal{C}$. The horizontal arrows in Figure 2.1 are compositions.

Let us give a couple of examples of functors.

Example 2.18. Let $\varphi\colon R \to S$ be a homomorphism of rings. For each object $M \in \operatorname{Mod} S$, one can construct $M_R \in \operatorname{Mod} R$ by setting $M_R = M$ as sets and $m \cdot r := m \cdot \varphi(r)$ for $m \in M_R$ and $s \in S$.

One can also define a functor from $\operatorname{Mod} R$ to $\operatorname{Mod} S$ which sends a module N to $N \otimes_{R,\varphi} S$.

Exercise 2.19. Let $\mathcal{C}, \mathcal{D}$ be categories. A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{op}$ to $\mathcal{D}$ in the sense of Definition 2.17.

Spell out the definition of contravariant functor as in Definition 2.17.

Example 2.20. Let $\mathcal{C}$ be a category. For each object $x \in \mathcal{C}$, one can define a functor

$$\mathcal{C}(x,-) : \mathcal{C} \to \mathcal{S}et \tag{2.14}$$

which sends an object y to the set $\mathcal{C}(x,y)$ and a morphism $f\colon y \to z$ to the map

$$f_* := \mathcal{C}(x,-)(f) : \mathcal{C}(x,y) \to \mathcal{C}(x,z); \quad (g\colon x \to y) \mapsto f \circ g. \tag{2.15}$$

Similarly, one can define the functor

$$\mathcal{C}(-,x) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et \tag{2.16}$$

which sends an object y to the set $\mathcal{C}^{op}(x,y) = \mathcal{C}(y,x)$ and a morphism $f\colon y \to z$ to the map

$$f^* := \mathcal{C}(-,x)(f) : \mathcal{C}(z,x) \to \mathcal{C}(y,x); \quad (g\colon z \to x) \mapsto g \circ f. \tag{2.17}$$

Definition 2.21. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A *natural transformation* θ from F to G is a collection of morphism

$$\theta(x) : F(x) \to G(x) \in \mathcal{D}(F(x), G(x)) \tag{2.18}$$

for $x \in \operatorname{ob} \mathcal{C}$ such that for each $f \in \mathcal{C}(x,y)$ the diagram is commutative.

Exercise 2.22. Let $\mathcal{C}, \mathcal{D}$ be two categories. Define the *functor category*

$$\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})\,, \tag{2.19}$$

whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations. Show that it actually is a category.

Exercise 2.23. Define compositions of functors, and show the associativity $(F \circ G) \circ H = F \circ (G \circ H)$.

Definition 2.24. An isomorphism from the category $\mathcal{C}$ to $\mathcal{D}$ is a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$, where $\mathrm{id}_{\bullet}$ is the *identity functor* of the category $\bullet$.

An *equivalence* from the category $\mathcal{C}$ to $\mathcal{D}$ is a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are *isomorphic to* identity functors.

Exercise 2.25. Show that a category $\mathcal{C}$ is canonically *isomorphic to* $(\mathcal{C}^{op})^{op}$.

Exercise 2.26. Show that the two categories $\mathfrak{Ab}$ and $\mathrm{Mod}\,\mathbb{Z}$ are isomorphic.

Exercise 2.27. Let G be a finite group whose order is coprime to the characteristic of $\mathbf{k}$. Show that the category $\mathrm{Rep}_{\mathbf{k}}(G)$ of representations over $\mathbf{k}$ is equivalent to the category $\coprod_{\mathrm{Irrep}\, G} \mathrm{Vect}\, \mathbf{k}$, the disjoint union of $\#\,(\mathrm{Irrep}\, G)$-copies of the category of $\mathbf{k}$-vector spaces, as $\mathbf{k}$-linear additive categories. Show that the equivalence does not respect the monoidal structures.

Show that $\mathrm{Rep}_{\mathbf{k}}(G)$ is equivalent to the category $\mathbf{k}G\,\mathrm{Mod}$ of left $\mathbf{k}G$-modules as monoidal $\mathbf{k}$-linear categories, where $\mathbf{k}G$ is the group algebra (definition of the monoidal structure on $\mathbf{k}G\,\mathrm{Mod}$ is part of the exercise. Note that $g \mapsto \sum_{h \in G} h \otimes h^{-1} g$ extends to a $\mathbf{k}$-algebra homomorphism $\mathbf{k}G \to \mathbf{k}G \otimes \mathbf{k}G$).

Finally, for $G = \mathbb{Z}/n\mathbb{Z}$, show that $\mathrm{Rep}_{\mathbf{k}}(G)$ is equivalent to the category of n-graded $\mathbf{k}$-vector spaces as monoidal $\mathbf{k}$-linear additive category.

Definition 2.28. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is *faithful* (resp. *full*) if $F\,(x, y) : \mathcal{C}\,(x, y) \to \mathcal{D}\,(F(x), F(y))$ is injective (resp. surjective) for all $x, y \in \mathcal{C}$.

It is called *essentially surjective* if for each $y \in \mathcal{D}$ there exists an object $x \in \mathcal{C}$ such that $F(x) \simeq y$.

Exercise 2.29. Show that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if F is fully faithful and essentially surjective.

Definition 2.30. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of the following data.

- A subcollection $\mathrm{ob}\,\mathcal{D} \subset \mathrm{ob}\,\mathcal{C}$ of objects.
- For each pair of objects $x, y \in \mathrm{ob}\,\mathcal{D}$, a subset $\mathcal{D}\,(x, y) \subset \mathcal{C}\,(x, y)$. They have to be closed under compositions and should satisfy $\mathrm{id}_x \in \mathcal{D}\,(x, x)$ for all $x \in \mathrm{ob}\,\mathcal{D}$.

Exercise 2.31. Show that a subcategory $\mathcal{D}$ is again a category equipped with the canonical faithful functor $\mathcal{D} \subset \mathcal{C}$.

Definition 2.32. A subcategory $\mathcal{D} \subset \mathcal{C}$ is said to be full if the embedding functor $\mathcal{D} \subset \mathcal{C}$ is full in the usual sense.

Note that a full subcategory is specified by the data $\mathrm{ob}\,\mathcal{D} \subset \mathrm{ob}\,\mathcal{C}$.

Exercise 2.33. Fix a field $\mathbf{k}$, and consider the category of finite dimensional $\mathbf{k}$-vector spaces $\mathrm{vect}_{\mathbf{k}}$. Let $\iota\colon \mathcal{N} \hookrightarrow \mathrm{vect}_{\mathbf{k}}$ be the full subcategory consisting of spaces of numerical vectors; i.e.,

$$\mathrm{ob}\,\mathcal{N} = \left\{0, \mathbf{k}, \mathbf{k}^2, \dots\right\}. \tag{2.20}$$

Show that ι is an equivalence of categories which is *not* an isomorphism.

Exercise 2.34. Show that

$$f \mapsto \Gamma_f \coloneqq \{(x, f(x)) \,| x \in X\} \tag{2.21}$$

yields a faithful functor $\mathcal{S}et \to \mathcal{C}\mathrm{orr}$.

Exercise 2.35 (米田の補題)**.** Show that the functor

$$\mathcal{C}^{op} \to \mathrm{Fun}\left(\mathcal{C}, \mathcal{S}et\right); \quad x \mapsto \mathcal{C}\left(x, -\right), \quad [f\colon x \to y] \mapsto [f^*\colon \mathcal{C}\left(y, -\right) \to \mathcal{C}\left(x, -\right)] \tag{2.22}$$

is full and faithful. Show the same for the functor

$$\mathcal{C} \to \mathrm{Fun}\left(\mathcal{C}^{op}, \mathcal{S}et\right); \quad x \mapsto \mathcal{C}\left(-, x\right), \quad [f\colon x \to y] \mapsto [f_*\colon \mathcal{C}\left(-, x\right) \to \mathcal{C}\left(-, y\right)]. \tag{2.23}$$

These are called the *Yoneda embeddings*.

Definition 2.36. Fix a functor $F \in \mathrm{Fun}\left(\mathcal{C}^{op}, \mathcal{S}et\right)$. A *fine moduli space* of F is an object $\mathcal{M} \in \mathcal{C}$ such that there exists an isomorphism of functors

$$\theta\colon F \xrightarrow{\sim} \mathcal{C}\left(-, \mathcal{M}\right). \tag{2.24}$$

The *universal object* $u \in F\left(\mathcal{M}\right)$ is defined by

$$u = \theta\left(\mathcal{M}\right)^{-1}\left(\mathrm{id}_{\mathcal{M}}\right). \tag{2.25}$$

For each object $T \in \mathcal{C}$ and $u' \in F\left(T\right)$, the map

$$\theta\left(T\right)\left(u'\right) : T \to \mathcal{M} \tag{2.26}$$

is called the *classifying morphism*.

Exercise 2.37. Show that the pair $(\mathcal{M}, u \in F\left(\mathcal{M}\right))$ is a fine moduli space and the universal object of the functor F if and only if for any object $T \in \mathcal{C}$ and an object $u' \in F\left(T\right)$ there exists a unique morphism $f\colon T \to \mathcal{M}$ such that $u' = F\left(f\right)\left(u\right)$.

Example 2.38. Let $\mathbf{k}$ be a field, and V a finite dimensional vector space over $\mathbf{k}$. Set $\mathbb{P} \coloneqq \mathbb{P}V \coloneqq \mathrm{Proj}\,\mathrm{Sym}^\bullet V$ (here we follow the Grothendieck convention, so that $\mathbb{P}\left(\mathbf{k}\right) = \{W \subset V |\, \dim V/W = 1\}$). Let

$$F\colon\ \left(\mathcal{S}ch/\mathbf{k}\right)^{op} \to \mathcal{S}et \tag{2.27}$$

be the functor defined by

$$F\left(T\right) = \{\mathcal{E} \subset \mathcal{O}_T \otimes_{\mathbf{k}} V | \text{quotient is an invertible sheaf}\} \tag{2.28}$$

and pull-backs, where $(\mathcal{S}ch/\mathbf{k})$ is the category of schemes over $\mathbf{k}$. Show that $\mathbb{P}$ is a fine moduli space of F and

$$\ker\left(\mathcal{O}_{\mathbb{P}} \otimes_{\mathbf{k}} V \twoheadrightarrow \mathcal{O}_{\mathbb{P}}\left(1\right)\right) \subset \mathcal{O}_{\mathbb{P}} \otimes_{\mathbf{k}} V \tag{2.29}$$

is the universal family.

Definition 2.39. Let $\mathcal{C}, \mathcal{D}$ be categories, and $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be a pair of functors. The functor F is said to be the *left adjoint* of G (equivalently, G is said to be the *right adjoint* of F) if there exists a natural isomorphism of functors

$$\mathcal{D}\left(-, -\right) \circ \left(F \times \mathrm{id}_{\mathcal{D}}\right) \xrightarrow{\sim} \mathcal{C}\left(-, -\right) \circ \left(\mathrm{id}_{\mathcal{C}} \times G\right) : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et. \tag{2.30}$$

It will be denoted by the symbol $F \dashv G$.

Exercise 2.40. Let R, S be (not necessarily commutative) ring.

(1) Recall that an (R, S)-module is a left R-module K which is also equipped with a right S-module structure such that $(rk)\, s = r\,(ks)$ holds for any $r \in R,\ k\ \in K,\ s \in S$. The category of (R, S)-modules is denoted by $R - \mathrm{Mod} - S$.

Show that for any $M \in \mathrm{Mod}\, S$, $\mathrm{Hom}_{\mathrm{Mod}\, S}(K, M)$ has a canonical structure of a right R-modules.

(2) Suppose that R and S shares a common subring T. Show that the category $\mathrm{Mod}\,(R^{op} \otimes_T S)$ is isomorphic to the full subcategory of $R - \mathrm{Mod} - S$ consisting of those modules K satisfying $t \cdot k = k \cdot t$ for all $t \in T$ and $k \in K$.

Exercise 2.41. Let R, S be (not necessarily commutative) rings and K be an (R, S)-module. Set

$$\mathcal{C} = \mathrm{Mod}\, R, \quad \mathcal{D} = \mathrm{Mod}\, S, \quad F = \otimes_R K, \quad G = \mathcal{D}\,(K, -)\,. \tag{2.31}$$

Show $F \dashv G$.

Remark 2.42. If R is a commutative ring, then for any $M, N \in \mathrm{Mod}\, R$ $\mathrm{Hom}_{\mathrm{Mod}\, R}(M, N)$ has a canonical structure of an R-module. So does $M \otimes_R N$.

On the other hand, if R is not commutative $\mathrm{Hom}_{\mathrm{Mod}\, R}(M, N)$ is merely an abelian group. Similarly if $M \in \mathrm{Mod}\, R$ and $N \in R - \mathrm{Mod}$, then $M \otimes_R N$ is merely an abelian group.

3. Additive category and abelian category

Definition 3.1. A *monoidal category*

$$\mathcal{C} = (\mathcal{C}, \otimes, I, a_{\bullet,\bullet,\bullet}, l, r) \tag{3.1}$$

consists of the following data.

- A category $\mathcal{C}$,
- a functor
$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \tag{3.2}$$
called the *tensor product*,
- an object $I \in \mathcal{C}$ called the *unit object*,
- a natural isomorphism of functors
$$a\colon\ ((-) \otimes (-)) \otimes (-) \xrightarrow{\sim} (-) \otimes ((-) \otimes (-)) \tag{3.3}$$
called the *associator*,
- natural isomorphisms of functors
$$l\colon I \otimes (-) \xrightarrow{\sim} (-)\,, \quad r\colon\ (-) \otimes I \xrightarrow{\sim} (-) \tag{3.4}$$
called the *left unitor* (resp. *right unitor*),

satisfying the *triangle identity* Figure 3.1 and the *pentagon identity* Figure 3.2, where x, y, z, w are any objects of $\mathcal{C}$.

Example 3.2. (1) $\mathcal{S}et$ has a standard monoidal structure with $\otimes = \times$, the direct product. The unit object is any set with one element.

(2) One can similarly define monoidal structures on the category of reasonable spaces, such as that of topological spaces, differentiable manifolds, complex manifolds, ...

(3) For a commutative ring R, the category $\mathrm{Mod}\, R$ has a standard monoidal structure with $\otimes = \otimes_R$ and $I = R$. In particular, $\mathfrak{Ab}$ is a monoidal category.

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a_{x,I,y}"] \arrow[dr, "r(x)\otimes \mathrm{id}_y"'] & & x \otimes (I \otimes y) \arrow[dl, "\mathrm{id}_x \otimes l(y)"] \\
& x \otimes y &
\end{tikzcd}$$

FIGURE 3.1. triangle identity

$$\begin{tikzcd}
& (x \otimes y) \otimes (z \otimes w) \arrow[dr, "a_{x,y,z\otimes w}"] & \\
((x \otimes y) \otimes z) \otimes w \arrow[ur, "a_{x\otimes y,z,w}"] \arrow[d, "a_{x,y,z}\otimes \mathrm{id}_w"'] & & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr, "a_{x,y\otimes z,w}"'] & & x \otimes ((y \otimes z) \otimes w) \arrow[u, "\mathrm{id}_x \otimes a_{y,z,w}"']
\end{tikzcd}$$

FIGURE 3.2. pentagon identity

(4) Moreover, one can introduce a monoidal structure on the category of S-schemes $\mathcal{S}ch/S$ by using fiber products over the base S as tensor product. $\mathrm{id}_S \colon S \to S$ serves as the unit object.
(5) Let $\mathcal{C}$ be a category and consider the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. It comes with a standard monoidal structure for which the tensor product is the composition of functors and $\mathrm{id}_{\mathcal{C}}$ is the unit object.

Exercise 3.3. Check that each of Example 3.2 is actually a monoidal category.

[Kel05] is a reference for monoidal categories.

Definition 3.4. Let $\mathcal{D}$ be a monoidal category. A *category enriched over* $\mathcal{D}$, say $\mathcal{C}$, consists of the following data.

- Collection of objects $\mathrm{ob}\,\mathcal{C}$.
- For any objects $x, y \in \mathcal{C}$, an object $\mathcal{C}(x, y) \in \mathrm{ob}\,\mathcal{D}$.
- For any objects $x, y, z \in \mathcal{C}$, a morphism
$$M = M_{x,y,z} \colon \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z) \tag{3.5}$$
in the category $\mathcal{D}$ called the *composition law*.
- For any object $x \in \mathcal{C}$, a morphism $j_x \colon I \to \mathcal{C}(x, x)$ called the *identity element*.

They should satisfy the associativity Figure 3.3 and the unit condition Figure 3.4, where x, y, z, w are any objects in $\mathcal{C}$.

Definition 3.5. Let $\mathcal{A}$ and $\mathcal{B}$ be enriched categories over the monoidal category $\mathcal{D}$.
A $\mathcal{D}$-enriched functor $F \colon \mathcal{A} \to \mathcal{B}$ consists of the following data:

- A correspondence
$$F \colon \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}. \tag{3.6}$$
- For any objects $x, y \in \mathcal{A}$, a morphism
$$F(x, y) : \mathcal{A}(x, y) \to \mathcal{B}(F(x), F(y)). \tag{3.7}$$

$$
\begin{tikzcd}
\left(\mathcal{C}\left(z,w\right)\otimes\mathcal{C}\left(y,z\right)\right)\otimes\mathcal{C}\left(x,y\right) \arrow[rr, "a"] \arrow[d, "M\otimes\mathrm{id}"'] & & \mathcal{C}\left(z,w\right)\otimes\left(\mathcal{C}\left(y,z\right)\otimes\mathcal{C}\left(x,y\right)\right) \arrow[d, "\mathrm{id}\otimes M"] \\
\mathcal{C}\left(y,w\right)\otimes\mathcal{C}\left(x,y\right) \arrow[dr, "M"'] & & \mathcal{C}\left(z,w\right)\otimes\mathcal{C}\left(x,z\right) \arrow[dl, "M"] \\
& \mathcal{C}\left(x,w\right) &
\end{tikzcd}
$$

FIGURE 3.3. associativity

$$
\begin{tikzcd}
\mathcal{C}\left(y,y\right)\otimes\mathcal{C}\left(x,y\right) \arrow[r, "M"] & \mathcal{C}\left(x,y\right) & \mathcal{C}\left(x,y\right)\otimes\mathcal{C}\left(x,x\right) \arrow[l, "M"'] \\
I\otimes\mathcal{C}\left(x,y\right) \arrow[u, "j\otimes"] \arrow[ur, "l"'] & & \mathcal{C}\left(x,y\right)\otimes I \arrow[u, "\otimes j"'] \arrow[ul, "r"]
\end{tikzcd}
$$

FIGURE 3.4. unit condition

$$
\begin{tikzcd}
\mathcal{A}\left(y,z\right)\otimes\mathcal{A}\left(x,y\right) \arrow[r] \arrow[d, "F(y,z)\otimes F(x,y)"'] & \mathcal{A}\left(x,z\right) \arrow[d, "F(x,z)"] \\
\mathcal{B}\left(F(y),F(z)\right)\otimes\mathcal{B}\left(F(x),F(y)\right) \arrow[r] & \mathcal{B}\left(F(x),F(z)\right)
\end{tikzcd}
$$

FIGURE 3.5. functoriality of enriched functor

$$
\begin{tikzcd}
& \mathcal{A}\left(x,x\right) \arrow[dd, "F(x,x)"] \\
I \arrow[ur, "j_x^{\mathcal{A}}"] \arrow[dr, "j_{F(x)}^{\mathcal{B}}"'] & \\
& \mathcal{B}\left(F(x),F(x)\right)
\end{tikzcd}
$$

FIGURE 3.6. compatibility with the identity

They should satisfy the functoriality Figure 3.5 and the compatibility with the identity Figure 3.6.

Definition 3.6. A *pre-additive category* $\mathcal{C}$ is a category enriched over the monoidal category $(\mathfrak{Ab}, \otimes_{\mathbb{Z}})$.

For pre-additive categories $\mathcal{A}$ and $\mathcal{B}$, an *additive functor* $F\colon \mathcal{A}\to\mathcal{B}$ is an enriched functor over $\mathfrak{Ab}$.

Following [Pop73], we will assume that a pre-additive category admits a zero object. Recall that an object 0 of a category is called a *zero object* if it is simultaneously an initial object and a final object. Recall, in turn, that an object i of a category $\mathcal{C}$ is *initial* if for any object $x \in \mathcal{C}$ it holds that $|\mathcal{C}(i,x)| = 1$. Dually, an object f is *final* if $|\mathcal{C}(x,f)| = 1$ holds for any $x \in \mathcal{C}$.

Example 3.7. The trivial abelian group is a zero object of $\mathfrak{Ab}$.

Definition 3.8. Let $\mathcal{C}$ be a pre-additive category and take objects $x, y \in \mathcal{C}$. The *coproduct* $x \oplus y \in \mathcal{C}$ is an object which represents the functor

$$\mathcal{C}(x,-) \times \mathcal{C}(y,-) : \mathcal{C} \to \mathfrak{Ab}; \tag{3.8}$$

namely, an object $x \oplus y$ which admits an isomorphism of functors

$$\mathcal{C}(x \oplus y,-) \simeq \mathcal{C}(x,-) \times \mathcal{C}(y,-). \tag{3.9}$$

Dually, the *product* $x \times y \in \mathcal{C}$ is an object which admits an isomorphism of functors

$$\mathcal{C}(-,x \times y) \simeq \mathcal{C}(-,x) \times \mathcal{C}(-,y) : \mathcal{C}^{op} \to \mathfrak{Ab}. \tag{3.10}$$

Definition 3.9. An *additive category* is a pre-additive category $\mathcal{C}$ such that for any pair of object $x, y \in \mathcal{C}$ there exists a coproduct $x \oplus y \in \mathcal{C}$.

If one replace $\mathfrak{Ab}$ with $\operatorname{Mod} R$, where R is a commutative ring, then one obtains the definition of an *R-linear category.*

Exercise 3.10. Let $\mathcal{C}$ be an additive category. Show that $\mathcal{C}$ admits arbitrary finite coproducts and finite products. Show moreover that they are canonically isomorphic.

Exercise 3.11. Let $\mathcal{C}$ be a pre-additive category. Show that a morphism $f\colon x \to y$ satisfies $f = 0 \in \mathcal{C}(x,y)$ if and only if f factors through 0; i.e., if there exists a decomposition $f = [0 \to y] \circ [x \to 0]$.

Definition 3.12. Let $\mathcal{C}$ be a pre-additive category and $f\colon x \to y$ be a morphism in it. The *kernel* of f is an object $\ker f \in \mathcal{C}$ which admits an isomorphism of functors

$$\mathcal{C}(-,\ker f) \xrightarrow{\sim} \ker f_* = \{g \in \mathcal{C}(-,x) \,|\, f \circ g = 0\} : \mathcal{C}^{op} \to \mathfrak{Ab}. \tag{3.11}$$

Dually, the *cokernel* of f is an object $\operatorname{coker} f \in \mathcal{C}$ which admits an isomorphism of functors

$$\mathcal{C}(\operatorname{coker} f,-) \xrightarrow{\sim} \ker f^* = \{g \in \mathcal{C}(y,-) \,|\, g \circ f = 0\} : \mathcal{C} \to \mathfrak{Ab}. \tag{3.12}$$

Moreover, the *image* and the *coimage* of f are defined as

$$\operatorname{im} f = \ker(x \to \operatorname{coker} f), \operatorname{coim} f = \operatorname{coker}(\ker f \to x). \tag{3.13}$$

Exercise 3.13. Recall that a morphism $f\colon x \to y$ is called a *monomorphism* if for any morphisms $g, h\colon y \rightrightarrows z$, $f \circ g = f \circ h$ implies $g = h$ (i.e., if f is left cancellable). Show that f is a monomorphism if and only if $\ker f = 0$. Similarly, show that f is an epimorphism (i.e., right cancellable) if and only if $\operatorname{coker} f = 0$.

Exercise 3.14. Show that there exists a unique morphism $\overline{f}\colon \operatorname{coim} f \to \operatorname{im} f$ such that

$$f = [\operatorname{im} f \to y] \circ \overline{f} \circ [x \to \operatorname{coim} f]. \tag{3.14}$$

Definition 3.15. An additive category is called *pre-abelian* if any morphism admits a kernel and a cokernel (hence also an image and a coimage).

Remark 3.16. Objects such as kernel, cokernel, etc. are defined as objects which represent specific functors. Hence they are unique up to the unique canonical isomorphism. Some people use the article "the" because of this strong uniqueness. In the author's opinion, it is merely a matter of taste.

Example 3.17. Let $\mathcal{C}$ be the category of locally free sheaves on a scheme X. One can easily show that $\mathcal{C}$ is an additive category, but is almost always not pre-abelian. For example, consider the case $X = \mathbb{P}^1$ over any field $\mathbf{k}$ and consider a non-trivial map $\varphi\colon \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)$. It follows that there is no cokernel of this morphism in the category $\mathcal{C}$. If one enlarge the category to $\operatorname{coh}\mathbb{P}^1$, the category of coherent sheaves on $\mathbb{P}^1$, then it is in fact an abelian category. A cokernel of the map φ in $\operatorname{coh}\mathbb{P}^1$ is given by the skyscraper sheaf $\mathcal{O}_x$ at the point $x \in \mathbb{P}^1$ defined by the linear equation which corresponds to the map φ (it is a good exercise on scheme theory to fill in the details of this argument).

In general, $\operatorname{coh} X$ is an abelian category (defined below) when X is a Noetherian scheme.

Definition 3.18. A pre-abelian category is called an *abelian category* if for any morphism f the canonical map $\overline{f}$ of (3.14) is an isomorphism.

One can refer to [Pop73] for basics of additive and abelian categories. An English translation of Grothendieck's Tohoku paper is available at

`http://www.math.mcgill.ca/barr/papers/gk.pdf`

4. Recap on algebraic geometry

5. (Some) homological algebra

Definition 5.1. Let $\mathcal{C}$ be an abelian category.

(1) A sequence of morphisms

$$E \xrightarrow{f} F \xrightarrow{g} G \tag{5.1}$$

in $\mathcal{C}$ is called a *complex* if

$$g \circ f = 0. \tag{5.2}$$

This is equivalent to saying

$$\operatorname{im} f \subset \ker g; \tag{5.3}$$

i.e., if the canonical map $\operatorname{im} f \to G$ factors through $\ker g \to G$.

(2) Let us assume that (5.1) is a complex. Then its *cohomology* is defined to be

$$h\left(E \xrightarrow{f} F \xrightarrow{g} G\right) = \operatorname{coker}\left(\operatorname{im} f \to \ker g\right). \tag{5.4}$$

(3) Let us assume that (5.1) is a complex. It is said to be *exact* (at F) if

$$h\left(E \xrightarrow{f} F \xrightarrow{g} G\right) = 0. \tag{5.5}$$

(4) More generally, consider a sequence of morphisms in $\mathcal{C}$ of any length:

$$\cdots \xrightarrow{f^{i-1}} E^i \xrightarrow{f^i} E^{i+1} \xrightarrow{f^{i+1}} E^{i+2} \xrightarrow{f^{i+2}} \cdots \tag{5.6}$$

It is said to be a complex if the truncation

$$E^i \xrightarrow{f^i} E^{i+1} \xrightarrow{f^{i+1}} E^{i+2} \tag{5.7}$$

is a complex as defined above for any index i. Similarly, it is said to be exact if it is exact at any E^i.

(5) A *short exact sequence* is an exact sequence of the form

$$0 \to E \xrightarrow{f} F \xrightarrow{g} G \to 0. \tag{5.8}$$

Remark 5.2. The complex (5.1) has an interpretation as follows[3]. The morphism $F \xrightarrow{g} G$ can be regarded as a system of equations, so that $\ker g$ is "the set of solutions". On the other hand, $\operatorname{im} f$ can be regarded as "the set of obvious solutions". Therefore $h((5.1)) = \operatorname{coker}(\operatorname{im} f \to \ker g)$ should be interpreted as "the set of non-trivial solutions". The definition of singular or simplicial homology group of a space or de Rham cohomology of a smooth manifold is a convincing example [BT82].

Exercise 5.3. Show that

$$0 \to E \xrightarrow{f} F \tag{5.9}$$

is exact at E if and only if $\ker f = 0$. Similarly, show that

$$F \xrightarrow{g} G \to 0 \tag{5.10}$$

is exact at G if and only if $\operatorname{coker} g = 0$.

Definition 5.4. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is said to be *left exact* if for any exact sequence

$$0 \to x \xrightarrow{f} y \xrightarrow{g} z, \tag{5.11}$$

the sequence

$$0 \to F(x) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z) \tag{5.12}$$

is again exact. Dually, F is said to be *right exact* if for any exact sequence

$$x \xrightarrow{f} y \xrightarrow{g} z \to 0, \tag{5.13}$$

the sequence

$$F(x) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z) \to 0 \tag{5.14}$$

is again exact. Finally, F is said to be *exact* if it is both left and right exact.

Exercise 5.5. Show that an additive functor between abelian categories is exact if and only if it sends an arbitrary exact sequence to an exact sequence.

Exercise 5.6. Let $\mathcal{C}$ be an abelian category. For an object $x \in \mathcal{C}$, consider functors

$$\mathcal{C}(x, -) : \mathcal{C} \to \mathfrak{Ab} \tag{5.15}$$

and

$$\mathcal{C}(-, x) : \mathcal{C}^{op} \to \mathfrak{Ab}. \tag{5.16}$$

Show that both of them are additive and left exact, but neither of them is not necessarily exact.

Exercise 5.7. Let R be a ring, and fix $M \in R\operatorname{Mod}$. Show that the functor

$$- \otimes_R M\colon \operatorname{Mod} R \to \mathfrak{Ab} \tag{5.17}$$

is additive and right exact, but not necessarily exact.

Although the functor (5.15) nor (5.16) is exact in general, they do become exact for some special objects x.

Definition 5.8. (1) An object $P \in \mathcal{C}$ is *projective* if the functor $\mathcal{C}(P, -)$ is exact.
(2) Dually, an object $I \in \mathcal{C}$ is *injective* if the functor $\mathcal{C}(-, I)$ is exact.
(3) Similarly, $M \in R\operatorname{Mod}$ is *left flat* if the functor $- \otimes_R M$ is exact.

[3]The author learned this from Takeshi Saito in one of his lectures on linear algebra.

Example 5.9. (1) Let R be a ring. Then R itself, namely $R_R \in \operatorname{Mod} R$ is projective.

(2) Let R be a commutative integral domain. Then the field of fractions $Q \in \operatorname{Mod} R$ is injective. In particular, $\mathbb{Q} \in \mathfrak{Ab} = \operatorname{Mod} \mathbb{Z}$ is injective. Note that it is *not* finitely generated as a module over $\mathbb{Z}$.

(3) Let R be a ring and $M \in R \operatorname{Mod}$ be projective. Then it is automatically left flat. This follows from

Exercise 5.10. (1) Let $(M_i)_{i \in I}$ be a set of objects such that $\bigoplus_{i \in I} M_i$ exists in $\mathcal{C}$. Show that it is projective if and only if each M_i is projective for all $i \in I$. As a corollary, show that when $\mathcal{C} = \operatorname{Mod} R$, projective objects are nothing but direct summands of free modules.

(2) Show that there is no injective object in $\operatorname{mod} \mathbb{Z}$, the category of *finitely generated* abelian groups, except for 0.

(3) Let R be a commutative ring and $S \subset R$ a multiplicatively closed set (i.e., a subset such that $s, t \in S \Rightarrow st \in R$). Show that the localization functor

$$- \otimes_R S^{-1}R \colon \operatorname{Mod} R \to \operatorname{Mod} S^{-1}R \tag{5.18}$$

is exact. Moreover, show that it restricts to a functor between the subcategories of finitely generated modules

$$\operatorname{mod} R \to \operatorname{mod} S^{-1}R \tag{5.19}$$

and is essentially surjective(!) if R is Noetherian (hint: use the fact that any finitely generated module over a Noetherian ring is finitely presented).

Definition 5.11. An abelian category $\mathcal{C}$ is said to have *enough injectives* if for any $x \in \mathcal{C}$ there exists an exact sequence

$$0 \to x \to I \tag{5.20}$$

such that $I \in \mathcal{C}$ is injective.

Dually, $\mathcal{C}$ is said to have *enough projectives* if for any $x \in \mathcal{C}$ there exists an exact sequence

$$P \to x \to 0 \tag{5.21}$$

such that $P \in \mathcal{C}$ is projective.

Example 5.12. (1) For a ring R, the category $\operatorname{Mod} R$ always has enough projectives. This is just because R_R is projective and any object admits an epimorphism from a free module.

(2) On the other hand, if X is a scheme which is not affine, then quite often $\operatorname{coh} X$ does not have enough projectives. For example, suppose that X is a smooth projective variety of dimension $n \geq 1$ over a field $\mathbf{k}$ and F is a non-zero coherent sheaf on X. Then by the Serre duality

$$\operatorname{Ext}^n(F, L) \simeq H^0\left(X, F \otimes L^{-1} \otimes \omega_X\right)^\vee \tag{5.22}$$

for any line bundle L. In particular, if one chooses sufficiently anti-ample L, then (5.22) is not zero since $F \neq 0$. This implies that F is not projective.

Remark 5.13. Let R be a commutative ring. It is easy to see that an R-algebra with a unit is nothing but an R-linear category with only one object. Conversely, one could say that an R-linear category is an "R-algebra with many objects (R-algebroid)".

Definition 5.14. An *injective resolution* of an object $x \in \mathcal{C}$ is an exact sequence

$$0 \to x \to I^0 \to I^1 \to I^2 \to \cdots \tag{5.23}$$

such that I^i are injective for all $i \in \mathbb{Z}_{\geq 0}$. Dually, a *projective resolution* of x is an exact sequence

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to x \to 0 \tag{5.24}$$

such that P^i are projective for all $i \in \mathbb{Z}_{\leq 0}$.

Exercise 5.15. Suppose that $\mathcal{C}$ has enough injectives. Show that any object of $\mathcal{C}$ admits an injective resolution. Dually, show that any object admits a projective resolution provided that $\mathcal{C}$ has enough projectives.

(hint: To obtain an injective resolution of $x \in \mathcal{C}$, take a monomorphism f to an injective object $0 \to x \xrightarrow{f} I^0 \to \operatorname{coker} f \to 0$. Then embed $\operatorname{coker} f$ into an injective object I^1, so that $0 \to x \to I^0 \to I^1$ is still exact. Check that one can repeat this process again and again, so as to obtain an injective resolution. Note that particular properties of injective objects are not used in this argument!)

Definition 5.16. Let $E^\bullet, F^\bullet$ be two complexes of objects in $\mathcal{C}$.

(1) A *(chain) homotopy*

$$H \colon E^\bullet \to F^\bullet \tag{5.25}$$

is a collection of morphisms

$$H^i \colon E^i \to F^{i-1} \tag{5.26}$$

for each $i \in \mathbb{Z}$.

(2) A *(chain) homotopy* between two morphism $f, g \colon E^\bullet \rightrightarrows F^\bullet$ of complexes is a homotopy H as defined above such that

$$f - g = dH + Hd, \tag{5.27}$$

in the sense that

$$f^i - g^i = d_F^{i-1} \circ H^i + H^{i+1} \circ d_E^i \tag{5.28}$$

holds for any $i \in \mathbb{Z}$.

Definition 5.17. Let $E^\bullet, F^\bullet$ be two complexes. A *quasi-isomorphism* is a morphism $f^\bullet \colon E^\bullet \to F^\bullet$ such that the induced maps

$$h^i(f) : h^i(E^\bullet) \to h^i(F^\bullet) \tag{5.29}$$

on the cohomology[4] objects are isomorphisms for any $i \in \mathbb{Z}$.

Exercise 5.18. Let $f \colon I^\bullet \to J^\bullet$ be a quasi-isomorphism between complexes of injective objects bounded below (i.e. $I^i = J^i = 0$ if $i << 0$). Show that f is homotopy equivalent to an isomorphism.

Definition 5.19. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a left exact functor between abelian categories. Assume that $\mathcal{C}$ has enough injectives. For each object $x \in \mathcal{C}$, fix an injective resolution $I_x^\bullet$. Then the i-th *(right) derived functor*

$$R^i F \colon \mathcal{C} \to \mathcal{D} \tag{5.30}$$

[4] "cohomology" is an uncountable noun, so one should not say "cohomologies".

for each $i \in \mathbb{Z}_{\geq 0}$ is defined as

$$h^i \left(F \left(I_x^\bullet \right) \right). \tag{5.31}$$

Dually, if F is right exact, then the i-th *(left) derived functor*

$$L_i F \colon \mathcal{C} \to \mathcal{D} \tag{5.32}$$

for each $i \in \mathbb{Z}_{\geq 0}$ is defined as

$$h^{-i} \left(F \left(P_x^\bullet \right) \right). \tag{5.33}$$

Here again one fixes a projective resolution $P_x^\bullet$ for each $x \in \mathcal{C}$ in advance.

Note that the isomorphism class of the derived functor is independent of the choice of injective resolutions because of Exercise 5.18. It is obvious from the definition that $R^0 F \simeq F$.

Example 5.20. (1) For an abelian category $\mathcal{C}$ with enough injectives and an object $x \in \mathcal{C}$, the i-th derived functor of $\mathcal{C}(x, -) : \mathcal{C} \to \mathfrak{Ab}$ is denoted by

$$\operatorname{Ext}^i_{\mathcal{C}} (x, -) \tag{5.34}$$

and called the *i-th extension group*. Dually, if $\mathcal{C}$ has enough projectives, then the i-th derived functor of $\mathcal{C}(-, x) : \mathcal{C} \to \mathfrak{Ab}$ is denoted by

$$\operatorname{Ext}^i_{\mathcal{C}} (-, x) \tag{5.35}$$

and also called the *i-th extension group*. Note that an object of $\mathcal{C}$ is projective if and only if it is injective as an object of $\mathcal{C}^{op}$.

(2) Fix a ring R and consider a left R-module M. Then the i-th left derived functor of the right exact functor $- \otimes_R M \colon R\,\mathrm{Mod} \to \mathfrak{Ab}$ is denoted by

$$\operatorname{Tor}^R_i (-, M) \tag{5.36}$$

and called the *i-th torsion* of M.

(3) Let $f \colon X \to Y$ be a morphism of schemes. Then the i-th derived functor of the left exact functor $f_* \colon \operatorname{Qcoh} X \to \operatorname{Qcoh} Y$ is called the *i-th higher direct image of f* and is denoted by

$$R^i f_*. \tag{5.37}$$

When X, Y are Noetherian schemes and f is proper, then the functor restricts to

$$R^i f_* \colon \operatorname{coh} X \to \operatorname{coh} Y \tag{5.38}$$

(the Grauert's theorem).

6. Derived category and the theorem of Bondal

7. Artin-Schelter algebras and AS-regular $\mathbb{Z}$-algebras

Definition 7.1. Let I be a set and $\mathcal{I}$ a monoidal category. An *I-algebra over $\mathcal{I}$* is a category $\mathcal{C}$ which is enriched over $\mathcal{I}$ such that

$$\operatorname{ob} \mathcal{C} = I. \tag{7.1}$$

From now on we only consider the case $\mathcal{I} = \operatorname{Mod} R$ for a commutative ring R, unless otherwise stated. In this case, an I-algebra is equivalently described as a doubly indexed associative R-algebra

$$A = \bigoplus_{i,j \in I} A_{i,j}, \tag{7.2}$$

where

$$A_{i,j} = \mathcal{C}(j,i) \tag{7.3}$$

and the product is naturally defined by the composition law of the category $\mathcal{C}$; in particular, the map

$$A_{i,j} \otimes_R A_{j',k} \to A_{i,k} \tag{7.4}$$

is non-trivial only if $j = j'$.

A does not admit a unit in general, but it comes with *local units*

$$e_i = \mathrm{id}_i \in A_{i,i} \tag{7.5}$$

for each $i \in I = \mathrm{ob}\,\mathcal{C}$.

Conversely, any I-algebra $\mathcal{C}$ is obtained from such an algebra A by an obvious manner.

Example 7.2.

(1) When $|I| = 1$, then I-algebra is nothing but an associative unital R-algebra.

(2) Let

$$S = \bigoplus_{i \in \mathbb{Z}} S_i \tag{7.6}$$

be a graded ring. Then the *associated* $\mathbb{Z}$*-algebra*

$$\check{S} = \bigoplus_{i,j \in \mathbb{Z}} \check{S}_{i,j} \tag{7.7}$$

is defined by $\check{S}_{i,j} = S_{j-i}$.

(3) Let X be a projective variety such that $\mathrm{Pic}\,X$ is a free abelian group of finite rank ρ (just for simplicity). Choose invertible sheaves $L_1, L_2, \ldots, L_\rho$ on X whose classes generate the Picard group and consider the full subcategory

$$\mathcal{C} = \left\{ L_1^{\otimes m_1} \otimes L_2^{\otimes m_2} \otimes \cdots \otimes L_\rho^{\otimes m_\rho} \,\middle|\, (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho \right\} \tag{7.8}$$

of $\mathrm{coh}\,X$. One can regard it as a $\mathbb{Z}^\rho$-algebra, and in fact it is the one associated to the *Cox ring*

$$\mathrm{Cox}\,(X; L_1, L_2, \ldots, L_\rho) = \bigoplus_{(m_1,\ldots,m_\rho) \in \mathbb{Z}^\rho} H^0\left(X, L_1^{\otimes m_1} \otimes L_2^{\otimes m_2} \otimes \cdots \otimes L_\rho^{\otimes m_\rho}\right). \tag{7.9}$$

Cox ring is not necessarily of finite type, and X is called a *Mori dream space* if it is of finite type. Typical examples are varieties of Fano type (hence in particular Schubert varieties and spherical (hence in particular toric) varieties) and K3 surfaces whose automorphism groups are finite.

If X is a Mori dream space, one can describe X as a GIT quotient of the stable locus of $\mathrm{Spec}\,\mathrm{Cox}\,X$ by the algebraic torus $\mathrm{Hom}_{gp}\left(\mathrm{Pic}(X), \mathbb{G}_m\right)$ with respect to an "ample" stability condition. In particular, one can describe $\mathrm{coh}\,X$ as $\mathrm{qgr}\,\mathrm{Cox}\,X$ or $\mathrm{qgr}\,\mathcal{C}$; here the torsion modules are defined as those supported on the unstable locus of $\mathrm{Cox}\,X$.

See [HK00, Oka16] for details.

Exercise 7.3. For a $\mathbb{Z}$-algebra A, one can define the shifted $\mathbb{Z}$-algebra $A\,(1)$ by

$$A\,(1)_{i,j} \coloneqq A_{i+1,j+1}. \tag{7.10}$$

A is said to be 1-*periodic* if $A \simeq A\,(1)$ as $\mathbb{Z}$-algebras. Show that A is 1-periodic if and only if there exists a $\mathbb{Z}$-graded algebra S such that $A \simeq \check{S}$.

Definition 7.4. Let $\mathcal{C}$ be an I-algebra. A $\mathcal{C}$-module is an object of $\operatorname{Fun}(\mathcal{C}, \mathfrak{Ab})$ (N.B. Fun is the category of *additive* functors). Morphisms between two modules over $\mathcal{C}$ are natural transformations of functors.

In short, the category of modules over $\mathcal{C}$ is nothing but the (additive) functor category $\operatorname{Fun}(\mathcal{C}, \mathfrak{Ab})$. This category will also be denoted by $\operatorname{Grmod} A$.

Exercise 7.5. If A is the algebra corresponding to the $\mathcal{I}$-algebra defined as a category $\mathcal{C}$ as above, then a module over A is a right module M of A in the usual sense such that $M \cdot A = M$. Show this. (hint: the assumption implies that there exists a decomposition $M = \bigoplus_{i \in I} M \cdot e_i = \bigoplus_{i \in I} M_i$. Check that one can define the functor $\mathcal{C} \to \mathfrak{Ab}$ by $i \mapsto M_i$.)

Let $\mathcal{C}$ be an I-algebra, and A be the corresponding description as an associative ring. Then for each $i \in I$ there exists a projective module (i.e. a projective object)

$$P_i := e_i A \in \operatorname{Grmod} A. \tag{7.11}$$

Also, there exists the simple module S_i defined by the following short exact sequence

$$0 \to \bigoplus_{j \neq i} A_{i,j} \otimes P_j \to P_i \to S_i \to 0. \tag{7.12}$$

Let us know define the notions of AS-regular $\mathbb{Z}$-algebras.

Definition 7.6. Fix a commutative base ring R. A $\mathbb{Z}$-algebra A over R is called an *Artin-Schelter regular $\mathbb{Z}$-algebra* if it satisfies the following conditions.

(1) A is *connected* in the sense that $A_{i,i}$ is a free R-module spanned by $e_i \in A_{i,i}$ and $A_{i,j} = 0$ if $i > j$; i.e.,

$$A = \bigoplus_{i \leq j} A_{i,j}. \tag{7.13}$$

(2) Each $A_{i,j}$ is finite and flat (i.e. "locally free") over R, and there exists a polynomial $P(t) \in \mathbb{Z}[t]$ independent of i, j such that

$$\operatorname{rank}_R A_{i,j} \leq P(|i - j|) \tag{7.14}$$

holds for any i, j. This condition is sometimes rephrased as the finite dimensionality of the Gelfand-Kirillov dimension.

(3) The projective dimensions of the simples S_i are uniformly bounded from above.

(4) For any object $i \in I$

$$\sum_{j \in I, k \in \mathbb{Z}} \operatorname{rank}_R \operatorname{Ext}^k_{\operatorname{Grmod} A}(S_i, P_j) = 1. \tag{7.15}$$

This is called the *AS-Gorenstein* condition.

Example 7.7. consider a polynomial ring in $n+1$ variables $S = \boldsymbol{k}[x_0, x_1, \ldots, x_n]$ over a field $\boldsymbol{k}$. To make the arguments natural, let us fix a vector space V of dimension $n+1$ and consider

$$S = \operatorname{Sym}^\bullet V = \bigoplus_{d \geq 0} \operatorname{Sym}^d V. \tag{7.16}$$

Then the associated $\mathbb{Z}$-algebra $A = \check{S}$ is naturally isomorphic to the $\mathbb{Z}$-algebra

$$\mathcal{C} = \{\mathcal{O}_{\mathbb{P}V}(-d) \,|\, d \in \mathbb{Z}\} \subset \operatorname{coh} \mathbb{P}V. \tag{7.17}$$

The isomorphism is given by

$$A_{i,j} = S_{j-i} = \operatorname{Sym}^{j-i} V \simeq H^0(\mathbb{P}V, \mathcal{O}(j-i)) \simeq \operatorname{coh}_{\mathbb{P}V}(\mathcal{O}(-j), \mathcal{O}(-i)). \tag{7.18}$$

Let us check that A is an AS-regular algebra of dimension $n+1$. In fact, Artin and Schelter defined the notion of AS-regular algebras (for graded algebras) as a generalization of commutative polynomial rings.

(1) This is obvious.

(2) Suppose $i \le j$. Then $\dim_{\boldsymbol{k}} A_{i,j} = \dim \operatorname{Sym}^{j-i} V = P(j-i)$, where

$$P(t) = \binom{n+1+t-1}{n+1} = \frac{(n+t)!}{(n+1)!\,(t-1)!} \tag{7.19}$$

is a polynomial in t of degree $n+1$. Hence the GK-dimension of A is $n+1$.

(3) In fact, one can explicitly compute the projective resolution of S_i as follows:

$$0 \to \wedge^{n+1} V \otimes P_{i+n+1} \to \wedge^n V \otimes P_{i+n} \to \cdots \to \wedge^2 V \otimes P_{i+2} \to V \otimes P_{i+1} \to P_i \to S_i \to 0. \tag{7.20}$$

This is merely the $\mathbb{Z}$-algebra version of the famous *Koszul* resolution. In fact, under the equivalence of categories $\operatorname{grmod} S \simeq \operatorname{grmod} A$, the sequence (7.20) is sent to the standard Koszul resolution of the simple module $\boldsymbol{k} = S/S_{>0}$.

(4) If $i \ge j$, applying the functor $\operatorname{grmod} A\,(-, P_j)$ to (7.20), one obtains the sequence

$$\begin{aligned} 0 \to \operatorname{Sym}^{i-j} V \to V^\vee \otimes \operatorname{Sym}^{i-j+1} V \to \wedge^2 V^\vee \otimes \operatorname{Sym}^{i-j+2} V \to \\ \cdots \to \wedge^{n+1} V^\vee \otimes \operatorname{Sym}^{i-j+n+1} V \to 0. \end{aligned} \tag{7.21}$$

Combined with the canonical (up to a universal constant) isomorphism

$$\wedge^i V \simeq \left(\wedge^{n+1-i} V\right)^\vee \simeq \wedge^{n+1-i} V^\vee, \tag{7.22}$$

one can identify (7.21) with the degree $i-j+n+1$ part of the Koszul resolution

$$\begin{aligned} 0 \to \wedge^{n+1} V \otimes S\,(-n-1) \to \wedge^n V \otimes S\,(-n) \to \\ \cdots \to \wedge^2 V \otimes S\,(-2) \to V \otimes S\,(-1) \to S \to S/S_{>0} \to 0. \end{aligned} \tag{7.23}$$

Hence it is exact if $i \ge j$.

If $i < j-n-1$, then we obtain a trivial complex.

If $j > i > j-n-1$, then we obtain the truncated version of the sequence:

$$\begin{aligned} 0 \to \wedge^{j-i} V^\vee \otimes \operatorname{Sym}^0 V \to \wedge^{j-i+1} V^\vee \otimes \operatorname{Sym}^1 V \to \\ \cdots \to \wedge^{n+1} V^\vee \otimes \operatorname{Sym}^{i-j+n+1} V \to 0, \end{aligned} \tag{7.24}$$

which is again exact by the same reason as above.

Finally, if $i = j-n-1$, then the complex is

$$0 \to \wedge^0 V^\vee \otimes \operatorname{Sym}^0 V \to 0, \tag{7.25}$$

where the unique nontrivial entry sits in degree $n+1$. Thus we obtain the conclusions.

Definition 7.8. (1) An AS-regular $\mathbb{Z}$-algebra is *quadratic of dimension* $n+1$ if each simple S_i admits a resolution of the form

$$0 \to P_{i+n+1}^{\oplus m_{i+n+1}} \to P_{i+n}^{\oplus m_{i+n}} \to \cdots \to P_{i+2}^{\oplus m_{i+2}} \to P_{i+1}^{\oplus m_{i+1}} \to P_i^{\oplus m_i} \to S_i \to 0, \tag{7.26}$$

where

$$m_{i+\ell} = \binom{n+1}{\ell}. \tag{7.27}$$

In particular when $n=2$, the resolution has the following form:

$$0 \to P_{i+3} \to P_{i+2}^{\oplus 3} \to P_{i+1}^{\oplus 3} \to P_i \to S_i \to 0. \tag{7.28}$$

(2) An AS-regular $\mathbb{Z}$-algebra is *cubic of dimension* 3 if each simple S_i admits a resolution of the form

$$0 \to P_{i+4} \to P_{i+3}^{\oplus 2} \to P_{i+1}^{\oplus 2} \to P_i \to S_i \to 0. \tag{7.29}$$

8. Classification of 3-dimensional AS-regular algebras

Definition 8.1. Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{D} \subset \mathcal{C}$ is called a *Serre subcategory* if it is closed under extensions and subquotients. If it is also closed under arbitrary coproducts, then it is called a *localizing subcategory.*

Note that a Serre subcategory of an abelian category is automatically an abelian category with the same kernels and cokernels.

Definition 8.2. Let $\mathcal{D} \subset \mathcal{C}$ be as in 8.1. Then the *quotient category* $\mathcal{C}/\mathcal{D}$ is defined as follows.

- $\operatorname{ob}(\mathcal{C}/\mathcal{D}) = \operatorname{ob}\mathcal{C}$.
- Set

  $$\mathcal{S} = \{s \in \hom \mathcal{C} | \ker s, \operatorname{coker} s \in \mathcal{D}\} \subset \hom \mathcal{C}. \tag{8.1}$$

  For objects $E, F \in \mathcal{C}/\mathcal{D}$ set

  $$\mathcal{C}/\mathcal{D}(E,F) = \left\{ E \xleftarrow{s} G \xrightarrow{f} F | s \in \mathcal{S}, f \in \mathcal{C}(E,F) \right\} / \sim, \tag{8.2}$$

  where

  $$\left(E \xleftarrow{s} G \xrightarrow{f} F\right) \sim \left(E' \xleftarrow{s'} G' \xrightarrow{f'} F'\right) \tag{8.3}$$
  $$\iff \exists G'' \in \operatorname{ob}\mathcal{C}, \exists s_1 \colon G'' \to G, s_2 \colon G'' \to G' \tag{8.4}$$
  $$\text{such that } ss_1 = s's_2 \text{ and } fs_1 = f's_2. \tag{8.5}$$

- The composition of morphisms $\left[E \xleftarrow{s} G \xrightarrow{f} F\right]$ and $\left[F \xleftarrow{s'} G' \xrightarrow{f'} H\right]$ is defined as follows. Find $s'' \colon G'' \to G \in \mathcal{S}$ and $f'' \colon G'' \to G'$ such that $fs'' = s'f''$. Then define the composition as

  $$\left[E \xleftarrow{ss''} G'' \xrightarrow{f'f''} H\right]. \tag{8.6}$$

Exercise 8.3. Show that the above construction yields a well-defined category.

Theorem 8.4. *(1) The Serre quotient category $\mathcal{C}/\mathcal{D}$ is abelian. Moreover, the natural functor*

$$\pi \colon \mathcal{C} \to \mathcal{C}/\mathcal{D} \tag{8.7}$$

is essentially surjective and exact.

(2) $\mathcal{D}$ is localizing if and only if the functor π admits a right adjoint $\omega \colon \mathcal{C}/\mathcal{D} \to \mathcal{C}$.

Definition 8.5.
- A *linear triple* is a triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$.
- An *admissible triple* is a triple

  $$(Y, L_0, L_1) \tag{8.8}$$

  of a curve Y and very ample line bundles L_0, L_1 on Y such that
  (1) Y is isomorphic to a plane cubic curve,
  (2) the complete linear system associated to L_i embeds Y into $\mathbb{P}^2$ as a cubic curve for $i = 0, 1$,
  (3) $\deg L_0|_C = \deg L_1|_C$ holds for any irreducible component $C \subset Y$,

(4) $L_0 \not\simeq L_1$.

- An *elliptic triple* is a linear triple or an admissible triple.
- An isomorphism from an elliptic triple (Y, L_0, L_1) to another elliptic triple (Y', L_0', L_1') is a triple
$$(\varphi, \psi_0, \psi_1), \tag{8.9}$$
where
  - $\varphi\colon Y \xrightarrow{\sim} Y'$ is an isomorphism of schemes,
  - $\psi_i\colon L_i \xrightarrow{\sim} \varphi^* L_i'$ is an isomorphism of line bundles for $i = 0, 1$.
- The category $\mathcal{E}\ell\ell$ of *elliptic triples* is the category whose objects are elliptic triples and whose morphisms are isomorphisms of elliptic triples. In particular, the category of elliptic triples is a groupoid.

Let $\mathcal{Q}uad$ be the category whose objects are quadratic 3-dimensional regular $\mathbb{Z}$-algebras and morphisms are isomorphisms of $\mathbb{Z}$-algebras.

Theorem 8.6. *The categories $\mathcal{E}\ell\ell$ and $\mathcal{Q}uad$ are canonically equivalent to each other.*

We describe a pair of functors
$$\alpha\colon \mathcal{E}\ell\ell \to \mathcal{Q}uad \tag{8.10}$$
and
$$\beta\colon \mathcal{Q}uad \to \mathcal{E}\ell\ell \tag{8.11}$$
which are quasi-inverses to each other.

α**:** Take an elliptic triple (Y, L_0, L_1). Define a sequence of line bundles $(L_i)_{i\in\mathbb{Z}}$ by the recurrence relation
$$L_i \otimes L_{i+1}^{\otimes -2} \otimes L_{i+2} \simeq \mathcal{O}_Y. \tag{8.12}$$
Then for each $i \in \mathbb{Z}$ one can show that the natural map
$$H^0(L_i) \otimes H^0(L_{i+1}) \to H^0(L_i \otimes L_{i+1}) \tag{8.13}$$
is surjective. Let R_i be the kernel.

Set $A_{i,i+1} := H^0(L_i)$ and consider the $\mathbb{Z}$-algebra freely generated by them. Then the $\mathbb{Z}$-algebra $\alpha(Y, L_0, L_1)$ is obtained by taking the quotient by the two-sided ideal generated by R_i.

β**:** Take a 3-dimensional quadratic regular $\mathbb{Z}$-algebra A. Then from the assumptions one can deduce that the multiplication map
$$m\colon A_{0,1} \otimes A_{1,2} \to A_{0,2} \tag{8.14}$$
is surjective, so as to obtain a 3-dimensional linear subspace
$$\ker m \subset A_{0,1} \otimes A_{1,2} \simeq H^0(\mathbb{P}A_{0,1} \times \mathbb{P}A_{1,2}, \mathcal{O}(1,1)). \tag{8.15}$$

Now let Y be the closed subscheme $Y \subset \mathbb{P}A_{0,1} \times \mathbb{P}A_{1,2} \simeq \mathbb{P}^2 \times \mathbb{P}^2$ defined by $\ker m$. One can then show that Y is projectively equivalent to the diagonal $\mathbb{P}^2$ if A is isomorphic to the $\mathbb{Z}$-algebra associated to $\boldsymbol{k}[x_0, x_1, x_2]$.

Otherwise Y is a complete intersection, so that Y is a curve of arithmetic genus 1. Moreover in this case one can show that the line bundles $L_0 := \mathcal{O}(1,0)|_Y$ and $L_1 := \mathcal{O}(0,1)|_Y$ satisfy the required assumptions.

It should be now obvious how to send the morphisms by the functors. See [VdB11] and [BP93] for details.

9. Deformation theory of abelian categories

It is nowadays a common sense that *any* deformation theory in *characteristic zero* is described by an appropriate differential graded Lie algebra. In particular, deformation theory of associative algebras over a field $\boldsymbol{k}$ of characteristic zero is described by the so-called *Hochschild dgla* of the original algebra. It turns out that the deformation theory of abelian categories can also be described by a suitable (even rather straightforward) generalization of the Hochschild dgla. This is a nice example of the philosophy "additive categories are merely a ring with many object".

The following exposition largely owe to [Man09] and [Kel03] ; the first one is a very nice introductory article on deformation theory *à la* dgla. The second one treats the deformation theory of associative algebras and also deformation quantization.

Definition 9.1. Let

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \tag{9.1}$$

be a graded vector space over a field $\boldsymbol{k}$ of characteristic 0, which is equipped with a differential

$$d \colon V \to V, \quad d^2 = 0, \tag{9.2}$$

of degree 1 and a bilinear form

$$[\ ,\ ] \colon V \otimes V \to V \tag{9.3}$$

of degree 0. Then the data $(V, d, [\ ,\ ])$ is called a *differential graded Lie algebra* (*dgla* for short) if the following conditions are satisfied. The degree of a homogeneous element $\bullet$ will be denoted by $|\bullet|$.

(1) (graded skew symmetry) For homogeneous $a, b \in V$,

$$[a, b] = -(-1)^{|a||b|}[b, a]. \tag{9.4}$$

(2) (graded Jacobi identity) For homogeneous $a, b, c \in V$,

$$(-1)^{|c||a|}[a, [b, c]] + (-1)^{|a||b|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0 \tag{9.5}$$

$$\iff [a, [b, c]] = (-1)^{|a||b|}[b, [a, c]] + [[a, b], c]. \tag{9.6}$$

(3) (graded Leibniz rule) For homogeneous $a, b \in V$,

$$d[a, b] = [da, b] + (-1)^{|a|}[a, db]. \tag{9.7}$$

For a dgla L, the equation

$$dx + \frac{1}{2}[x, x] = 0 \tag{9.8}$$

for $x \in L^1$ is called the *Maurer-Cartan equation.*

Let (A, m) be a local Artinian $\boldsymbol{k}$-algebra such that $A/m \simeq \boldsymbol{k}$. Then one can construct a new dgla $L \otimes_{\boldsymbol{k}} m$ equipped with the differential and the Lie bracket

$$d(x \otimes r) = dx \otimes r, \tag{9.9}$$

$$[x \otimes r, y \otimes s] = [x, y] \otimes rs. \tag{9.10}$$

If n is an integer such that $m^n = 0$, then for any $a \in L \otimes m$ it holds that $[a, -]^n = 0$.

Notations 9.2. The category of local Artinian $\boldsymbol{k}$-algebras (A, m) such that $A/m \simeq \boldsymbol{k}$ will be denoted by **Art**.

For a dgla L and $(A,m) \in \mathbf{Art}$, set

$$\mathbf{MC}_L(m) := \{\text{solutions of the MC equations of } L \otimes m\} \tag{9.11}$$

Definition 9.3. A dgla L is *nilpotent*[5] if for any $a \in L$ there exists a positive integer $n > 0$ such that $[a, -]^n = 0$.

Let L be a nilpotent dgla. Then for any $a \in L$ the exponential

$$e^a := \sum_{i=0}^{\infty} \frac{1}{n!} a^n \in L \tag{9.12}$$

is well-defined. Similarly for any $a \in L^0$ one can define the exponential automorphism

$$e^{[a,-]} := \sum_{i=0}^{\infty} \frac{1}{n!} [a,-]^n \colon L \xrightarrow{\sim} L. \tag{9.13}$$

Exercise 9.4.
- Show that
$$e^{[a,-]}(b) = e^a b e^{-a} \tag{9.14}$$
for any $a \in L^0$ and $b \in L$.
- Use the formula (9.14) to show that $e^{[a,-]} \colon L \xrightarrow{\sim} L$ is in fact an automorphism dgla. Namely, show that
$$e^{[a,-]}(db) = d\left(e^{[a,-]} b\right), \tag{9.15}$$
$$[e^{[a,-]}b, e^{[a,-]}c] = e^{[a,-]}[b,c]. \tag{9.16}$$

It follows from the exercise that $e^{[a,-]} \colon L \xrightarrow{\sim} L$ preserve the solutions of the Maurer-Cartan equation (9.8). Moreover, since for any $a, b \in L^0$ there exists the *Campbell-Hausdorff product*[6] $a \bullet b \in L^0$ satisfying

$$e^a e^b = e^{a \bullet b}, \tag{9.17}$$

it holds that

$$e^{[a,-]} \circ e^{[b,-]} = e^{[a \bullet b, -]}. \tag{9.18}$$

Combined with $e^{[0,-]} = \mathrm{id}_L$, we obtain the following proposition.

Proposition 9.5. *The image of the map*

$$e^{[-,-]} \colon L^0 \to \mathrm{Aut}_{\mathrm{dgla}}(L) \tag{9.19}$$

is a subgroup.

The prototypical example is given by the deformation theory of a differential $\overline{\partial}$ on the graded vector space (9.1).

Example 9.6. Let $\left(V, \overline{\partial}\right)$ be a complex. One can associate to it a dgla as follows.

- Consider the graded vector space
$$\mathrm{Hom}^\bullet(V,V) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(V,V), \tag{9.20}$$
where $\mathrm{Hom}^i(V,V)$ is the linear space of linear maps of degree i; namely, the linear space of linear maps $f \colon V \to V$ satisfying $|f(v)| = |v| + i$ for any homogeneous element $v \in V$.

[5] "Locally nilponent" sounds more reasonable, but I am not so sure.

[6] See [Ser06, Part I, Section 7] for details.

- One can introduce a differential d on $\mathrm{Hom}^\bullet(V,V)$ by

$$df := \overline{\partial} f - (-1)^{|f|} f\overline{\partial} \tag{9.21}$$

for any homogeneous f.
- The (graded) Lie bracket $[\ ,\ ]$ is defined by

$$[f,g] := fg - (-1)^{|f||g|} gf \tag{9.22}$$

for any homogeneous f, g.

For any $(A,m) \in \mathbf{Art}$, consider the nilpotent dgla $L = \mathrm{Hom}^\bullet(V,V) \otimes_{\boldsymbol{k}} m$.

Then the action restricts to the space of solutions of the Maurer-Cartan equation (9.8) of L by

$$e^{[a,-]} \colon \xi \mapsto e^{[a,-]}\left(\overline{\partial} + \xi\right) - \overline{\partial} = \xi + \sum_{i=0}^{\infty} \frac{1}{(i+1)!}[a,-]^i \left(d\xi + [a, \overline{\partial}]\right) \tag{9.23}$$

$$= \xi + \sum_{i=0}^{\infty} \frac{1}{(i+1)!}[a,-]^i (d\xi - da). \tag{9.24}$$

In fact, this action is nothing but the natural one on the shifted space of solutions

$$\overline{\partial} + \mathbf{MC}_L(m), \tag{9.25}$$

under the identification

$$\overline{\partial}+ \colon \mathbf{MC}_L(m) \xrightarrow{\sim} \overline{\partial} + \mathbf{MC}_L(m). \tag{9.26}$$

Note also that the description (9.24) (rather than (9.23)) of the action above is given solely by the structure of the dgla L.

Two solutions $\xi, \xi' \in L^1$ are said to be *gauge equivalent* if they belong to the same group orbit. Set

$$\mathbf{Def}_{\mathrm{Hom}^\bullet(V,V)}(m) := \frac{\mathbf{MC}_{\mathrm{Hom}^\bullet(V,V)}(m)}{\text{gauge equivalences}}. \tag{9.27}$$

It is easy to observe that we thus obtain a covariant functor

$$\mathbf{Def}_{\mathrm{Hom}^\bullet(V,V)} \colon \mathbf{Art} \to \mathcal{S}et. \tag{9.28}$$

On the other hand, we obviously obtain a functor of deformations of $\left(V, \overline{\partial}\right)$, which will be denoted by

$$\mathbf{Def}_{\left(V,\overline{\partial}\right)} \colon \mathbf{Art} \to \mathcal{S}et; (A,m) \mapsto \frac{\left\{\text{deformations of } \overline{\partial} \text{ over } A\right\}}{\text{isomorphisms}}. \tag{9.29}$$

Now we can state the main point.

Theorem 9.7. *There exists a standard natural isomorphism of functors*

$$\mathbf{Def}_{\mathrm{Hom}^\bullet(V,V)} \xrightarrow{\sim} \mathbf{Def}_{\left(V,\overline{\partial}\right)}. \tag{9.30}$$

Proof. Fix $(A,m) \in \mathbf{Art}$ as above and take $[\xi] \in \mathbf{Def}_{\mathrm{Hom}^\bullet(V,V)}(A,m)$. Then the *integrability condition*

$$\overline{\partial}_A^2 = 0 \tag{9.31}$$

for

$$\overline{\partial}_A := \overline{\partial} + \xi \colon V \to V \tag{9.32}$$

can be rephrased as

$$\overline{\partial}_A^2 = 0 \iff \overline{\partial}\xi + \xi\overline{\partial} + \xi^2 = 0 \iff d\xi + \frac{1}{2}[\xi, \xi] = 0. \tag{9.33}$$

For ξ, ξ', set $\overline{\partial}_A = \overline{\partial}+\xi, \overline{\partial}'_A = \overline{\partial}+\xi'$. Then the complexes $\left(V \otimes A, \overline{\partial}_A\right)$ and $\left(V \otimes A, \overline{\partial}'_A\right)$ are isomorphic deformations if and only if there exists an isomorphism

$$\varphi = \mathrm{id}_V + \eta \colon \left(V \otimes A, \overline{\partial}_A\right) \xrightarrow{\sim} \left(V \otimes A, \overline{\partial}'_A\right), \tag{9.34}$$

where $\eta \in \operatorname{Hom}(V, V) \otimes m$. By setting $a := \log \varphi$, we see

$$\varphi\overline{\partial}_A = \overline{\partial}'_A\varphi \iff \overline{\partial}'_A = e^a\overline{\partial}_A e^{-a} = e^{[a,-]}\overline{\partial}_A \iff \xi' = e^{[a,-]}\left(\overline{\partial} + \xi\right) - \overline{\partial}. \tag{9.35}$$

Thus we conclude the proof. □

Exercise 9.8. Show that the dgla defined in Example 9.6 actually satisfies the axioms of dglas.

Now we describe the main example; namely, the *Hochschild dgla* associated to an associative $\boldsymbol{k}$-algebra A, which describes the deformations of the associative product of A.

10. Blowup of noncommutative surfaces and SOD

11. Compact moduli of marked noncommutative del Pezzo surfaces

12. Noncommutative Hirzebruch surfaces

13. Relation to generalized complex geometry

14. Noncommutative $\mathbb{P}^3$–4-dimensional Sklyanin algebras and central extensions of 3-dimensional Sklyanin algebras

15. Comments on references

Standard introductory textbooks on schemes and algebraic geometry are [Har77] and [Mum99]. The unpublished textbook entitled 'Foundations of algebraic geometry' by Ravi Vakil is available online and good. The stacks project is extremely useful to look up theorems on commutative algebras and schemes, algebraic spaces, and stacks.

Noncommutative algebraic geometry was initiated in the paper [AS87], in which Artin-Schelter regular algebra was defined. The AS-regular algebras of dimension 3 which are generated in degree 1 were classified in [ATVdB90]. The amazing relationship to curves of genus 1 first appeared in [ATVdB90].

On the other hand, the same classification was independently recovered from the point of view of derived categories, exceptional collections, and $\mathbb{Z}$-algebras in [BP93]. This is the first place where the importance of the notion of I-algebras, where I is a set, was recognized. This point of view was later exploited by Van den Bergh to understand noncommutative quadrics [VdB11] and noncommutative Hirzebruch surfaces [VdB12].

[AZ94] is a foundational paper on 'noncommutative projective schemes' based on graded algebras. [SvdB01] is a great survey article on noncommutative curves and surfaces.

Notion of flat deformation of abelian category was established in [LVdB06]. Deformation theory was established in [LVdB05].

For general category theory, one can refer to nLab. A standard reference on homological algebra is [Wei94] and [GM03]. A standard reference on derived category and its application to algebraic geometry is [Huy06]. For higher category theory, one should refer to [Lur09]. PDF file of the book is also available from the author's website.

The standard reference on generalized complex geometry is [Gua11]. The coincidence of deformation theory of abelian categories and that of generalized complex structures was pointed in [VdB07].

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