ON GLOBAL OKOUNKOV BODIES OF MORI DREAM SPACES

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ABSTRACT. The author gives an approach and partial answer to a problem posed by Lazarsfeld and Mustaţă on the rational polyhedrality of the global Okounkov body of a Mori Dream Space.

CONTENTS

1. INTRODUCTION

Let X be a projective variety of dimension n. A flag $Y_\bullet$ on X is a sequence

$$Y_\bullet = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n = \{pt\}$$

of closed subvarieties such that each Y_i is smooth at the point Y_n.

For a flag $Y_\bullet$ and a line bundle L on X, we can define the Okounkov body $\Delta_{Y_\bullet}(X, L)$, which is a compact convex set in $\mathbb{R}^n$. It is known that the Euclidian volume of this body coincides with the volume of the line bundle L (up to $n!$). Therefore we can regard Okounkov body as a geometric refinement of the volume function for line bundles.

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We can also define the notion of global Okounkov body $\Delta_{Y_\bullet}(X)$, which is a closed convex cone in $\mathbb{R}^n \times N^1(X)_\mathbb{R}$ whose fiber over a big line bundle $L \in N^1(X)_\mathbb{R}$ coincides with the Okounkov body $\Delta_{Y_\bullet}(X, L)$ of L.

In [LM], Lazarsfeld and Mustaţă asked the following problem ([LM, Problem 7.1]):

Problem 1.1. Does a Mori dream space admit a flag with respect to which the global Okounkov body is rational polyhedral?

Mori dream space is a class of projective varieties which contains log Fano varieties. In particular it can be regarded as a generalization of toric varieties. It is known that for a Mori dream space X the volume function $vol(\cdot) : \mathrm{Eff}\,(X) \to \mathbb{R}$ is piecewise polynomial (this follows from Proposition 3.5 and the fact that the volume of a nef line bundle equals to its self-intersection number). Problem 1.1 can be regarded as a refinement of this fact.

Problem 1.1 is known to be true for toric varieties ([LM, Proposition 6.1 (ii)]); in that case, we choose the flag consisting of torus invariant strata.

The purpose of this article is to propose an approach toward the problem, give some partial answers.

We first establish a formula (see Lemma 4.1) which describes slices of an Okounkov body as the Okounkov body of certain line bundles on Y_1, the first piece of the flag $Y_\bullet$. This enables us to calculate Okounkov bodies inductively.

As a first application of the formula, we obtain the following result:

Lemma 1.2. *Problem 1.1 is true for surfaces.*

To deal with higher dimensional cases, we define the notion of a good flag:

Definition 1.3. Let X be a MDS. A flag $Y_\bullet = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{pt\}$ is said to be good if the following conditions hold:

- Y_i is the birational image of a MDS, say $\tilde{Y}_i$, for $i = 1, \ldots, n-2$, such that there exists a sequence of closed immersions
$$\tilde{Y}_1 \supset \tilde{Y}_2 \supset \cdots \supset \tilde{Y}_{n-2}$$
compatible with the projections to Y_i's.
- Y_i is not contained in the base loci of line bundles on Y_{i-1} for all $i = 1, 2, \ldots, n-1$
- Y_0 is not contained in the images of the exceptional loci of small $\mathbb{Q}$-factorial modifications of $\tilde{Y}_i's$ or exceptional loci of birational morphisms $\tilde{Y}_i \to Y_i$ appearing in the first condition.

Note that the last two conditions are fulfilled if Y_i is a general member of a base point free linear system.

Remark 1.4.

- Since only $\mathbb{P}^1$ is a Mori dream curve, we cannot expect in general that Y_1 is a Mori dream curve. But we do not need this because line bundles on a curve behaves quite nicely. This is the reason why we do not assume that Y_{n-1} is a MDS.

- We should not assume that Y_i itself is a MDS. In fact, when we think of a rational homogeneous variety (which is a MDS since it is a Fano), a natural candidate for a good flag is the one consisting of Schubert subvarieties. Schubert varieties are not necessarily $\mathbb{Q}$-factorial, hence we have to pass to their Bott-Samelson resolutions.

With this notion, we can show

Theorem 1.5. *Let X be a MDS and $Y_\bullet$ is a good flag . Then $\Delta_{Y_\bullet}(X)$ is a rational polyhedral cone.*

The final part of the article is devoted to a discussion of how to construct such a flag.

Jow Shin-Yao proved ([S, Theorem 6]) that a sufficiently ample and very general divisor of a MDS of dimension at least three again is a MDS, provided that the ambient variety satisfies certain GIT condition. Therefore, the following naive expectation arises:

Problem 1.6. Let X be a MDS of dimension at least three (not necessarily satisfying the GIT condition above). Let A be a sufficiently ample and very general divisor of X. Then A also is a MDS.

It turns out that this problem is not at all clear, even for quite simple cases.

Therefore we are forced to construct good flags on a case-by-case basis. We discuss two special cases. The first one is a Mori dream 3-fold given in [KLM, Proposition 3.5]. It behaves badly to a class of flags, but still we can find a good flag such that the global Okounkov body is rational polyhedral.

The second one is rational homogeneous varieties. There is a natural candidate for the flag $Y_\bullet$ for such varieties so that the global Okounkov body is rational polyhedral, but there still remains some difficulty.

Here is a historical remark. The notion of Okounkov body first appeared in the works of Andrei Okounkov. His aim was to describe the multiplicities of irreducible representations appearing in a representation in terms of the volume of certain convex bodies so that he can prove that the log concavity holds for the multiplicities by using the Brunn-Minkowski inequality for convex bodies ([O] is an interesting survey).

Later Lazarsfeld and Mustaţă defined similar convex bodies (which they called the Okounkov body) for big line bundles and did a foundational work ([LM]). In this case, the volume of the convex body associated to a big line bundle coincides with the volume of the line bundle as mentioned above.

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2. (Global) Okounkov body

In this section we recall the definition and first properties of (global) Okounkov bodies. Most of the subjects in this section was taken from [LM].

Consider any divisor D on X. We begin by defining a function

$$\nu \ = \nu_{Y_\bullet} \ : \ H^0\big(X, \mathcal{O}_X(D)\big) \setminus \{0\} \longrightarrow \mathbf{Z}^n \quad , \quad s \mapsto \nu(s) \ = \ \big(\nu_1(s), \dots, \nu_n(s)\big).$$

Given

$$0 \neq s \in H^0\big(X, \mathcal{O}_X(D)\big),$$

set to begin with

$$\nu_1 \ = \ \nu_1(s) \ = \ \mathrm{ord}_{Y_1}(s).$$

After choosing a local equation for Y_1 in X, say t_1, s determines a section

$$\tilde{s}_1 = s \otimes t_1^{-\nu_1} \ \in \ H^0\big(X, \mathcal{O}_X(D - \nu_1 Y_1)\big)$$

that does not vanish (identically) along Y_1, and so we get by restricting a non-zero section

$$s_1 \in H^0\big(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)\big).$$

Then take

$$\nu_2 \ = \ \nu_2(s) \ = \ \mathrm{ord}_{Y_2}(s_1).$$

In general, given integers $a_1, \dots, a_i \geq 0$ denote by $\mathcal{O}(D - a_1 Y_1 - a_2 Y_2 - \dots - a_i Y_i)_{|Y_i}$ the line bundle

$$\mathcal{O}_X(D)_{|Y_i} \otimes \mathcal{O}_X(-a_1 Y_1)_{|Y_i} \otimes \mathcal{O}_{Y_1}(-a_2 Y_2)_{|Y_i} \otimes \dots \ \otimes \mathcal{O}_{Y_{i-1}}(-a_i Y_i)_{|Y_i}$$

on Y_i. Suppose inductively that for $i \leq k$ one has constructed non-vanishing sections

$$s_i \ \in \ H^0\Big(Y_i, \mathcal{O}(D - \nu_1 Y_1 - \nu_2 Y_2 - \dots - \nu_i Y_i)_{|Y_i}\Big),$$

with $\nu_{i+1}(s) = \mathrm{ord}_{Y_{i+1}}(s_i)$, so that in particular

$$\nu_{k+1}(s) \ = \ \mathrm{ord}_{Y_{k+1}}(s_k).$$

Dividing by the appropriate power of a local equation of Y_{k+1} in Y_k yields a section

$$\tilde{s}_{k+1} \ \in \ H^0\Big(Y_k, \mathcal{O}\big(D - \nu_1 Y_1 - \nu_2 Y_2 - \dots - \nu_k Y_k\big)_{|Y_k} \otimes \mathcal{O}_{Y_k}(-\nu_{k+1} Y_{k+1})\Big)$$

not vanishing along Y_{k+1}. Then take

$$s_{k+1} \ = \ \tilde{s}_{k+1}|Y_{k+1} \ \in \ H^0\Big(Y_{k+1}, \mathcal{O}\big(D - \nu_1 Y_1 - \nu_2 Y_2 - \dots - \nu_{k+1} Y_{k+1}\big)_{|Y_{k+1}}\Big)$$

to continue the process. Note that while the sections $\tilde{s}_i$ and s_i will depend on the choice of a local equation of each Y_i in Y_{i-1}, the values $\nu_i(s) \in \mathbf{N}$ do not.

Definition 2.1. (Graded semigroup of a divisor). The *graded semigroup* of D is the sub-semigroup

$$\Gamma(D) \;=\; \Gamma_{Y_\bullet}(D) \;=\; \big\{ (\nu_{Y_\bullet}(s), m) \,\big|\, 0 \neq s \in H^0\big(X, \mathcal{O}_X(mD)\big)\,,\, m \geq 0 \,\big\}$$

of $\mathbf{N}^n \times \mathbf{N} = \mathbf{N}^{n+1}$. □

Writing $\Gamma = \Gamma(D)$, denote by

$$\Sigma(\Gamma) \;\subseteq\; \mathbf{R}^{n+1}$$

the intersection of all the closed convex cones containing Γ. The Okounkov body of D is then the slice of this cone at the level one:

Definition 2.2. (Okounkov body). The *Okounkov body* of D (with respect to the fixed flag $Y_\bullet$) is the compact convex set

$$\Delta_{Y_\bullet}(X, D) \;=\; \Sigma(\Gamma) \,\cap\, \big(\mathbf{R}^n \times \{1\}\big).$$

We view $\Delta(D)$ in the natural way as a closed convex subset of $\mathbf{R}^n$.

We can show that it is compact, and that it depends only on the numerical class of the divisor D (see [LM] for detail).

Remark 2.3. $\Delta_{Y_\bullet}(X, L)$ does depend on the choice of the flag $Y_\bullet$. For example, even if X is a toric Fano and L is ample, we have to chose a suitable flag $Y_\bullet$ to make $\Delta_{Y_\bullet}(X, L)$ rational polyhedral (see [KLM, Example 3.4]).

There is a globalization of this notion (see [LM, §4.2]):

Theorem-Definition 2.4. *There exists a closed convex cone*

$$\Delta_{Y_\bullet}(X) \;\subseteq\; \mathbf{R}^n \times N^1(X)_{\mathbf{R}}$$

characterized by the property that the fibre of $\Delta_{Y_\bullet}(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta_{Y_\bullet}(X, \xi)$, i.e.

$$\mathrm{pr}_2^{-1}(\xi) \cap \Delta(X) \;=\; \Delta_{Y_\bullet}(X, \xi) \;\subseteq\; \mathbf{R}^n \times \{\xi\} = \mathbf{R}^n.$$

The cone $\Delta_{Y_\bullet}(X)$ is called the global Okounkov body of X with respect to the flag $Y_\bullet$.

The following theorem says that the global Okounkov body is a refinement of the volume function:

Theorem 2.5. *If D is any big divisor on X, then*

$$\mathrm{vol}_{\mathbf{R}^n}\big(\Delta(D)\big) \;=\; \frac{1}{n!} \cdot \mathrm{vol}_X(D).$$

The quantity on the right is the *volume* of D, defined as the limit

$$\mathrm{vol}_X(D) \;=_{\mathrm{def}}\; \lim_{m\to\infty} \frac{h^0\big(X, \mathcal{O}_X(mD)\big)}{m^n/n!}.$$

3. Basics of Mori Dream Spaces

In this section we summarize some basic facts about Mori dream spaces. For detail, see [HK].

First of all, we recall the definition of Mori dream space.

Definition 3.1. Let X be a normal projective variety. A small $\mathbb{Q}$-factorial modification (SQM, for short) of X is a small (i.e. isomorphic in codimension one) birational map $f : X \dashrightarrow Y$ to another $\mathbb{Q}$-factorial projective variety Y.

Definition 3.2. A normal projective variety X is called a Mori Dream Space provided that the following conditions hold:

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}\,(X)_{\mathbb{Q}} \cong \mathrm{N}^1\,(X)_{\mathbb{Q}} holds.$
(2) $\mathrm{Nef}\,(X)$ is the affine hull of finitely many semi-ample line bundles.
(3) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (1) and (2), and $\mathrm{Mov}\,(X)$ is the union of the $f_i^*(\mathrm{Nef}\,(X_i))$.

The following is a basic fact on MDS (see [HK, Proposition 1.11 (2)] and the proof given there):

Proposition 3.3. *The f_i of (3.2) are the only SQMs of X. There are finitely many birational contractions $g_j : X \dashrightarrow Y_j$ with Y_j MDSs, such that*

$$\mathrm{Eff}\,(X) = \bigcup_j \left(g_j^*(\mathrm{Nef}\,(Y_j)) * \mathrm{ex}(g_j)\right)$$

is a decomposition of $\mathrm{Eff}\,(X)$ *into closed convex chambers with disjoint interiors (these chambers are called Mori chambers in* [HK]*). Here* $\mathrm{ex}(g_j)$ *denotes the cone spanned by the irreducible components of the exceptional locus of g_j, and $*$ denotes the join. They are in one-to-one correspondence with birational contractions of X having $\mathbb{Q}$-factorial image.*

Moreover, pushing out by a suitable SQM $f_i : X \dashrightarrow X_i$, every contracting rational map $X \dashrightarrow X'$ becomes a genuine morphism $X_i \to X'$.

Using Proposition 3.3, we can see that the Zariski decompositions of divisors on MDS's are quite easy to describe.

Definition 3.4 (Zariski decomposition)**.** Let X be a normal projective variety. Let D be a pseudo-effective Cartier divisor. A Zariski decomposition of D (in the sense of Cutkosky-Kawamata-Moriwaki) is a decomposition

$$D \sim_{\mathbb{Q}} P + N$$

where P, N are $\mathbb{Q}$-divisors on X and $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence such that

- P is nef and N is effective
- for all sufficiently divisible $m \in \mathbb{N}_{>0}$ the map $H^0(X, mP) \to H^0(X, mD)$ (multiplication by the section corresponding to mN) is isomorphic.

When $\dim X \geq 3$, Zariski decomposition (in the sense above) does not exist in general. However MDS possesses quite nice properties concerning Zariski decompositions. This is an easy application of Proposition 3.3 (for a proof, see [Ok]):

Proposition 3.5. *Let X be a MDS. Then there exists a decomposition of the effective cone of X*

$$\mathrm{Eff}\,(X) = \bigcup_{\mathit{finite}} C$$

into finitely many rational polyhedral cones C such that for each chamber C there exists a small $\mathbb{Q}$-factorial modification $f_i : X \dashrightarrow X_i$ of X and two $\mathbb{Q}$-linear maps

$$P, N : C \to \mathrm{Eff}\,(X)$$

such that for any $\mathbb{Z}$-divisor $D \in C$, $D \sim_{\mathbb{Q}} P(D) + N(D)$ gives a Zariski decomposition of D as a divisor on X_i; i.e.

- *$P(D) \in \mathrm{SA}\,(X_i)$.*
- *$N(D) \geq 0$.*
- *The natural map*

$$H^0(X, P(D)^{\otimes m}) \to H^0(X, \mathcal{O}_X(D)^{\otimes m}),$$

which is defined by a multiplication of a non-zero global section of the line bundle $\mathcal{O}_X(mN(D))$ is isomorphic for every sufficiently divisible positive integer m.

Conversely, a normal projective variety satisfying Definition 3.2 (1) and having the property above actually is a MDS[1].

4. Inductive calculation of Okounkov bodies

4.1. **Slices of Okounkov bodies.** We recall some facts on the slices of Okounkov bodies.

Lemma 4.1. *Let X be a normal projective variety and L be a big line bundle on X. Let $Y_\bullet$ be a flag on X such that $Y_1 \not\subset \mathbb{B}_+(L)$, where $\mathbb{B}_+(L)$ denotes the augmented base locus of the line bundle L (see* [L, Definition 10.3.2]*). Take some rational number $t \in \mathbb{Q}_{\geq 0}$ which satisfies the following properties:*

- *$L - tY_1$ is big*
- *$L - tY_1$ admits a Zariski decomposition*
- *$Y_1 \not\subset \mathbb{B}_+(P(L - tY_1))$,*

where $P(L - tY_1)$ is the positive part of $L - tY_1$.

Then

$$\Delta_{Y_\bullet}(L)_{\nu_1 = t} := \Delta_{Y_\bullet}(L) \cap \{\nu_1 = t\} = \Delta_{Y_\bullet}(Y_1, P(L - tY_1)|_{Y_1}) \tag{1}$$

[1] the author would like to thank Prof. Y.-H. Kiem for asking him if it could be the case.

holds up to the parallel transportation by the valuation vector of the restriction of the section of $N(L - tY_1)$*.*

Proof. Since $Y_1 \not\subset \mathbb{B}_+(L)$, the following holds

$$\Delta_{Y_\bullet}(L)_{\nu_1=t} = \Delta_{Y_\bullet}(Y_1|X, L - tY_1),$$

where the right hand side denotes the restricted Okounkov body (see [LM, Theorem 4.24]). By a property of the Zariski decomposition, the right hand side of (1) is a parallel transportation of $\Delta_{Y_\bullet}(Y_1|X, P(L - tY_1))$ by the valuation vector mentioned in the statement of the lemma.

By [LM, Corollary 4.25 (i)], $(n-1)!$ times the volume of $\Delta_{Y_\bullet}(Y_1|X, P)$ equals to the restricted volume of P. By [ELMNP, Corollary 2.17], the restricted volume of P equals to $(P^{n-1} \cdot Y_1)$ since $Y_1 \not\subseteq \mathbb{B}_+(P)$. On the other hand, since P is nef, the volume of $P|_{Y_1}$, which in turn equals the $(n-1)!$ times the volume of the Okounkov body $\Delta_{Y_\bullet}(Y_1, P|_{Y_1})$, equals to $(P^{n-1} \cdot Y_1)$. Summing up, we see that $\Delta_{Y_\bullet}(Y_1|X, P) \subseteq \Delta_{Y_\bullet}(Y_1, P|_{Y_1})$ are closed convex bodies of the same volume. Hence they must coincide. □

4.2. **A decomposition of the Effective cone.** Fix a flag $Y_\bullet$ on a MDS X such that Y_1 avoids the base loci of effective line bundles on X. Let

$$\pi : \mathrm{Pic}(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2, \ldots, \nu_n) \mapsto (D, \nu_1)$$

be the projection.

The purpose of this subsection is to prove the existence of a decomposition of $\pi(\Delta_{Y_\bullet}(X)) \subset \mathrm{Eff}(X) \times \mathbb{R}_{\geq 0}$ into finitely many rational polyhedral cones such that on each of the cones the function $\varphi : (D, t) \mapsto P(D - tY_1)$ is rationally linear.

Consider the linear mapping

$$T : \mathrm{Eff}(X) \times \mathbb{R}_{\geq 0} \to \mathrm{Pic}(X)_{\mathbb{R}};\ (D, t) \mapsto D - tY_1.$$

By Lemma A.2 for each Mori chamber $C \subset \mathrm{Pic}(X)_{\mathbb{R}}$, $\mathcal{C} = T^{-1}(C)$ is a rational polyhedral cone. Combined with Proposition 3.5, we see that φ is rationally linear on $\mathcal{C}$. Therefore we have the decomposition $T^{-1}(\mathrm{Eff}(X)) = \bigcup \mathcal{C}$ into rational polyhedral subcones. Now we can show the following

Lemma 4.2. $\pi(\Delta_{Y_\bullet}(X)) = T^{-1}(\mathrm{Eff}(X))$ *holds.*

Proof. $T^{-1}(\mathrm{Eff}(X)) \subseteq \pi(\Delta_{Y_\bullet}(X))$ follows from Lemma 4.1. Conversely, choose a point (D, t) from the interior of $\pi(\Delta_{Y_\bullet}(X))$.

Then we can apply [LM, Corollary A.3] to obtain

$$\pi^{-1}(D, t) \bigcap \Delta_{Y_\bullet}(X) = \Delta_{Y_\bullet}(Y_1|X, D - tY_1).$$

Moreover we know that the left hand side has an interior point. Hence $D - tY_1 \in \mathrm{Eff}(X)$ must hold. Since $T^{-1}(\mathrm{Eff}(X))$ is closed, this is enough to show the lemma. □

Thus we obtain the desired decomposition of $\pi(\Delta_{Y_\bullet}(X))$.

4.3. **Proof of Lemma 1.2.**

Proof of Lemma 1.2. Let $Y_\bullet$ be a good flag. Consider the following projection

$$\pi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^2 \to \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2) \mapsto (D, \nu_1).$$

Let $\mathcal{C} \subset \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0}$ be a chamber as in §4.2. By Lemma A.1, it is enough to show that $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X)$ is a rational polyhedral cone. To prove it, it is enough to show that the function

$$(D, t) \mapsto \mathrm{Vol}\,(\Delta_{Y_\bullet}(D)_{\nu_1 = t}) \tag{2}$$

is rationally linear on $\mathcal{C}$, since

$$\Delta_{Y_\bullet}(D)_{\nu_1=t} = [0, \mathrm{Vol}\,(\Delta_{Y_\bullet}(D)_{\nu_1=t})]$$

holds.

But we know from Lemma 4.1 that the right hand side of (2) equals to $\deg\,(\Delta_{Y_\bullet}(Y_1, P(D - tY_1)|_{Y_1})) = Y_1.(P(D - tY_1))$, which is clearly rationally linear in (D, t). □

4.4. **Proof of Theorem 1.5.**

Proof of Theorem 1.5. Consider the following projection

$$\pi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2, \ldots, \nu_n) \mapsto (D, \nu_1).$$

Let $\mathcal{C} \subset \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0}$ be a chamber as in §4.2. By Lemma A.1, it is enough to show that $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X)$ is a rational polyhedral convex cone.

Consider the following linear mapping:

$$\varphi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}\,(Y_1)_{\mathbb{R}} \times \mathbb{R}^{n-1}; (D, t, \nu') \mapsto (P(D - tY_1)|_{Y_1}, \nu' - \nu'(N(D - tY_1))),$$

where $\nu' = (\nu_2, \nu_3, \ldots, \nu_n)$. Let $\varphi_{\mathcal{C}}$ be the restriction of φ to $\pi^{-1}(\mathcal{C})$. Recall that $\varphi_{\mathcal{C}}$ is rationally linear. Now since $\Delta_{Y_\bullet}(X, D)_{\nu_1=t} = \Delta_{Y_\bullet}(Y_1, P(D - tY_1)|_{Y_1}) + \nu'(N(D - tY_1))$, for each $(D, t, \nu') \in \pi^{-1}(\mathcal{C})$

$$(D, t, \nu') \in \Delta_{Y_\bullet}(X) \iff \varphi_{\mathcal{C}}(D, t, \nu') \in \Delta_{Y_\bullet}(Y_1).$$

Therefore $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X) = \varphi_{\mathcal{C}}^{-1}(\Delta_{Y_\bullet}(Y_1))$. Since we assumed that $Y_\bullet$ is a good flag, Y_1 is the image of a birational morphism from a MDS $\tilde{Y}_1$ and we can identify $\Delta_{Y_\bullet}(Y_1)$ with a subcone of the global Okounkov body of $\tilde{Y}_1$ (with respect to the flag obtained by $\tilde{Y}_i$'s). Hence it is a rational polyhedral cone. By Lemma A.2, it follows that $\varphi_{\mathcal{C}}^{-1}(\Delta_{Y_\bullet}(Y_1))$ also is a rational polyhedral cone. □

5. Constructing good flags

In this section, we discuss the existence problem of good flags.

As mentioned before, Problem 1.6 is true if X satisfies the small unstable locus condition (see [S]). An easiest example of a MDS which does not satisfy the condition is $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Let $A \subset X$ be a general ample divisor of type (a, b). If a is sufficiently large, $A \to \mathbb{P}^{n-1}$ is a finite map. Therefore $\mathcal{O}_A(p, -1)$ is ample for $p >> 1$. Therefore there exists a nef divisor N on A which is not the restriction of a nef divisor on X (this does not happen if X satisfies the small unstable locus condition; see [S]). It does not seem to be easy (at least to the author) to check if N is semi-ample or not. Therefore it seems to be hopeless to solve Proposition 1.6.

Next we construct a good flag for a MDS given in [KLM, Proposition 3.5]. We use the same notations as in the paper.

Let H be a sufficiently general member. We see that $H \to \pi(H)$ is a blow-up of $\pi(H) \cong \mathbb{P}^2$ in $\pi(H) \bigcap (C_1 \bigcup C_2) = \{8 \text{ points}\}$. By [TVV, Example 1.1(a)], H turns out to be a MDS. Set $Y_1 = H$, and choose Y_2, Y_3 sufficiently general. This gives a good flag for X, and $\Delta_{Y_\bullet}(X)$ is a rational polyhedral cone by Theorem 1.5.

Finally we discuss the case when X is a rational homogeneous variety. For such an X, the flag $Y_\bullet$ consisting of Schubert varieties seems to be the most natural one. We cannot expect that Schubert varieties are $\mathbb{Q}$-factorial, but still we can take their Bott-Samelson resolutions. The author heard from Dave Anderson that it has been proven that Bott-Samelson varieties are log Fano, hence are MDSs. Therefore if we could show that each Y_i is in a general position in Y_{i-1}, we are done. Unfortunately this seems to be difficult, since Y_i is contained in the complement of the dense open orbit of the action of Borel group on Y_{i-1}. We need more information on the geometry of Schubert varieties to overcome this problem.

Appendix A. Some combinatorial lemmas

In this section, we recall some elementary combinatorial facts which we need.

Lemma A.1. *Let $\Delta \subset \mathbb{R}^{p+q}$ be a closed cone. Let $\pi : \mathbb{R}^{p+q} \to \mathbb{R}^p$ be the natural projection, and assume that $\pi(\Delta)$ is a rational polyhedral cone. Suppose furthermore that $\pi(\Delta)$ is decomposed into finitely many rational polyhedral cones, and for each cone C $\pi^{-1}(C) \bigcap \Delta$ is rational polyhedral. Then Δ itself is a rational polyhedral cone.*

Lemma A.2. *Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear mapping defined over $\mathbb{Q}$, and let $\Delta \subset \mathbb{R}^q$ be a rational polyhedral cone. Then so is $T^{-1}(\Delta)$. In particular, if we restrict T to another rational polyhedral cone $\Delta' \subset \mathbb{R}^p$, the same conclusion holds: i.e. $(T|_{\Delta'})^{-1}(\Delta) = \Delta' \bigcap T^{-1}(\Delta)$ is a rational polyhedral cone.*

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