EXTENSIONS OF TWO CHOW STABILITY CRITERIA TO POSITIVE CHARACTERISTICS

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ABSTRACT. We extend two results on Chow (semi-)stability to positive characteristics. One is on the stability of non-singular projective hypersurfaces of degree greater than 2, and the other is the criterion by Y.Lee in terms of log canonical thresholds. Some properties of log-canonicity in positive characteristics are discussed with a couple of examples, in connection with the proof of the latter one. It is also proven in appendix that the sum of Chow (semi-)stable cycles are again Chow (semi-)stable.

1. INTRODUCTION

We work over an algebraically closed field k of arbitrary characteristic.

Let $X \subset \mathbb{P}^n_k$ be an effective cycle of dimension r and degree d in a projective space of dimension n. Analysis of the Chow (semi-)stability of X is one of the basic problems in Geometric Invariant Theory (GIT). Contrary to the asymptotic Chow (semi-)stability, the precise classification of Chow (semi-)stable cycles is quite a subtle problem, and is known only for few cases, even for projective hypersurfaces. To name a few, J.Shah studied the case of plane sextics ([Sh]) and recently R.Laza did the case of cubic fourfolds ([L]), both in relation with period maps.

On the other hand, there are two sufficient conditions for Chow (semi-)stability in terms of the singularity of X or that of Chow divisor $Z(X) \subset \mathbb{G} = \text{Grass}_k(n - r, n + 1)$, which deal with general situations. Both have been proven in characteristic zero, and the purpose of this paper is to extend them to arbitrary characteristics. Namely we prove the following two theorems:

Theorem 1.1 (= Theorem 3.1). *If $d \geq 3$, any non-singular projective hypersurface of degree d is Chow stable.*

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Theorem 1.2 (= Theorem 4.1). *Let X be an effective cycle of dimension r and degree d in $\mathbb{P}^n_k$. Let $(\mathbb{G}, Z(X))$ be the log pair defined by the Chow divisor $Z(X)$ of X. If $\mathrm{lct}(\mathbb{G}, Z(X)) > \frac{n+1}{d}$ (resp. $\geq \frac{n+1}{d}$), then X is Chow stable (resp. Chow semi-stable).*

In the statement of Theorem 1.2 $\mathrm{lct}(\mathbb{G}, Z(X))$ stands for the log canonical threshold of $(\mathbb{G}, Z(X))$, which measures how good the singularity of $Z(X)$ is (see §2.2 for detail).

Characteristic zero case of Theorem 1.1 is due to Mumford ([GIT, Chapter 4 §2]), and that of Theorem 1.2 is due to Y.Lee ([Le]).

The original proof of Theorem 1.1 works only when the characteristic of the base field does not divide d (see §3). To prove the general case, we depend on the corresponding result in characteristic zero.

We sketch the proof of Theorem 1.1 in positive characteristics. First we take a suitable lift of the equation of given hypersurface over the ring of Witt vectors. This defines a family of projective hypersurfaces over the ring. We are assuming that the closed fiber is non-singular, hence the geometric generic fiber is again non-singular. Since we know that Theorem 1.1 holds in characteristic zero, we obtain some inequalities for the Hilbert-Mumford numerical functions of the lift. By the choice of the lift, those numerical functions coincide with those of the original hypersurface. Thus we obtain the inequalities for the numerical functions of the original one, concluding the proof.

The point is that the singularity of the hypersurface over the generic point is better than that of the special fiber, so that we can use the corresponding stability criterion in characteristic zero. This method seems to be applicable to other stability problems (see the remark at the beginning of §4).

In §3.2 it will also be shown that the complement of the locus of non-singular hypersurfaces is an irreducible divisor, even when p divides d. In general some multiple of the defining equation of this divisor lifts to the usual discriminant in characteristic zero. These are shown by a version of standard geometric arguments (see, for example, [Mu, Chapter 5, §2]).

Theorem 1.2 will be proven along the same line as the proof given in [Le], but we must modify several points. This is due to the fact that some properties of log canonicity which hold in characteristic zero fail in positive characteristics, because of the existence of wild ramifications and inseparable morphisms.

We can prove that the property of log canonicity which we need still holds for finite separable morphisms. It turns out that this is enough

for our purpose, for we can use a perturbation technique so that we need not to deal with the inseparable morphisms (see §4).

In §4 we also discuss some other properties of log canonicity, with a couple of (counter-)examples.

In Appendix A we reprove Proposition 4.9, which is a key step for the proof of Theorem 1.2, via the theory of F-singularities, in characteristic zero. The main tool of the proof is the Fedder-type criterion (Lemma A.4) due to [HW].

In Appendix B we prove the following

Propostion 1.3 (= Proposition B.1). *Let Y, Z be Chow semi-stable cycles of the same dimension in a projective space $\mathbb{P}_k^n$. Then $Y + Z$ is again Chow semi-stable. Furthermore if Y is Chow stable, so is $Y + Z$.*

This proposition may be well-known to experts, but the author could not find it in the literature. The proof is a simple application of the fact that the stability can be checked 1-PS wise, which is essentially same as the numerical criterion. But the conclusion itself seems to be rather surprizing: if we have two Chow stable cycles, the sum of them is always Chow stable no matter how badly they touch.

Proposition 1.3 will be used to give a family of stable projective hypersurfaces whose stability can not be detected by Theorem 1.2 (see Example B.5).

2. PRELIMINARY

2.1. **Notations from scheme theory.** We need some notations from [Ha].

Let R be an $\mathbb{N}$-graded ring. For a homogeneous ideal I of R we denote by $V(I)$ the corresponding closed subscheme $\mathrm{Proj}(R/I)$ of $\mathrm{Proj}R$.

For a homogeneous element $f \in R$, we denote by $D_+(f)$ the open subscheme of $\mathrm{Proj}(R)$ defined by the non-vanishing of f. This subscheme is known to be affine, with coordinate ring

$$R_{(f)} = \left\{ \frac{r}{f^n} | r \in R, \ \deg(r) = n \cdot \deg(f) \right\}.$$

2.2. **Notions of singularities.** In this subsection, we summarize the notions of singularities of pairs which we need later.

Definition 2.1 (discrepancy, log canonical, Kawamata log terminal). Let X be a normal variety over k and Δ be an effective $\mathbb{R}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Let $\pi : Y \to X$ be a birational morphism from another normal variety Y over k and $E \subset Y$ be a prime divisor. Then in a neighborhood of the generic point of E, the following canonical bundle formula holds:

$$K_Y = \pi^*(K_X + \Delta) + aE.$$

The real number a in the above equation is called the *discrepancy* of E with respect to (X, Δ), and denoted by $a(E; X, \Delta)$. It is independent of the choice of Y and π, depending only on the valuation of $k(X)$ corresponding to E.

We say that the log pair (X, Δ) is *log canonical* (*lc*, for short) (resp. *Kawamata log terminal, klt*) if $a(E; X, \Delta) \geq -1$ (resp. > -1) holds for all E.

A finer version is:

Definition 2.2. Let $x \in X$ be a point. We say that the log pair (X, Δ) is *log canonical at x* if the restriction of (X, Δ) to an open neighborhood of x is log canonical (similar for klt).

Definition 2.3 (log canonical threshold). Let (X, Δ) be a log canonical pair and D be an effective $\mathbb{R}$-Cartier divisor on X. The *log canonical threshold* of D with respect to (X, Δ) is defined as follows:

$\mathrm{lct}(X, \Delta; D) = \sup\{t \in \mathbb{R} | (X, \Delta + tD) \text{ is log canonical}\}$.

For a point $x \in X$, we set

$\mathrm{lct}_x(X, \Delta; D) = \sup\{t \in \mathbb{R} | (X, \Delta + tD) \text{ is log canonical at } x\}$.

When we consider the case $\Delta = 0$, we write $\mathrm{lct}(X, \Delta; D) = \mathrm{lct}(X, D)$ for short (resp. $\mathrm{lct}_x(X, \Delta; D) = \mathrm{lct}_x(X, D)$).

The following is a basic fact ([KoM, Corollary 2.35(5)]):

Propostion 2.4. *Let k be a field of characteristic zero. Then*

$\mathrm{lct}(X, \Delta) = \sup\{t \in \mathbb{R} | (X, t\Delta) \text{ is klt}\}$.

2.3. **Chow stability and the numerical criterion.** Let $X \subset \mathbb{P}_k^n$ be an effective r-dimensional cycle of degree d. We associate to X its *Chow divisor* $Z(X)$, which is a hypersurface of degree d of the Grassmannian $\mathbb{G} = \mathrm{Grass}_k(n - r, n + 1)$, as follows (one may consult either [Ko2] or [GKZ] for detail). If X is a variety, set $Z(X) = \{L \in \mathbb{G} | L \cap X \neq \phi\}$. For a general cycle X, define $Z(X)$ additively. The defining equation of $Z(X)$ is called the *Chow form* of X (Chow form is

determined by X only up to scalar multiplication). The homogeneous coordinate ring of $\mathbb{G}$ with respect to the Plücker embedding is denoted by $\mathcal{B} = \sum_{d \geq 0} \mathcal{B}_d$. This is the subring of the polynomial ring of $(n - r)(n + 1)$ indeterminants $U_i^{(j)}$'s, where (i, j) runs through the range $i = 0, \ldots, n$ and $j = 1, \ldots, n - r$, generated by all the $(n - r) \times (n - r)$ minors of the matrix $(U_i^{(j)})$. The Chow form of a cylce X is an element of $\mathcal{B}_d$ (up to scalar multiplication), so that the Chow divisor $Z(X)$ of X can be regarded as an element of the projective space $\mathbb{P}_* \mathcal{B}_d$. The canonical action of $SL(n + 1, k)$ on $\mathbb{P}_k^n$ naturally induces a linear action on $\mathcal{B}_d$, hence we can discuss the GIT (semi-)stability of an element of $\mathbb{P}_* \mathcal{B}_d$ (here we are using the terminology "stable" in the sense of "properly stable" in [GIT], which requires the finiteness of the stabilizer subgroup. We heavily rely on the numerical criterion, so we follow this terminology [1]). Chow (semi-)stability of X is defined to be the (semi-)stability of $Z(X)$ in the above sense.

Next we recall the *Hilbert-Mumford numerical criterion* (*numerical criterion*, for short) for stability, explicitly describing the numerical function μ in our case following [GIT, Proposition 2.3]. We start with some preparations.

For a non-negative integer n, set

$$(1) \qquad\qquad [n] = \{0, 1, \ldots, n\}.$$

For a subset $I \subset [n]$ with $\#I = n - r$, let Δ_I be the $(n - r) \times (n - r)$ minor of the matrix $(U_i^{(j)})$ obtained by picking out the $n - r$ rows according to I. Recall that $\mathcal{B}_d$ is a k-vector space generated by the set $\{\Delta_{I_1} \ldots \Delta_{I_d} | I_\ell \subset [n], \#I_\ell = n - r \text{ for all } \ell = 1, \ldots, d\}$.

Now fix X. Take any $g \in SL(n + 1, k)$ and let F be the Chow form of $g^* X$ ($=$ the defining equation of $g^* Z(X)$). Set
(2)

$$\mathcal{R} = \{\vec{r} = (r_0, \ldots, r_n) \in \mathbb{Z}^{n+1} \setminus \{0\} | \sum_{i=0}^{n} r_i = 0, \quad r_0 \leq r_1 \leq \cdots \leq r_n\}.$$

An element $\vec{r}$ of $\mathcal{R}$ corresponds to a non-trivial one-parameter subgroup (1-PS for short) $\lambda : \mathbb{G}_m \to SL(n + 1, k)$ of $SL(n + 1, k)$ which is defined by $\lambda(t) = \mathrm{diag}(t^{r_0}, \ldots, t^{r_n})$. If we regard $\mathcal{B}_d$ as a representation of $\mathbb{G}_m$ via λ, the subspace of $\mathcal{B}_d$ generated by $\Delta_{I_1} \cdots \Delta_{I_d}$ is an eigenspace of weight

$$\mathrm{wt}(I_1, \ldots, I_d) = \sum_{\ell=1}^{d} \sum_{i \in I_\ell} r_i = \sum_{i=0}^{n} r_i \cdot \#\{\ell | i \in I_\ell\}.$$

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[1] The author would like to thank Dr. S. Ma for this remark.

Given $g \in SL(n+1, k)$ and $\vec{r} \in \mathcal{R}$, the numerical function of X associated to them is defined as follows:

Definition 2.5 (numerical function). Let $\mathcal{I}(F)$ be the set of such d-tuples $(I_1, \ldots, I_d)$ that the coefficient of $\Delta_{I_1} \cdots \Delta_{I_d}$ in F is not zero. Then set

$$\mu(Z(X), g, \vec{r}) = \mu(V(F), \mathrm{id}, \vec{r}) = \min_{(I_1, \ldots, I_d) \in \mathcal{I}(F)} \mathrm{wt}(I_1, \ldots, I_d).$$

Remark 2.6. $\mu(Z(X), g, \vec{r})$ depends only on $g, \vec{r}$ and the set $\mathcal{I}(F)$.

Now the numerical criterion is:

Propostion 2.7. *X is Chow stable (resp. semi-stable) if and only if $\mu(Z(X), g, \vec{r}) < 0$ (resp. ≤ 0) holds for any $g \in SL(n+1, k)$ and $\vec{r} \in \mathcal{R}$.*

Next we rephrase Proposition 2.7 in such a way as to prove Theorem 4.1. This reinterpretation is just a generalization of [Le, Lemma 2.1]. Before that, we need some preparations. Take an arbitrary $g \in SL(n+1, k)$ and let F be the Chow form of g^*X.

Let f be the local equation of F on $D_+(\Delta_{[n-r-1]}) \simeq \mathrm{Spec}\, \mathcal{B}_{(\Delta_{[n-r-1]})}$. Recall that $\mathcal{B}_{(\Delta_{[n-r-1]})}$ is the polynomial ring over k with the set of indeterminants $\left\{ x_I = \frac{\Delta_I}{\Delta_{[n-r-1]}}; I \right\}$ where I runs through those subsets of $[n]$ (see (1)) satisfying the following two conditions:

$$\#I = n - r$$
(3)
$$\#(I \cap [n-r-1]) = n - r - 1.$$

Therefore f is a polynomial in x_I's. Now assign nontrivial integral weights $\vec{r} = (r_0, \ldots, r_n) \in \mathcal{R}$ to $X_0, \ldots, X_n$, so that the induced weight $w(x_I)$ on x_I satisfies

(4)
$$w(x_I) = \sum_{i \in I} r_i - \sum_{i=0}^{n-r-1} r_i,$$

which is non-negative by the assumption $r_0 \leq r_1 \leq \cdots \leq r_n$.

Now Proposition 2.7 is equivalent to

Lemma 2.8. *A cycle X is Chow stable (resp. semi-stable) if and only if*

(5)
$$\frac{w(f)}{\sum_I w(x_I)} < \frac{d}{n+1}$$

(resp. $\leq \frac{d}{n+1}$) holds for all $g \in SL(n+1, k)$ and $\vec{r} \in \mathcal{R}$ (see (2) for the definition of $\mathcal{R}$).

In the left hand side of (5), $w(f)$ denotes the weighted multiplicity of f (= the lowest weight of the monomials occurring in f) with respect to the weight $(w(x_I))_I$.

Proof. We only discuss the stable case. Semi-stable case can be proven similarly.

The inequality (5) is equivalent to

$$(6) \qquad d\sum_I w(x_I) - (n+1)w(f) > 0.$$

Combining the calculation of $w(x_I)$ (see (4)) with the definition of $w(f)$, we see that the left hand side of (6) equals to

$$d\left(\sum_I \sum_{i\in I} r_i - (n-r)(r+1)\sum_{i=0}^{n-r-1} r_i\right) - (n+1)\left(\mu(X,g,\vec{r}) - d\sum_{i=0}^{n-r-1} r_i\right).$$

Recalling the conditions (3) posed on I's we see

$$\sum_I \sum_{i\in I} r_i = (n-r-1)(r+1)\sum_{i=0}^{n-r-1} r_i + (n-r)\sum_{i=n-r}^{n+1} r_i.$$

A little calculation shows that the left hand side of (6) boils down to

$$d(n-r)\sum_{i=0}^{n} r_i - (n+1)\mu(X,g,\vec{r}) = -(n+1)\mu(X,g,\vec{r}),$$

since we assumed that $\sum_{i=0}^{n} r_i = 0$.

Therefore (5) is equivalent to the condition $\mu(X,g,\vec{r}) < 0$. $\qquad \square$

2.4. Chow stability in characteristic p from characteristic zero.
Let k be a field of characteristic $p > 0$ and X be a cycle in $\mathbb{P}^n_k$. In this subsection we want to propose a method to deduce the Chow (semi-)stability of X from the corresponding results in characteristic zero.

From now on, we denote by $W = W(k)$ the ring of Witt vectors. This is a discrete valuation ring (DVR for short) of characteristic zero, whose residue field is isomorphic to k (see [S, Chapter 2 §5 Theorem 5]). Actually these are all the properties of W which we need in this paper. We denote by K the field of fractions of W and by m_W the unique maximal ideal of W.

Take $g \in SL(n+1,k)$ and let F be the Chow form of g^*X as in the previous subsection. Let F_W be a lift of F over W such that a monomial which does not appear in F never appears in F_W, which is equivalent to the assumption $\mathcal{I}(F) = \mathcal{I}(F_W)$ (see Definition 2.5 for the definition of

$\mathcal{I}$). Note that F_W defines a hypersurface $V(F_W) \subset \text{Grass}_{\overline{K}}(n-r, n+1)$ of degree d, where $\overline{K}$ is the algebraic closure of K.

Theorem 2.9. *Assume that for any $g \in SL(n+1, k)$ we can take F_W such that $\mathcal{I}(F) = \mathcal{I}(F_W)$ holds and $V(F_W)$ is stable (resp. semi-stable) with respect to the induced action of $SL(n+1, \overline{K})$. Then X is Chow stable (resp. Chow semi-stable).*

Proof. Since F_W is (semi-)stable, $\mu(V(F_W), \text{id}, \vec{r}) < 0$ (resp. ≤ 0) holds for any $\vec{r} \in \mathcal{R}$ (see (2) in the previous subsection for the definition of $\mathcal{R}$). But it holds that $\mu(V(F_W), \text{id}, \vec{r}) = \mu(Z(X), g, \vec{r})$, since $\mathcal{I}(F) = \mathcal{I}(F_W)$ (see Remark 2.6). Therefore $\mu(Z(X), g, \vec{r})$ holds for all $g \in SL(n+1, k)$ and $\vec{r} \in \mathcal{R}$, hence we see the Chow (semi-)stability of X by Proposition 2.7. $\qquad\square$

Remark 2.10. By the result of C. S. Seshadri ([Se, Proposition 6], see also [GIT, Appendix to Chapter 1, §G]), the converse of Theorem 2.9 also holds: if X is Chow stable (resp. Chow semi-stable), any lift F_W of F is also Chow stable (resp. Chow semi-stable) with respect to the induced action of $SL(n+1, \overline{K})$.

3. Chow stability of non-singular hypersurfaces

In this section X denotes a hypersurface of degree d in $\mathbb{P}^n_k$.

In §3.1, we prove the stability of non-singular hypersurfaces of degree greater than 2. This is an easy application of Theorem 2.9. In §3.2 we study the complement of the locus of non-singular hypersurfaces via geometric arguments. It turns out that the complement is an irreducible divisor and that some multiple of its defining equation lifts to the usual discriminant in characteristic zero.

3.1. **A proof via lifting to characteristic zero.** First of all we recall that the characteristic zero case of Theorem 3.1 was settled in [GIT, Chapter 4 §2]. Thanks to a theorem by Matsumura and Monsky, the proof given there also works for characteristic p cases if p does not divide d. We briefly recall the proof and see why it does not work for the cases when p do divide d.

Let $F(X_0, X_1, \ldots, X_n)$ be a homogeneous polynomial of degree d. We have the Euler's lemma:

$$dF = \sum_{i=0}^{n} X_i \frac{\partial F}{\partial X_i}.$$

Therefore we see that

$$(7) \qquad V\left(F, \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}\right) = V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}\right),$$

provided that p does not divide d. The emptiness of the latter is equivalent to the vanishing of the discriminant of F when $d \geq 2$. This shows the semi-stability of non-singular hypersurfaces of degree greater than 1. Furtheremore, when $d \geq 3$, it is known (see [MM, Theorem 1]) that only finitely many projective linear transformations preserve the given non-singular hypersurface. This means that any non-singular hypersurface is stable, provided $d \geq 3$ and $p \nmid d$.

The above argument does not work in general, for the equality (7) may break down when p divides d. Actually when p divides d the right hand side of the equality (7) can not be empty. This will be proven in the next subsection (see Proposition 3.2).

Even when p divides d, a closer look at the numerical criterion shows that non-singular hypersurfaces are always (semi-)stable if $d > n + 1$ (resp. $d \geq n + 1$) (see [N, Lemma 4.2]. This may also be deduced from Theorem 4.1, for $(\mathbb{P}^n_k, X)$ is log canonical when X is a non-singular hypersurface).

Now we prove that the stability is always the case:

Theorem 3.1. *If $d \geq 3$, any non-singular projective hypersurface of degree d is Chow stable.*

Proof. The theorem is already established when $\operatorname{char} k = 0$, so we assume $\operatorname{char} k > 0$. We use Theorem 2.9. Let $X \subset \mathbb{P}^n_k$ be an non-singular projective hypersurface of degree greater than 2. Take any $g \in SL(n + 1, k)$ and let F_k be an equation of $g^* X$. Note that in this case F_k itself is the Chow form of $g^* X$. Take a lift F_W of F_k over the ring of Witt vectors W satisfying $\mathcal{I}(F_k) = \mathcal{I}(F_W)$ (see Definition 2.5 for the definition of $\mathcal{I}$). Then it is easy to see the

Claim. *$V(F_W)$ is an integral scheme.*

Proof. Since W is a DVR, $W[X_0, \ldots, X_n]$ is a UFD. So it is enough to show that F_W is an irreducible element of $W[X_0, \ldots, X_n]$. Suppose for a contradiction that $F_W = G \cdot H$ holds for some $G, H \in W[X_0, \ldots, X_n]$ such that neither G nor H is a unit. Note that both G and H are homogeneous, since F_W is. $\overline{G} \cdot \overline{H} = \overline{F_W} = F_k \neq 0$ (here $\overline{F}$ denotes the reduction $\mod m_W$ of $\overline{F}$), so neither G nor H is contained in m_W. This means that $\deg G, \deg H \geq 1$. Therefore $\deg \overline{G}, \deg \overline{H} \geq 1$ must hold, contradicting the irreducibility of F_k. $\square$

Note that $V(F_W)$ dominates the generic point of $\operatorname{Spec} W$, hence the above claim means that $V(F_W)$ is flat over $\operatorname{Spec} W$ (see [Ha, Chapter III, Proposition 9.7]). Also it is projective over $\operatorname{Spec} W$.

The closed fiber of $V(F_W) \to \operatorname{Spec} W$ is $g^* X$, which is non-singular. Therefore the geometric generic fiber is also non-singular (see [EGA IV,

(12.2.4)(iii)]). Since the characteristic of the generic fiber is zero and $\deg(F_W) \geq 3$, we already know that it is stable. By Theorem 2.9, we see that X is stable too. $\qquad\square$

3.2. **The defining equation.** Let $Hyp_d(n)$ be the projective space of degree d hypersurfaces in $\mathbb{P}_k^n$, and $U_{ns} \subset Hyp_d(n)$ be the locus of non-singular hypersurfaces. In this subsection we study the defining equation for the complement of the locus of non-singular hypersurfaces, $Hyp_d(n) \setminus U_{ns}$. This is already well-known when p does not devide d, so we are interested in the cases when p divide d. First we show the following

Propostion 3.2. *Assume d is divided by p and let $X = V(F) \subset \mathbb{P}_k^n$ be an arbitrary hypersurface of degree d. Then*

$$V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}\right) \neq \emptyset$$

holds.

Proof. Set

$$Z = \left\{(x, X); x \in V\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}\right)\right\} \subset \mathbb{P}_k^n \times Hyp_d(n)$$

and $p : Z \to \mathbb{P}_k^n$ and $q : Z \to Hyp_d(n)$ be the natural projections. First we show the

Claim. *p is a smooth morphism with connected fibers.*

Proof. Let $x : \operatorname{Spec}\Omega \to \mathbb{P}_k^n$ be a geometric point. By the definition of Z above, it is easy to see that $Z_x \subset \mathbb{P}_\Omega^n$ is a linear subspace. To see that the dimension of the linear subspace is independent of x, we show that Z_x is isomorphic to $Z_{(1:0:\cdots:0)}$, where $(1 : 0 : \cdots : 0) \in \mathbb{P}_\Omega^n$. Consider the action of $SL_\Omega(n+1)$ on $\mathbb{P}_\Omega^n \times Hyp_d(n)_\Omega$ which is defined by $g \cdot (x, X) = (gx, g_*X)$ for $g \in SL_\Omega(n+1)$. It can be easily checked that this action preserves Z and that we obtain an isomorphism between $Z_{(1:0:\cdots:0)}$ and Z_x via this action. $\qquad\square$

Next we calculate the dimension of $Z_{(1:0:\cdots:0)}$. Note that if we write

$$F(X) = \sum_{|\alpha|=d} C_\alpha X^\alpha$$

by using multi-indices, $(C_\alpha; |\alpha| = d)$ gives a system of coordinates for the projective space $Hyp_d(n)$. Then

$$Z_{(1:0:\cdots:0)} = V(C_{(010\cdots0)}, C_{(0010\cdots0)}, \ldots, C_{(00\cdots01)}) \subset Hyp_d(n)$$

holds. Therefore

$$
\begin{aligned}
\dim Z &= \dim \mathbb{P}^n_k + \dim Z_{(1:0:\cdots:0)} \\
&= n + (\dim Hyp_d(n) - n) \\
&= \dim Hyp_d(n).
\end{aligned}
$$

Now all we have to show is that $p : Z \to p(Z)$ is generically finite, since then we see that $\dim p(Z) = \dim Z = \dim Hyp_d(n)$, hence $p(Z) = Hyp_d(n)$. To see this, we can easily check the finiteness of the fiber of p at

$$
F(X) = X_0^{d-1} X_1 + X_1^{d-1} X_2 + \cdots + X_{n-1}^{d-1} X_n + X_n^{d-1} X_0.
$$

$\square$

Next we study $Hyp_d(n) \setminus U_{ns}$. Recall that the non-singularity of $X = V(F)$ is equivalent to the emptiness of the left hand side of (7). By using this fact, we show the following

Theorem 3.3. *Assume p divides d. Then*

$$
Hyp_d(n) \setminus U_{ns}
$$

is an irreducible divisor. Moreover some multiple of its defining equation lifts to the discriminant in characteristic zero.

Example 3.4. Consider the case $(n, d) = (1, 4)$. Let

$$
X = V(F), F = a_0 X_0^4 + a_1 X_0^3 X_1 + a_2 X_0^2 X_1^2 + a_3 X_0 X_1^3 + a_4 X_1^4
$$

be an hypersurface in $\mathbb{P}^1_k$. When $\mathrm{char}\, k \neq 2$, the defining equation for $Hyp_4(1) \setminus U_{ns}$ is given by $D = 4S^3 - T^2$, where

$$
\begin{aligned}
S &= 2^2 \cdot 3 a_0 a_4 - 3 a_1 a_3 + a_2^2 \\
T &= 2^3 \cdot 3^2 a_0 a_2 a_4 - 3^3 a_0 a_3^2 + 3^2 a_1 a_2 a_3 - 3^3 a_1^2 a_4 - 2 a_2^3.
\end{aligned}
$$

When $\mathrm{char}\, k = 2$, $D \mod 2 = (T \mod 2)^2$ and the defining equation for $Hyp_4(1) \setminus U$ is given by $T \mod 2 = a_0 a_3^2 + a_1 a_2 a_3 + a_1^2 a_4$.

Proof of Theorem 3.3. Let $W = W(k)$ be the ring of Witt vectors. Set

$$
I = \left\{ (x, X); x \in V \left(F, \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n} \right) \right\} \subset \mathbb{P}^n_W \times_{\mathrm{Spec}\, W} Hyp_d(n),
$$

where $Hyp_d(n) = |\mathcal{O}_{\mathbb{P}^n_W}(d)|$ is the projective space of families of degree d projective hypersurfaces over $\mathrm{Spec}\, W$. Let $p : I \to \mathbb{P}^n_W, q : I \to Hyp_d(n)$ be the natural projections. As in the proof of Proposition 3.2, we can show the following

Claim. *p is a smooth morphism with connected fibers.*

Therefore we see that both I and I_k, restriction of I over the closed point $\operatorname{Spec} k \subset \operatorname{Spec} W$, are integral schemes.

Now consider the integral closed subscheme $q(I) \subset Hyp_d(n)$. Note that the defining equation for $q(I)$ is the usual discriminant, and that $q(I)_k$, restriction of $q(I)$ over $\operatorname{Spec} k \subset \operatorname{Spec} W$, coincides with $q(I_k)$ as sets. $\qquad\square$

4. Y.LEE'S CRITERION IN CHARACTERISTIC p

Next we prove the following

Theorem 4.1. *Let X be an effective cycle of dimension r and degree d in $\mathbb{P}^n_k$. Let $(\mathbb{G}, Z(X))$ be the log pair defined by the Chow divisor $Z(X)$ of X. If $\operatorname{lct}(\mathbb{G}, Z(X)) > \frac{n+1}{d}$ (resp. $\geq \frac{n+1}{d}$), then X is Chow stable (resp. Chow semi-stable).*

See §2.3 for notations. Our proof goes along the same line as the original one by Y.Lee ([Le]), but we need to modify several points.

Before the proof, we point out that we might prove Theorem 4.1 via Theorem 2.9 as in the previous section, provided that the following conjecture would be true (below W is the ring of Witt vectors and K, k are the field of fractions and the residue field of W, respectively):

Conjecture 4.2. *Let $X_W \to \operatorname{Spec} W$ be a smooth proper morphism where X_W is an integral scheme. Let D_W be an effective $\mathbb{R}$-divisor on X_W, such that no irreducible component is contained in a fiber of the projection to $\operatorname{Spec} W$. By X_K and D_K we denote the restrictions of X_W and D_W over the generic point of $\operatorname{Spec} W$. Similarly X_k, D_k denote the restrictions of X_W and D_W over the closed point of $\operatorname{Spec} W$. Then if (X_k, D_k) is log canonical, so is (X_K, D_K). In particular $\operatorname{lct}(X_k, D_k) \leq \operatorname{lct}(X_K, D_K)$.*

Note that the lower semi-continuity of log canonical thresholds in a smooth family is already established over $\mathbb{C}$ (see [Laz, Corollary 9.5.39]).

4.1. **Log canonicity in positive characteristics.** In this subsection, we discuss how the log canonicity of log pairs are preserved under finite morphisms. Some properties of log canonicity which hold in characteristic zero fail in characteristic $p > 0$, but we can circumvent those difficulties and obtain Proposition 4.9, which is the key for the proof Theorem 4.1.

When the characteristic of the base field is zero, it is well known that log canonicity is preserved under finite dominant morphisms (see [KoM, Proposition 5.20(4)]). Namely:

Theorem 4.3. *Let $g : X' \to X$ be a finite dominant morphism of normal varieties over a field of characteristic zero. Let Δ (resp. Δ') be a $\mathbb{Q}$-divisor on X (resp. X') such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $g^*(K_X + \Delta) = K_{X'} + \Delta'$. Then (X, Δ) is log canonical if and only if (X', Δ') is.*

We should note that the canonical divisors K_X and $K_{X'}$ in Theorem 4.3 above are chosen in such a way that $K_{X'} = g^*K_X + R$ holds, where R is the ramification divisor of g.

When the characteristic of the base field is positive, we need to modify Theorem 4.3. First we consider the case when g is separable. In this case we may have wild ramifications, so we only have a weaker version of the ramification formula:

Lemma 4.4. *Let $g : X \to Y$ be a finite separable morphism between normal varieties over k. Let $E \subset X$ be a prime divisor on X and r be the ramification index of g along E. Then there exists a non-negative integer $b \geq r - 1$ such that $K_X = g^*K_Y + bE$ holds around the generic point of E.*

Proof. Set $V = Y \setminus \operatorname{Sing} Y$ and $U = g^{-1}(V) - \operatorname{Sing} X$. Note that the closed subsets we have through away have codimension greater than 1. Over U we have the following exact sequence:

$$(8) \qquad\qquad g^*\Omega_V \xrightarrow{f} \Omega_U \to \Omega_{U/V} \to 0.$$

Since g is separable, $\Omega_{U/V}$ generically vanishes (see [M, Theorem 59]). Hence f is generically isomorphic.

Let $F : g^*\mathcal{O}_V(K_V) \to \mathcal{O}_U(K_U)$ be the highest exterior product of the morphism f in (8) above. This is also generically isomorphic. Therefore $\ker F$ is a torsion subsheaf of the torsion free sheaf $g^*\mathcal{O}_V(K_V)$, so is trivial. Hence we see that F is injective.

Take a generic closed point e of $E \cap U$ which is contained in no other irreducible component of $\operatorname{Supp}\Omega_{U/V}$ except for E. Set $e' = g(e)$, $E' = g(E)$. Choose systems of local coordinates $x_1, \dots, x_n$ at e and $y_1, \dots, y_n$ at e', satisfying the following conditions:

(a) $E = \operatorname{div}(x_1)$ near e (resp. $E' = \operatorname{div}(y_1)$).
(b) $g^*y_i = x_i$ holds for all $i = 2, \dots, n$.
(c) there exists an invertible function u at e such that $g^*y_1 = u \cdot x^r$.

In (c), r denotes the ramification index of g along E. Now

$$\begin{aligned} F(g^*(dy_1 \wedge \cdots \wedge dy_n)) &= d(u \cdot x_1^r) \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n \\ (9) &= \left(\frac{\partial u}{\partial x_1} x_1 + ru \right) x_1^{r-1} dx_1 \wedge dx_2 \cdots \wedge dx_n, \end{aligned}$$

so there exists some non-negative integer b that

$$K_X = g^* K_Y + bE$$

holds in a neighborhood of e.

If $r \not\equiv 0 \pmod{p}$, $b = r - 1$. Now assume the contrary, i.e. assume E is wildly ramifying. Then

$$\begin{aligned} (9) &= \frac{\partial u}{\partial x_1} x_1^r dx_1 \wedge dx_2 \cdots \wedge dx_n \\ &\neq 0, \end{aligned}$$

since otherwise F is not generically isomoprhic. In this case we see that

$$b = \mathrm{val}_E \left(\frac{\partial u}{\partial x_1} \right) + r \geq r,$$

where val_E denotes the valuation corresponding to E. $\qquad \square$

Remark 4.5. Ramification formula for inseparable morphisms are discussed in [RS]. In this case the ramification divisor is defined only up to linear equivalence. If we adopt this version of ramification formula, the 'only if' part of Theorem 4.3 does not hold in general. For our purpose we need not to deal with inseparable cases.

With the weaker version of ramification formula above, we can prove that the 'only if' part of the Theorem 4.3 still holds for separable morphisms:

Propostion 4.6. *Let k be an algebraically closed field of characteristic $p > 0$. Let $g : X' \to X$ be a finite separable morphism of normal varieties over k. Let Δ (resp. Δ') be a $\mathbb{Q}$-divisor on X (resp. X') such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $g^*(K_X + \Delta) = K_{X'} + \Delta'$. Then if (X, Δ) is log canonical, so is (X', Δ').*

Proof. The proof goes along the same line as the proof of ([KoM, Prop 5.20(4)]), once we replace the ramification formula by the weaker version given above. $\qquad \square$

Remark 4.7. In general, the 'if' part of Theorem 4.3 holds only when there exists no wildly ramifying divisor. In such a case, the proof goes as in characteristic zero. If some of the ramification divisors are wildly ramifying it may not hold. An example is:

Example 4.8. Let $X = X' = \mathbb{A}^1_k$. Set $g : X' \to X; g(x) = x^p(x+1)$, $\Delta = \frac{p+1}{p} \operatorname{div}(x)$ and $\Delta' = \operatorname{div}(x)$. Since $g^* dx = x^p dx$, we obtain

$$g^*(K_X + \Delta) = K_{X'} + \Delta'.$$

Note that (X, Δ) is not lc, but (X', Δ') is.

Using Proposition 4.6, we can extend [Ko, Proposition 8.13] over arbitrary fields:

Propostion 4.9. *Take any $f \in k[x_1, \ldots, x_n]$. Assign a weight $w = (w(x_i))_{i=1,\ldots,n} \in (\mathbb{Z}_{\geq 0})^n \setminus \{0\}$ to the variables $x_1, \ldots, x_n$ and let $w(f)$ be the weighted multiplicity of f (= the lowest weight of the monomials occurring in f). Then*

$$\frac{1}{\operatorname{lct}_0(\mathbb{A}^n, \operatorname{div}(f))} \geq \frac{w(f)}{\sum_{i=1}^n w(x_i)}.$$

Proof.

Step 1. First we establish the inequality for those w's such that $w(x_i) > 0$ holds for all $i = 1, \ldots, n$, and p divides none of the $w(x_i)$'s.

In this case the inequality can be established along the same line as the original proof, since we have Proposition 4.6. For the sake of completeness, we reestablish the argument.

Consider $g : \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $g(x_i) = x_i^{w(x_i)}$. By the assumptions on $w(x_i)$'s, g is dominant and separable. Take a real number $c \in \mathbb{R}_{\geq 0}$ and assume $(\mathbb{A}^n_k, c \cdot \operatorname{div}(f))$ is lc at 0. Now calculate the pull-back of $K_{\mathbb{A}^n_k} + c \cdot \operatorname{div}(f)$ by g:

$$
\begin{aligned}
& g^*(K_{\mathbb{A}^n_k} + c \cdot \operatorname{div}(f)) \\
= {} & K_{\mathbb{A}^n_k} + \sum_{i=1}^n (1 - w(x_i)) \operatorname{div}(x_i) + c \cdot \operatorname{div}(f(x_1^{w(x_1)}, \ldots, x_n^{w(x_n)})) \\
=: {} & K_{\mathbb{A}^n_k} + \Delta'.
\end{aligned}
$$

By Proposition 4.6, we see $(\mathbb{A}^n_k, \Delta')$ is lc at 0. Let E be the exceptional divisor of the blow-up of $\mathbb{A}^n_k$ at the origin. We know that $a(E; \mathbb{A}^n_k, \Delta') \geq -1$ holds. With a calculation we see that $a(E; \mathbb{A}^n_k, \Delta')$ equals to $-1 + \sum_i w(x_i) - cw(f)$, obtaining the inequality.

Step 2. Now consider the continuous function $\varphi : (\mathbb{Q}_{\geq 0})^n \setminus \{0\} \to \mathbb{Q}$ defined by

$$\varphi(w) = \frac{w(f)}{\sum_i w(x_i)},$$

as in the case when $w(x_i)$'s are integers. If we replace w by some positive multiple of it, the value of φ never changes. Therefore φ factors through the quotient space

$$S := (\mathbb{Q}_{\geq 0})^n \setminus \{0\} / \mathbb{Q}_{>0},$$

inducing the continuous function $\overline{\varphi} : S \to \mathbb{Q}$.

The set of points represented by those w's satisfying the assumptions in Step 1 is dense in S. Hence, by the continuity of $\overline{\varphi}$, we see that

$$\overline{\varphi}(s) \leq \frac{1}{\mathrm{lct}_0(\mathbb{A}^n, \mathrm{div}(f))}$$

holds for arbitrary $s \in S$. We finish the proof.

$\square$

Remark 4.10. Step 2 in the proof above is inevitable, for $(\mathbb{A}_k^n, \Delta')$ need not be lc if g is inseparable. For example, consider the case $n = 2$, $w(x_1) = w(x_2) = p$ and $f(x_1, x_2) = x_1 - x_2$. In this case $\mathrm{lct}_0(\mathbb{A}_k^2, \mathrm{div}(f)) = 1$. On the other hand

$$\Delta' = (1 - p)\, \mathrm{div}(x_1 x_2) + p \cdot \mathrm{div}(x_1 - x_2),$$

hence $(\mathbb{A}_k^2, \Delta')$ is not lc at the origin.

4.2. **Proof of Theorem 4.1.**

Proof of Theorem 4.1. We only discuss stable case. Semi-stable case can be proven exactly in the same way. We have only to confirm the inequality (5) of Lemma 2.8. On the other hand, by the assumption and Proposition 4.9, the inequality clearly holds. $\square$

Appendix A. Proposition 4.9 via F-singularities

Proposition 4.9 was the main ingredient of the proof of Theorem 4.1. In this Appendix A we reprove it in characteristic zero, using the notion of F-singularities.

A.1. **Notion of F-singularities.** In this subsection, we summarize some definitions and results about F-singularities which we need. For detail one can consult [HW], [TW]. Below k denotes a field of characteristic $p > 0$, until otherwise stated.

Definition A.1. Let A be an integral domain of charateristic p, $F :$ $A \to A$ be the Frobenius map defined by $F(x) = x^p$. For an A-module M, we denote by $F_*^e M$ (or $^e M$ for short) the A-module which is obtained by pulling back the action of A on M via the ring homomorphism $F^e : A \to A$, the e-times iteration of F. A is said to be *F-finite* if $F_* A$ is a finite A-module.

If k is a perfect field, all the rings which are essentially of finite type over k are F-finite.

Definition A.2. Let A be an F-finite Noetherian local normal ring. Let Δ be an effective $\mathbb{R}$-divisor on $\operatorname{Spec} A$. Set $A(\Delta) = \{0\} \cup \{f \in K(A)^* | \operatorname{div}(f) + \Delta \geq 0\}$. (A, Δ) is said to be *F-pure* if the A-module homomorphism $A \to F_*^e A \to F_*^e (A((q-1)\Delta))$ splits for all $q = p^e$, where the first arrow is the e-times iteration of the Frobenius map and the second one is the natural inclusion. Similarly we say that (A, Δ) is *strongly F-regular* if for any non-zero element $c \in A$ there exists $q = p^e$ such that $A \to F_*^e A \to F_*^e (A((q-1)\Delta)) \xrightarrow{\times c} F_*^e (A((q-1)\Delta))$ splits as an A-module homomorphism.

Remark A.3. It can be shown that strong F-regularity implies F-purity (see [HW, Proposition 2.2(1)]).

Take a normal variety X over k and an effective $\mathbb{R}$-divisor Δ on X. We say (X, Δ) is F-pure if $(\mathcal{O}_{x,X}, \Delta)$ is F-pure for all $x \in X$.

For such a pair, the *F-pure threshold* is defined and denoted by $\operatorname{Fpt}(X, \Delta) = \sup\{t \in \mathbb{R} | (X, t\Delta) \text{ is F-pure}\}$.

F-pure threshold measures how bad the singularity of a log pair (X, Δ) is. Larger Fpt means that the singularity of (X, Δ) is better.

Now we state a useful consequence of F-purity, which actually characterizes F-purity in most cases. Below $\lfloor \cdot \rfloor$ denotes the round down of $\cdot$.

Lemma A.4. *Let $k = \overline{k}$. Set $A = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}, m = (x_1, \ldots, x_n)A$. Take any non-zero element $f \in A \setminus \{0\}$ and a non-negative rational number $t \in \mathbb{Q}_{\geq 0}$. Then if $(A, t \cdot \operatorname{div}(f))$ is F-pure, $f^{\lfloor (q-1)t \rfloor} \notin (x_1^q, \ldots, x_n^q)$ holds for all $q = p^e$.*

Proof. The 'only if ' part of ([HW, Proposition 2.6(1)]) holds without the assumption that $\operatorname{div}(f)$ is reduced (the proof given in that paper works for non-reduced cases too). The lemma above is its special case. $\square$

Next we recall the relationship between klt singularities and F-singularities.

Definition A.5. Let k be a field of characteristic zero. Let X be a normal variety over k, Δ be an effective $\mathbb{R}$-divisor on X. Let A be a subring of k which is of finite type over $\mathbb{Z}$. A *model of (X, Δ) over A* is a pair $(\mathcal{X}, \delta)$ of a normal scheme $\mathcal{X}$ of finite type and flat over $\operatorname{Spec} A$ and an effective $\mathbb{R}$-divisor δ on $\mathcal{X}$ such that $(\mathcal{X}, \delta) \times_{\operatorname{Spec} A} \operatorname{Spec} k \simeq (X, \Delta)$ over $\operatorname{Spec} k$. Note that A/P is a finite field for all $0 \neq P \in \operatorname{Spec} A$. We say that (X, Δ) is *of strongly F-regular type* if there exists a dense open subset $U \subset \operatorname{mSpec} A$ such that $(\mathcal{X}, \delta) \times_{\operatorname{Spec} A} \operatorname{Spec}(A/P)$ is strongly F-regular for all $P \in U$. This notion does not depend on the choice of A and models on it.

Theorem A.6. *Let (X, Δ) be a log pair over $\operatorname{Spec} k$, where k is a field of characteristic zero. Then (X, Δ) is klt if and only if it is of strongly F-regular type.*

A.2. **Proof of Proposition 4.9 in characteristic zero.** Let k be a field of characteristic zero.

Proof of Proposition 4.9. Assume for a contradiction that

$$(10) \qquad \frac{1}{\operatorname{lct}_0(\mathbb{A}^n_k, \operatorname{div}(f))} < \frac{w(f)}{\sum_i w(x_i)}$$

holds for some $w = (w(x_i))_i$. Note that $w(f) > 0$ holds in (10). Then there exists some rational number $t > \frac{\sum_i w(x_i)}{w(f)}$ such that $(\mathbb{A}^n_k, t \cdot \operatorname{div}(f))$ is klt (see Proposition 2.4). Let A be the ring generated over $\mathbb{Z}$ by the coefficients of f. By Theorem A.6, we can take a maximal ideal m of A such that the reduction of $(\mathbb{A}^n_A, t \cdot \operatorname{div}(f))$ at m is strongly F-regular (therefore F-pure) at the origin (see Remark A.3). Now we replace t, if necessary, so that $p := \operatorname{char}(A/m)$ does not divide the denominator of t. Then we can take some $q = p^e$ such that $t(q-1) \in \mathbb{Z}$. By Lemma A.4, we see that

$$(11) \qquad f^{t(q-1)} \notin (x_1^q, \ldots, x_n^q).$$

On the other hand, the equation of f is of the form

$$(12) \qquad f = f(x_1, \ldots, x_n) = \sum_\alpha C_\alpha x^\alpha.$$

From (11), we see that there exist $\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(t(q-1))}$ satisfying

$$C_{\boldsymbol{\alpha}^{(j)}} \neq 0$$

for all $j = 1, \ldots, t(q-1)$, and

$$(13) \qquad \sum_{j=1}^{t(q-1)} \alpha_i^{(j)} \leq q - 1$$

for all $i = 1, \ldots, n$. By the definition of $w(f)$, we see that

$$(14) \qquad \sum_{i=1}^{n} w(x_i)\alpha_i^{(j)} \geq w(f) \left(> \frac{\sum_i w(x_i)}{t} \right)$$

holds for all j. Now

$$
\begin{aligned}
(q-1)\sum_{i=1}^{n} w(x_i) \; &= \; t(q-1) \cdot \frac{\sum_i w(x_i)}{t} \\
&\underset{(14)}{<} \; \sum_{j=1}^{t(q-1)} \left(\sum_{i=1}^{n} w(x_i)\alpha_i^{(j)} \right) \\
&= \; \sum_{i=1}^{n} w(x_i) \left(\sum_{j=1}^{t(q-1)} \alpha_i^{(j)} \right) \\
&\underset{(13)}{\leq} \; (q-1)\sum_{i=1}^{n} w(x_i),
\end{aligned}
$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark A.7. When the characteristic of k is positive, via a similar argument we may prove the following inequality:

$$(15) \qquad \frac{1}{\mathrm{Fpt}_0(\mathbb{A}^n, \mathrm{div}(f))} \geq \frac{w(f)}{\sum_i w(x_i)}.$$

(15) is a direct consequence of (and is slightly weaker than) Proposition 4.9, since we have the following general implication

$$\text{F-pure} \Rightarrow \text{log canonical}$$

(see [HW, Theorem 3.3] for detail. Examples of singularities whose lct are strictly greater than Fpt can be found in [TW, Example 2.5]).

APPENDIX B. CHOW STABILITY OF THE SUM

In this section we show that the sum of two Chow semi-stable cycles of the same dimension are again Chow semi-stable. Moreover if one of them is stable, it follows that the sum also becomes stable.

Propostion B.1. *Let Y, Z be Chow semi-stable cycles of the same dimension in a projective space $\mathbb{P}_k^n$. Then $Y + Z$ is again Chow semi-stable. Furthermore if Y is Chow stable, so is $Y + Z$.*

In the proof we freely use the notation like $\lim_{t \to 0} \lambda(t) \cdot F$, as in [GIT], since the idea becomes clearer. To be logically complete, we of course need to replace the argument suitably. It is a routine work, so we omit the detail.

Proof. Let d, e be the degrees of Y and Z respectively. Let $F \in \mathcal{B}_d$, $G \in \mathcal{B}_e$ be the Chow forms of Y and Z, respectively. Then the Chow form of $Y + Z$ is given by $F \cdot G \in \mathcal{B}_{d+e}$.

Choose a non-trivial 1-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \to SL(n+1, k)$. Via λ we pull back the canonical actions of $SL(n + 1, k)$ onto $\mathcal{B}_d, \mathcal{B}_e$ and $\mathcal{B}_{d+e}$ to $\mathbb{G}_m$. Now consider the natural multiplication map $\mu : \mathcal{B}_d \times \mathcal{B}_e \to \mathcal{B}_{d+e}$, given by $(F, G) \mapsto F \cdot G$. If we pose the diagonal action of $\mathbb{G}_m$ on the left hand side, μ becomes equivariant.

Assume that Y, Z are both Chow semi-stable. Then both $\lim_{t \to 0} \lambda(t) \cdot F \neq 0$ and $\lim_{t \to 0} \lambda(t) \cdot G \neq 0$ holds. Now since we know that μ is continuous,

$$\lim_{t \to 0} \lambda(t) \cdot (F \cdot G)$$
$$= (\lim_{t \to 0} \lambda(t) \cdot F) \cdot (\lim_{t \to 0} \lambda(t) \cdot G)$$
$$\neq 0,$$

since $\mathcal{B} = \oplus_{d \geq 0} \mathcal{B}_d$ is an integral domain. Therefore $Y + Z$ is again Chow semi-stable.

Second, assume further that Y is Chow stable. Then $\lim_{t \to 0} \lambda(t) \cdot F = \infty$, so that

$$\lim_{t \to 0} \lambda(t) \cdot (F \cdot G)$$
$$= (\lim_{t \to 0} \lambda(t) \cdot F) \cdot (\lim_{t \to 0} \lambda(t) \cdot G)$$
$$= \infty \cdot (\lim_{t \to 0} \lambda(t) \cdot G)$$
$$= \infty,$$

since $(\lim_{t \to 0} \lambda(t) \cdot G)$ is not 0. Therefore $Y + Z$ is Chow stable. $\square$

Remark B.2. We can not expect the converse of Proposition B.1 at all. There exists a semi-stable cycle such that all its subcycles are unstable:

Example B.3. Take the union of three lines on a plane which are in a general position. The union itself is Chow semi-stable (see [GIT, §4-2]), but lines and reducible conics on a plane are Chow unstable.

However, the following holds:

Propostion B.4. *Let Z be a cycle of $\mathbb{P}_k^n$. Then the followings are equivalent:*

(1) Z is Chow (semi-)stable.
(2) mZ is Chow (semi-)stable for any positive integer $m \in \mathbb{Z}_{>0}$.
(3) mZ is Chow (semi-)stable for some positive integer $m \in \mathbb{Z}_{>0}$.

Proof. We have only to prove (3)$\Rightarrow$(1). Let G be the Chow form of Z. Then G^m gives the Chow form of mZ. Assume that mZ is Chow semi-stable. Take any 1-PS λ as in the proof of the Proposition B.1. Then $0 \neq \lim_{t\to 0} \lambda(t) \cdot G^m = (\lim_{t\to 0} \lambda(t) \cdot G)^m$, hence $\lim_{t\to 0} \lambda(t) \cdot G \neq 0$. Therefore Z is semi-stable. Stable case can also be shown via a similar argument. $\square$

Example B.5. Let $Y \subset \mathbb{P}_k^n$ be a non-singular hypersurface of degree 3, which is Chow stable by Theorem 3.1. By Proposition B.1, mY is also Chow stable for all the positive integers m. On the other hand, $\mathrm{lct}(\mathbb{P}_k^n, mY) = \frac{1}{m}$ and hence $\frac{1}{m} < \frac{n+1}{3m}$ if $n \geq 3$. Thus we obtain a sequence of examples of Chow stable hypersurfaces whose stability can not be detected by Theorem 4.1.

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