Calculation and Visualization of 3-braids

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0 Contents

ThreeBraidClass: a python module for 3-braids

1 3-braids in ThreeBraidClass

In the **ThreeBraidClass** module...

3-braid is given by

AB-word, σ-word, ST-word, $\mathrm{PSL}_2(\mathbb{Z})$,
bdpq-word, RL-word, Syzygy sequence

$$B_3 = \text{the 3-strand Artin braid group}$$
$$= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

$$\mathcal{B}_3 = B_3/Z(B_3) \simeq \mathrm{PSL}_2(\mathbb{Z})$$
$$Z(B_3) = \langle (\sigma_1\sigma_2)^3 \rangle \simeq \mathbb{Z}$$

$$\mathcal{B}_3 = \langle \bar{\boldsymbol{\sigma}}_1, \bar{\boldsymbol{\sigma}}_2 \mid \cdots \rangle \qquad (\bar{\sigma}_i : \text{ the image of } \sigma_i)$$

$$\bar{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \bar{\sigma}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\simeq \mathbf{PSL_2(\mathbb{Z})}$$

$$= \langle \boldsymbol{S}, \boldsymbol{T} \mid \boldsymbol{S}^2 = (\boldsymbol{ST})^3 = \mathbf{1} \rangle$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \langle \boldsymbol{A}, \boldsymbol{B} \mid \boldsymbol{A}^2 = \boldsymbol{B}^3 = \mathbf{1} \rangle \simeq \mathbb{Z}/\mathbf{2}\mathbb{Z} * \mathbb{Z}/\mathbf{3}\mathbb{Z}$$

$$\bar{\sigma}_1 = STS = BBA \qquad \bar{\sigma}_1^{-1} = TST = AB$$

$$\bar{\sigma}_2 = T = ABB \qquad \bar{\sigma}_2^{-1} = STSTS = BA$$

$$A = S = \bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_1 \qquad B = STST = \bar{\sigma}_1\bar{\sigma}_2$$

Modular groups:

$\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \quad (\simeq \mathcal{B}_3)$

$\Gamma(2) = \{M \in \Gamma \mid M \equiv 1 \pmod 2\}$

$$\mathrm{PSL}_2(\mathbb{Z})/\Gamma(2) \simeq S_3$$

$\bar{\Gamma}^2$: the index 2 subgroup of Γ

(the equianharmonic modular group)

$$\Gamma \rhd \bar{\Gamma}^2 \rhd \Gamma(2)$$

$$\bar{\Gamma} = \langle \bar{\sigma}_1 \bar{\sigma}_2 \rangle * \langle \bar{\sigma}_2 \bar{\sigma}_1 \rangle \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

$$\Gamma = \bar{\Gamma}^2 \cup \bar{\Gamma}^2 A$$

$$\bar{\Gamma}^2 = \{\mathbf{id}, \mathbf{b}, \mathbf{p}\} * \{\mathbf{id}, \mathbf{d}, \mathbf{q}\} \quad \simeq \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

$$\mathbf{b} = \bar{\sigma}_2^{-1}\bar{\sigma}_1^{-1} = BB \qquad \mathbf{p} = \bar{\sigma}_1\bar{\sigma}_2 = B$$

$$\mathbf{d} = \bar{\sigma}_1^{-1}\bar{\sigma}_2^{-1} = ABBA \qquad \mathbf{q} = \bar{\sigma}_2\bar{\sigma}_1 = ABA$$

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \mathbf{R}, \mathbf{L} \mid \cdots \rangle$$

$$\mathbf{R} = \bar{\sigma}_2 = ABB \qquad \mathbb{R} = \bar{\sigma}_2^{-1} = BA$$

$$\mathbf{L} = \bar{\sigma}_1^{-1} = AB \qquad \mathbb{L} = \bar{\sigma}_1 = BBA$$

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{R} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mathbb{L} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

2 Syzygy sequences

3-braid $=$ a collision-free motion of triangles

Shape function of triangles (ordered triples):

$$\mathrm{Conf}^3(\mathbb{C}) = \{(a,b,c) \in \mathbb{C}^3 \mid a \neq b \neq c \neq a\}$$

For $\Delta = (a,b,c) \in \mathrm{Conf}^3(\mathbb{C})$,

$$\psi_j(\Delta) = \tfrac{1}{3}(a + b\omega^{-j} + c\omega^{j}) \quad (j = 0,1,2)$$

$$\omega = \exp(2\pi i/3)$$

$$\Psi(\Delta) = (\psi_0(\Delta), \psi_1(\Delta), \psi_2(\Delta))$$

$$\psi(\Delta) = \frac{\psi_2(\Delta)}{\psi_1(\Delta)} \in \mathbb{P}(\mathbb{C})$$: **the shape** of Δ

$\boldsymbol{B_3}$ and $\boldsymbol{\mathcal{B}_3}$ as fundamental groups:

S_3 and $\mathbb{C}^\times$ ast on $\mathrm{Conf}^3(\mathbb{C})$

$$\mathrm{Conf}^3(\mathbb{C}) \overset{\hookleftarrow}{\twoheadrightarrow} C(\Psi) : \Delta \mapsto \begin{pmatrix} \psi_2(\Delta) \\ \psi_1(\Delta) \end{pmatrix}$$

$\rightarrow S_3$ and $\mathbb{C}^\times$ ast on $C(\varPsi)$

$$C(\varPsi) = \{ \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \in \mathbb{C}^2 \mid \psi_1^3 \neq \psi_1^3 \}$$

$$C(\Psi)/\mathbb{C}^\times \xrightarrow{\sim} \mathbb{P}^1_\psi(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$$

$$\begin{pmatrix} \psi_2 \\ \psi 1 \end{pmatrix} \quad \mapsto \quad \frac{\psi_2}{\psi_1}$$

the shape (the similarity class) of triangles

We call $\mathbb{P}^1_\psi(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$ **the shape sphere**.

$$\begin{array}{ccccc} \mathrm{Conf}^3(\mathbb{C}) & \to & C(\Psi) & \to & \mathbb{P}^1_\psi(\mathbb{C}) \setminus \{1, \omega, \omega^2\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Conf}^3(\mathbb{C})/S_3 & \to & C(\Psi)/S_3 & \to & \mathbb{P}^1_{\infty 23} \end{array}$$

$\mathbb{P}^1_{\infty 23}$: the orbifold quotient

∞ $\cdots$ one missing point

$2\,3$ $\cdots$ two elliptic point of order 2, 3

$$B_3 \simeq \pi_1(\mathrm{Conf}^3(\mathbb{C})/S_3) = \pi_1(C(\Psi)/S_3)$$

$$\mathcal{B}_3 \simeq \pi_1^{\mathrm{orb}}(\mathbb{P}^1_{\infty 23}, *)$$

Motion of triangles

Let $\Delta(t) \in \mathrm{Conf}^3(\mathbb{C})$ be smooth on $t_0 \leq t \leq t_1$
with $\{\Delta(t_0)\} = \{\Delta(t_1)\}$
where $\{\Delta\}$ means the set of vertices of Δ.

The motion of $\Delta(t)$ forms a 3-braid in $\mathbb{C} \times \mathbb{R}$.

$\gamma_\Delta = \{\psi(\Delta(t))\}_t \ \subset \mathbb{P}^1_\psi(\mathbb{C}) \setminus \{1, \omega, \omega^2\}$
the corresponding path on the shape sphere

$\bar{\gamma}_\Delta$ (the image of γ_Δ) $\in \pi_1^{\mathrm{orb}}(\mathbb{P}^1_{\infty 23}) \simeq \mathcal{B}_3$
the loop on the orbifold $\mathbb{P}^1_{\infty 23}$

What is syzygy?

syzygy = a state in which at least three points are aligned on a straight line

(Syzygy is an astronomical term. Similar astronomical phenomena include eclipse, conjunction, and opposition.)

$\Delta = (a, b, c) \in \mathrm{Conf}^3(\mathbb{C})$ is **a syzygy**

$\Leftrightarrow$ $\psi(\Delta)$ is on the equator of the shape sphere

(i.e. $|\psi(\Delta)| = 1$)

1, ω, ω^2 divide the equator into 3 sections:

$[0]$ = the arc from 1 to ω : $\boldsymbol{a\,b\,c}$

$[1]$ = the arc from ω to ω^2 : $\boldsymbol{c\,a\,b}$

$[2]$ = the arc from ω^2 to 1 : $\boldsymbol{b\,c\,a}$

Syzygy sequence of the motion:

Let $\Delta(t) \in \mathrm{Conf}^3(\mathbb{C})$ $(t_0 \le t \le t_1)$ be smooth with $\{\Delta(t_0)\} = \{\Delta(t_1)\}$ is a syzygy. ※

$K := \{t \in [t_0, t_1] \mid |\psi(\Delta(t))| = 1\}$

Assume that $\sharp \boldsymbol{K} < \infty$.

$t_0 = k_0 < k_1 < \cdots < k_r = t_1$: all members of K

$\{s_j\}_j \in (\mathbb{Z}/3\mathbb{Z})^{r+1}$ s.t. $\psi(\Delta(k_j)) \in [s_j]$ (for $^\forall j$)

$\mathcal{S}_\Delta = \{s_j\}_j$: **the syzygy sequence** of $\Delta(t)$

Rem. $s_j = \left\lfloor \frac{3}{2\pi} \arg(\psi(\Delta(k_j))) \right\rfloor \pmod 3$

3 Lissajous 3-braids

Let $m, n \in \mathbb{Z}$, $\varphi \in \mathbb{R}$ with

$$m \equiv n \equiv 1 \pmod 3 \qquad (\gcd(m,n) = 1)$$

Lissajous curve of type $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{\varphi})$: $(t \in \mathbb{R}/\mathbb{Z})$

$$L(t) = A \sin(2\pi m t) + i\, B \sin(2\pi n t + \varphi)$$

$$\Delta_{m,n,\varphi}(t) := (L(t - 1/3), L(t), L(t + 1/3))$$

Lissajous (toric) 3-braid:

the motion of $\Delta_{m,n,\varphi}(t)$ $(t \in \mathbb{R}/\mathbb{Z})$

in $\mathbb{C} \times \mathbb{R}/\mathbb{Z}$ or in $\mathbb{C} \times \mathbb{R}$

Rem.

Kin-Nakamura-O. studied the Lissajous 3-braids ([KNO], [KNO2]). The first purpose of ThreeBraidClass module was to efficiently compute examples of Lissajous 3-braids. This module provides methods for outputting computation results for the purpose. One can get them by specifying 'KNO' as an argument to the 'show()' method of the LissajousBraid class in ThreeBraidClass module.

Rem.

In the case of a Lissajous 3-braid, the corresponding path $\gamma_{\Delta_{m,n,\varphi}}$ is a loop on the shape sphere. And the syzygy sequence $\mathcal{S}_{\Delta_{m,n,\varphi}}$ should be treated as a circular word.

Prop. Put $\ell = \dfrac{m-n}{3} \in \mathbb{Z}$.

The length of $\mathcal{S}_{\Delta_{m,n,\varphi}} = 6\ell$

$^{\exists}\alpha, \beta \in \mathbb{R}$ s.t. $\mathcal{S}_{\Delta_{m,n,\varphi}} = \{\lfloor \alpha j + \beta \rfloor \pmod 3\}_j$

Rem. We may take $\alpha \equiv \dfrac{-n}{|\ell|} \pmod{3\mathbb{Z}}$.

Linear 3-braid:

br : 3-braid (= the motion of $\Delta(t) \in \mathrm{Conf}^3(\mathbb{C})$)

$\mathcal{S}_{\mathbf{br}} := \mathcal{S}_\Delta$: the syzygy sequence of **br**

The extension of syzygy sequence:

extend finite length syzygy sequence $\boldsymbol{S}$

to infinite length syzygy sequence $\tilde{\boldsymbol{S}}$

explain with examples: (connect **red** to **blue**)

$(\mathbf{0}\cdots\mathbf{1})$–$(\mathbf{0}\cdots\mathbf{1}) \to (\mathbf{0}\cdots\mathbf{1})$–$(\mathbf{1}\cdots\mathbf{2}) \to (\mathbf{0}\cdots 1\cdots\mathbf{2})$

$(\mathbf{0}\cdots\mathbf{1})$–$(\mathbf{0}\cdots\mathbf{1}) \to (\mathbf{2}\cdots\mathbf{0})$–$(\mathbf{0}\cdots\mathbf{1}) \to (\mathbf{2}\cdots\mathbf{0}\cdots\mathbf{1})$

align connect

$\tilde{\mathcal{S}}_{\mathbf{br}}$: the extension of $\mathcal{S}_{\mathbf{br}}$

br is **linear**

$\overset{\text{def}}{\Longleftrightarrow}$ $^\exists \alpha, \beta \in \mathbb{R}$ s.t.

$$\tilde{\mathcal{S}}_{\mathbf{br}} = \{\lfloor \alpha j + \beta \rfloor \pmod 3\}_{j \in \mathbb{Z}}$$

Theorem (O.)

- Each Lissajous 3-braid is linear.
- For any linear 3-braid **br**,
  there exists a Lissajous type (m, n, φ) s.t.

$$\tilde{\mathcal{S}}_{\mathbf{br}} = \tilde{\mathcal{S}}_{\Delta_{m,n,\varphi}}$$

4 Drawing 3-braids

Aim: Draw 3D images of 3-braids given by
AB-words, σ-words, $\mathrm{PSL}_2(\mathbb{Z})$, ...

- In the case of a Lissajous 3-braid:
  use the Lissajous curve as the frame ...

- In the general case:
  [step 1] construct a path on the shape sphere
  corresponding to a given 3-braid
  [step 2] find ordered triples with given shapes

[step 1] construct a path on the shape sphere

br : a 3-braid given by a AB-word, a σ-word ...

$\mathcal{S}_{\textbf{br}} = \{s_j\}_j$: the syzygy seq. of **br** (length $r+1$)

γ : the corresponding path on the shape sphere

The poler coodinate:

$$\gamma(t) = \boldsymbol{R(t)} \exp(\frac{2\pi i}{3}\boldsymbol{\theta(t)})$$

(1) $\sharp\{t \in [0,1] \mid |R(t)| = 1\} = r+1$

(2) $k_0 = 0 < \cdots < k_r = 1$ s.t. $|R(k_j)| = 1$

$$\lfloor \theta(k_j) \rfloor \pmod 3 = s_j \quad \text{(for } {}^\forall j\text{)}$$

(1) Put $\boldsymbol{R(t) = \frac{2+\sin(\pi r t)}{2-\sin(\pi r t)}}$

then $\{t \in \mathbb{R}/\mathbb{Z} \mid \ |R(t)| = 1\} = \frac{1}{r}\mathbb{Z}/\mathbb{Z}$

(2) $\{u_j\}_j \in \mathbb{Z}^{r+1}$: a lift of $\mathcal{S}_{\mathbf{br}}$

$u_j \pmod 3 = s_j$ (for $^\forall j$)

$$f_{\mathbf{br}}(t) = \alpha t + \beta + \sum_h a_h \sin(2\pi h t) + a'_h \cos(2\pi h t)$$

the Fourier expansion ($t \in [0,1]$)

of the line graph through $\{(j/r, u_j)\}_j$

truncate:

$$f_{\mathbf{br}}^{(k)}(t) = \alpha t + \beta + \sum_{h \le k} a_h \sin(2\pi ht) + a'_h \cos(2\pi ht)$$

error evaluation:

$$\mathrm{err}_k = \max\{|f_{\mathbf{br}}^{(k)}(j/r) - u_j| \mid 0 \le j < r\}$$

Take $k_0 \in \mathbb{N}$ satisfying that $\boldsymbol{\mathrm{err}_{k_0} < \frac{1}{2}}$

and put $\boldsymbol{\theta_{\mathbf{br}}(t) = f_{\mathbf{br}}^{(k_0)}(t) + \frac{1}{2}}$

Then $\lfloor \theta_{\mathbf{br}}(j/r) \rfloor = u_j$ (for $\forall j \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$)

[step 2] find ordered triples with given shapes

Find $\Delta_{\mathbf{br}}(t) \in \mathrm{Conf}^3(\mathbb{C})$ s.t.
$\psi(\Delta_{\mathbf{br}}(t)) = \gamma_{\mathbf{br}}(t)$.

$$\begin{aligned}\gamma_{\mathbf{br}}(t) &= R(t)\exp(\frac{2\pi i}{3}\theta_{\mathbf{br}}(t)) \\ &= \frac{(2+\sin(\pi r t))\exp(\ \ \frac{\pi i}{3}\theta_{\mathbf{br}}(t))}{(2-\sin(\pi r t))\exp(-\frac{\pi i}{3}\theta_{\mathbf{br}}(t))}\end{aligned}$$

$$\psi(\Delta_{\mathbf{br}}(t)) = \frac{\psi_2(\Delta_{\mathbf{br}}(t))}{\psi_1(\Delta_{\mathbf{br}}(t))}$$

We may solve the following linear equation:

$$\psi_0(\Delta_{\mathsf{br}}(t)) = 0$$

$$\psi_1(\Delta_{\mathsf{br}}(t)) = (2 - \sin(\pi r t)) \exp(-\frac{\pi i}{3}\theta_{\mathbf{br}}(t))$$

$$\psi_2(\Delta_{\mathsf{br}}(t)) = (2 + \sin(\pi r t)) \exp(\quad \frac{\pi i}{3}\theta_{\mathbf{br}}(t))$$

Theorem (O.)

$\{\psi(\Delta_{\mathsf{br}}(t))\}_{t\in\mathbb{R}}$ $(= \{\gamma_{\mathsf{br}}(t)\}_{t\in\mathbb{R}})$ is a loop on the shape sphere following the extention of the syzygy sequence of **br** through the equator.

5 Demonstration

6 References

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