MORI DREAM SPACES OF CALABI-YAU TYPE AND THE LOG CANONICITY OF THE COX RINGS

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ABSTRACT. We prove that a Mori dream space over a field of characteristic zero is of Calabi-Yau type if and only if its Cox ring has at worst log canonical singularities. By slightly modifying the arguments we also reprove the characterization of varieties of Fano type by log terminality of their Cox rings.

1. INTRODUCTION

Throughout the paper we work over a field k of characteristic zero. A normal projective variety X is said to be of Fano type (respectively of Calabi-Yau type) if there exists a boundary $\mathbf{Q}$-divisor Δ on X such that (X, Δ) has klt singularities (resp. log canonical singularities) and $-(K_X + \Delta)$ is ample (resp. numerically trivial). [BCHM10] proved that a $\mathbf{Q}$-factorial variety of Fano type is a Mori dream space. Then [GOST12, Theorem 1.1] proved that a $\mathbf{Q}$-factorial variety of Fano type can be characterized as a Mori dream space whose Cox ring has at worst log terminal singularities. In [GOST12] they also considered the Calabi-Yau version and proved Theorem 1.1 below, assuming some conjecture (see [GOST12, Theorem 4.13]). The conjecture has been partially verified in [FT11, Theorem 3.4], so that their arguments work in the corresponding cases [GOST12, Theorem 4.10]. Unfortunately it seems to be difficult to solve the conjecture in full generality at this moment.

In this paper we give a conjecture-free proof to the Calabi-Yau version. Namely, we verify the following theorem.

Theorem 1.1. *Let X be a Mori dream space over a field of characteristic zero. Then X is of Calabi-Yau type if and only if the Cox ring of X has at worst log canonical singularities.*

A normal projective variety X is said to be of Calabi-Yau type if (X, Δ) has log canonical singularities and $K_X + \Delta$ is numerically trivial for some boundary $\mathbf{Q}$-divisor Δ. Note that a variety of Calabi-Yau type is not necessarily a Mori dream space. For example, a K3 surface is a Mori dream space if and only if its automorphism group is finite (see [AHL10, Theorem 2.11]).

By slightly modifying the arguments, we can also reprove [GOST12, Theorem 1.1] in Theorem 4.1. During the preparation of this paper, we received a preprint version of [B12] from Morgan Brown in which he independently gave a different proof to the "only if" directions of Theorem 1.1 and Theorem 4.1.

Here we give the idea of the proof of Theorem 1.1. For the "only if" direction, we first take a set of $r = \operatorname{rank} \operatorname{Pic}(X)$ ample divisors $A_1, \ldots, A_r$ which are linearly independent in $\operatorname{Pic}(X)_{\mathbf{Q}}$, and take the affine space bundle $\pi \colon Y = \operatorname{Spec}_X \operatorname{Sym}(\bigoplus_i \mathcal{O}_X(A_i)) \to X$. From a boundary $\mathbf{Q}$-divisor Δ on X which makes the pair (X, Δ) log Calabi-Yau, we construct a boundary divisor Δ_Y on Y such that the pair (Y, Δ_Y) is also log canonical and $K_Y + \Delta_Y$ is $\mathbf{Q}$-linearly trivial. By contracting the zero section of π, we obtain a birational morphism $f \colon Y \to Z$ such that (Z, Δ_Z) is log canonical and $K_Z + \Delta_Z$ is $\mathbf{Q}$-linearly trivial for some $\mathbf{Q}$-divisor Δ_Z. It turns out that there exists a small birational morphism from the spectrum of the Cox ring of X to Z. Thus we prove the log canonicity of the Cox ring.

In our proof of the "if" direction of Theorem 1.1, we first derive the $\mathbf{Q}$-effectivity of the anti-canonical divisor of X from the log canonicity of the Cox ring, by using a similar construction as above and applying the numerical characterization of the pseudo-effective divisors due to [BDPP13] (for a Mori dream space, this result is much easier to prove; see Proposition 2.6). Since X is assumed to be a Mori dream space, we can apply the anti-canonical MMP which terminates in a semi-ample model in the same way as in [GOST12]. By the standard facts from [HK00], the semi-ample model is the quotient of an open subset of the spectrum of the Cox ring by the dual torus action of $\operatorname{Pic}(X)$. We can assume that the action is free, so that the log canonicity descends to the quotient. Therefore the semi-ample model is of Calabi-Yau type. As demonstrated in [GOST12, Proof of Theorem 1.2], we can trace back the anti-canonical MMP to prove that X itself is of Calabi-Yau type.

2. Preliminaries

We start with the definitions of varieties of Fano type and Calabi-Yau type.

Definition 2.1 (cf. [PS09, Lemma-Definition 2.6]). Let X be a projective normal variety over a field and Δ an effective $\mathbf{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier.

  (i) We say that (X, Δ) is a *klt Fano pair* if $-(K_X + \Delta)$ is ample and (X, Δ) is klt. We say that X is of *Fano type* if there exists an effective $\mathbf{Q}$-divisor Δ on X such that (X, Δ) is a klt Fano pair.
  (ii) We say that X is of *Calabi–Yau type* if there exits an effective $\mathbf{Q}$-divisor Δ such that $K_X + \Delta \sim_{\mathbf{Q}} 0$ and (X, Δ) is log canonical.

Next we give the definition of Mori dream spaces.

Definition 2.2 (cf. [HK00]). A normal projective variety X over a field is called a *Mori dream space* if X satisfies the following three conditions:

  (i) X is $\mathbf{Q}$-factorial, $\mathrm{Pic}\,(X)$ is finitely generated, and $\mathrm{Pic}\,(X)_{\mathbf{Q}} \simeq \mathrm{N}^1(X)_{\mathbf{Q}}$,
  (ii) The nef cone $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles,
  (iii) there exists a finite collection of small birational maps $f_i \colon X \dashrightarrow X_i$ such that each X_i satisfies (i) and (ii), and that the closed movable cone $\mathrm{Mov}(X)$ is the union of the $f_i^*(\mathrm{Nef}(X_i))$.

The following is one of the most important properties of Mori dream spaces.

Proposition 2.3. *([HK00, Proposition 1.11]) Let X be a Mori dream space. Then for any divisor D on X, a D-MMP can be run and terminates.*

The notion of multi-section rings and the Cox rings are repeatedly used in the studies of Mori dream spaces.

Definition 2.4 (Multi-section rings and Cox rings)**.** Let X be an integral normal scheme. For a semi-group Γ of Weil divisors on X, the Γ-graded ring

$$R(X, \Gamma) = \bigoplus_{D \in \Gamma} \mathrm{H}^0(X, \mathcal{O}_X(D))$$

is called the *multi-section ring* of Γ.

Suppose that the divisor class group $\mathrm{Cl}(X)$ is finitely generated. For such X, choose a group Γ of Weil divisors on X such that $\Gamma_{\mathbf{Q}} \to \mathrm{Cl}(X)_{\mathbf{Q}}$ is an isomorphism. Then the multi-section ring $R(X, \Gamma)$ is called a *Cox ring* of X.

Remark 2.5 (See [GOST12, Remark 2.18] for details)**.** As seen above, the definition of Cox rings depends on the choice of the group Γ. We can prove that the ambiguity does not affect the basic properties of rings, such as finite generation, log terminality (log canonicity), etc. In fact, there is a canonical way to define Cox rings without ambiguity (up to isomorphisms) due to Hausen [Hau08]. We can check that the properties of rings mentioned above holds for his Cox ring if and only if ours has the same properties. Since our definition of Cox rings as multi-section rings is easier for calculation, we adopt our definition in this paper.

If X is not $\mathbf{Q}$-factorial, our definition of Cox ring is different from that of [HK00], since we also take not-necessarily-$\mathbf{Q}$-Cartier Weil divisors into account. If we assume the finite generation of Cox rings in our sense, we can show the existence of a small $\mathbf{Q}$-factorization and Cox rings do not change under this operation. This is why we think our definition is more natural. See [AHL10, Theorem 2.3] for details.

We need the following intersection-theoretic characterization of effective (resp. big) divisors on a Mori dream space.

Proposition 2.6. *Let D be a divisor on a Mori dream space X. Then X is $\mathbf{Q}$-effective (resp. big) if and only if its intersection number with a general member of any covering family of curves is at least (strictly greater than) zero.*

Proof. Note that the $\mathbf{Q}$-effectivity and the pseudo-effectivity are equivalent for divisors on a Mori dream space. It is enough to show the following statement. $\square$

Claim. *Let X be a Mori dream space and D a not big but $\mathbf{Q}$-effective (resp. not $\mathbf{Q}$-effective) divisor on X. Then there exists a curve C in a covering family on X such that $C \cdot D = 0$ (resp. $C \cdot D < 0$).*

Proof. We run a D-MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N,$$

so that the pushforward D_N of D on X_N is semi-ample (resp. there exists a D_N-Mori fiber space from X_N).

Note that all the birational maps above are surjective in codimension one. Therefore we can find an open subset U of X_N whose complement has codimension at least two such that all the birational maps above are identities on U.

If D is not big (resp. not $\mathbf{Q}$-effective), we take D_N-Iitaka fiber space (resp. D_N-Mori fiber space) $f \colon X_N \to Y$. Since the relative dimension of f is positive, there is a curve C' in a covering family which is contained in a fiber of f. Since $X_N \setminus U$ has codimension at least two, we can choose C' so that it is contained in U. Now let C be the strict transform of C' on X. We see that this C has the desired properties. $\qquad\square$

Remark 2.7. The intersection theoretic characterization of pseudo-effective divisors was first proven in [BDPP13, Theorem 0.2]. On the other hand, the characterization of big divisors in Proposition 2.6 does not hold for an arbitrary variety. For example, it is known that there exists a strictly nef divisor on a smooth projective surface which is not $\mathbf{Q}$-effective. See [Har70, Example 10.6].

3. Proof of Theorem 1.1

3.1. **Cox rings of Calabi-Yau Mori dream spaces.** In this subsection, we prove the "only if" direction of Theorem 1.1.

Theorem 3.1. *Let X be a Mori dream space of Calabi-Yau type. Then Cox rings of X have at worst log canonical singularities.*

Proof. Choose ample line bundles $A_1, \ldots, A_r$ on X which are linearly independent in $\mathrm{Pic}\,(X)_{\mathbf{Q}}$, where $r = \mathrm{rank}\,\mathrm{Pic}\,(X)$. Consider the following natural morphism

$$f \colon Y = \mathrm{Spec}_X \mathrm{Sym}(\bigoplus_i \mathcal{O}_X(A_i)) \to Z = \mathrm{Spec}\,R(X, \Gamma'),$$

where Γ' is the semi-group generated by the classes of A_i, and let $\pi \colon Y \to X$ be the structure morphism. Since A_i are ample, f is a

birational projective morphism which contracts the zero section of π, which we denote again by X (see [Har66, Proposition 3.5]).

If (X, Δ) is a log canonical pair, then it follows that

$$\left(Y, \sum_{i=1}^{r} E_i + \pi^* \Delta \right)$$

is also a log canonical pair, where $E_j \subset Y$ is the divisor corresponding to the projection $\bigoplus_i \mathcal{O}_X(A_i) \to \bigoplus_{i \neq j} \mathcal{O}_X(A_i)$. Set $\Delta_Y = \sum_{i=1}^{r} E_i + \pi^* \Delta$. Then it holds that $(K_Y + \Delta_Y)|_X = K_X + \Delta \sim_{\mathbf{Q}} 0$. Since the restriction map $\mathrm{Pic}\,(Y)_{\mathbf{Q}} \to \mathrm{Pic}\,(X)_{\mathbf{Q}}$ is bijective, this implies that $K_Y + \Delta_Y$ is $\mathbf{Q}$-linearly trivial. Thus we see that $K_Z + \Delta_Z = f_*(K_Y + \Delta_Y)$ is also $\mathbf{Q}$-linearly trivial, where $\Delta_Z = f_* \Delta_Y$. Hence we get the equality

$$K_Y + \Delta_Y = f^*(K_Z + \Delta_Z),$$

concluding that (Z, Δ_Z) is a log canonical pair.

We shall derive the log canonicity of the Cox rings of X.

Set $\Gamma = \Gamma' + (-\Gamma')$. Then the multi-section ring $R(X, \Gamma)$ is a Cox ring of X. Consider the natural injective ring homomorphism

$$R(X, \Gamma') \to R(X, \Gamma).$$

Claim. *The corresponding morphism*

$$\mathrm{Spec}\,R(X, \Gamma) \to \mathrm{Spec}\,R(X, \Gamma')$$

is birational and contracts no divisor.

Proof. Choose an ample divisor A_0 from the interior of the cone spanned by Γ'. Then for any positive integer $m > 0$ and a non-zero global section $s \in \mathrm{H}^0(X, \mathcal{O}_X(mA_0))$, the natural ring homomorphism

$$R(X, \Gamma')_s \to R(X, \Gamma)_s$$

is an isomorphism.

By [HK00, Lemma 2.7], if we take two global sections s_1 and s_2 of mA_0 such that the corresponding divisors on X have no common component, then $\{s_1, s_2\} \subset R(X, \Gamma)$ is a regular sequence. Therefore the divisors of s_1 and s_2 have no common irreducible components, and this concludes the proof. $\square$

From the claim above, we see that the pair $(\mathrm{Spec}\,R(X, \Gamma), \Delta')$ is also log canonical, where Δ' is the effective $\mathbf{Q}$-divisor naturally defined from Δ_Z. Since $\mathrm{Spec}\,R(X, \Gamma)$ is $\mathbf{Q}$-factorial by [HK00, Proposition 2.9], we see that $\mathrm{Spec}\,R(X, \Gamma)$ itself is log canonical. $\square$

3.2. **Mori dream spaces with log canonical Cox rings.** In this subsection we prove the "if" direction of Theorem 1.1.

Theorem 3.2. *Let X be a Mori dream space whose Cox rings have at worst log canonical singularities. Then X is of Calabi-Yau type.*

Proof.

Step 1. We prove that $-K_X$ is $\mathbf{Q}$-effective.

Let C be an arbitrary movable curve on X. We take a set of divisors $D_1, \ldots, D_r$ on X with the following properties.

- They form a basis of $\mathrm{Cl}(X)_{\mathbf{Q}}$, so that if we set $\Gamma = \bigoplus_{i=1}^r \mathbf{Z}D_i$, then $R(X, \Gamma)$ is a Cox ring of X.
- The effective cone of X is contained in the cone spanned by $D_1, \ldots, D_r$, so that

$$R(X, \Gamma) = \bigoplus_{d_1, \ldots, d_r \geq 0} \mathrm{H}^0(X, \sum_i d_i D_i)$$

  holds.
- $C \cdot D_i \geq 0$ holds for $i = 1, \ldots, r$.

We note that the choice of the divisors D_i depends on C.

Let Y be the total space of the direct sum $\bigoplus_i \mathcal{O}_X(-D_i)$:

$$Y = \mathrm{Spec}_X \mathrm{Sym}(\bigoplus_i \mathcal{O}_X(D_i))$$

and regard $X \subset Y$ as the zero section. Then we have $\mathrm{H}^0(Y, \mathcal{O}_Y) \cong R(X, \Gamma)$. Therefore we have a natural birational morphism $f \colon Y \to Z = \mathrm{Spec}\, R(X, \Gamma)$ which is not necessarily proper.

Lemma 3.3. *Let E_i be the divisors on Y corresponding to the divisors D_i. Then the exceptional locus of f is contained in the union of the E_i.*

Proof. If the assertion does not hold, then there exists a curve C' on Y which is not contained in the union of the E_i and mapped to a point by f. Since the morphism f is defined by the elements of $R(X, \Gamma)$, for any $s \in R(X, \Gamma)$, the value of s is constant on C'. We note that C' is not necessarily complete.

Take a positive linear combination $D = \sum d_i D_i$ which is very ample. For a global section $0 \neq s \in \mathrm{H}^0(X, D)$, we see that $\mathrm{div}_Y(\tilde{s}) = \pi^* \mathrm{div}_X(s) + \sum d_i E_i$, where $\pi \colon Y \to X$ is the structure morphism and $\tilde{s}$ is s regarded as a global function on Y. This equality can be checked locally as follows. Take an open subset $U \subset X$ on which $\mathcal{O}_X(D_i)$ are trivial. By fixing its nowhere vanishing section $s_i \in \mathrm{H}^0(U, \mathcal{O}_X(D_i))$ for

each i, we obtain an isomorphism $\pi^{-1}(U) \cong U \times \mathbb{A}^r$. Under this isomorphism, the divisors E_i are identified with i-th coordinate hyperplanes and $\tilde{s}$ is identified with a function of the form $f x_1^{d_1} \cdots x_r^{d_r}$, where x_i are coordinates of $\mathbb{A}^r$ and $f \in \mathrm{H}^0(U, \mathcal{O}_U)$ is given by $s = f(\otimes_i s_i^{\otimes d_i})$. Thus

$$\mathrm{div}_Y(\tilde{s})|_{\pi^{-1}(U)} = \left(\pi^*(\mathrm{div}_X(s)) + \sum d_i E_i \right) |_{\pi^{-1}(U)}$$

holds.

Suppose that $\pi(C')$ is not a point. Then we can take a section $s \in \mathrm{H}^0(X, D)$ such that the divisor $\mathrm{div}_Y(s)$ does not contain C' but intersect C'. Hence s is not constant on C', a contradiction. If $\pi(C')$ is a point, C' is a curve in the fiber $\pi^{-1}(\pi(C'))$, which is identified with $\mathbf{A}^r$ by choosing a nowhere-vanishing local section of $\mathcal{O}_X(D_i)$ around $\pi(C')$ for each i. If $\pi(C')$ is not contained in $\mathrm{div}_X(s)$, we see that the restriction of s to the fiber is a non-zero monomial of exponent $(d_1, \ldots, d_r)$ under the identification. Since C' is assumed to be not contained in the coordinate hyperplanes, we can find a suitable exponent $(d_1, \ldots, d_r)$ such that the monomial is not constant on C'. $\qquad\square$

Since Z has only log canonical singularities, we can write $f^* K_Z = K_Y + \sum_i e_i E_i$ such that $e_i \le 1$ for all i. Then $(K_Y + \sum_i e_i E_i)|_X \sim_{\mathbf{Q}} 0$ because X is mapped to a point by f. By the adjunction formula, we have $K_X \sim_{\mathbf{Q}} -\sum(1 - e_i)D_i$.

It follows that $(K_X \cdot C) \le 0$. Since C was arbitrary, we conclude that $-K_X$ is $\mathbf{Q}$-effective by Proposition 2.6.

Step 2. Since X is a Mori dream space, we have a $(-K_X)$-MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N,$$

where each step is a birational map and $-K_{X_N}$ is semi-ample, because $-K_X$ is $\mathbf{Q}$-effective.

Fix a Cox ring $R(X, \Gamma)$ of X. We recall some facts on the GIT of Cox rings from [HK00, Proposition 2.9]. First of all, there exists a canonical action of the torus $T = \mathrm{Hom}(\Gamma, k^*)$ on the affine variety $\mathrm{Spec}\, R(X, \Gamma)$, and X_N is the categorical quotient by T of the semistable locus $U \subset \mathrm{Spec}\, R(X, \Gamma)$ with respect to a character of T which corresponds to an ample divisor on X_N ([HK00, Proof of Proposition 2.9]).

Since X is $\mathbf{Q}$-factorial and X_N is obtained from a MMP starting from X, we see that X_N also is $\mathbf{Q}$-factorial. $\mathbf{Q}$-factoriality of X_N, in turn, implies that the quotient $U \to U/T = X_N$ is the geometric quotient (see [HK00, Proposition 1.11(2)] and [HK00, Corollary 2.4]). Moreover, by replacing Γ with its subgroup of finite index if necessary,

we can assume that the torus T acts on U freely ([HK00, Proposition 2.9]). Therefore the quotient morphism $U \to U/T = X_N$ is smooth.

Since Spec $R(X, \Gamma)$ is log canonical, so is its open subset U. Hence we see that $X_N = U/T$ has log canonical singularities. By taking a general effective $\mathbf{Q}$-divisor Δ_N on X_N which is $\mathbf{Q}$-linearly equivalent to $-K_{X_N}$, we obtain a log Calabi-Yau pair (X_N, Δ_N) due to the semi-ampleness of $-K_{X_N}$.

Finally we trace back the $(-K_X)$-MMP as in [GOST12, Proof of Theorem 1.2], showing that X itself is of Calabi-Yau type.

$\square$

Remark 3.4. The choice of the divisors $D_1, \ldots, D_r$ in the proof of Theorem 3.2 is essential, and the proof does not work if we naively choose a collection of divisors corresponding to the extremal rays of the effective cone.

For example[1], let X be a toric Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at three points which are not co-linear. The complete fan $\Sigma \subset \mathbb{R}^2$ corresponding to X is spanned by six primitive vectors u_i ($i = 1, 2, \ldots, 6$), where

$$(u_1, u_2, \ldots, u_6) = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

X has six (-1)-curves $C_1, \ldots, C_6$ corresponding to the u_i, and they span the effective cone of X. Since the effective cone has dimension four, it is not simplicial.

Set

$$S = \bigoplus_{c_1, \ldots, c_6 \in \mathbb{Z}_{\geq 0}} \mathrm{H}^0(X, \mathcal{O}_X(\sum_i c_i C_i))$$

and $Z = \mathrm{Spec}\, S$. Z is an affine toric variety corresponding to the cone $\sigma_Z \subset \mathbb{R}^2 \times \mathbb{R}^6$ which is spanned by twelve vectors $w_i = u_i + e_i$ ($i = 1, 2, \ldots, 6$) and $w_{i+6} = e_i$ ($i = 1, 2, \ldots, 6$), where $e_1, \ldots, e_6$ are the standard basis of $\mathbb{R}^6$. The singularity of Z does not have an obvious relationship with that of the Cox ring. In fact Z is not $\mathbf{Q}$-factorial since σ_Z is not simplicial, despite that the Cox ring of X is a polynomial ring.

We have a simplicial partial decomposition $\sum_{i=1}^6 \sigma_i \subset \sigma_Z$, where

$$\sigma_i = \mathrm{cone}\,(u_i + e_i, u_{i+1} + e_{i+1}, e_1, e_2, \ldots, e_6).$$

Correspondingly, there is a birational morphism

$$f \colon Y = \mathrm{Spec}_X \mathrm{Sym}(\bigoplus_i \mathcal{O}_X(C_i)) \to Z,$$

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[1]We thank Osamu Fujino for drawing our attention to this example.

which is similar to the one which appeared in the proof of Theorem 3.2. There is no divisor which is contracted by f, but the zero section, which is isomorphic to X, is contracted to a point.

4. Characterization of varieties of Fano type revisited

Using the similar arguments as above, we can reprove the characterization of varieties of Fano type (=[GOST12, Theorem 1.1]).

Theorem 4.1. *Let X be a $\mathbf{Q}$-factorial normal projective variety over k. X is of Fano type if and only if the Cox ring of X is of finite type and log terminal.*

Proof. For "only if" direction it is enough to prove that the Cox ring of X is log terminal, since the finite generation is proved in [BCHM10, Corollary 1.3.2]. We only point out which part of the proof of Theorem 3.1 should be modified.

Instead of thinking of the pair

$$\left(Y, \ \sum_{i=1}^{r} E_i + \pi^* \Delta \right),$$

we should think of the following klt pair

$$\left(Y, \ \Delta_Y := (1 - \epsilon) \sum_{i=1}^{r} E_i + \pi^* \Delta \right)$$

for some positive number $\epsilon < 1$.

Then it holds that $(K_Y + \Delta_Y)|_X = K_X + \Delta + \epsilon \sum_i A_i$, since we have $\mathcal{O}_Y(E_i)|_X \simeq \mathcal{O}_X(-A_i)$. By replacing A_i from the beginning if necessary, we can assume that there exists ϵ such that $-(K_X + \Delta) \sim_{\mathbf{Q}} \epsilon \sum_i A_i$, so that $(K_Y + \Delta_Y)|_X \sim_{\mathbf{Q}} 0$.

Therefore we get the equality

$$K_Y + \Delta_Y = f^*(K_Z + \Delta_Z)$$

as before, where $\Delta_Z = f_* \Delta_Y$. Hence the pair (Z, Δ_Z) is klt. The rest of the proof is the same.

For the proof of the "if" direction, we first prove that $-K_X$ is big. If we carry out the same arguments as in Step 1 of the proof of Theorem 3.2, by the log terminality of the Cox ring we see that $e_i < 1$ holds for all i. Since some positive linear combination of the D_i is ample, we deduce that $C \cdot (-K_X) = C \cdot (\sum (1 - e_i) D_i) > 0$. Since X is a Mori dream space, this implies the bigness of $-K_X$ by Proposition 2.6.

Next, by arguing as in Step 2, we arrive at a model X_N such that $-K_{X_N}$ is semi-ample and big. We can also show that X_N is log terminal

from the log terminality of the Cox ring, concluding that X_N is of Fano type. The rest is the same. □

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