ADDENDUM TO “ON IMAGES OF MORI DREAM SPACES”

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ABSTRACT. In this note we discuss the following question: *when is the image of a dominant rational map from a Mori dream space again a Mori dream space?*

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1. INTRODUCTION

It was shown in [Oka15] that if

$$f\colon X \to Y \tag{1.1}$$

is a surjective morphism of projective $\mathbb{Q}$-factorial varieties over a field $\mathbf{k}$ and X is a Mori dream space, then so is Y. As one can easily observe in the following example, this is not the case any more if we replace f with an arbitrary *dominant rational map*.

Example 1.1. Set $X = \mathbb{P}^2$ and take nine points $p_1, \ldots, p_9 \in X$ in a very general position. Let Y be the blowup of X in those nine points, and $f\colon X \dashrightarrow Y$ the inverse of the blowup morphism $Y \to X$. Then Y is not a Mori dream space, although X is.

Note that the rational map f of Example 1.1 is dominant, birational, and defined on an open subset of X whose complement is a small closed subset of X.

Let us state it explicitly as a question.

Question 1.2. When is the image of a dominant rational map from a Mori dream space again a Mori dream space?

The aim of this note is to point out the following claim.

Proposition 1.3. *Let $f\colon X \dashrightarrow Y$ be a* contracting rational map *between projective $\mathbb{Q}$-factorial varieties over* $\mathbf{k}$. *If X is a Mori dream space, so is Y.*

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The notion of *contracting rational map* plays an important role in [HK00].

One can use [Oka15, Theorem 1.1] to prove that something is *not* a MDS, by constructing a surjective morphism to a non-MDS. This idea was effectively used in the remarkable papers [CT15] and [GK14] to show that $\overline{M}_{0,n}$ is not a MDS for $n \geq 13$ in characteristic zero. The author hope that Proposition 1.3 would be useful to produce more results in that direction.

Notations and conventions. Unless otherwise stated, all the varieties in this paper are normal and projective over a field $\mathbf{k}$.

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2. Contracting birational map

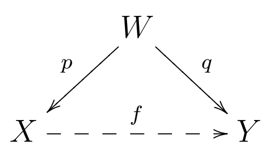
In this section we give details of some arguments of [HK00, Section 1].

Let $f\colon X \dashrightarrow Y$ be a dominant rational map. It is standard to think of a pair of morphisms

$$p\colon W \to X, \quad q\colon W \to Y, \tag{2.1}$$

satisfying the following properties:

- W is a normal projective variety as well.
- p is a birational morphism.
- $f \circ p = q$.



For a $\mathbb{Q}$-Cartier divisor D on Y, its pull-back is defined by

$$f^*D = p_*q^*D. \tag{2.2}$$

It is easy to observe that the definition of f^*D is independent of the choice of W, p, q.

Definition 2.1 (= [HK00, 1.1 Definition])**.** A dominant rational map f as above is said to be *contracting* if for any p-exceptional effective divisor $E \geq 0$ on W, the natural map

$$\mathcal{O}_Y \to q_*\mathcal{O}_W(E) \tag{2.3}$$

is an isomorphism.

It is not so difficult to see that the notion of contracting rational map does not depend on the choice of W, p, q. Note that a contracting rational map has to be an algebraic fiber space, since by taking $E = 0$, we obtain the isomorphism $\mathcal{O}_Y \xrightarrow{\sim} q_*\mathcal{O}_W$.

Remark 2.2. A birational map f is contracting if and only if it is surjective in codimension one. In particular, the birational map of Example 1.1 is not contracting.

For a general dominant rational map f, it is not so easy to check if it is contracting or not. The author only knows the following technical sufficient condition for that.

Claim 2.3. *Suppose that one could find the resolution W, p, q such that for any p-exceptional prime divisor E on W, we have* $\operatorname{codim}_Y q(E) \geq 2$. *Then f is contracting.*

One caution. Suppose that the rational map f is defined on an open set $U \subset X$ whose complement is small, and that the closure of the subset $Y \setminus f(U)$ is also small in Y. This is not sufficient to guarantee that the map f is contracting. In fact, consider the projection from a point $p \in \mathbb{P}^2$, $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. It is not contracting, but f is defined on a big open subset $U = \mathbb{P}^2 \setminus \{p\}$ and $f(U) = \mathbb{P}^1$.

The following lemma directly follows from the definition of contracting rational map.

Lemma 2.4. *Let $f\colon X \dashrightarrow Y$ be a contracting birational map and $g\colon Y \dashrightarrow Z$ a contracting rational map. Then $g \circ f\colon X \dashrightarrow Z$ is also contracting.*

The following lemmas are used in the proof of [HK00, 1. 11 Proposition (2)] without proof. For the sake of completeness, we include a detailed proof here.

Lemma 2.5. *Let $f\colon X \dashrightarrow Y$ be a contracting rational map, and assume X is $\mathbf{Q}$-factorial. Suppose that A is an ample divisor on Y and f^*A is semi-ample on X. Then f is a regular morphism.*

Proof. Without loss of generality, we can replace A with mA for any integer $m > 0$. In the arguments below, we tacitly do this so that all the relevant divisors on X will be integral and Cartier. In particular, we can assume that A is a very ample divisor on Y. Hence we can choose A in such a way that q^*A contains no p-exceptional divisor. Then it is clear that $E := p^*p_*q^*A - q^*A$ is p-exceptional. Since f is assumed to be contracting, we have the isomorphism $\mathcal{O}_Y \xrightarrow{\sim} q_*\mathcal{O}_W(E)$ and hence $q_*(p^*p_*q^*\mathcal{O}_Y(A)) \xrightarrow{\sim} \mathcal{O}_Y(A) \otimes q_*\mathcal{O}_W(E) \xrightarrow{\sim} \mathcal{O}_Y(A)$. From this, we obtain the canonical isomorphism

$$H^0(Y, \mathcal{O}_Y(A)) \xrightarrow{\sim} H^0(W, \mathcal{O}_W(p^*p_*q^*A)) \xrightarrow{\sim} H^0(X, \mathcal{O}_X(p_*q^*A)). \tag{2.4}$$

Assume for a contradiction that f is not a regular morphism. Then there exists a curve $C \subset W$ such that $p(C)$ is a point and $q(C)$ is a curve in Y (to see this, consider the image of the morphism $p \times q\colon W \to X \times Y$. If we do not have such a curve, then it would imply that the projection $\mathrm{Im}(p \times q) \to X$ is an isomorphism by the Zariski's main theorem and hence we would obtain the morphism $X \xrightarrow{\sim} \mathrm{Im}(p \times q) \xrightarrow{pr_Y} Y$). Since q_*p^*A is semi-ample on X, it is clear that $C.q^*q_*p^*A = q_*[C].q_*p^*A = 0$. On the other hand, since q_*p^*A is assumed to be semi-ample, we can replace A with a $\mathbb{Q}$-linearly equivalent divisor $B \geq 0$ such that $p^*p_*q^*B = q^*B$, using the surjectivity of (2.4). Then we see $C.q^*q_*p^*B = C.p^*B = p_*[C].B > 0$, which is a contradiction. □

Lemma 2.6. *Let X be a Mori dream space, and $f\colon X \dashrightarrow Y$ a contracting rational map. Then there exists a small $\mathbb{Q}$-factorial modification $f_i\colon X \dashrightarrow X_i$ such that $f \circ f_i^{-1}\colon X_i \dashrightarrow Y$ is a morphism.*

Proof. Since the indeterminancy locus of p^{-1} is small in X by the ZMT, f is defined on an open subset of X whose complement is small. Hence for any ample divisor A on Y, the divisor f^*A is movable. Hence there exists at least one SQM of X on which f^*A is semi-ample. Then the rest follows from Lemma 2.4 and Lemma 2.5. □

3. Proof of the main statement

Proof of Proposition 1.3. By Lemma 2.6, one can find a SQM X' of X from which f becomes a regular morphism. Since X' is also a MDS, one can apply [Oka15, Theorem 1.1] to conclude that Y is also a MDS. □

Remark 3.1. As observed in [Oka15], we can drop the assumption of $\mathbb{Q}$-factoriality from the definition of MDSs and still obtain Proposition 1.3 in that generality. In fact, we can take small $\mathbb{Q}$-factorizations to reduce the proof to the $\mathbb{Q}$-factorial case.

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