Multi-section rings and surjective morphisms

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Multi-section ring

1. Introduction – Multi-section ring
2. Properties of Cox rings and geometry of line bundles
   - Finite generation/ Mori dream space
   - VGIT/ Geometry of line bundles
3. Multi-section rings and surjective morphisms
4. Geometric implications

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- for $f \in \mathrm{H}^0(X, \mathcal{O}_X(D))$ and $g \in \mathrm{H}^0(X, \mathcal{O}_X(E))$, set $f \cdot g := f \otimes g \in \mathrm{H}^0(X, \mathcal{O}_X(D+E))$.

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(A resolution of) φ_D is an algebraic fiber space.

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cf) arXiv:1104.1326 ‘On images of Mori dream spaces’

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Remark

Finite generation of $R_X(\Gamma)$ does not depend on the choice of Γ.

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$\Rightarrow$ follows from the theory of VGIT.

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Remark

We expect the similar result for Calabi-Yau manifolds and projective complex symplectic varieties in general.

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For this action, we consider the Variation of GIT quotients (VGIT).

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Remark

If $\chi = \mathrm{ev}_D$, then $R_\chi = R_X(D)$.

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Relation to the geometry of line bundles

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\begin{array}{ccccc}
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\downarrow{\scriptstyle /T} & & \downarrow{\scriptstyle /T} & & \downarrow{\scriptstyle //T} \\
V^{ss}(\mathrm{ev}_A)/T & \xleftarrow{\ \supset\ } & V^{ss}(\mathrm{ev}_A) \bigcap V^{ss}(\mathrm{ev}_D)/T & \longrightarrow & V^{ss}(\mathrm{ev}_D)//T \\
\downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\
X & & \overset{\varphi_D}{\dashrightarrow} & & \operatorname{Proj} R_X(D)
\end{array}
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($\subset$ is an open immersion)

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- $\varphi_D = \varphi_E$ *and*
- $\mathbb{B}(D) = \mathbb{B}(E)$ *($\mathbb{B}$: stable base locus).*

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$\operatorname{Eff}(X)$ *has a finite fan structure such that the relative interior of a cone of the fan is an equivalence class in two senses.*

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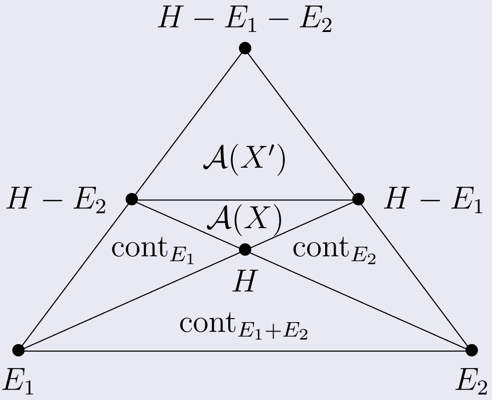

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A general hypersurface of degree $(2, \ldots, 2)$ in $(\mathbb{P}^1)^{n+1}$ is a Calabi-Yau manifold of this kind, and is studied by Oguiso in detail.

Multi-section rings

Multi-section rings and surjective morphisms

Consider a surjective morphism

$$f : X \to Y,$$

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$$f^* : R_Y(\Gamma) \to R_X(f^*\Gamma),$$

a Γ-graded k-algebra homomorphism.

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Proposition

$$f^* : R_Y(\Gamma) \to R_X(f^*\Gamma)$$

is an integral extension.
Moreover $R_Y(\Gamma)$ is finitely generated if and only if $R_X(f^*\Gamma)$ is, and in this case f^* is finite.

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  $\Rightarrow$ "Use δ instead of the Galois group."

□

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Since X is a Mori dream space, any multi-section ring is finitely generated (relatively easy). □

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Corollary

The fan of Y is the same as the restriction of the fan of X to $\mathrm{Pic}\,(Y)_{\mathbb{R}}$ via $f^* : \mathrm{Pic}\,(Y)_{\mathbb{R}} \subset \mathrm{Pic}\,(X)_{\mathbb{R}}$.

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If $k[V] \to k[U]$ is an integral extension, then equality holds.

OMAKE

Log terminality of Cox ring

Sannai gave the following conjecture:

Conjecture

Let X be a MDS over $\mathbb{C}$. There exists an effective $\mathbb{Q}$-divisor Δ on X such that

- (X, Δ) *is klt*
- $-(K_X + \Delta)$ *is ample*

(i.e. (X, Δ) is log Fano) if and only if the singularity of the Cox ring of X is at worst log terminal.

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Proposition (Sannai (2011))

Suppose $\mathrm{char}\,(k) > 0$. *Then a MDS* X *over* k *is globally* F*-regular if and only if the Cox ring of* X *is strongly* F*-regular.*