CHARACTERIZATION OF VARIETIES OF FANO TYPE VIA SINGULARITIES OF COX RINGS

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Dedicated to Professor Yujiro Kawamata on the occasion of his sixtieth birthday.

ABSTRACT. We show that every Mori dream space of globally F-regular type is of Fano type. As an application, we give a characterization of varieties of Fano type in terms of the singularities of their Cox rings.

CONTENTS

1. INTRODUCTION

The notion of Cox rings was defined in [HK], generalizing Cox's homogeneous coordinate ring [Co] of projective toric varieties.

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Let X be a normal ($\mathbb{Q}$-factorial) projective variety over an algebraically closed field k. Suppose that the divisor class group $\mathrm{Cl}(X)$ is finitely generated and free, and let $D_1, \cdots, D_r$ be Weil divisors on X which form a basis of $\mathrm{Cl}(X)$. Then the ring

$$\bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^n} H^0(X, \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)) \subseteq k(X)[t_1^{\pm}, \cdots, t_r^{\pm}]$$

is called the Cox ring of X (for the case when $\mathrm{Cl}(X)$ has torsion, see Definition 2.17 and Remark 2.18). If the Cox ring of a variety X is finitely generated over k, X is called a ($\mathbb{Q}$-factorial) Mori dream space. This definition is equivalent to the geometric one given in Definition 2.1 ([HK, Proposition 2.9]). Projective toric varieties are Mori dream spaces and their Cox rings are isomorphic to polynomial rings [Co]. The converse also holds [HK], characterizing toric varieties via properties of Cox rings.

We say that X is of *Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and (X, Δ) is klt. It is known by [BCHM] that $\mathbb{Q}$-factorial varieties of Fano type are Mori dream spaces. Since projective toric varieties are of Fano type, this result generalizes the fact that projective toric varieties are Mori dream spaces. Therefore, in view of the characterization of toric varieties mentioned above, it is natural to expect a similar result for varieties of Fano type. The purpose of this paper is to give a characterization of varieties of Fano type in terms of the singularities of their Cox rings.

Theorem 1.1 (=Theorem 4.7)**.** *Let X be a $\mathbb{Q}$-factorial normal projective variety over an algebraically closed field of characteristic zero. Then X is of Fano type if and only if its Cox ring is finitely generated and has only log terminal singularities.*

Our proof of Theorem 1.1 is based on the notion of global F-regularity, which is defined for projective varieties over a field of positive characteristic via splitting of Frobenius morphisms. A projective variety over a field of characteristic zero is said to be of globally F-regular type if its modulo p reduction is globally F-regular for almost all p (see Definition 2.14 for the precise definition). Schwede–Smith [SS] proved that varieties of Fano type are of globally F-regular type, and they asked whether the converse is true. We give an affirmative answer to their question in the case of Mori dream spaces.

Theorem 1.2. *Let X be a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero. Then X is of Fano type if and only if it is of globally F-regular type.*

Theorem 1.2 is a key to the proof of Theorem 1.1, so we outline its proof here. The only if part was already proved by [SS, Theorem 5.1], so we explain the if part. Since X is a $\mathbb{Q}$-factorial Mori dream space, we can run a $(-K_X)$-MMP which terminates in finitely many steps. A $(-K_X)$-MMP $X_i \dashrightarrow X_{i+1}$ usually makes the singularities of X_i worse as i increases, but in our setting, we can check that each X_i is also of globally F-regular type. This means that each X_i has only log terminal singularities, so that a $(-K_X)$-minimal model becomes of Fano type. Finally we trace back the $(-K_X)$-MMP above and show that in each step the property of being of Fano type is preserved, concluding the proof.

In order to prove Theorem 1.1, we also show that if X is a $\mathbb{Q}$-factorial Mori dream space of globally F-regular type, then modulo p reduction of a multi-section ring of X is the multi-section ring of modulo p reduction X_p of X for almost all p (Lemma 2.22). The proof is based on the finiteness of contracting rational maps from a fixed Mori dream space, vanishing theorems for globally F-regular varieties and cohomology-and-base-change arguments. This result enables us to apply the theory of F-singularities to a Cox ring of X and, as a consequence, we see that that a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero is of globally F-regular type if and only if its Cox ring has only log terminal singularities. Thus, Theorem 1.1 follows from Theorem 1.2.

As an application of Theorem 1.1, we give an alternative proof of [FG1, Corollary 3.3], [FG2, Corollary 5.2] and [PS, Theorem 2.9].

Corollary 1.3 (=Theorem 5.5)**.** *Let $f : X \to Y$ be a projective morphism between normal projective varieties over an algebraically closed field k of characteristic zero. If X is of Fano type, then Y is of Fano type.*

A normal projective variety X over a field of characteristic zero is said to be of *Calabi–Yau type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X, Δ) is log canonical, and is said to be of dense globally F-split type if its modulo p reduction is Frobenius split for infinitely many p. Using arguments similar to the proof of Theorem 1.2, we show an analogous statement for varieties of Calabi–Yau type.

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero. If X is of dense globally F-split type, then it is of Calabi–Yau type.*

The equivalence of log canonical singularities and singularities of dense F-pure type is still a conjecture (see Conjecture 4.9), but once we admit it we can prove the Calabi-Yau version of Theorem 1.1.

Theorem 1.5 (= Theorem 4.10)**.** *Let X be a $\mathbb{Q}$-factorial Mori dream space over an algebraically closed field of characteristic zero. Suppose that Conjecture 4.9 is true. Then X is of Calabi–Yau type if and only if its Cox rings have only log canonical singularities.*

Before this joint work started, the first version of [Br] appeared on arXiv in which Brown proved that the Cox rings of $\mathbb{Q}$-factorial Fano varieties with only log terminal singularities have only log terminal singularities. His proof is based on a different argument from ours.

After this work was done, Kawamata and the second author found ([KO]) a proof of Theorem 4.7 which avoids the reduction modulo p arguments but uses the same strategy as ours of running an MMP for the anti-canonical divisor. Their arguments work equally well for Calabi-Yau cases, and they gave a conjecture-free proof to Theorem 1.5. Also after this paper appeared on arXiv, the authors received an updated version of [Br] in which he also proved the "only if" directions of Theorems 1.1 and 1.5, applying his method to those cases.

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We will freely use the standard notations in [KM] and [BCHM, 3.1. Notation and conventions].

2. Preliminaries and lemmas

2.1. **Mori dream spaces.** Mori Dream Spaces were first introduced by Hu and Keel [HK].

Definition 2.1. A normal projective variety X over a field is called a $\mathbb{Q}$*-factorial Mori dream space* (or *Mori dream space* for short) if X satisfies the following three conditions:

(i) X is $\mathbb{Q}$-factorial, $\mathrm{Pic}(X)$ is finitely generated, and the natural map $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{N}^1(X)_{\mathbb{Q}}$ is isomorphic,
(ii) $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles,
(iii) there exists a finite collection of small birational maps $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (i) and (ii), and that $\mathrm{Mov}(X)$ is the union of the $f_i^*(\mathrm{Nef}(X_i))$.

Remark 2.2. Over the complex number field, the finite generation of $\mathrm{Pic}(X)$ is equivalent to the condition $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$.

On a Mori dream space, as its name suggests, we can run an MMP for any divisor.

Proposition 2.3. *(*[HK, Proposition 1.11]*) Let X be a $\mathbb{Q}$-factorial Mori dream space. Then for any divisor D on X, a D-MMP can be run and terminates.*

Moreover, we know the finiteness of models for a $\mathbb{Q}$-factorial Mori dream space as follows:

Proposition 2.4 ([HK, Proposition 1.11])**.** *Let X be a $\mathbb{Q}$-factorial Mori dream space. Then there exists finitely many dominant rational contractions $f_i : X \dashrightarrow Y_i$ to a normal projective variety Y_i such that, for any dominant rational contraction $f : X \dashrightarrow Y$, there exists i such that $f \simeq f_i$, i.e. there exists an isomorphism $g : Y \to Y_i$ such that $g \circ f = f_i$.*

2.2. **Varieties of Fano type and of Calabi–Yau type.** In this paper, we use the following terminology.

Definition 2.5 (cf. [KM, Definition 2.34],[SS, Remark 4.2])**.** Let X be a normal variety over a field k of *arbitrary characteristic* and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : \widetilde{X} \to X$ be a birational morphism from a normal variety $\widetilde{X}$. Then we can write

$$K_{\widetilde{X}} = \pi^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E,$$

where E runs through all the distinct prime divisors on $\widetilde{X}$ and the $a(E, X, \Delta)$ are rational numbers. We say that the pair (X, Δ) is *log canonical* (resp. *klt*) if $a(E, X, \Delta) \geq -1$ (resp. $a(E, X, \Delta) > -1$) for every prime divisor E over X. If $\Delta = 0$, we simply say that X has only log canonical singularities (resp. log terminal singularities).

Definition 2.6 (cf. [PS, Lemma-Definition 2.6])**.** Let X be a projective normal variety over a field and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(i) We say that (X, Δ) is a *log Fano pair* if $-(K_X+\Delta)$ is ample and (X, Δ) is klt. We say that X is of *Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is a log Fano pair.

(ii) We say that X is of *Calabi–Yau type* if there exits an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X, Δ) is log canonical.

Remark 2.7. If there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is nef and big, then X is of Fano type. See [PS, Lemma-Definition 2.6].

2.3. **Globally F-regular and F-split varieties.** In this subsection, we briefly review the definitions and basic properties of *global F-regularity* and *global F-splitting*.

A scheme X of prime characteristic p is *F-finite* if the Frobenius morphism $F : X \to X$ is finite. A ring R of prime characteristic p is called F-finite if $\operatorname{Spec} R$ is F-finite. For each integer $e \geq 1$, the e-th iterated Frobenius pushforward F^e_*R of a ring R of prime characteristic p is R endowed with an R-module structure given by the e-th iterated Frobenius map $F^e : R \to R$.

Definition 2.8. Let R be an F-finite integral domain of characteristic $p > 0$.

(i) We say that R is *F-pure* if the Frobenius map

$$F : R \to F_*R \quad a \to a^p$$

splits as an R-module homomorphism.

(ii) We say that R is *strongly F-regular* if for every nonzero element $c \in R$, there exists an integer $e \geq 1$ such that

$$cF^e : R \to F^e_*R \quad a \to ca^{p^e}$$

splits as an R-module homomorphism.

An F-finite integral scheme X has only F-pure (resp. strongly F-regular) singularities if $\mathcal{O}_{X,x}$ is F-pure (resp. strongly F-regular) for all $x \in X$.

Definition 2.9. Let X be a normal projective variety defined over an F-finite field of characteristic $p > 0$.

(i) We say that X is *globally F-split* if the Frobenius map
$$\mathcal{O}_X \to F_*\mathcal{O}_X$$
splits as an $\mathcal{O}_X$-module homomorphism.

(ii) We say that X is *globally F-regular* if for every effective divisor D on X, there exists an integer $e \geq 1$ such that the composition map
$$\mathcal{O}_X \to F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$$
of the e-times iterated Frobenius map $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ with a natural inclusion $F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(D)$ splits as an $\mathcal{O}_X$-module homomorphism.

Remark 2.10. Globally F-regular (resp. globally F-split) varieties have only strongly F-regular (resp. F-pure) singularities.

Let X be a normal projective variety over a field. For any ample Cartier divisor H on X, we denote the corresponding *section ring* by
$$R(X, H) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mH)).$$

Proposition 2.11 ([Sm, Proposition 3.1 and Theorem 3.10])**.** *Let X be a normal projective variety over an F-finite field of characteristic $p > 0$. The following conditions are equivalent to each other:*

(1) *X is globally F-split* (*resp. globally F-regular*)*,*
(2) *the section ring $R(X, H)$ with respect to some ample divisor H is F-pure* (*resp. strongly F-regular*)*,*
(3) *the section ring $R(X, H)$ with respect to every ample divisor H is F-pure* (*resp. strongly F-regular*)*.*

Theorem 2.12 ([SS, Theorem 4.3])**.** *Let X be a normal projective variety over an F-finite field of characteristic $p > 0$. If X is globally F-regular* (*resp. globally F-split*)*, then X is of Fano type* (*resp. Calabi–Yau type*)*.*

Lemma 2.13. *Let $f : X \dashrightarrow X_1$ be a small birational map or an algebraic fiber space of normal varieties over an F-finite field of characteristic $p > 0$. If X is globally F-regular (resp. globally F-split), then so is X_1.*

Proof. When f is an algebraic fiber space, the globally F-split case follows from [MR, Proposition 4] and the globally F-regular case does from [HWY, Proposition 1.2 (2)].

When f is a small birational map, X and X_1 are isomorphic in codimension one. In general, a normal projective variety Y is globally

F-regular (resp. globally F-split) if and only if so is $Y \setminus E$, where $E \subseteq Y$ is a closed subset of codimension at least two (see [BKu, 1.1.7 Lemma (iii)] for the globally F-split case and [Has, Lemma 2.9] for globally F-regular case). Thus, we obtain the assertion. □

Now we briefly explain how to reduce things from characteristic zero to characteristic $p > 0$. The reader is referred to [HH2, Chapter 2] and [MS, Section 3.2] for details.

Let X be a normal variety over a field k of characteristic zero and $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on X. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra A of k, we can construct a scheme X_A of finite type over A and closed subschemes $D_{i,A} \subsetneq X_A$ such that there exists isomorphisms

$$\begin{array}{ccc} X & \xrightarrow{\cong} & X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k \\ \uparrow & & \uparrow \\ D_i & \xrightarrow{\cong} & D_{i,A} \times_{\operatorname{Spec} A} \operatorname{Spec} k. \end{array}$$

Note that we can enlarge A by localizing at a single nonzero element and replacing X_A and $D_{i,A}$ with the corresponding open subschemes. Thus, applying the generic freeness [HH2, (2.1.4)], we may assume that X_A and $D_{i,A}$ are flat over $\operatorname{Spec} A$. Enlarging A if necessary, we may also assume that X_A is normal and $D_{i,A}$ is a prime divisor on X_A. Letting $D_A := \sum_i d_i D_{i,A}$, we refer to (X_A, D_A) as a *model* of (X, D) over A.

Given a closed point $\mu \in \operatorname{Spec} A$, we denote by X_μ (resp., $D_{i,\mu}$) the fiber of X_A (resp., $D_{i,A}$) over μ. Then X_μ is a scheme of finite type over the residue field $k(\mu)$ of μ, which is a finite field. Enlarging A if necessary, we may assume that X_μ is a normal variety over $k(\mu)$, $D_{i,\mu}$ is a prime divisor on X_μ and consequently $D_\mu := \sum_i d_i D_{i,\mu}$ is a $\mathbb{Q}$-divisor on X_μ for all closed points $\mu \in \operatorname{Spec} A$.

Let Γ be a finitely generated group of Weil divisors on X. We then refer to a group Γ_A of Weil divisors on X_A generated by a model of a system of generators of Γ as a *model* of Γ over A. After enlarging A if necessary, we denote by Γ_μ the group of Weil divisors on X_μ obtained by restricting divisors in Γ_A over μ.

Given a morphism $f : X \to Y$ of varieties over k and a model (X_A, Y_A) of (X, Y) over A, after possibly enlarging A, we may assume that f is induced by a morphism $f_A : X_A \to Y_A$ of schemes of finite type over A. Given a closed point $\mu \in \operatorname{Spec} A$, we obtain a corresponding morphism $f_\mu : X_\mu \to Y_\mu$ of schemes of finite type over $k(\mu)$. If f is projective (resp. finite), after possibly enlarging A, we may assume that f_μ is projective (resp. finite) for all closed points $\mu \in \operatorname{Spec} A$.

Definition 2.14. Let the notation be as above.

(i) A projective variety (resp. an affine variety) X is said to be of *globally F-regular type* (resp. *strongly F-regular type*) if for a model of X over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense open subset $S \subseteq \operatorname{Spec} A$ of closed points such that X_μ is globally F-regular (resp. strongly F-regular) for all $\mu \in S$.

(ii) A projective variety (resp. an affine variety) X is said to be of *dense globally F-split type* (resp. *dense F-pure type*) if for a model of X over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset $S \subseteq \operatorname{Spec} A$ of closed points such that X_μ is globally F-split (resp. F-pure) for all $\mu \in S$.

Remark 2.15. (1) The above definition is independent of the choice of a model.

(2) If X is of globally F-regular type (resp. strongly F-regular type), then we can take a model X_A of X over some A such that X_μ is globally F-regular (resp. strongly F-regular) for all closed points $\mu \in \operatorname{Spec} A$.

Proposition 2.16. *Let X be a normal projective variety over a field of characteristic zero.*

(1) *If X is $\mathbb{Q}$-Gorenstein and of globally F-regular type* (*resp. dense globally F-split type*), *then it has only log terminal singularities* (*resp. log canonical singularities*).

(2) *If X is of Fano type, then X is of globally F-regular type.*

Proof. (2) is nothing but [SS, Theorem 5.1]. So, we will prove only (1).

Since X is of globally F-regular type (resp. dense globally F-regular type), then it has only singularities of strongly F-regular type (resp. dense F-pure type). It then follows from [HW, Theorem 3.9] that X has only log terminal singularities (resp. log canonical singularities). □

2.4. **Cox rings and their reductions to positive characteristic.** In this paper, we define Cox rings as follows:

Definition 2.17 (Multi-section rings and Cox rings)**.** Let X be an integral normal scheme. For a semi-group Γ of Weil divisors on X, the Γ-graded ring

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))$$

is called the *multi-section ring* of Γ.

Suppose that $\mathrm{Cl}(X)$ is finitely generated. For such X, choose a group Γ of Weil divisors on X such that $\Gamma_{\mathbb{Q}} \to \mathrm{Cl}(X)_{\mathbb{Q}}$ is an isomorphism. Then the multi-section ring $R_X(\Gamma)$ is called a *Cox ring* of X.

Remark 2.18. As seen above, the definition of a Cox ring depends on a choice of the group Γ. When $\mathrm{Cl}(X)$ is a free group, it is common to take Γ so that the natural map $\Gamma \to \mathrm{Cl}(X)$ is an isomorphism. In this case, the corresponding multi-section ring does not depend on the choice of such a group Γ, up to isomorphisms. In general Cox rings are not unique. Here we note that the basic properties of Cox rings are not affected by the ambiguity.

Let m be a positive integer. Then the natural inclusion $R_X(m\Gamma) \subset R_X(\Gamma)$ is an integral extension. Therefore $R_X(\Gamma)$ is of finite type if $R_X(m\Gamma)$ is. Conversely, we can represent $R_X(m\Gamma)$ as an invariant subring of $R_X(\Gamma)$ under an action of a finite group scheme. Therefore $R_X(m\Gamma)$ is of finite type if $R_X(\Gamma)$ is. This shows that the finite generation of a Cox ring does not depend on the choice of Γ.

Suppose that m is not divisible by the characteristic of the base field. Then $R_X(m\Gamma) \subset R_X(\Gamma)$ is étale in codimension one (this follows from [SS, Lemma 5.7.(1)]. See also [Br, Lemma 5.2.]). This shows that in characteristic zero the log-canonicity (resp. log-terminality) of a Cox ring does not depend on the choice of Γ, provided that they are of finite type ([KM, Proposition 5.20]).

Finally, F-purity (resp. quasi-F-regularity) of a Cox ring is also independent of the choice of Γ. We prove it for F-purity, and the arguments for quasi-F-regularity are the same. Suppose that $R_X(\Gamma)$ is a Cox ring and is F-pure. Take an ample divisor $H \in \Gamma$. Then $R(X, H) = R_X(\mathbb{N}H)$ is also F-pure, since $\mathbb{N}H$ is a sub-semigroup of Γ (use the argument in the proof of Lemma 4.1 below). By Proposition 2.11, this implies that X is globally F-split. By Lemma 4.5, the multi-section ring $R_X(\Gamma')$ of any semigroup Γ' of Weil divisors on X is F-split.

Remark 2.19. In [Hau, Section 2], Hausen gave a canonical (up to graded isomorphisms) way to define Cox rings so that they are graded by the class group $\mathrm{Cl}(X)$, including the torsion part. Hausen's definition seems to be the genuine one, but it is more involved than ours since it requires the step of taking a quotient by a relation. Our definition of Cox rings as multi-section rings are easier to handle in this sense.

Below we prove that Hausen's Cox ring and ours differ only by a finite extension which is étale in codimension one (in characteristic zero). Therefore log-terminality (resp. log-canonicity) is not affected by the choice of a definition.

Lemma 2.20. *Let X be a Mori dream space defined over a field of characteristic zero. The Cox ring of X in the sense of* [Hau] *is log terminal (resp. log canonical) if and only if one (hence any) Cox ring $R_X(\Gamma)$ of X has the same property.*

Proof. Choose a group Γ of Weil divisors on X such that the natural map

$$\Gamma \to \mathrm{Cl}\,(X)/\,\mathrm{Cl}\,(X)_{tors}$$

is an isomorphism. Below we denote the image of the injection $\Gamma \to \mathrm{Cl}\,(X)$ by the same letter Γ.

Let $\mathcal{R}$ be the Cox ring of X in the sense of [Hau]. If we set $H = \mathrm{Spec}\, k[\mathrm{Cl}\,(X)]$, it acts on $\mathrm{Spec}\,\mathcal{R}$ via the $\mathrm{Cl}\,(X)$ grading. Since we have the canonical projection $\mathrm{Cl}\,(X) \to \mathrm{Cl}\,(X)_{tors}$ defined by the splitting $\mathrm{Cl}\,(X) = \Gamma \bigoplus \mathrm{Cl}\,(X)_{tors}$, the finite group $G = \mathrm{Spec}\, k[\mathrm{Cl}\,(X)_{tors}] \subset H$ naturally acts on $\mathrm{Spec}\,\mathcal{R}$. Note that the subring $R_X(\Gamma) \subset \mathcal{R}$ is the invariant subring with respect to the action of G.

By [Hau, Proposition 2.2, (iii)], there exists an H invariant open subset U of $\mathrm{Spec}\,\mathcal{R}$, whose complement has codimension at least two, on which H (hence G) acts freely. Therefore the finite surjective morphism

$$\pi : \mathrm{Spec}\,\mathcal{R} \to \mathrm{Spec}\, R_X(\Gamma)$$

is étale over the open subset $U/G \subset \mathrm{Spec}\, R_X(\Gamma)$, showing that π is étale in codimension one. Thus the conclusion follows. □

The following is a basic fact on the finite generation of Cox rings.

Remark 2.21 ([HK, Proposition 2.9])**.** Let X be a normal projective variety satisfying (i) of Definition 2.1. Then X is a Mori dream space if and only if its Cox rings are finitely generated over k.

If the variety is a $\mathbb{Q}$-factorial Mori dream space of globally F-regular type, then we can show that taking multi-section rings commutes with reduction modulo p.

Lemma 2.22. *Let X be a $\mathbb{Q}$-factorial Mori dream space defined over a field k of characteristic zero and Γ be a finitely generated group of Cartier divisors on X. Suppose that X is of globally F-regular type* (*resp. dense globally F-split type*)*. Then, replacing Γ with a suitable positive multiple if necessary, we can take a model (X_A, Γ_A) of (X, Γ) over a finitely generated $\mathbb{Z}$-subalgebra A of k and a dense open subset* (*resp. a dense subset*) *$S \subseteq \mathrm{Spec}\, A$ of closed points such that*

(1) X_μ is globally F-regular (resp. globally F-split),
(2) one has

$$(R_X(\Gamma))_\mu = R_{X_A}(\Gamma_A) \otimes_A k(\mu) \simeq R_{X_\mu}(\Gamma_\mu)$$

for every $\mu \in S$.

Proof. We will show that there exists an integer $m \geq 1$, a model (X_A, Γ_A) of (X, Γ) over a finitely generated $\mathbb{Z}$-subalgebra A of k and a dense open subset (resp. a dense subset) $S \subseteq \operatorname{Spec} A$ of closed points such that for every $\mu \in S$ and every divisor $D_A \in m\Gamma_A$,

(1) X_μ is globally F-regular (resp. globally F-split),
(2) one has

$$H^0(X_A, \mathcal{O}_{X_A}(D_A)) \otimes_A k(\mu) \simeq H^0(X_\mu, \mathcal{O}_{X_\mu}(D_\mu)).$$

First note that for every divisor $D \in \Gamma$, a D-MMP can be run and terminates by Lemma 2.3. It follows from Proposition 2.4 that there exist finitely many birational contractions $f_i : X \dashrightarrow Y_i$ and finitely many projective morphisms $g_{ij} : Y_i \to Z_{ij}$ with connected fibers, where the Y_i are $\mathbb{Q}$-factorial Mori dream spaces and the Z_{ij} are normal projective varieties, satisfying the following property: for every divisor $D \in \Gamma$, there exist i and j such that $f_i : X \dashrightarrow Y_i$ is isomorphic to a composition of D-flips and D-divisorial contractions and $g_{ij} : Y_i \to Z_{ij}$ is

- the D-canonical model: i.e. the morphism defined by the complete linear system $|m f_{i,*}D|$ for sufficiently divisible m, if D is pseudo-effective. Or
- the D-Mori fiber space, otherwise.

Then by Lemma 2.13, all Y_i are of globally F-regular type (resp. dense globally F-split type).

Suppose given models $f_{i,A} : X_A \dashrightarrow Y_{i,A}$ and $g_{ij,A} : Y_{i,A} \to Z_{ij,A}$ of the f_i and g_{ij} over a finitely generated $\mathbb{Z}$-subalgebra A of k, respectively. Enlarging A if necessary, we may assume that $Y_{i,\mu}, Z_{ij,\mu}$ are normal varieties for all closed points $\mu \in \operatorname{Spec} A$. In addition, after possibly enlarging A again, we can assume that all $f_{i,A}$ and $f_{i,\mu}$ are compositions of D_A (resp. D_μ) flips and D_A (resp. D_μ) divisorial contractions, and all $g_{ij,A}$ and $g_{ij,\mu}$ are algebraic fiber spaces for all closed points $\mu \in \operatorname{Spec} A$.

By Definition 2.1 (ii), we take a sufficiently large m so that for each pseudo-effective divisor $D \in \Gamma$, there exists some i, j and an ample Cartier divisor H_{mD} on Z_{ij} such that $m f_{i_*} D \sim g_{ij}^* H_{mD}$. By Definition 2.1 (ii) again, enlarging A if necessary, we may assume that models $H_{mD,A}$ of the H_{mD} are given over A.

Now we fix an effective divisor $D \in \Gamma$, and choose f_i, g_{ij} for this divisor D as above. Then

$$\begin{aligned} H^0(X_A, \mathcal{O}_{X_A}(mD_A)) &\simeq H^0(Y_{i,A}, \mathcal{O}_{Y_{i,A}}(mf_{i,A*}D_A)) \\ &\simeq H^0(Z_{ij,A}, \mathcal{O}_{Z_{ij,A}}(H_{mD,A})). \end{aligned}$$

The first isomorphism follows from the effectivity of the divisor $D_A - f_{i,A}^* f_{i,A*} D_A$, which follows from the fact that $f_{i,A}$ is a D_A-MMP.

Similarly, for all closed points $\mu \in \operatorname{Spec} A$, we have

$$H^0(X_\mu, \mathcal{O}_{X_\mu}(mD_\mu)) \simeq H^0(Z_{ij,\mu}, \mathcal{O}_{Z_{ij,\mu}}(H_{mD,\mu})).$$

We then use the following claim.

Claim 2.23. *Let W_A be a normal projective variety over a finitely generated $\mathbb{Z}$-algebra A such that $W_\mu := W_A \otimes_A k(\mu)$ is globally F-split for all closed points μ in a dense subset S of* $\operatorname{Spec} A$*, and let H_A be an ample Cartier divisor on W_A. Then*

$$H^0(W_A, \mathcal{O}_{W_A}(H_A)) \otimes_A k(\mu) \simeq H^0(W_\mu, \mathcal{O}_{W_\mu}(H_\mu)).$$

for all closed points $\mu \in S$.

Proof of Claim 2.23. First note that $H^1(W_\mu, \mathcal{O}_{W_\mu}(H_\mu)) = 0$ for all closed points $\mu \in S$. In fact, since H_μ is ample and X_μ is globally F-split, this follows from [MR, Proposition 3]. By [Har, Chapter III, Corollary 12.9], we see that $H^1(W_A, \mathcal{O}_{W_A}(H_A)) = 0$. Applying [Har, Chapter III, Theorem 12.11 (b)] for $i = 1$, and then [Har, Chapter III, Theorem 12.11 (a)] for $i = 0$, we get the conclusion. □

Applying the above claim to $Z_{ij,A}$ and $H_{mD,A}$, we see that

$$H^0(X_A, \mathcal{O}_{X_A}(mD_A)) \otimes_A k(\mu) \simeq H^0(X_\mu, \mathcal{O}_{X_\mu}(mD_\mu))$$

for all effective divisors $D_A \in \Gamma_A$ and all closed points $\mu \in S$.

Next we consider the case when a divisor $D \in \Gamma$ is not effective. In particular, $H^0(X, \mathcal{O}_X(D)) = 0$. Choose f_i, g_{ij} for this divisor D as above, and we take a g_{ij}-contracting curve C_{ij}. Note that the class $[C_{ij}] \in \mathrm{NE}^1(Y_i)$ is a movable class. After possibly enlarging A, we can take a model $C_{ij,A}$ of C_{ij} over A such that the class $[C_{ij,\mu}] \in \mathrm{NE}^1(Y_{i,\mu})$ is also a movable class for all closed points $\mu \in \operatorname{Spec} A$. Then

$$f_{i,\mu*}D_\mu . C_{i,j,\mu} = f_{i*}D . C_{i,j} < 0,$$

which implies

$$H^0(X_\mu, \mathcal{O}_{X_\mu}(mD_\mu)) \simeq H^0(Y_{i,\mu}, \mathcal{O}_{Y_{i,\mu}}(mf_{i,\mu*}D_\mu)) = 0$$

for all closed points $\mu \in \operatorname{Spec} A$. Thus,

$$H^0(X_A, \mathcal{O}_{X_A}(mD_A)) \otimes_A k(\mu) = H^0(X_\mu, \mathcal{O}_{X_\mu}(mD_\mu))$$

holds for all divisors $D_A \in \Gamma_A$ and all closed points $\mu \in \operatorname{Spec} A$. $\square$

3. Proofs of Theorems 1.2 and 1.4

In this section, we give proofs of Theorems 1.2 and 1.4.

3.1. **Globally F-regular case.** The following lemma is a special case of [FG1, Corollary 3.3] and [PS, Theorem 2.9], which follow from Kawamata's semi-positivity theorem and Ambro's canonical bundle formula (cf. [Am]), respectively. We, however, do not need any semi-positivity type theorem for the proof of Lemma 3.1.

Lemma 3.1 (cf. [FG1, Theorem 3.1]). *Let X be a normal variety over a field of characteristic zero and $f : X \to Y$ be a small projective birational contraction. Then X is of Fano type if and only if so is Y.*

Proof. First we assume that X is of Fano type, that is, there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is a log Fano pair. Let H be a general ample divisor on Y, and take a sufficiently small rational number $\epsilon > 0$ so that $-(K_X + \Delta + \epsilon f^*H)$ is ample and $(X, \Delta + \epsilon f^*H)$ is klt. We also take a general effective ample $\mathbb{Q}$-divisor A on X such that $(X, \Delta + \epsilon f^*H + A)$ is klt and

$$K_X + \Delta + \epsilon f^*H + A \sim_{\mathbb{Q}} 0.$$

Then

$$K_Y + f_*\Delta + \epsilon H + f_*A = f_*(K_X + \Delta + \epsilon f^*H + A) \sim_{\mathbb{Q}} 0.$$

On the other hand, since f is small,

$$f^*(K_Y + f_*\Delta + \epsilon H + f_*A) = K_X + \Delta + \epsilon f^*H + A$$

Therefore, $(Y, f_*\Delta + f_*A)$ is klt and $-(K_Y + f_*\Delta + f_*A) \sim_{\mathbb{Q}} \epsilon H$, which means that Y is of Fano type.

Conversely, we assume that Y is of Fano type. Let Γ be an effective $\mathbb{Q}$-divisor on Y such that (Y, Γ) is a log Fano pair and let Γ_X denote the strict transform of Γ on X. Since f is small, we see that

$$K_X + \Gamma_X = f^*(K_Y + \Gamma).$$

Thus, (X, Γ_X) is klt and $-(K_X + \Gamma_X)$ is nef and big. It then follows from Remark 2.7 that X is of Fano type. $\square$

Now we give a proof of Theorem 1.2.

Theorem 3.2 (=Theorem 1.2)**.** *Let X be a $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero. Then X is of Fano type if and only if it is of globally F-regular type.*

Proof. The only if part follows from Proposition 2.16 (2), so we will prove the if part.

First we remark that if Y is any $\mathbb{Q}$-factorial Mori dream space of globally F-regular type, then $-K_Y$ is big. Indeed, choosing a suitable integer $m \geq 1$, by Lemma 2.22, we can take a model Y_A of Y over a finitely generated $\mathbb{Z}$-subalgebra A such that

(1) $Y_\mu = Y_A \times_{\operatorname{Spec} A} \operatorname{Spec} k(\mu)$ is globally F-regular,
(2) $R(Y, -mK_Y)_\mu \cong R(Y_\mu, -mK_{Y_\mu})$

for all closed points $\mu \in \operatorname{Spec} A$. Since $-K_{Y_\mu}$ is big by (1) and Theorem 2.12, it follows from (2) that $-K_Y$ is also big.

Since X is a $\mathbb{Q}$-factorial Mori dream space, we can run a $(-K_X)$-MMP:

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{l-2}}{\dashrightarrow} X_{l-1} \overset{f_{l-1}}{\dashrightarrow} X_l = X',$$

where each X_i is a $\mathbb{Q}$-factorial Mori dream space and X' is a $(-K_X)$-minimal model. Note that each X_i is of globally F-regular type by Lemma 2.13. In particular, $-K_{X'}$ is nef and big, and X' has only log terminal singularities by Lemma 2.16 (1). It then follows from Remark 2.7 that X' is of Fano type.

Now we show that X_{l-j} is of Fano type by induction on j. When $j = 0$, we have already seen that $X' = X_l$ is of Fano type. Suppose that X_{l-j+1} is of Fano type. Let Δ_{l-j+1} be an effective $\mathbb{Q}$-divisor on X_{l-j+1} such that $(X_{l-j+1}, \Delta_{l-j+1})$ is a log Fano pair.

When $f := f_{l-j}$ is a divisorial contraction, $K_{X_{l-j}}$ is f-ample and $\Delta_{l-j} := f_*^{-1}\Delta_{l-j+1}$ is f-nef. In particular, $K_{X_{l-j}} + \Delta_{l-j}$ is f-ample. It then follows from the negativity lemma that

$$-(K_{X_{l-j}} + \Delta_{l-j}) = -f^*(K_{X_{l-j+1}} + \Delta_{l-j+1}) + aE,$$

where a is a positive rational number and E is the f-exceptional prime divisor on X_{l-j}. We see from this that the pair $(X_{l-j}, \Delta_{l-j} + aE)$ is klt and $-(K_{X_{l-j}} + \Delta_{l-j} + aE)$ is nef and big, which implies by Remark 2.7 that X_{l-j} is of Fano type.

When f_{l-j} is a $(-K_{X_{l-j}})$-flip, we consider the following flipping diagram:

$$\begin{array}{ccc} X_{l-j} & \overset{f_{l-j}}{\dashrightarrow} & X_{l-j+1} \\ & {\scriptstyle \psi_{l-j}} \searrow \quad \swarrow {\scriptstyle \psi^{+}_{l-j}} & \\ & Z_{l-j} & \end{array}$$

Applying Lemma 3.1 to ψ_{l-j} and ψ^{+}_{l-j}, we see that X_{l-j} is of Fano type.

Thus, we conclude that $X = X_0$ is of Fano type. □

Theorem 1.2 has an application to spherical varieties defined over fields of characteristic zero. It was proven in [BKn] that projective $\mathbb{Q}$-factorial spherical varieties are Mori dream spaces. Among them, projective equivariant embeddings of connected reductive groups are known to be of globally F-regular type by [BT, Theorem 4.4]. Hence we obtain the following corollary of Theorem 1.2.

Corollary 3.3. *Any projective $\mathbb{Q}$-factorial compactification of a connected reductive group in characteristic zero is of Fano type.*

3.2. **Globally F-split case.** In this subsection, we start with the following lemma. An analogous statement for klt Calabi–Yau pairs follows from [Am, Theorem 0.2], but our proof of Lemma 3.4 is easier.

Lemma 3.4. *Let X be a normal variety over a field of characteristic zero and $f : X \to Y$ be a small projective birational contraction. Then X is of Calabi–Yau type if and only if so is Y.*

Proof. Suppose that X is of Calabi–Yau type, that is, there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is log canonical and $K_X + \Delta \sim_{\mathbb{Q}} 0$. Letting $\Delta_Y := f_*\Delta$, one has

$$K_Y + \Delta_Y = f_*(K_X + \Delta) \sim_{\mathbb{Q}} 0.$$

On the other hand, since f is small,

$$f^*(K_Y + \Delta_Y) = K_X + \Delta,$$

which implies that (Y, Δ_Y) is log canonical.

Conversely, we assume that Y is of Calabi–Yau type. Let Γ be an effective $\mathbb{Q}$-divisor on Y such that (Y, Γ) is log canonical and $K_Y + \Gamma \sim_{\mathbb{Q}} 0$. Let Γ_X denote the strict transform of Γ on X. Since f is small, we see that

$$K_X + \Gamma_X = f^*(K_Y + \Gamma),$$

which implies that (X, Γ_X) is log canonical and $K_X + \Gamma_X \sim_{\mathbb{Q}} 0$. □

Proof of Theorem 1.4. First we remark that if Y is a $\mathbb{Q}$-factorial Mori dream space of dense globally F-split type, then $-K_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor on Y. Indeed, choosing a suitable integer $m \geq 1$, by Lemma 2.22, we can take a model Y_A of Y over a finitely generated $\mathbb{Z}$-subalgebra A and a dense subset $S \subseteq \operatorname{Spec} A$ such that

(1) $Y_\mu = Y_A \times_{\operatorname{Spec} A} \operatorname{Spec} k(\mu)$ is globally F-split,
(2) $R(Y, -mK_Y)_\mu \cong R(Y_\mu, -mK_{Y_\mu})$

for all closed points $\mu \in S$. Since $-K_{Y_\mu}$ is $\mathbb{Q}$-linearly equivalent to some effective $\mathbb{Q}$-divisor by (1) and Theorem 2.12, it follows from (2) that so is $-K_Y$.

Since X is a $\mathbb{Q}$-factorial Mori dream space, we can run a $(-K_X)$-MMP:

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{l-2}}{\dashrightarrow} X_{l-1} \overset{f_{l-1}}{\dashrightarrow} X_l = X',$$

where each X_i is a $\mathbb{Q}$-factorial Mori dream space and X' is a $(-K_X)$-minimal model. Note that each X_i is of dense globally F-split type by Lemma 2.13. In particular, X' has only log canonical singularities by Lemma 2.16 (1). Then X' is of Calabi–Yau type, because $-K_{X'}$ is semi-ample.

Now we show that X_{l-j} is of Calabi–Yau type by induction on j. When $j = 0$, we have already seen that $X' = X_l$ is of Calabi–Yau type. Suppose that X_{l-j+1} is of Calabi–Yau type. Let Δ_{l-j+1} be an effective $\mathbb{Q}$-divisor on X_{l-j+1} such that $(X_{l-j+1}, \Delta_{l-j+1})$ is log canonical and $K_{X_{l-j+1}} + \Delta_{l-j+1} \sim_{\mathbb{Q}} 0$.

When $f := f_{l-j}$ is a divisorial contraction, by an argument similar to the proof of Theorem 1.2, we have

$$-(K_{X_{l-j}} + \Delta_{l-j}) = -f^*(K_{X_{l-j+1}} + \Delta_{l-j+1}) + aE,$$

where Δ_{l-j} is the strict transform of Δ_{l-j+1} on X_{l-j}, E is the f-exceptional prime divisor on X_{l-j} and a is a positive rational number. It then follows that $(X_{l-j}, \Delta_{l-j} + aE)$ is log canonical and $K_{X_{l-j}} + \Delta_{l-j} + aE \sim_{\mathbb{Q}} 0$, that is, X_{l-j} is of Calabi–Yau type.

When f_{l-j} is a $(-K_{X_{l-j}})$-flip, by an argument similar to the proof of Theorem 1.2, Lemma 3.4 implies that X_{l-j} is of Calabi–Yau type.

Thus, we conclude that $X = X_0$ is of Calabi–Yau type. □

4. Characterization of varieties of Fano type

In this section, we give a characterization of varieties of Fano type in terms of the singularities of their Cox rings.

Lemma 4.1. *Let X be a normal projective variety over a field k of characteristic zero. Let Γ be a finitely generated semigroup of Weil divisors on X and $\Gamma' \subset \Gamma$ be a sub-semigroup. If $R_X(\Gamma)$ is of strongly F-regular type, so is $R_X(\Gamma')$.*

Proof. Let $(X_A, \Gamma_A, \Gamma'_A)$ be a model of (X, Γ, Γ') over a finitely generated $\mathbb{Z}$-subalgebra A of k.

Note that the natural inclusion $\iota : R_{X_A}(\Gamma'_A) \subset R_{X_A}(\Gamma_A)$ splits. In fact we have the natural $R_{X_A}(\Gamma'_A)$ module homomorphism

$$\varphi : R_{X_A}(\Gamma_A) \to R_{X_A}(\Gamma'_A),$$

which is defined as follows: for $f \in R_{X_A}(\Gamma_A)$, write $f = \sum_{D_A \in \Gamma_A} f_{D_A}$. Define

$$\varphi(f) = \sum_{D_A \in \Gamma'_A} f_{D_A}.$$

It is easy to see that φ is a $R_{X_A}(\Gamma'_A)$-linear map and $\varphi \circ \iota = \mathrm{id}_{R_{X_A}(\Gamma'_A)}$. Once we have such a splitting, it is clear that for any closed point $\mu \in \operatorname{Spec} A$, $R_{X_A}(\Gamma'_A) \otimes_A k(\mu)$ is a split subring of $R_{X_A}(\Gamma_A) \otimes_A k(\mu)$. Now the conclusion follows from the fact that the strong F-regularity descends to a direct summand (see [HH, Theorem 3.1]). ◻

Definition 4.2 ([Has, (2.1)])**.** Let Γ be a finitely generated torsion free abelian group. Let R be a (not necessarily Noetherian) Γ-graded integral domain of characteristic $p > 0$. For each integer $e \geq 1$, $F^e_* R$ is just R as an abelian group, but its R-module structure is determined by $r \cdot x := r^{p^e} x$ for all $r \in R$ and $x \in F^e_* R$. We give $F^e_* R$ a $\frac{1}{p^e}\Gamma$-module structure by putting $[F^e_* R]_{n/p^e} = [R_n]$.

We say that R is *quasi-F-regular* if for any homogeneous nonzero element $c \in R$ of degree n, there exists an integer $e \geq 1$ such that

$$cF^e : R \to F^e_* R(n)$$

splits as a $\frac{1}{p^e}\Gamma$-graded R-linear map, where $R(n)$ denotes the degree shifting of R by n.

Remark 4.3. When R is a Noetherian F-finite Γ-graded integral domain, R is quasi-F-regular if and only if R is strongly F-regular.

Remark 4.4. The notion of F-purity can be defined for non-Noetherian rings. Let R be a (not necessarily Noetherian) ring of prime characteristic p. We say R is F-pure if the Frobenius map $R \to F_* R$ is pure, that is, $M \to F_* R \otimes_R M$ is injective for every R-module M. When R is a Noetherian and F-finite, this definition coincides with that given in Definition 2.8.

Lemma 4.5 (cf. [Has, Lemma 2.10])**.** *Let X be a normal projective variety defined over an F-finite field of characteristic $p > 0$ and Γ be a semigroup of Weil divisors on X. If X is globally F-regular (resp. globally F-split), then $R_X(\Gamma)$ is quasi-F-regular (resp. F-pure).*

Proof. The globally F-regular case follows from [Has, Lemma 2.10] and the globally F-split case also follows from essentially the same argument. □

Proposition 4.6. *Let X be a normal projective variety over an F-finite field of characteristic $p > 0$. Then X is globally F-regular if and only if its Cox rings are quasi-F-regular.*

Proof. If X is globally F-regular, then by Lemma 4.5, any multisection ring of X is quasi-F-regular, and so are the Cox rings of X.
Conversely, suppose that a cox ring $R_X(\Gamma)$ is quasi-F-regular. Since Γ contains an ample divisor H on X, its section ring $R(X, H)$ is a graded direct summand of $R_X(\Gamma)$. Since $R(X, H)$ is Noetherian and F-finite, this implies that $R(X, H)$ is strongly F-regular. It then follows from Proposition 2.11 that X is globally F-regular. □

Theorem 4.7 (= Theorem 1.1)**.** *Let X be a $\mathbb{Q}$-factorial projective variety over an algebraically closed field k of characteristic zero. Then X is of Fano type if and only if it is a Mori dream space and its Cox rings have only log terminal singularities.*

Proof of Theorem 4.7. Let Γ be a group of Cartier divisors on X which defines a Cox ring of X.

First assume that X is of Fano type. Then by [BCHM, Corollary 1.3.2], $R_X(\Gamma)$ is a finitely generated algebra over k. Also by Proposition 2.16, X is of globally F-regular type. Replacing Γ with a suitable positive multiple if necessary, by Lemma 2.22, we can take a model (X_A, Γ_A) of (X, Γ) over a finitely generated $\mathbb{Z}$-subalgebra A of k such that

(1) $X_\mu = X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k(\mu)$ is globally F-regular,
(2) $R_X(\Gamma)_\mu = R_{X_A}(\Gamma_A) \otimes_A k(\mu) \cong R_{X_\mu}(\Gamma_\mu)$

for all closed points $\mu \in \operatorname{Spec} A$.

It follows from Lemma 4.5 and (1) that $R_{X_\mu}(\Gamma_\mu)$ is strongly F-regular for all closed points $\mu \in \operatorname{Spec} A$, which means by (2) that $R_X(\Gamma)$ is of strongly F-regular type. Since $\operatorname{Spec} R_X(\Gamma)$ is $\mathbb{Q}$-Gorenstein, we can conclude from [HW, Theorem 3.9] that $\operatorname{Spec} R_X(\Gamma)$ has only log terminal singularities.

Conversely, suppose that the Cox ring $R_X(\Gamma)$ of X is finitely generated over k and has only log terminal singularities. Then we see that $R_X(\Gamma)$ is of strongly F-regular type by [Ha, Theorem 5.2]. Take an ample divisor $H \in \Gamma$ on X. Since $R(X, H) = R_X(\mathbb{Z}H)$ and $\mathbb{Z}H$ is a subsemigroup of Γ, by Lemma 4.1, $R(X, H)$ is also of strongly F-regular type. By replacing H with its positive multiple and enlarging A if necessary, we may assume that

$$R(X_A, H_A) \otimes_A k(\mu) \cong R(X_\mu, H_\mu)$$

holds for any closed point $\mu \in \operatorname{Spec} A$ (use the Serre vanishing theorem and the Grauert theorem [Har, Corollary 12.9]). It then follows from Proposition 2.11 that X is of globally F-regular type, which implies by Theorem 1.2 that X is of Fano type. Thus, we finish the proof of Theorem 4.7. □

Remark 4.8. Suppose that X is a variety of Fano type defined over an algebraically closed field of characteristic zero. If X is in addition locally factorial and $\mathrm{Cl}(X)$ is free, then its Cox ring is a UFD ([**?**, Corollary 1.2.]) and hence a Gorenstein domain. Hence it has only Gorenstein canonical singularities.

In the rest of this section we consider the Calabi-Yau version of Theorem 4.7.

The notion of F-purity is defined also for a pair of a normal variety X and an effective $\mathbb{Q}$-divisor Δ on X (the reader is referred to [HW, Definition 2.1] for the definition of F-pure pairs). It is conjectured that modulo p reduction of a log canonical pair is F-pure for infinitely many p :

Conjecture 4.9 (cf. [HW, Problem 5.1.2])**.** *Let X be a normal variety over an algebraically closed field of characteristic zero and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the pair (X, Δ) is log canonical if and only if it is of dense F-pure type.*

If Conjecture 4.9 is true, then we can give a characterization of Mori dream spaces of Calabi–Yau type in terms of the singularities of their Cox rings, using an argument similar to the proof of Theorem 4.7.

Theorem 4.10. *Let X be a $\mathbb{Q}$-factorial Mori dream space over an algebraically closed field of characteristic zero. Suppose that Conjecture 4.9 is true. Then X is of Calabi–Yau type if and only if its Cox rings have only log canonical singularities.*

Proof. The proof is similar to that for Theorem 4.7. Suppose that (X, Δ) is a log Calabi-Yau pair and X is a Mori dream space. Let H be any ample Cartier divisor on X. By [SS, Proposition 5.4], the pair $(\operatorname{Spec} R(X, H), \Delta_{\operatorname{Spec} R(X,H)})$ has only log canonical singularities. It then follows from Conjecture 4.9 that $R(X, H)$ is of dense F-pure type, which implies by Proposition 2.11 that X is of dense globally F-split type.

Let Γ be a group of Cartier divisors on X which defines a Cox ring of X. Replacing Γ with a subgroup of finite index if necessary, by Lemma 2.22, we can take a model X_A of X and Γ_A of Γ over a finitely generated $\mathbb{Z}$-subalgebra A of k and a dense subset $S \subseteq \operatorname{Spec} A$ of closed points such that

(1) X_μ is globally F-split,
(2) $R_X(\Gamma)_\mu = R_{X_A}(\Gamma_A) \otimes_A k(\mu) \cong R_{X_\mu}(\Gamma_\mu)$

for all $\mu \in S$. It then follows from Lemma 4.5 and (1) that $R_{X_\mu}(\Gamma_\mu)$ is F-pure for all closed points $\mu \in S$, which means by (2) that $R_X(\Gamma)$ is of dense F-pure type. Since $R_X(\Gamma)$ is $\mathbb{Q}$-Gorenstein, we can conclude from [HW, Theorem 3.9] that $\operatorname{Spec} R_X(\Gamma)$ has only log canonical singularities.

Conversely, suppose that a Cox ring $R_X(\Gamma)$ of X is log canonical. By Conjecture 4.9, we see that $R_X(\Gamma)$ is of dense F-pure type. For an ample divisor $H \in \Gamma$ the split subring $R(X, H) \subset R_X(\Gamma)$ is also dense F-pure type. By Proposition 2.11, X is of dense globally F-split type. Finally we apply Theorem 1.4 to conclude that X is of Calabi-Yau type. □

Since Conjecture 4.9 was partially confirmed in [FT, Corollary 3.6], we can show that if a Calabi–Yau surface X is a Mori dream space, then the Cox rings of X have only log canonical singularities.

Corollary 4.11. *Let X be a klt projective surface over an algebraically closed field k of characteristic zero such that $K_X \sim_{\mathbb{Q}} 0$. If X is a Mori dream space, then its Cox rings have only log canonical singularities.*

Proof. Trace the proof for Theorem 4.10 above using [FT, Corollary 3.6]. □

5. Case of Non-$\mathbb{Q}$-factorial Mori dream space

In this section we generalize our results to not-necessarily $\mathbb{Q}$-factorial Mori dream spaces (see [Ok, Section 10]). These varieties admit a small $\mathbb{Q}$-factorial modification by a Mori dream space, so that we can apply our results obtained so far.

Definition 5.1. Let X be normal projective variety whose divisor class group $\operatorname{Cl}(X)$ is finitely generated. Choose a finitely generated group of Weil divisors Γ on X such that the natural map

$$\Gamma_{\mathbb{Q}} \to \operatorname{Cl}(X)_{\mathbb{Q}}$$

is an isomorphism. X is said to be a *not-necessarily $\mathbb{Q}$-factorial Mori dream space* if the multi-section ring $R_X(\Gamma)$ is of finite type over the base field.

When X is $\mathbb{Q}$-factorial, this coincides with an ordinary Mori dream space. The following is quite useful:

Proposition 5.2. *For a not necessarily $\mathbb{Q}$-factorial Mori dream space X we can find a small birational morphism $X' \to X$ from a $\mathbb{Q}$-factorial Mori dream space X'.*

Proof. This is essentially proven in [AHL, Proof of Theorem 2.3]. See also [Ok, Remark 10.3]. □

Corollary 5.3 (not necessarily $\mathbb{Q}$-factorial version of Theorem 4.7)**.** *Let X be a normal projective variety over a field k of characteristic zero. Then X is of Fano type if and only if it is a (not necessarily $\mathbb{Q}$-factorial) Mori dream space and its Cox rings have only log terminal singularities.*

Proof. Suppose that X is of Fano type. Then we can take a small $\mathbb{Q}$-factorization $f : \tilde{X} \to X$ (see [BCHM, Corollary 1.4.3]) and show that $\tilde{X}$ also is of Fano type by Lemma 3.1. By Theorem 4.7, we see that $\tilde{X}$ is a Mori dream space and its Cox rings have only log terminal singularities. Since f is small, we see that X is a not necessarily $\mathbb{Q}$-factorial Mori dream space, and its Cox rings are the same as those of $\tilde{X}$.

Conversely, suppose that X is a not necessarily $\mathbb{Q}$-factorial Mori dream space and its Cox rings have only log terminal singularities. Take a small $\mathbb{Q}$-factorization $f : \tilde{X} \to X$ as in Proposition 5.2. Again by Theorem 4.7 we see that $\tilde{X}$ is of Fano type. By Lemma 3.1, we see that X is of Fano type. □

Corollary 5.4 (not necessarily $\mathbb{Q}$-factorial version of Theorem 1.2)**.** *Let X be a not necessarily $\mathbb{Q}$-factorial Mori dream space over a field of characteristic zero. Then X is of Fano type if and only if it is of globally F-regular type.*

Proof. Since both of the notions are preserved by taking a small $\mathbb{Q}$-factorization and taking a small birational contraction, we can prove the equivalence by taking the small $\mathbb{Q}$-factorization of X as in Proposition 5.2 and then apply Theorem 1.2. □

As an application of Corollary 5.4, we show that the image of a variety of Fano type again is of Fano type, which was first proven in [FG2] (see also [FG1]).

Corollary 5.5. *Let $f : X \to Y$ be a surjective morphism between normal projective varieties over an algebraically closed field k of characteristic zero. If X is of Fano type, then Y is of Fano type.*

Proof. Taking the Stein factorization of f, we can assume either f is an algebraic fiber space or a finite morphism. When f is finite, it is dealt with in [FG2]. Therefore we consider the case when f is an algebraic fiber space. By [BCHM, Corollary 1.3.2] and Theorem 5.4, X is a not necessarily $\mathbb{Q}$-factorial Mori dream space and of globally F-regular type. By [Ok, Theorem 1.1] and Lemma 2.13, we see that Y is a a not necessarily $\mathbb{Q}$-factorial Mori dream space of globally F-regular type. Again by Theorem 1.2, we conclude that Y is of Fano type. □

Remark 5.6. In this paper, the algebraic closedness of the ground field k is used only where we use the results of [BCHM].

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