AN EXAMPLE OF A STRICTLY NEF DIVISOR WITH KODAIRA DIMENSION $-\infty$ ON A SMOOTH RATIONAL SURFACE

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Abstract. In this short note we give an example of a strictly nef divisor D on a smooth rational surface X such that the linear system $|mD|$ is empty for all $m > 0$. I learned this example from Frédéric Campana, so it is not my original result.

1. introduction

We work over an algebraically closed field k. In this note, we construct an example of a surface with $q = 0$ over which there exists a strictly nef divisor D with $H^0(X, \mathcal{O}_X(mD)) = 0$ for all $m > 0$. I learned this example from Frédéric Campana, through Yoshinori Gongyo. I would like to thank both of them. I would also like to thank Professor Campana for kindly answering my question about the proof.

The construction of the example is based on the work of Nagata ([N]). To be precise, X is a blow-up of $\mathbb{P}^2$ in $r = s^2$ very general points, where s is an integer greater than three.

Original motivation for the author was to find an example of a pseudo-effective divisor D on a smooth projective variety with $q = h^1(X, \mathcal{O}_X) = 0$ such that $|mD| = \emptyset$ for all $m > 0$. There were several examples which says that pseudo-effectiveness need not imply non-vanishing, but all of them were divisors on varieties with $q > 0$. On such a variety one can use the fact that numerical equivalence does not necessarily imply linear equivalence.

The author naively thought that there exists no such divisor on a variety with $q = 0$; for example, this expectation holds on K3 surfaces. The purpose of this note is to disprove this naive expectation.

2. the example

Let s be an integer at least 4, and set $r = s^2$. We prove the following

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Theorem 2.1. *Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in a very general set of r points $(P_1, \ldots, P_r)$. Let $H = \pi^*\mathcal{O}(1)$ and $E = \sum E_i$ be the sum of the exceptional divisors. Set $D = sH - E$. Then D is strictly nef.*

In order to prove the above theorem, it is enough to show the following claim:

Lemma 2.2. *Let $(d, m_1, \ldots, m_r)$ be an $(r+1)$-tuple of non-negative integers such that*

$$sd \le \sum m_i. \tag{1}$$

Then there exists a set of points $(P_1, \ldots, P_r)$ such that

$$H^0\left(X, \mathcal{O}_X(dH - \sum m_i E_i)\right) = 0. \tag{2}$$

Proof of the lemma. Suppose the contrary: i.e. assume that there exists an $(r+1)$-tuple of non-negative integers $(d, m_1, \ldots, m_r)$ satisfying (1) and

$$H^0\left(X, \mathcal{O}_X(dH - \sum m_i E_i)\right) \ge 1 \tag{3}$$

holds for any choice of the set of points $(P_1, \ldots, P_r)$. Note that the same things also hold if we permute $m_1, \ldots, m_r$'s in an arbitrary manner. Summing up, we obtain an $(r+1)$-tuple of integers $(d', m', m', \ldots, m')$ such that the same things hold.

On the other hand, Lemma 2.2 for the case $m_1 = \cdots = m_r$ has been proven in [N, Proposition of §3]. Thus we obtain a contradiction. □

Remark 2.3. By the upper semi-continuity, the conclusion (2) of Lemma 2.2 holds for a generic choice of $(P_1, \ldots, P_r)$. Thus we obtain the theorem from Lemma 2.2.

Corollary 2.4. *Let D be the divisor as in the theorem. Then $|mD| = \emptyset$ for all $m > 0$.*

Proof. Note first that $D^2 = 0$. If mD is linearly equivalent to an effective divisor E, it must hold that $mD.E > 0$ since mD is strictly nef by the above theorem. This is a contradiction since $mD.E = mD.mD = 0$. □

Remark 2.5. This remark also is suggested by Prof. Campana. If S is an arbitrary projective surface, a blow-up of S in a set of points admits a strictly nef Cartier divisor with $\kappa = -\infty$. In fact, first make a finite morphism from S to $\mathbb{P}^2$ (the geometric Noether normalization!). By blowing up a finitely many points, it lifts to a finite morphism to X (a resolution of indeterminancy). Just pulling back our D to the blow-up of S, we obtain the divisor.

References

[N] M. Nagata, *On the* 14*-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

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