

# Calculation and Visualization of 3-braids

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**ThreeBraidClass**: a python module for 3-braids

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# 1 3-braids in ThreeBraidClass

In the **ThreeBraidClass** module...

**3-braid** is given by

$AB$ -word,  $\sigma$ -word,  $ST$ -word,  $\mathrm{PSL}_2(\mathbb{Z})$ ,  
bdpq-word, RL-word, Syzygy sequence

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$$\begin{aligned} B_3 &= \text{the 3-strand Artin braid group} \\ &= \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \end{aligned}$$

$$\begin{aligned} \mathcal{B}_3 &= B_3 / Z(B_3) \simeq \mathrm{PSL}_2(\mathbb{Z}) \\ Z(B_3) &= \langle (\sigma_1 \sigma_2)^3 \rangle \simeq \mathbb{Z} \end{aligned}$$

$$\mathcal{B}_3 = \langle \bar{\sigma}_1, \bar{\sigma}_2 \mid \cdots \rangle \quad (\bar{\sigma}_i : \text{the image of } \sigma_i)$$

$$\bar{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \bar{\sigma}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\simeq \mathbf{PSL}_2(\mathbb{Z})$$

$$= \langle S, T \mid S^2 = (ST)^3 = 1 \rangle$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \langle A, B \mid A^2 = B^3 = 1 \rangle \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

$$\bar{\sigma}_1 = STS = BBA \quad \bar{\sigma}_1^{-1} = TST = AB$$

$$\bar{\sigma}_2 = T = ABB \quad \bar{\sigma}_2^{-1} = STSTS = BA$$

$$A = S = \bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_1 \quad B = STST = \bar{\sigma}_1 \bar{\sigma}_2$$

## Modular groups:

$$\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \quad (\simeq \mathcal{B}_3)$$

$$\Gamma(2) = \{M \in \Gamma \mid M \equiv 1 \pmod{2}\}$$

$$\mathrm{PSL}_2(\mathbb{Z})/\Gamma(2) \simeq S_3$$

$\bar{\Gamma}^2$  : the index 2 subgroup of  $\Gamma$   
(the equianharmonic modular group)

$$\Gamma \triangleright \bar{\Gamma}^2 \triangleright \Gamma(2)$$

$$\bar{\Gamma} = \langle \bar{\sigma}_1 \bar{\sigma}_2 \rangle * \langle \bar{\sigma}_2 \bar{\sigma}_1 \rangle \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

$$\Gamma = \bar{\Gamma}^2 \cup \bar{\Gamma}^2 A$$

$$\bar{\Gamma}^2 = \{\mathbf{id}, \mathbf{b}, \mathbf{p}\} * \{\mathbf{id}, \mathbf{d}, \mathbf{q}\} \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

$$\begin{aligned} \mathbf{b} &= \bar{\sigma}_2^{-1} \bar{\sigma}_1^{-1} = BB & \mathbf{p} &= \bar{\sigma}_1 \bar{\sigma}_2 = B \\ \mathbf{d} &= \bar{\sigma}_1^{-1} \bar{\sigma}_2^{-1} = ABBA & \mathbf{q} &= \bar{\sigma}_2 \bar{\sigma}_1 = ABA \end{aligned}$$

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \mathbf{R}, \mathbf{L} \mid \cdots \rangle$$

$$\begin{aligned} \mathbf{R} &= \bar{\sigma}_2 = ABB & \mathbb{R} &= \bar{\sigma}_2^{-1} = BA \\ \mathbf{L} &= \bar{\sigma}_1^{-1} = AB & \mathbb{L} &= \bar{\sigma}_1 = BBA \end{aligned}$$

$$\begin{aligned} \mathbf{R} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \mathbb{R} &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \mathbf{L} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \mathbb{L} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

## 2 Syzygy sequences

3-braid = a collision-free motion of triangles

**Shape function of triangles (ordered triples):**

$$\text{Conf}^3(\mathbb{C}) = \{(a, b, c) \in \mathbb{C}^3 \mid a \neq b \neq c \neq a\}$$

For  $\Delta = (a, b, c) \in \text{Conf}^3(\mathbb{C})$ ,

$$\psi_j(\Delta) = \frac{1}{3}(a + b\omega^{-j} + c\omega^j) \quad (j = 0, 1, 2)$$

$$\omega = \exp(2\pi i/3)$$

$$\Psi(\Delta) = (\psi_0(\Delta), \psi_1(\Delta), \psi_2(\Delta))$$

$$\psi(\Delta) = \frac{\psi_2(\Delta)}{\psi_1(\Delta)} \in \mathbb{P}(\mathbb{C}) : \text{the shape of } \Delta$$

## **$B_3$ and $\mathcal{B}_3$ as fundamental groups:**

$S_3$  and  $\mathbb{C}^\times$  act on  $\text{Conf}^3(\mathbb{C})$

$$\text{Conf}^3(\mathbb{C}) \xrightarrow{\sim} C(\Psi) : \Delta \mapsto \begin{pmatrix} \psi_2(\Delta) \\ \psi_1(\Delta) \end{pmatrix}$$

$\rightarrow S_3$  and  $\mathbb{C}^\times$  act on  $C(\Psi)$

$$C(\Psi) = \left\{ \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \in \mathbb{C}^2 \mid \psi_1^3 \neq \psi_2^3 \right\}$$

$$C(\Psi)/\mathbb{C}^\times \xrightarrow{\sim} \mathbb{P}_\psi^1(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$$

$$\begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \mapsto \frac{\psi_2}{\psi_1}$$

the shape (the similarity class) of triangles



We call  $\mathbb{P}_\psi^1(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$  **the shape sphere**.

$$\begin{array}{ccccc} \mathrm{Conf}^3(\mathbb{C}) & \rightarrow & C(\Psi) & \rightarrow & \mathbb{P}_\psi^1(\mathbb{C}) \setminus \{1, \omega, \omega^2\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Conf}^3(\mathbb{C})/S_3 & \rightarrow & C(\Psi)/S_3 & \rightarrow & \mathbb{P}_{\infty 23}^1 \end{array}$$

$\mathbb{P}_{\infty 23}^1$  : the orbifold quotient

$\infty \cdots$  one missing point

$2\,3 \cdots$  two elliptic point of order 2, 3

$$B_3 \simeq \pi_1(\mathrm{Conf}^3(\mathbb{C})/S_3) = \pi_1(C(\Psi)/S_3)$$

$$\mathcal{B}_3 \simeq \pi_1^{\mathrm{orb}}(\mathbb{P}_{\infty 23}^1, *)$$

## Motion of triangles

Let  $\Delta(t) \in \text{Conf}^3(\mathbb{C})$  be smooth on  $t_0 \leq t \leq t_1$   
with  $\{\Delta(t_0)\} = \{\Delta(t_1)\}$   
where  $\{\Delta\}$  means the set of vertices of  $\Delta$ .

The motion of  $\Delta(t)$  forms a 3-braid in  $\mathbb{C} \times \mathbb{R}$ .

$$\gamma_\Delta = \{\psi(\Delta(t))\}_t \subset \mathbb{P}_\psi^1(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$$

the corresponding path on the shape sphere

$$\bar{\gamma}_\Delta \text{ (the image of } \gamma_\Delta) \in \pi_1^{\text{orb}}(\mathbb{P}_{\infty 23}^1) \simeq \mathcal{B}_3$$

the loop on the orbifold  $\mathbb{P}_{\infty 23}^1$

# What is syzygy?

**syzygy** = a state in which at least three points are aligned on a straight line

(Syzygy is an astronomical term. Similar astronomical phenomena include eclipse, conjunction, and opposition.)

$\Delta = (a, b, c) \in \text{Conf}^3(\mathbb{C})$  is **a syzygy**

$\Leftrightarrow \psi(\Delta)$  is on the equator of the shape sphere  
(i.e.  $|\psi(\Delta)| = 1$ )

$1, \omega, \omega^2$  divide the equator into 3 sections:

$[0]$  = the arc from 1 to  $\omega$  :  $a \textcolor{red}{b} c$

$[1]$  = the arc from  $\omega$  to  $\omega^2$  :  $c \textcolor{red}{a} b$

$[2]$  = the arc from  $\omega^2$  to 1 :  $b \textcolor{red}{c} a$

## Syzygy sequence of the motion:

Let  $\Delta(t) \in \text{Conf}^3(\mathbb{C})$  ( $t_0 \leq t \leq t_1$ ) be smooth  
with  $\{\Delta(t_0)\} = \{\Delta(t_1)\}$  is syzygy.  $\times$

$$K := \{t \in [t_0, t_1] \mid |\psi(\Delta(t))| = 1\}$$

Assume that  $\#K < \infty$ .

$t_0 = k_0 < k_1 < \dots < k_r = t_1$  : all members of  $K$   
 $\{s_j\}_j \in (\mathbb{Z}/3\mathbb{Z})^{r+1}$  s.t.  $\psi(\Delta(k_j)) \in [s_j]$  (for  $\forall j$ )

$\mathcal{S}_\Delta = \{s_j\}_j$  : **the syzygy sequence** of  $\Delta(t)$

**Rem.**  $s_j = \left\lfloor \frac{3}{2\pi} \arg(\psi(\Delta(k_j))) \right\rfloor \pmod{3}$

### 3 Lissajous 3-braids

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Let  $m, n \in \mathbb{Z}$ ,  $\varphi \in \mathbb{R}$  with

$$m \equiv n \equiv 1 \pmod{3} \quad (\gcd(m, n) = 1)$$

**Lissajous curve of type  $(m, n, \varphi)$ :**  $(t \in \mathbb{R}/\mathbb{Z})$

$$L(t) = A \sin(2\pi m t) + i B \sin(2\pi n t + \varphi)$$

$$\Delta_{m,n,\varphi}(t) := (L(t - 1/3), L(t), L(t + 1/3))$$

**Lissajous (toric) 3-braid:**

the motion of  $\Delta_{m,n,\varphi}(t)$   $(t \in \mathbb{R}/\mathbb{Z})$

in  $\mathbb{C} \times \mathbb{R}/\mathbb{Z}$       or      in  $\mathbb{C} \times \mathbb{R}$

## Rem.

Kin-Nakamura-O. studied the Lissajous 3-braids ([KNO], [KNO2]). The first purpose of ThreeBraidClass module was to efficiently compute examples of Lissajous 3-braids. This module provides methods for outputting computation results for the purpose. One can get them by specifying 'KNO' as an argument to the 'show()' method of the LissajousBraid class in ThreeBraidClass module.

## Rem.

In the case of a Lissajous 3-braid, the corresponding path  $\gamma_{\Delta_{m,n,\varphi}}$  is a loop on the shape sphere. And the syzygy sequence  $\mathcal{S}_{\Delta_{m,n,\varphi}}$  should be treated as a circular word.

**Prop.** Put  $\ell = \frac{m-n}{3} \in \mathbb{Z}$ .

The length of  $\mathcal{S}_{\Delta_{m,n,\varphi}} = 6\ell$

$\exists \alpha, \beta \in \mathbb{R}$  s.t.  $\mathcal{S}_{\Delta_{m,n,\varphi}} = \{\lfloor \alpha j + \beta \rfloor \pmod{3}\}_j$

**Rem.** We may take  $\alpha \equiv \frac{-n}{|\ell|} \pmod{3\mathbb{Z}}$ .

## Linear 3-braid:

**br** : 3-braid (= the motion of  $\Delta(t) \in \text{Conf}^3(\mathbb{C})$ )

$\mathcal{S}_{\text{br}} := \mathcal{S}_{\Delta}$  : the syzygy sequence of **br**

## The extension of syzygy sequence:

extend finite length syzygy sequence  $\mathcal{S}$

to infinite length syzygy sequence  $\tilde{\mathcal{S}}$

explain with examples: (connect **red** to **blue**)

$$(\mathbf{0} \cdots \mathbf{1}) - (\mathbf{0} \cdots \mathbf{1}) \rightarrow (\mathbf{0} \cdots \mathbf{1}) - (\mathbf{1} \cdots \mathbf{2}) \rightarrow (\mathbf{0} \cdots \mathbf{1} \cdots \mathbf{2})$$

$$(\mathbf{0} \cdots \mathbf{1}) - (\mathbf{0} \cdots \mathbf{1}) \rightarrow (\mathbf{2} \cdots \mathbf{0}) - (\mathbf{0} \cdots \mathbf{1}) \rightarrow (\mathbf{2} \cdots \mathbf{0} \cdots \mathbf{1})$$

align

connect



$\tilde{\mathcal{S}}_{\mathbf{br}}$  : the extension of  $\mathcal{S}_{\mathbf{br}}$

**br** is **linear**

$\stackrel{\text{def}}{\iff} \exists \alpha, \beta \in \mathbb{R} \text{ s.t.}$

$$\tilde{\mathcal{S}}_{\mathbf{br}} = \{ \lfloor \alpha j + \beta \rfloor \pmod{3} \}_{j \in \mathbb{Z}}$$

## Theorem (O.)

- Each Lissajous 3-braid is linear.
- For any linear 3-braid **br**,  
there exists a Lissajous type  $(m, n, \varphi)$  s.t.

$$\tilde{\mathcal{S}}_{\mathbf{br}} = \tilde{\mathcal{S}}_{\Delta_{m,n,\varphi}}$$

## 4 Drawing 3-braids

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**Aim:** Draw 3D images of 3-braids given by  $AB$ -words,  $\sigma$ -words,  $\mathrm{PSL}_2(\mathbb{Z})$ , ...

- In the case of a Lissajous 3-braid:  
use the Lissajous curve as the frame ...
- In the general case:
  - [step 1] construct a path on the shape sphere corresponding to a given 3-braid
  - [step 2] find ordered triples with given shapes

**[step 1] construct a path on the shape sphere**

**br** : a 3-braid given by a AB-word, a  $\sigma$ -word ...

$\mathcal{S}_{\mathbf{br}} = \{s_j\}_j$  : the syzygy seq. of **br** (length  $r+1$ )

$\gamma$  : the corresponding path on the shape sphere

The poler coodinate:

$$\gamma(t) = \mathbf{R}(t) \exp\left(\frac{2\pi i}{3} \boldsymbol{\theta}(t)\right)$$

$$(1) \quad \#\{t \in [0, 1] \mid |R(t)| = 1\} = r + 1$$

$$(2) \quad k_0 = 0 < \cdots < k_r = 1 \text{ s.t. } |R(k_j)| = 1$$

$$[\theta(k_j)] \pmod{3} = s_j \quad (\text{for } \forall j)$$

(1) Put  $R(t) = \frac{2 + \sin(\pi r t)}{2 - \sin(\pi r t)}$

then  $\{t \in \mathbb{R}/\mathbb{Z} \mid |R(t)| = 1\} = \frac{1}{r}\mathbb{Z}/\mathbb{Z}$

(2)  $\{u_j\}_j \in \mathbb{Z}^{r+1}$  : a lift of  $\mathcal{S}_{\mathbf{br}}$   
 $u_j \pmod{3} = s_j \quad (\text{for } \forall j)$

$$f_{\mathbf{br}}(t) = \alpha t + \beta + \sum_h a_h \sin(2\pi h t) + a'_h \cos(2\pi h t)$$

**the Fourier expansion** ( $t \in [0, 1]$ )

of the line graph through  $\{(j/r, u_j)\}_j$

truncate:

$$f_{\mathbf{br}}^{(k)}(t) = \alpha t + \beta + \sum_{h \leq k} a_h \sin(2\pi ht) + a'_h \cos(2\pi ht)$$

error evaluation:

$$\text{err}_k = \max\{|f_{\mathbf{br}}^{(k)}(j/r) - u_j| \mid 0 \leq j < r\}$$

Take  $k_0 \in \mathbb{N}$  satisfying that  $\text{err}_{k_0} < \frac{1}{2}$

and put  $\theta_{\mathbf{br}}(t) = f_{\mathbf{br}}^{(k_0)}(t) + \frac{1}{2}$

Then  $\lfloor \theta_{\mathbf{br}}(j/r) \rfloor = u_j$  (for  $\forall j \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$ )

**[step 2] find ordered triples with given shapes**

Find  $\Delta_{\mathbf{br}}(t) \in \text{Conf}^3(\mathbb{C})$  s.t.

$$\psi(\Delta_{\mathbf{br}}(t)) = \gamma_{\mathbf{br}}(t).$$

$$\begin{aligned}\gamma_{\mathbf{br}}(t) &= R(t) \exp\left(\frac{2\pi i}{3} \theta_{\mathbf{br}}(t)\right) \\ &= \frac{(2 + \sin(\pi r t)) \exp\left(\frac{\pi i}{3} \theta_{\mathbf{br}}(t)\right)}{(2 - \sin(\pi r t)) \exp\left(-\frac{\pi i}{3} \theta_{\mathbf{br}}(t)\right)}\end{aligned}$$

$$\psi(\Delta_{\mathbf{br}}(t)) = \frac{\psi_2(\Delta_{\mathbf{br}}(t))}{\psi_1(\Delta_{\mathbf{br}}(t))}$$

We may solve the following linear equation:

$$\psi_0(\Delta_{\mathbf{br}}(t)) = 0$$

$$\psi_1(\Delta_{\mathbf{br}}(t)) = (2 - \sin(\pi r t)) \exp(-\frac{\pi i}{3} \theta_{\mathbf{br}}(t))$$

$$\psi_2(\Delta_{\mathbf{br}}(t)) = (2 + \sin(\pi r t)) \exp(\frac{\pi i}{3} \theta_{\mathbf{br}}(t))$$

## Theorem (O.)

$\{\psi(\Delta_{\mathbf{br}}(t))\}_{t \in \mathbb{R}}$  ( $= \{\gamma_{\mathbf{br}}(t)\}_{t \in \mathbb{R}}$ ) is a loop on the shape sphere following the extension of the syzygy sequence of  $\mathbf{br}$  through the equator.

# 5 Demonstration



# 6 References

- [KNO1] E.Kin, H.Nakamura, H.Ogawa, Lissajous 3-braids, J. Math. Soc. Japan (to appear)
- [KNO2] E.Kin, H.Nakamura, H.Ogawa, Lissajous 3-braids with phase differences, in preparation
- [M] K.Murasugi, On closed 3-braids, Memoirs of AMS 151 (1974), American Math. Soc.
- [MM] R.Moeckel, R.Montgomery, Realizing all reduced syzygy sequences in the planar three-body problem, Nonlinearity 28 (2015), 1919-1935
- [NO1] H.Nakamura, H.Ogawa, A family of geometric operators on triangles with two complex variables, J. of Geometry 111, 2 (2020)
- [NO2] H.Nakamura, H.Ogawa, On generalized median triangles and tracing orbits, Results in Math. 75, 144 (2020).
- [Og] H.Ogawa, Lissajous 3-braids as linear syzygy sequences, in preparation