How should we bet on prime number dice ?

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Abstruct

§1. Introduction

§2. Prime distributions with congruence conditions

In this paper, we generally use the letter n by a positive integer, the letter p by a prime number, and the letter x by a positive real number. We denote by $\mathbb{N}$ the set of positive integers, by P the set of prime numbers, and by $N_\delta(x)$ the open δ-neighborhood of x. We use the notations $\sum_{n\le x} f(x)$, $\prod_{p\ge x} f(p)$, $\prod_p f(p)$, and so on, to indicate sums or products over all positive integers n or all prime numbers p, within the specified ranges. We use symbols O, o and $\sim$ as below. $f(x) = O(g(x))$, $f(x) = o(g(x))$ and $f(x) \sim g(x)$ (as $x \to \infty$) mean respectively that $|f(x)/g(x)|$ is bounded for all sufficiently large x, that f(x)/g(x) converges to 0 as $x \to \infty$ and that $f(x)/g(x)$ converges to 1 as $x \to \infty$.

We denote by $\pi(x)$ the number of prime numbers at most x, and by $\mu(x)$ the average density of prime numbers in the neighbourhood of x. Namely,

$$\pi(x) = \sharp(P \cap (0,\, x)) \qquad \mu(x) = \frac{\sharp(P \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))},$$

where a positive number δ is small compared with x, but is large enough to get the meaningful statistical data. In our computational experiments (see §4), we take $10^8 \le x \le 10^{14}$, and δ to be $\sharp(P \cap N_\delta(x)) = 10^6$. It follows from the prime number theorem that $\mu(x) \sim \log(x)^{-1}$.

For a positive integer n, we define the next prime $\mathrm{np}(n)$ by the smallest prime number greater that n, and the gap to the next prime $\mathrm{gap}(n)$ by the distance to the next prime (i.e. $\mathrm{gap}(n) = \mathrm{np}(n) - n$). Let $\mu(x \,|\, \mathrm{gap}{=}d)$ be the average density of positive integers n satisfying $\mathrm{gap}(n) = d$ in the integers near x, $\mu_p(x \,|\, \mathrm{gap}{=}d)$ the average density of prime numbers p satisfying $\mathrm{gap}(p) = d$ in the prime numbers near x, and $\mu(x \,|\, \mathrm{prime} \wedge \mathrm{gap}{=}d)$ the average density of prime numbers p satisfying $\mathrm{gap}(p) = d$ in the integers near x;

$$\mu(x \,|\, \mathrm{gap}{=}d) = \frac{\sharp(\mathbb{N} \cap Gap(d) \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))},$$

$$\mu_p(x \,|\, \mathrm{gap}{=}d) = \frac{\sharp(P \cap Gap(d) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))},$$

$$\mu(x \,|\, \mathrm{prime} \wedge \mathrm{gap}{=}d) = \frac{\sharp(P \cap Gap(d) \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))},$$

where $Gap(d)$ is the set of the integers n satisfying $\mathrm{gap}(n) = d$. The subscript p in μ_p means to narrow parent population to prime numbers. By definition,

$$\mu(x \mid \mathrm{prime} \wedge \mathrm{gap}{=}d) = \mu(x)\, \mu_p(x \mid \mathrm{gap}{=}d).$$

We may consider the average density $\mu(n)$ as the probability of the event whether an integer n is a prime number. If we assume the independence on these events for all positive integers, we have

$$\mu_p(x \mid \mathrm{gap}{=}d) = \mu(x+d) \prod_{t=1}^{d-1} (1 - \mu(x+t)).$$

The prime number theorem $\mu(x) \sim \log(x)^{-1}$ implies that

$$\mu_p(x \mid \mathrm{gap}{=}d) \sim \log(x)^{-1}(1 - \log(x)^{-1})^{d-1},$$
$$\mu(x \mid \mathrm{prime} \wedge \mathrm{gap}{=}d) \sim \log(x)^{-2},$$

On the other hand, the Hardy-Littlewood conjecture (Conjecture 3.1) about the prime gaps leads us to

$$\mu(x \mid \mathrm{prime} \wedge \mathrm{gap}{=}d) \sim c_d \log(x)^{-2},$$

where $c_d = 2c \prod_{3 \le p \mid d} (p-1)/(p-2)$, $c = \prod_{p \ge 3} p(p-2)/(p-1)^2 = 0.66016\cdots$ (Corollary 3.3). The Hardy-Littlewood conjecture takes account of a sieve by congruence relations and a bias of the number of irreducible residue classes in small interval. The conjecture gives a good approximation to the number of prime numbers with a given gap. The first estimate is different to the estimate due to the Hardy-Littlewood conjecture, so we have to correct the path to get the estimate. However, we comment that $\log(x)^{-1}(1 - \log(x)^{-1})^{d-1}$ gives a good approximation to the average density $\mu(x \mid \mathrm{gap}{=}d)$ (not of $\mu_p(x \mid \mathrm{gap}{=}d)$), on experimental data.

Let m and m' be positive integers. For any $\alpha \in (\mathbb{Z}/m\mathbb{Z})^*$ and any $\beta \in (\mathbb{Z}/m'\mathbb{Z})^*$, we define the average density $\mu_p(x \mid \alpha)$ of prime numbers belonging to α in prime numbers near x, the average density $\mu_p(x \mid \mathrm{np}{\in}\beta)$ of prime numbers whose next prime belongs to β in prime numbers near x, and the average density $\mu_p(x \mid \alpha{\to}\beta)$ of prime numbers whose next prime belongs to β in prime numbers belonging to α near x, as below.

$$\mu_p(x \mid \alpha) = \frac{\sharp(P \cap \alpha \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))},$$
$$\mu_p(x \mid \mathrm{np}{\in}\beta) = \frac{\sharp(P \cap Np(\beta) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))},$$
$$\mu_p(x \mid \alpha{\to}\beta) = \frac{\sharp(P \cap \alpha \cap Np(\beta) \cap N_\delta(x))}{\sharp(P \cap \alpha \cap N_\delta(x))},$$

where $Np(\beta)$ is the set of integers n satisfying $\mathrm{np}(n) \in \beta$. If we assume independence the event on a congruence condition for a given prime p and the event on a congruence condition for the next prime $\mathrm{np}(p)$, then we have

$$\mu_p(x \mid \alpha{\to}\beta) = \mu_p(x \mid \mathrm{np}{\in}\beta).$$

We denote by $\pi_\alpha(x)$ the number of prime numbers belonging to α at most x, by $\pi_{\cdot,\beta}(x)$ the number of prime numbers at most x whose next prime belongs to β, and by $\pi_{\alpha,\beta}(x)$ the number of prime numbers belonging to α at most x whose next prime belongs to β. By the definitions of the average densities, we have

$$\pi_\alpha(x) = \sharp(P \cap \alpha \cap (0,\, x)) \sim \int_2^x \mu_p(t \,|\, \alpha)\, \mu(t)\, dt,$$
$$\pi_{\cdot,\beta}(x) = \sharp(P \cap Np(\beta) \cap (0,\, x)) \sim \int_2^x \mu_p(t \,|\, \mathrm{np}{\in}\beta)\, \mu(t)\, dt,$$
$$\pi_{\alpha,\beta}(x) = \sharp(P \cap \alpha \cap Np(\beta) \cap (0,\, x)) \sim \int_2^x \mu_p(t \,|\, \alpha{\to}\beta)\, \mu_p(t \,|\, \alpha)\, \mu(t)\, dt.$$

The Dirichlet prime number theorem implies that $\pi_\alpha(x)/\pi(x) \sim 1/\varphi(m)$, and $\pi_{\cdot,\beta}(x)/\pi(x) \sim 1/\varphi(m')$, and then

$$\mu_p(x \,|\, \alpha) \sim \frac{1}{\varphi(m)}, \qquad \mu_p(x \,|\, \mathrm{np}{\in}\beta) \sim \frac{1}{\varphi(m')}.$$

We call $\mu_p(x \,|\, \alpha{\to}\beta)$ the distribution of prime numbers with congruence conditions. Our goal is to get approximate expressions of prime distributions $\mu_p(x \,|\, \alpha{\to}\beta)$ and $\pi_{\alpha,\beta}(x)/\pi(x)$, with the evaluated error terms. We have only shown approximate expressions in elementary cases (Theorem 2.2), and conjectured the limit values (Conjecture 2.1) by experimental data.

Conjecture 2.1. *Let m and m' be positive integers. For any irreducible residue class α on modulo m and any irreducible residue class β on modulo m', we have*

$$\lim_{x\to\infty} \mu_p(x \,|\, \alpha{\to}\beta) = \frac{1}{\varphi(m')}, \qquad \lim_{x\to\infty} \frac{\pi_{\alpha,\beta}(x)}{\pi(x)} = \frac{1}{\varphi(m)\varphi(m')}.$$

Theorem 2.2. *The above conjecture holds, in the case that one of m and m' is a power of 2 and the other is not divisible by 4. Furthermore, the error terms are evaluated as much as well-known estimates on the prime number theorem and the Dirichlet's prime number theorem.*

Now, we will give the proof of theorem 2.2. Let m, m', α and β be as in the theorem. We put $m_0 = \mathrm{lcm}(m,\, m')$, $\alpha' = \{a \in \alpha \,|\, (a,\, m_0) = 1\}$, $\beta' = \{b \in \beta \,|\, (b,\, m_0) = 1\}$, and decompose $\alpha' = \alpha_1 \cup \cdots \cup \alpha_s$ and $\beta' = \beta_1 \cup \cdots \cup \beta_t$ into the disjoint union of irreducible residue classes on modulo m_0. We can split $\mu_p(x \,|\, \alpha{\to}\beta)$ to the sum of probabilities of exclusive events.

$$\mu_p(x \,|\, \alpha{\to}\beta) = \sum_{i,j} \mu_p(x \,|\, \alpha_i{\to}\beta_j).$$

We put $D_{i,j} = (\beta_j - \alpha_i) \cap (0,\, \infty) = \{b - a \,|\, a \in \alpha_i,\, b \in \beta_j,\, b > a\}$ for $1 \le i \le s$ and $1 \le j \le t$. For any prime number p belonging to α_i, $p + D_{i,j}$ is the set of all integers belonging to β_j more than p. The event that $\mathrm{np}(p) \in \beta_j$ is the disjoint union of the exclusive events that $\mathrm{gap}(p) = d$ for any $d \in D_{i,j}$. We have

$$\mu_p(x \,|\, \alpha_i{\to}\beta_j) = \sum_{d \in D_{i,j}} \mu_p(x \,|\, \mathrm{gap}{=}d),$$

and then

$$\pi_{\alpha,\beta}(x) = \sum_{i,j} \pi_{\alpha_i,\beta_j}(x) + O(1).$$

The error term of the last equation is evaluated by the number of prime factor of m_0, so it is bounded.

First, we treat the case that m is not divisible by 4 and m' is the power of 2. We put $m_1 = \mathrm{lcm}(m, 2)$. The parallel translation $T(n) = n + m_1$ for any integer n induces the translations on $(\mathbb{Z}/m\mathbb{Z})^*$, $(\mathbb{Z}/m'\mathbb{Z})^*$ and $(\mathbb{Z}/m_0\mathbb{Z})^*$. We write $T(\alpha_i) = \alpha_{i'}$ and $T(\beta_j) = \beta_{j'}$, then $D_{i,j} = D_{i',j'}$. For $x \gg m_0$, we have

$$\mu_p(x \,|\, T(\alpha_i) \to T(\beta_j)) \sim \mu_p(x \,|\, \alpha_i \to \beta_j).$$

The error term on this evaluation formula is able to be evaluated small than any positive numbers to take sufficiently large δ. We choosed δ in anticipation of the error evaluation of the evaluation formula. It follows from $T(\alpha) = \alpha$ and $T(\beta) = T(\beta_1) \cup \cdots T(\beta_t)$ that

$$\begin{aligned} \mu_p(x \,|\, \alpha \to T(\beta)) = \mu_p(x \,|\, T(\alpha) \to T(\beta)) &= \sum_{i,j} \mu_p(x \,|\, T(\alpha_i) \to T(\beta_j)) \\ &\sim \sum_{i,j} \mu_p(x \,|\, \alpha_i \to \beta_j) = \mu_p(x \,|\, \alpha \to \beta). \end{aligned}$$

Since m_1 is congruent to 2 on modulo 4, and m' is the power of 2, the action of the translation T on $(\mathbb{Z}/m'\mathbb{Z})^*$ is cyclic of order $m'/2$, that is equal to the order of $(\mathbb{Z}/m'\mathbb{Z})^*$. The translation T acts transitively on $(\mathbb{Z}/m'\mathbb{Z})^*$. Hence, $\mu_p(x \,|\, \alpha \to \beta)$ does not depend on the choice of β. The sum $\sum_\beta \mu_p(x \,|\, \alpha \to \beta)$ is the probability of whole events, so it is equal to 1. We obtain

$$\mu_p(x \,|\, \alpha \to \beta) \sim \frac{1}{\varphi(m')},$$

and from the Dirichlet's prime number theorem

$$\frac{\pi_{\alpha,\beta}(x)}{\pi(x)} \sim \frac{1}{\varphi(m)\varphi(m')}.$$

The error terms of the above evaluation formulae are given by the estimates of the prime number theorem and the Dirichlet's prime number theorem. Theorem 2.2 holds in the case that m' is the power of 2.

In the case that m is the power of 2, we can show that $\mu_p(x \,|\, \alpha \to \beta)$ does not depend on the choice of α.

$$\sum_\alpha \mu_p(x \,|\, \alpha \to \beta) \sim \varphi(m)\, \mu_p(x \,|\, \alpha \to \beta).$$

On the other hand,

$$\begin{aligned} \mu_p(x \,|\, \alpha \to \beta) &= \frac{\sharp(P \cap \alpha \cap Np(\beta) \cap N_\delta(x))}{\sharp(P \cap \alpha \cap N_\delta(x))} \\ &= \frac{\sharp(P \cap \alpha \cap Np(\beta) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))} \, \frac{\sharp(P \cap N_\delta(x))}{\sharp(P \cap \alpha \cap N_\delta(x))} \\ &\sim \frac{\sharp(P \cap \alpha \cap Np(\beta) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))} \, \varphi(m). \end{aligned}$$

The last evaluation follows from the Dirichlet's prime number theorem. Hence,

$$\mu_p(x\,|\,\alpha\to\beta) \sim \sum_{\alpha} \mu_p(x\,|\,\alpha\to\beta)/\varphi(m) \sim \sum_{\alpha} \frac{\sharp(P\cap\alpha\cap Np(\beta)\cap N_\delta(x))}{\sharp(P\cap N_\delta(x))}$$
$$= \frac{\sharp(P\cap Np(\beta)\cap N_\delta(x))}{\sharp(P\cap N_\delta(x))} = \mu_p(x\,|\,\mathrm{np}\in\beta) \sim \frac{1}{\varphi(m')},$$

and then

$$\frac{\pi_{\alpha,\beta}(x)}{\pi(x)} \sim \frac{1}{\varphi(m)\varphi(m')}.$$

The proof of Theorem 2.2 concluded.

§3. Hardy-Littlewood conjecture

We denote by $\pi_d(x)$ the number of prime numbers p at most x satisfying that $p+d$ is also prime. G. H. Hardy and J. E. Littlewood conjectured as below.

Conjecture 3.1 (Hardy-Littlewood, [4])**.**

$$\pi_d(x) \sim c_d \frac{x}{\log(x)^2} \quad \text{where } c = \prod_{p\geq 3} \frac{p(p-2)}{(p-1)^2} \text{ and } c_d = 2c \prod_{3\leq p\,|\,d} \frac{p-1}{p-2}$$

According to the book of G. H. Hardy-E. M. Wright ([5]), we introduce the deduction leading to the Conjecture 3.1. Let x be a positive real number, and N_x be the product of the prime numbers at most the square root of x. For a positive real number X, we put $S(X)$ the number of the integers at most X prime to N_x. The ratio $S(X)/X$ is the average density of integers prime to N_x at most X. In the case of $X = N_x$, we have $S(N_x)/N_x = \varphi(N_x)/N_x = \prod_{p\leq\sqrt{x}}(1-1/p)$. The right hand side is evaluated by the Mertens's formula.

Theorem 3.2 (Mertens)**.** *Let γ be the Euler constant.*

$$\prod_{p\leq x}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\log(x)}\{1+O(\frac{1}{\log(x)})\}.$$

Then, we have

$$\frac{S(N_x)}{N_x} = \prod_{p\leq\sqrt{x}}(1-\frac{1}{p}) \sim \frac{e^{-\gamma}}{\log(\sqrt{x})} = 2e^{-\gamma}\frac{1}{\log(x)}.$$

If X is a positive multiple of N_x, then the ratio $S(X)/X$ is equal to $S(N_x)/N_x$. We have $S(X)/X \sim 2e^{-\gamma}\log(x)^{-1}$ for any sufficiently large positive real number X. In the case of $X = x$, $S(x)$ is the number of the prime numbers more than $\sqrt{x}$ at most x. We have $S(x) = \pi(x) - \pi(\sqrt{x}) \sim x\log(x)^{-1}$ by the prime number theorem.

$$\frac{S(x)}{x} \sim \frac{1}{\log(x)} \sim \frac{e^{\gamma}}{2}\frac{S(N_x)}{N_x}.$$

The ratio $S(x)/x$ is $e^{\gamma}/2$ $(= 0.89\cdots)$ times than the ratio $S(N_x)/N_x$. The average density of integers prime to N_x at most x is $e^{\gamma}/2$ times that the average density of integers prime to N_x at most N_x.

For a positive real number X and a positive integer d, we put $S_d(X)$ the number of the integers n at most X so that both n and $n+d$ are prime to N_x. Since N_x is square-free, we have

$$S_d(N_x) = \prod_{p \nmid d}{}' (p-2) \prod_{p \mid d} (p-1) = N_x \prod_{p \nmid d}{}' \frac{p-2}{p-1} \prod_{p \leq \sqrt{x}} (1 - \frac{1}{p}),$$

where $\prod'_{p \nmid d}$ means a product over all odd primes at most $\sqrt{x}$ not deviding d. The ratio $S_d(X)/X$ is the average density that both an integer n at most X and $n+d$ are prime to N_x. In case of $X = x$, we consider that the ratio $S_d(x)/x$ is $(e^\gamma/2)^2$ times than the ratio $S(N_x)/N_x$.

$$\frac{S_d(x)}{x} \sim (\frac{e^\gamma}{2})^2 \frac{S(N_x)}{N_x} = (\frac{e^\gamma}{2})^2 \prod_{p \nmid d}{}' \frac{p-2}{p-1} \prod_{p \leq \sqrt{x}} (1 - \frac{1}{p}).$$

It followrs from the Mertens' formula $e^\gamma/2 \sim \prod_{p<\sqrt{x}}(1-1/p)^{-1} \log(x)^{-1}$ that

$$\begin{aligned}
\frac{S_d(x)}{x} &\sim \prod_{p<\sqrt{x}} (1-\frac{1}{p})^{-2} \frac{1}{\log(x)^2} \prod_{p \nmid d}{}' \frac{p-2}{p-1} \prod_{p \leq \sqrt{x}} (1-\frac{1}{p}) \\
&= \frac{1}{\log(x)^2} \prod_{p \nmid d}{}' \frac{p-2}{p-1} \prod_{p<\sqrt{x}} \frac{1}{1-1/p} \\
&= \frac{1}{\log(x)^2} \prod_{3 \leq p \mid d} \frac{p-1}{p-2} \prod_{3 \leq p \leq \sqrt{x}} \frac{p-2}{p-1} \prod_{p<\sqrt{x}} \frac{p}{p-1} \\
&= \frac{1}{\log(x)^2} \prod_{3 \leq p \mid d} \frac{p-1}{p-2} \prod_{3 \leq p < \sqrt{x}} \frac{p(p-2)}{(p-1)^2} \times \frac{2}{2-1} \\
&\sim \frac{1}{\log(x)^2}\, 2 \prod_{3 \leq p \mid d} \frac{p-1}{p-2} \prod_{3 \leq p} \frac{p(p-2)}{(p-1)^2} \\
&= c_d \frac{1}{\log(x)^2}.
\end{aligned}$$

It leads us to $S_d(x) = \pi_d(x) - \pi_d(\sqrt{x})$ that $S_d(x)$ is the number of prime numbers p more that $\sqrt{x}$ at most x satisfying that $p+d$ is also prime. Therefore,

$$\begin{aligned}
\pi_d(x) &= S_d(x) + S_d(\sqrt{x}) + S_d(\sqrt[4]{x}) + S_d(\sqrt[8]{x}) + \cdots \\
&\sim c_d \frac{x}{\log(x)^2}.
\end{aligned}$$

Corollary 3.3. *Suppose that the Conjecrure 3.1 holds, then*

$$\mu(x \mid \text{prime} \wedge \text{gap}{=}d) \sim c_d \log(x)^{-2}.$$

We denote by $\widetilde{\pi}_d(x)$ the number of prime numbers p at most x satisfying that $\text{gap}(p) = d$, for any positive number x and any even integer d. It is easy to show that

$$\widetilde{\pi}_d(x) = \pi_d(x) + O(\frac{x}{\log(x)^{-3}}),$$

from the above argument that lead us to the Hardy-Littlewood conjecture. Hence, Corollary 3.3 is concluded by

$$\widetilde{\pi}_d(x) \sim \int_2^x \mu(t \mid \text{prime} \wedge \text{gap}{=}d)\, dt.$$

§4. Numerical experiments

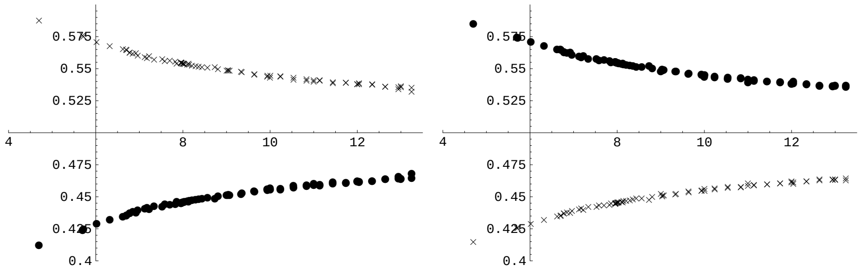
In this section, we give some experimental data and numerical calculations of statistic estimator of $\mu_p(x \,|\, \alpha \to \beta)$, to be expected establishment of the limit value of the prime distribution $\mu_p(x \,|\, \alpha \to \beta)$ with congruence conditions (Conjecture 2.1). In this section, a prime range means a finite set of consecutive prime numbers. We define

$$\varpi_{\alpha,\beta}(R) = \frac{\sharp(R \cap \alpha \cap Np(\beta))}{\sharp R}$$

for any prime range R. We put a prime range R_r consisting of odd prime numbers at most 2^r for each $20 \leq r \leq 28$, and a prime range $R_{s,0}$ consisting of the first ten million prime numbers greater than 2^s for each $20 \leq s \leq 44$, and split $R_{s,0}$ in ten disjoint prime ranges $\{R_{s,i}\}_{1 \leq i \leq 10}$ consisting of one million consecutive prime numbers, for each $20 \leq s \leq 44$. For example

$R_{20} = \{3, 5, 7, \cdots, 1048573\}$	(82024 primes)
$R_{28} = \{3, 5, 7, \cdots, 268435399\}$	(14630842 primes)
$R_{20,0} = \{1048583, \cdots, 180985369\}$	(ten million primes)
$R_{21,0} = \{2097169, \cdots, 182387531\}$	(ten million primes)
$R_{44,0} = \{17592186044423, \cdots, 17592491056057\}$	(ten million primes)
$R_{20,1} = \{1048583, \cdots, 16845919\}$	(one million primes)
$R_{20,2} = \{16845949, \cdots, 33871501\}$	(one million primes)
$R_{20,10} = \{162034127, \cdots, 180985369\}$	(one million primes)

For any even integer $4 \leq m \leq 30$ and any $\alpha, \beta \in (\mathbb{Z}/m\mathbb{Z})^*$, we plot a point whose x-coordinate is the common logarithm of the median of R (i.e. the digit number of a prime number) and y-coordinate is $\varpi_{\alpha,\beta}(R)$ where R runs through above prime ranges. In the case of $m = 4$, we draw two graphs for $\alpha = 1 \pmod 4$ and for $\alpha = 3 \pmod 4$. In the graphs, we take digit numbers on abscissa axis, put the intersection point of axes at the coordinate $(6, 1/\varphi(4))$, and indicate the experimental data of $\beta = 1 \pmod 4$ by small filled circles, and of $\beta = 3 \pmod 4$ by crosses.



These graphs are quite similar. This leads us to $\mu_p(x \,|\, \alpha \to \beta) \sim \mu_p(x \,|\, \beta \to \alpha)$ for any $\alpha, \beta \in (\mathbb{Z}/4\mathbb{Z})^*$. More generally, we can show the following symmetric property by the same way to show Theorem 2.2.

Theorem 4.1. *Let m be the power of 2, For any $\alpha, \beta \in (\mathbb{Z}/m\mathbb{Z})^*$, and any even integer d, we put $\alpha' = \alpha + d$, $\beta' = \beta + d \in (\mathbb{Z}/m\mathbb{Z})^*$. Then we have $\mu_p(x \,|\, \alpha \to \beta) \sim \mu_p(x \,|\, \alpha' \to \beta')$ and $\pi_{\alpha,\beta}(x) \sim \pi_{\alpha',\beta'}(x)$.*

Let m be an even positive integer, and $\alpha,\ \beta \in (\mathbb{Z}/m\mathbb{Z})^*$. We put $D = \{b-a \,|\, a \in \alpha,\ b \in \beta,\ b-a > 0\}$. We define the estimator of the average density $\mu_p(x \,|\, \alpha{\to}\beta)$ as the following three equations. For any positive number x,

$$\mu'_p(x \,|\, \alpha{\to}\beta) = \sum_{d \in D} \mu'_p(x \,|\, \text{gap}{=}d)$$
$$\mu'_p(x \,|\, \text{gap}{=}d) = \mu'_p(x, d) \prod_{1 \le t < d} (1 - \mu'_p(x,\ t))$$
$$\mu'_p(x,\ t) = \begin{cases} r_{t,m}\, c_t\, \mu(x+t) & (\text{if } t+\alpha \in (\mathbb{Z}/m\mathbb{Z})^*) \\ 0 & (\text{otherwise}) \end{cases}$$

where c_t is a Hardy-Littlewood constant defined in §3, and

$$r_{t,m} = \prod_{3 \le p | m,\ p \nmid t} \frac{p-1}{p-2} \quad (= \frac{c_{\gcd(m,t)}}{2c}).$$

We consider that $\mu'_p(x \,|\, \alpha{\to}\beta)$ (resp. $\mu'_p(x \,|\, \text{gap}{=}d)$ and $\mu'_p(x,\ t)$) is an estimator of $\mu_p(x \,|\, \alpha{\to}\beta)$ (reap. $\mu_p(x \,|\, \text{gap}{=}d)$ and $\mu_p(x,\ t)$), as below. The first equation means $\mu'_p(x \,|\, \alpha{\to}\beta)$ to divide into the sum of the exclusive events $\mu'_p(x \,|\, \text{gap}{=}d)$ for every $d \in D$.

The term $c_t\, \mu(x{+}t)$ estimates the average density of prime numbers p near n satisfying that $p{+}t$ is also prime, by the argument leading to Hardy-Littlewood conjecture. Let m_t be the number of irreducible residue classes α on modulo m satisfying that $t+\alpha$ is also irreducible residue. Then, we have

$$m_t = m\,(1-1/2) \prod_{3 \le p|m,\ p|t} (1-1/p) \prod_{3 \le p|m,\ p \nmid t} (1-2/p) = \varphi(m)/r_{t,m}$$

If we assume that prime numbers p satisfying that $p+t$ is also prime are uniformly distributed in possible irreducible residue classes on modulo m, then $\varphi(m)/m_t$ $(= r_{t,m})$ times of $c_t\, \mu(x+t)$ estimates the average density of prime numbers p satisfying that $p+t$ is also prime in prime numbers near x.

For any prime number p near x, suppose that independence on probability events that $p+t$ is prime for any posirive integer t. Then the probability of $\text{gap}(p) = d$ is equal to the product of the probability that $p+d$ is prime and the probabilities that $p+t$ is not prime for $1 \le t \le d-1$. This is a reason to define $\mu'_p(x \,|\, \text{gap}{=}d)$ by the second equation.

The estimator $\mu'_p(x \,|\, \alpha{\to}\beta)$ satisfies the same symmetric property as Theorem 4.1.

Theorem 4.2. *Let m be the power of 2, For any $\alpha,\ \beta \in (\mathbb{Z}/m\mathbb{Z})^*$, and any even integer d, we put $\alpha' = \alpha + d,\ \beta' = \beta + d \in (\mathbb{Z}/m\mathbb{Z})^*$. Then we have $\mu'_p(x \,|\, \alpha{\to}\beta) \sim \mu'_p(x \,|\, \alpha'{\to}\beta')$.*

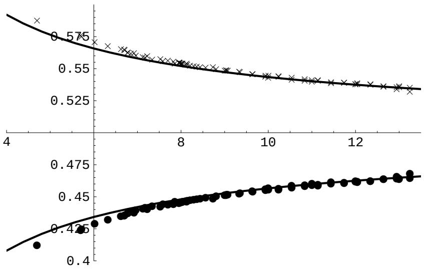
We want to calculate the value of the estimator $\mu'_p(x \,|\, \alpha{\to}\beta)$. The series $\sum_d \mu'_p(x \,|\, \text{gap}{=}d)$ converges slowly for the large x, so we device the summation. We substitute the ratio of sums cut off for a sufficiently large N. Namely, we put

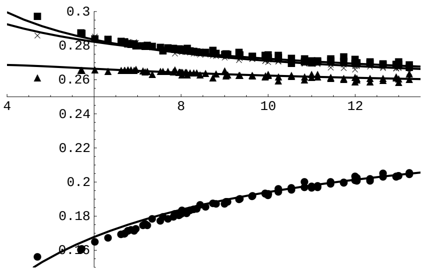
$$\tilde{\mu}'_p(x \,|\, \alpha{\to}\beta) = \frac{\sum_{d \in D_N} \mu'_p(x \,|\, \text{gap}{=}d)}{\sum_{d<N} \mu'_p(x \,|\, \text{gap}{=}d)}$$

where $D_N = \{b - a \,|\, a \in \alpha,\, b \in \beta,\, 0 < b - a < N\}$. Since

$$\sum_{\beta} \mu'_p(x \,|\, \alpha{\to}\beta) = \sum_{d} \mu'_p(x \,|\, \mathrm{gap}{=}d) = 1,$$

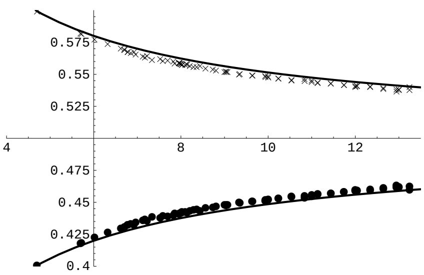
$\tilde{\mu}'_p(x \,|\, \alpha{\to}\beta)$ takes a value close to $\mu'_p(x \,|\, \alpha{\to}\beta)$ for any sufficiently large positive number N. In the scope of our experiments, it is possible to obtain a sufficient accuracy by taking the 100 times of m as N. We will show the graphs to overlap the value of the estimator in the graphs of experimental data, as below, where we put $\mu(x) = \log(x)^{-1}$.

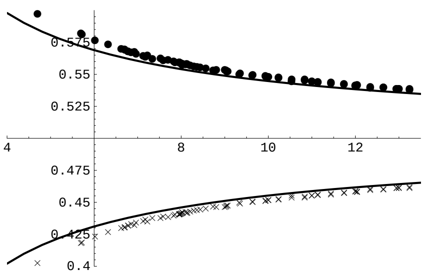




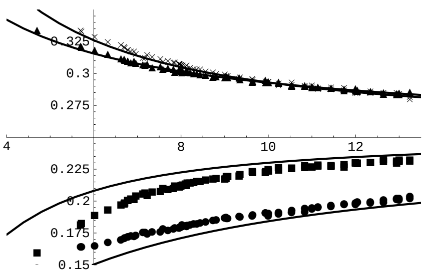
The above right graph is in the case of $\alpha = 1$ (mod 4), and right graph in the case of $\alpha = 1$ (mod 8). Although it is hard to see, the points of experimental data are arranged in order of $\beta = 7$, 3, 5 and 1 (mod 8) from the top. The experimental data of $\beta = 7$ and $\beta = 3$ (mod 8) are almost same values, and the points corresponding to them on the graph overlap considerably. The estimators is also arranged in the same order on β, and located in proximity to the experimental data. The graphs for another irreducible residue classes on modulo 4 and 8 are essentially the same as these graphs respectively, due to the symmetric properties (Theorem 4.1, Theorem 4.2).

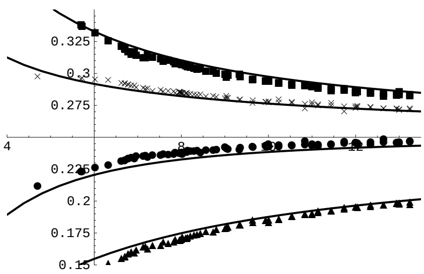
The next graphs are in the case of $\alpha = 1$ (mod 6) and $\alpha = 5$ (mod 6).

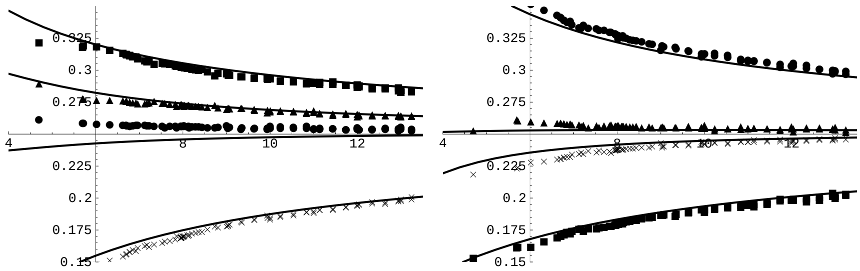




The following 4 graphs are in case of $\alpha = 1$, 3, 7, and 9 on modulo 10.

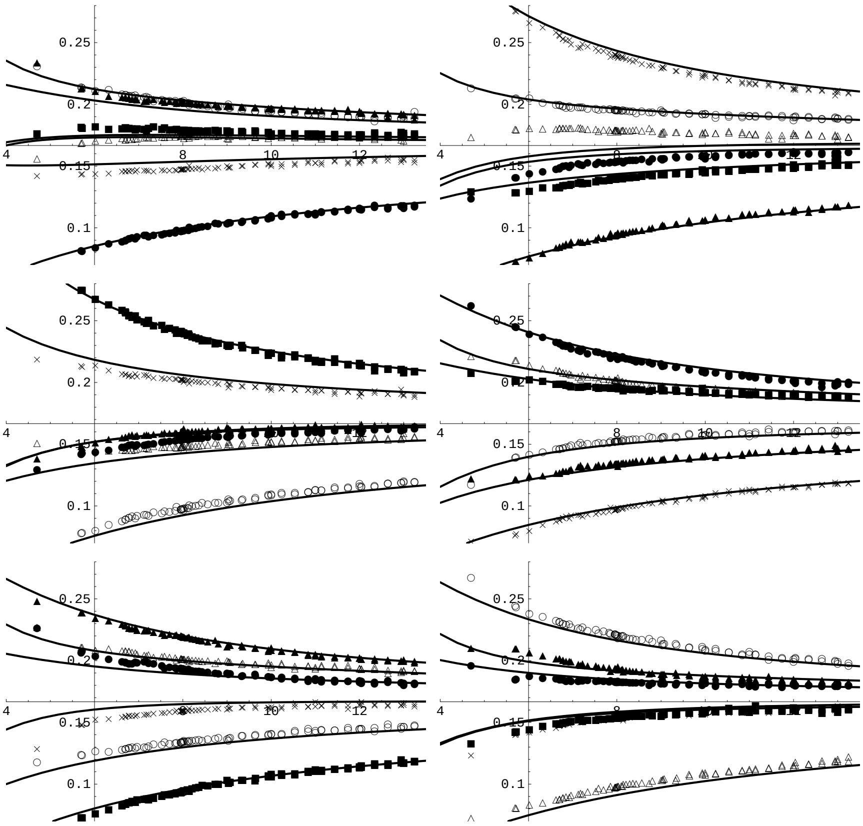




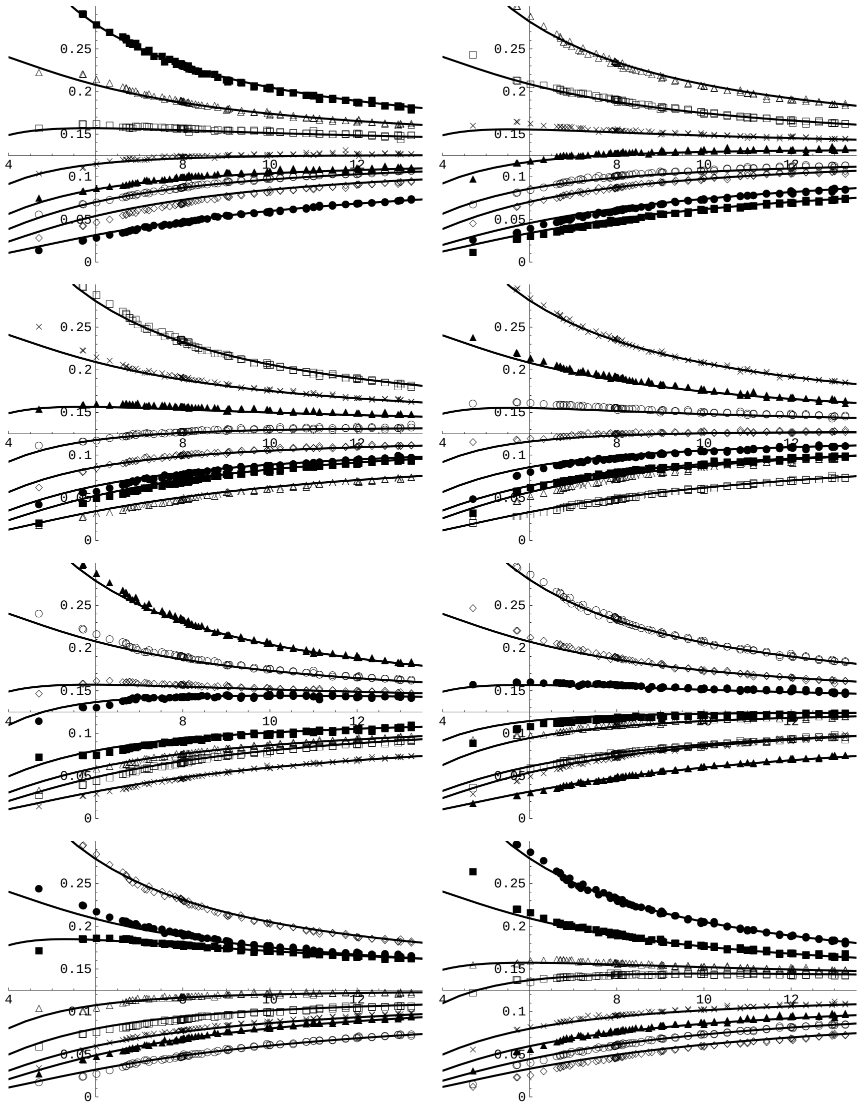


Any symmetric property is not seen in the graphs. However the estimators are located in proximity to the experimental data, and are a faithful reproduction of the bias of the prime distribution with congruence conditions.

The following graphs are in case of $\alpha = 1$, 3, 5, 9, 11 and 13 on modulo 14.



Finally, we draw eight graphs in case of $\alpha = 1$, 7, 11, 13, 17, 19, 23 and 29 on modulo 30 as below.



In all above graphs, the estimator $\mu'_p(x \mid \alpha \to \beta)$ is approximate the experimental data very well.

Conjecture 4.3. *Let m be an even positive integer, and α, $\beta \in (\mathbb{Z}/m\mathbb{Z})^*$. Then we have $\mu_p(x \mid \alpha \to \beta) \sim \mu'_p(x \mid \alpha \to \beta)$ and*

$$\pi_{\alpha,\beta}(x) \sim \int_2^x \mu'_p(t \mid \alpha \to \beta)\, \mu_p(t \mid \alpha)\, \mu(t)\, dt.$$

We can calculate numerically the limit of the estimator $\mu'_p(x \mid \alpha \to \beta)$, for an arbitrary even positive integer m and α, $\beta \in (\mathbb{Z}/m\mathbb{Z})^*$. In many numerical calculation, the limit of the estimator is equal to the expected value.

Conjecture 4.4. *Let m be an even positive integer, and α, $\beta \in (\mathbb{Z}/m\mathbb{Z})^*$. Then* *we have*

$$\lim_{x\to\infty} \mu'_p(x \mid \alpha{\to}\beta) = \frac{1}{\varphi(m)}$$

In generally, we give a conjecture of the limit of prime distributions with congruence cognitions with different moduli.

Conjecture 4.5. *Let m and m' be even positive integers, and $\alpha \in (\mathbb{Z}/m\mathbb{Z})^*$, $\beta \in (\mathbb{Z}/m'\mathbb{Z})^*$. Then we have*

$$\lim_{x\to\infty} \mu_p(x \mid \alpha{\to}\beta) = \frac{1}{\varphi(m')}$$
$$\lim_{x\to\infty} \frac{\pi_{\alpha,\beta}(x)}{\pi(x)} = \frac{1}{\varphi(m)\varphi(m')}$$

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