Congluence Conditional Prime Distributions

Hiroyuki Ogawa

Department of Mathematics, Osaka University

§1. Average Densities

The average density of prime numbers near x $(x \gg 0)$

$$\mu(x) = \frac{\sharp(P \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))}$$

where P : the set of primes

$N_\delta(x)$: δ-neighbourhood of x

Rem $x \gg \delta \gg 0$

e.g. $\sharp(P \cap N_\delta(x)) = 10^6$ for numerical experiments.

Prime Number Theorem

$$\mu(x) \sim \frac{1}{\log(x)}$$

The average densities on prime gaps

$$\mu(x \,|\, \mathrm{gap}{=}d) = \frac{\sharp(\mathbb{N} \cap \mathrm{Gap}(d) \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))}$$

$$\mu_p(x \,|\, \mathrm{gap}{=}d) = \frac{\sharp(P \cap \mathrm{Gap}(d) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))}$$

$$\mu(x \,|\, \mathrm{prime} \wedge \mathrm{gap}{=}d) = \frac{\sharp(P \cap \mathrm{Gap}(d) \cap N_\delta(x))}{\sharp(\mathbb{N} \cap N_\delta(x))}$$

where $\mathrm{Gap}(d) = \{n | \mathrm{gap}(n)(= \mathrm{np}(n) - n) = d\}$

$\mathrm{np}(n) =$ the next prime of n

$=$ the smallest prime number $> n$

Rem $\mu(x \,|\, \mathrm{prime} \wedge \mathrm{gap}{=}d) = \mu(x)\, \mu_p(x \,|\, \mathrm{gap}{=}d)$

Rem If assume independence on $\mu(x)$, $\mu(x+1)$, $\cdots$, then

$$\begin{aligned}\mu_p(x \,|\, \mathrm{gap}{=}d) &= \mu(x \,|\, \mathrm{gap}{=}d) \\ &= \mu(x+d) \prod_{1 \le t < d} (1 - \mu(x+t))\end{aligned}$$

Prime Number Theorem

$$\mu(x) \sim \frac{1}{\log(x)}$$

leads to

$$\mu(x \,|\, \mathrm{gap}{=}d) \sim \frac{1}{\log(x)}$$

$$\mu_p(x \,|\, \mathrm{gap}{=}d) \sim \frac{1}{\log(x)} \quad (\leftarrow \text{ wrong estimate!})$$

§2. Hardy-Littlewood Conjecture

Hardy-Littlewood Conjecture (*H-L conj*)

$$\pi_d(x) = \sharp\{p < x \,|\, p \text{ and } p+d \text{ are prime}\}$$

$$\sim c_d \, \frac{x}{\log(x)^2}$$

where $\quad c_d = 2\,c \displaystyle\prod_{3 \le p | d} \frac{p-1}{p-2}$ (Hardy-Littlewood constant)

$$c = \prod_{p \ge 3} \frac{p(p-2)}{(p-1)^2} = 0.66016\cdots$$

Rem Value of Hardy-Littlewood Constants c_d

$$c_2 = 1.320 \qquad c_4 = 1.320 \qquad c_6 = 2.640 \qquad c_8 = 1.320$$
$$c_{10} = 1.760 \qquad c_{12} = 2.640 \qquad c_{14} = 1.584 \qquad c_{16} = 1.320$$
$$c_{18} = 2.640 \qquad c_{20} = 1.760 \qquad c_{22} = 1.467 \qquad c_{24} = 2.640$$
$$c_{26} = 1.440 \qquad c_{28} = 1.584 \qquad c_{30} = 3.520 \qquad c_{32} = 1.320$$

$$c_{30} = c_{2\cdot 3\cdot 5} = 3.520 \qquad c_{210} = c_{2\cdot 3\cdot 5\cdot 7} = 4.225$$
$$c_{2310} = c_{2\cdot 3\cdot 5\cdot 7\cdot 11} = 4.694 \qquad c_{30030} = c_{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13} = 5.121$$

Cor $\mu_p(x|\mathrm{gap}{=}d) \sim \dfrac{c_d}{\log(x)}$ (under H-L conj)

§3. Average Densities — Congluence Conditional

$\alpha \in (\mathbb{Z}/m\mathbb{Z})^*$, $\beta \in (\mathbb{Z}/m'\mathbb{Z})^*$ (m, $m' \in \mathbb{N}$, even)

The average densities with congluence conditions

$$\mu_p(x \,|\, \alpha) = \frac{\sharp(P \cap \alpha \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))}$$

$$\mu_p(x \,|\, \mathrm{np}{\in}\beta) = \frac{\sharp(P \cap \mathrm{Np}(\beta) \cap N_\delta(x))}{\sharp(P \cap N_\delta(x))}$$

$$\boxed{\mu_p(x \,|\, \alpha{\to}\beta) = \frac{\sharp(P \cap \alpha \cap \mathrm{Np}(\beta) \cap N_\delta(x))}{\sharp(P \cap \alpha \cap N_\delta(x))}}$$

where $\mathrm{Np}(\beta) = \{n | \mathrm{np}(n) \in \beta\}$

$$\pi(x) = \sharp(P \cap (0,\, x)) \sim \sum_{2 \le t < x} \mu(t) \sim \int_2^x \mu(t)dt$$

$$\begin{aligned}\pi(x \,|\, \alpha) &= \sharp(P \cap \alpha \cap (0,\, x)) \\ &\sim \sum_{2 \le t < x} \mu_p(t \,|\, \alpha)\mu(t) \sim \int_2^x \mu_p(t \,|\, \alpha)\mu(t)dt\end{aligned}$$

$$\begin{aligned}\pi(x \,|\, \mathrm{np}{\in}\beta) &= \sharp(P \cap \mathrm{Np}(\beta) \cap (0,\, x)) \\ &\sim \sum_{2 \le t < x} \mu_p(t \,|\, \mathrm{np}{\in}\beta) \sim \int_2^x \mu_p(t \,|\, \mathrm{np}{\in}\beta)\mu(t)dt\end{aligned}$$

$$\begin{aligned}\pi(x \,|\, \alpha{\to}\beta) &= \sharp(P \cap \alpha \cap \mathrm{Np}(\beta) \cap (0,\, x)) \\ &\sim \sum_{2 \le t < x} \mu_p(t \,|\, \alpha{\to}\beta)\mu_p(t \,|\, \alpha)\mu(t) \\ &\sim \int_2^x \mu_p(t \,|\, \alpha{\to}\beta)\mu_p(t \,|\, \alpha)\mu(t)dt\end{aligned}$$

Dirichlet's Prime Number Theorem

$$\mu_p(x \mid \alpha) \sim \frac{1}{\varphi(m)} \qquad \left(\pi(x \mid \alpha) \sim \frac{1}{\varphi(m)}\pi(x)\right)$$

Rem $\pi(x \mid \mathrm{np}{\in}\beta) \sim \frac{1}{\varphi(m')}\pi(x), \quad \mu_p(x \mid \mathrm{np}{\in}\beta) \sim \frac{1}{\varphi(m')}$

Conjecture (Low of Large Numbers)

$$\mu_p(x \mid \alpha{\to}\beta) \sim \frac{1}{\varphi(m')}$$

Theorem $m = 2^\sharp$, $4 \nmid m'$ or $4 \nmid m$, $m' = 2^\sharp$

$\Longrightarrow$ The conjecture holds.

And the error term is evaluated as much as well-known estimates on prime number theorems.

§4. Estimator

$\alpha,\ \beta \in (\mathbb{Z}/m\mathbb{Z})^*$ $\quad m \in \mathbb{N}$: even

$$\mu'_p(x \,|\, \alpha \to \beta) = \sum_{d \in D} \mu'_p(x \,|\, \mathrm{gap}{=}d)$$

$$\mu'_p(x \,|\, \mathrm{gap}{=}d) = \mu'_p(x,\ d) \prod_{1 \le t < d} (1 - \mu'_p(x,\ t))$$

$$\mu'_p(x,\ t) = \begin{cases} r_{t,m}\, c_t\, \mu(x+t) & \text{(if } t + \alpha \in (\mathbb{Z}/m\mathbb{Z})^*\text{)} \\ 0 & \text{(otherwize)} \end{cases}$$

where $\quad D = \{b - a \,|\, a \in \alpha,\ b \in \beta,\ b - a > 0\}$

$c_t = 2\,c \displaystyle\prod_{3 \le p | t} \frac{p-1}{p-2}$: Hardy-Littlewood constant

$$r_{t,m} = \prod_{3 \le p | m,\ p \nmid t} \frac{p-1}{p-2}$$

Rem $\varphi(m)/r_{t,m} = \sharp\{\alpha \in (\mathbb{Z}/m\mathbb{Z})^* | t + \alpha \in (\mathbb{Z}/m\mathbb{Z})^*\}$

Rem $r_{t,m}\, c_t = c_m\, r_{m,t}$

Theorem (Low of Large Numbers on Estimator)

$$\lim_{x \to \infty} \mu'_p(x \,|\, \alpha \to \beta) = \frac{1}{\varphi(m)}$$

§5. Numerical Experiments

α, $\beta \in (\mathbb{Z}/m\mathbb{Z})^*$ $\quad m = 4$, 8, 6, 10, 14

prime range = a set of consective primes

$$\varpi(R \,|\, \alpha{\to}\beta) = \frac{\sharp(R \cap \alpha \cap \mathrm{Np}(\beta))}{\sharp R} \quad (R : \text{a prime range})$$

Prime Ranges (Sample for experiments) :

R_r : odd primes $< 2^r$ $\ (20 \leq r \leq 28)$

$R_{s,0}$: the first 10^7 primes $> 2^s$ $\ (20 \leq s \leq 44)$

$R_{s,0} = R_{s,1} \cup R_{s,2} \cup \cdots \cup R_{s,10}$ (disjoint) 10^6 primes

Data (Prime Ranges)

$$R_{20} = \{3,\, 5,\, 7,\, \cdots,\, 1048573\} \qquad (82024)$$

$$\vdots$$

$$R_{28} = \{3,\, 5,\, 7,\, \cdots,\, 268435399\} \qquad (14630842)$$

$$R_{20,0} = \{1048583,\, \cdots,\, 180985369\} \qquad (10^7)$$

$$R_{21,0} = \{2097169,\, \cdots,\, 182387531\} \qquad (10^7)$$

$$\vdots$$

$$R_{44,0} = \{17592186044423,\, \cdots,\, 17592491056057\} \qquad (10^7)$$

$$R_{20,1} = \{1048583,\, \cdots,\, 16845919\} \qquad (10^6)$$

$$R_{20,2} = \{16845949,\, \cdots,\, 33871501\} \qquad (10^6)$$

$$\vdots$$

$$R_{20,10} = \{162034127,\, \cdots,\, 180985369\} \qquad (10^6)$$

Table	$\mu_p(x \,|\, \alpha \to \beta)$	:	average density
	$\mu'_p(x \,|\, \alpha \to \beta)$	:	estimator
	$\varpi(R \,|\, \alpha \to \beta)$	:	experimantal data

Conjecture $\quad \mu_p(x \,|\, \alpha \to \beta) = \mu'_p(x \,|\, \alpha \to \beta)$

Compare $\varpi(R \,|\, \alpha \to \beta)$ *and* $\mu'_p($*median of* $R \,|\, \alpha \to \beta)$

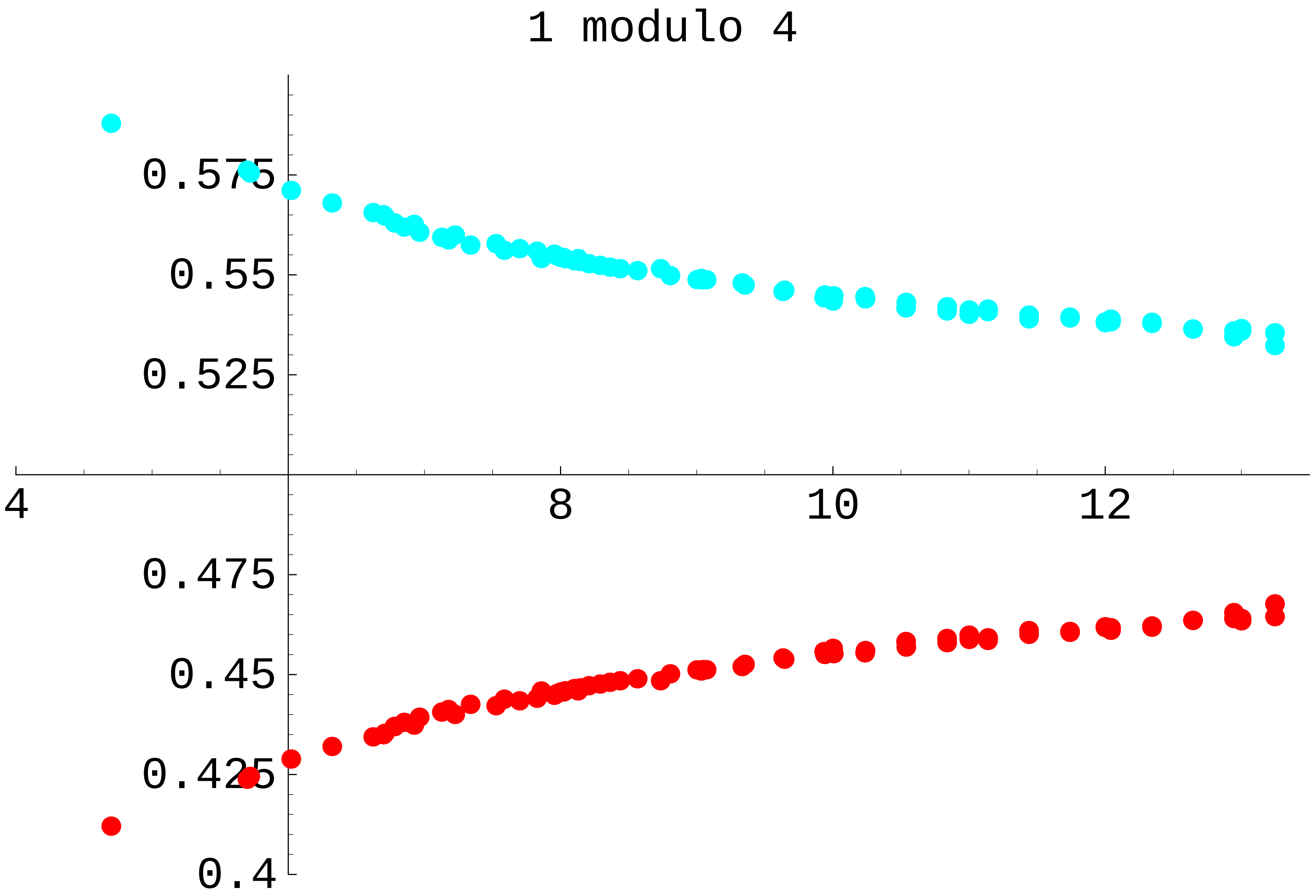
Graphs : x-axis $\cdots$ digit of median of R

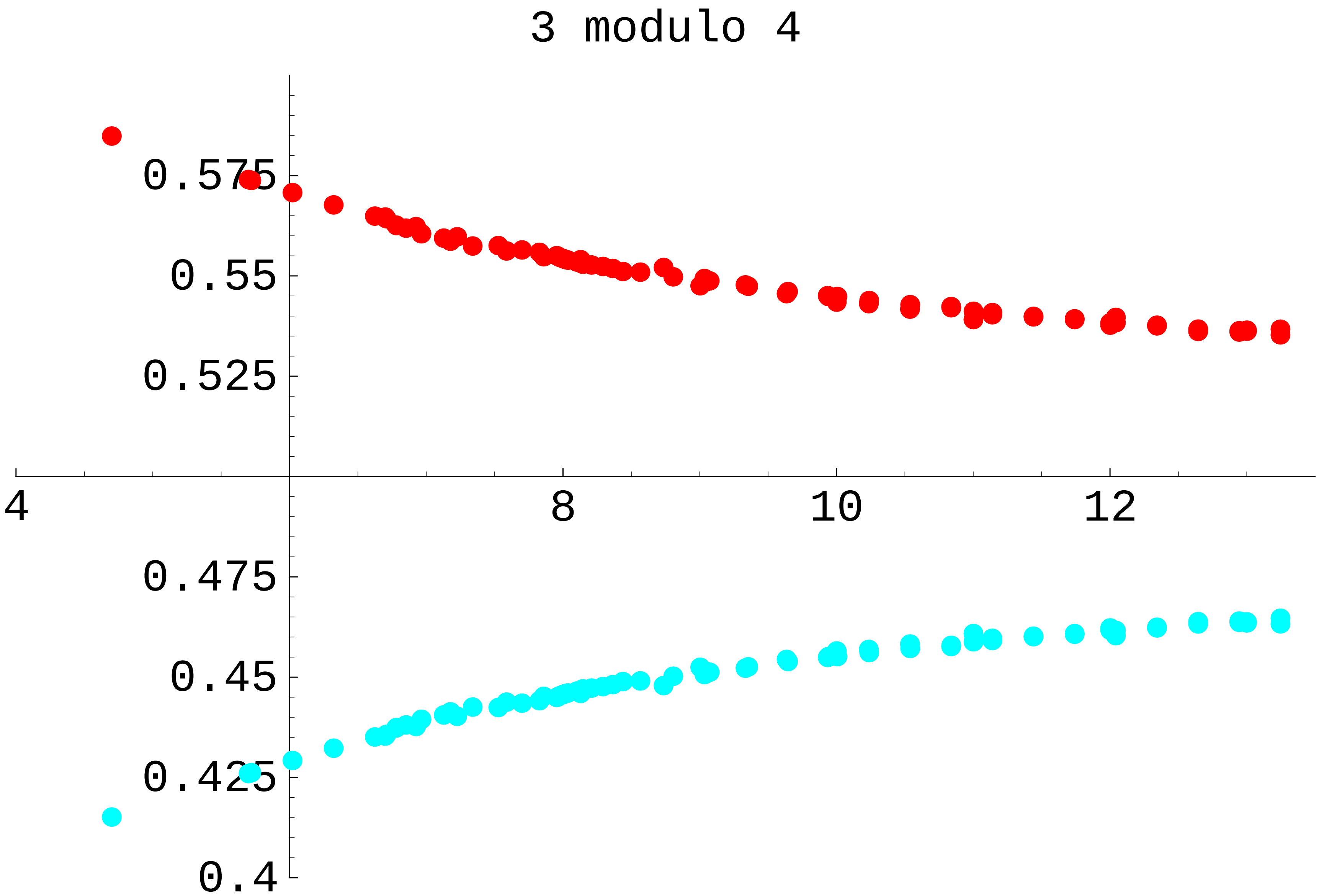
origin of coordinates $= (6,\, 1/\varphi(m))$

$\varpi(R \,|\, \alpha \to \beta)$ $\cdots$ coloured dots

$\mu_p(10^x \,|\, \alpha \to \beta)$ $\cdots$ coloured lines

modulo 4




3 modulo 4
0.575
0.55
0.525
0.475
0.45
0.425
0.4
4
8
10
12

Theorem (Symmetric Property)

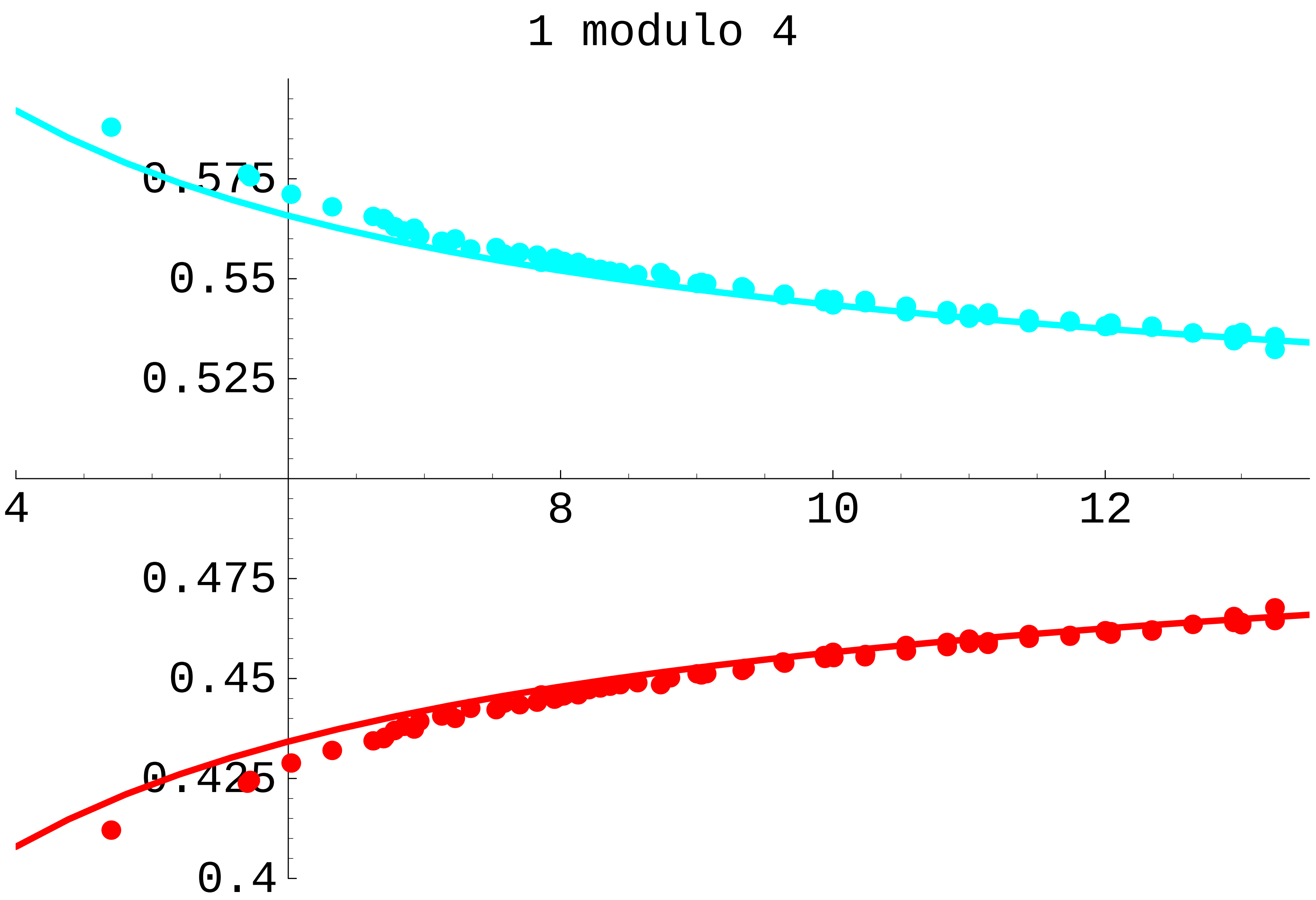
Let m be a power of 2.

For any α, $\beta \in (\mathbb{Z}/m\mathbb{Z})^*$ and any even $d \in \mathbb{Z}$,

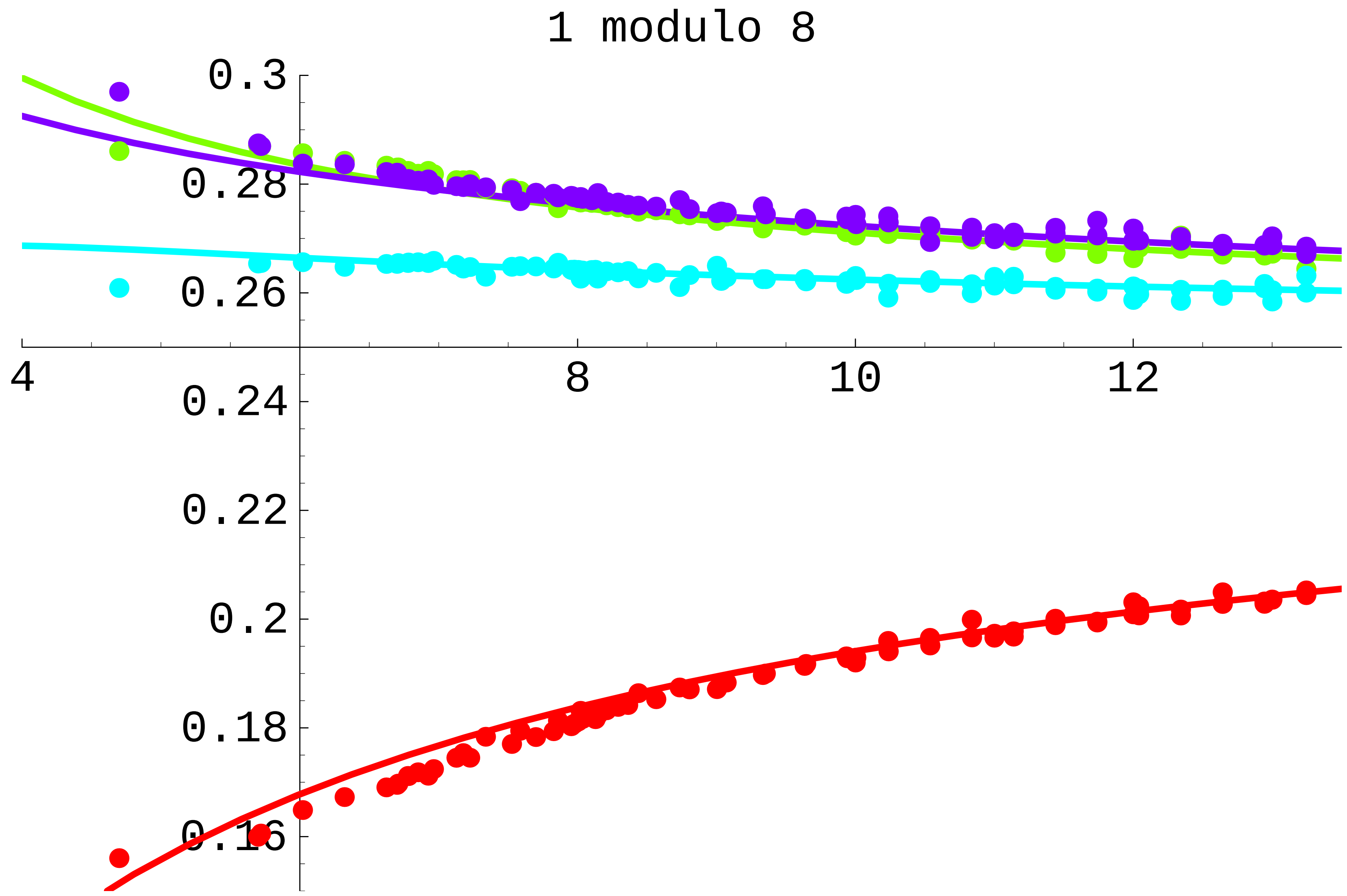
$$\mu_p(x \,|\, \alpha \to \beta) = \mu_p(x \,|\, \alpha+d \to \beta+d)$$

Theorem (Symmetric Property on Estimator)

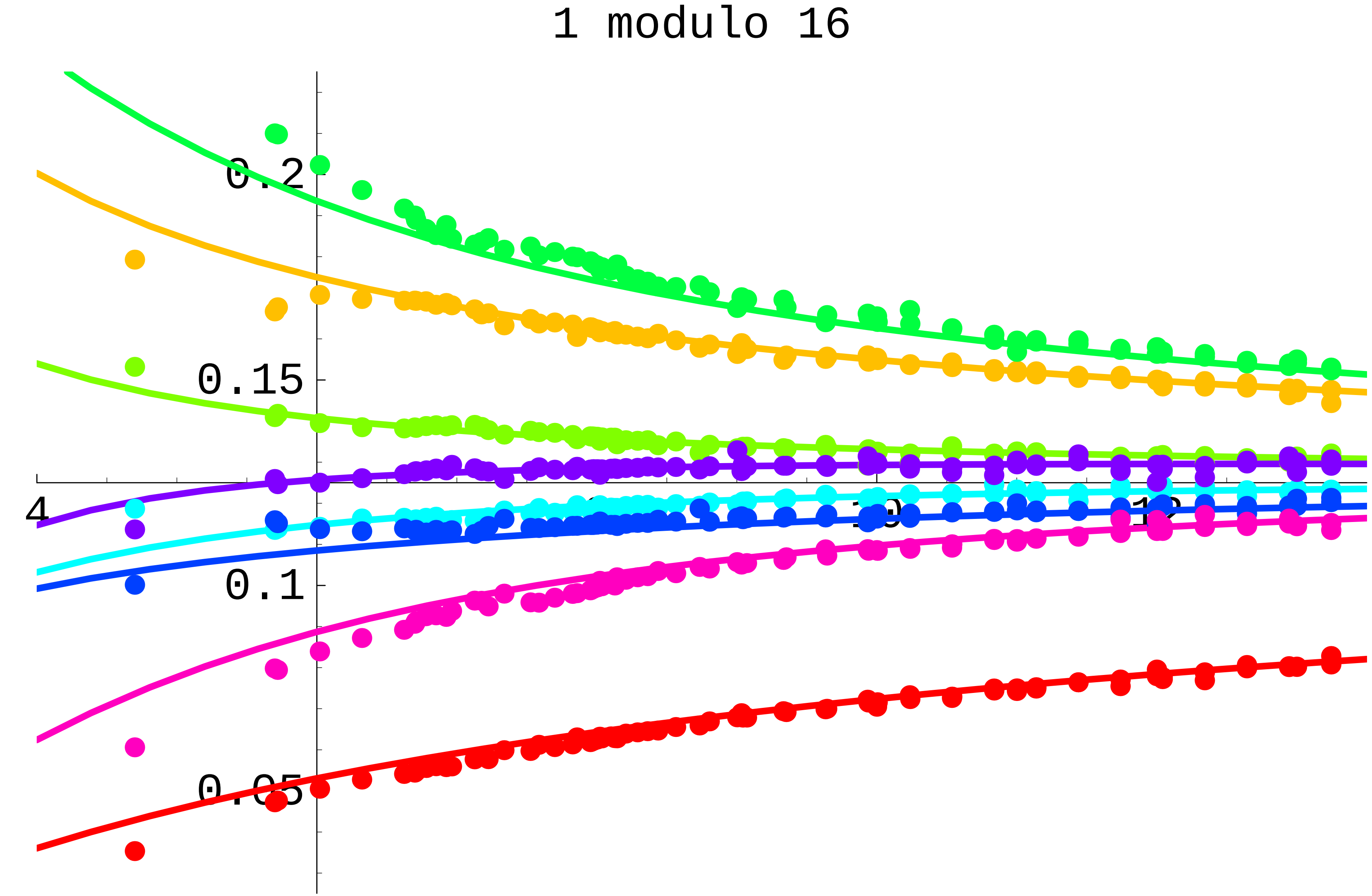
$$\mu'_p(x \,|\, \alpha \to \beta) = \mu'_p(x \,|\, \alpha+d \to \beta+d)$$


1 modulo 4
0.575
0.55
0.525
4
8
10
12
0.475
0.45
0.425
0.4

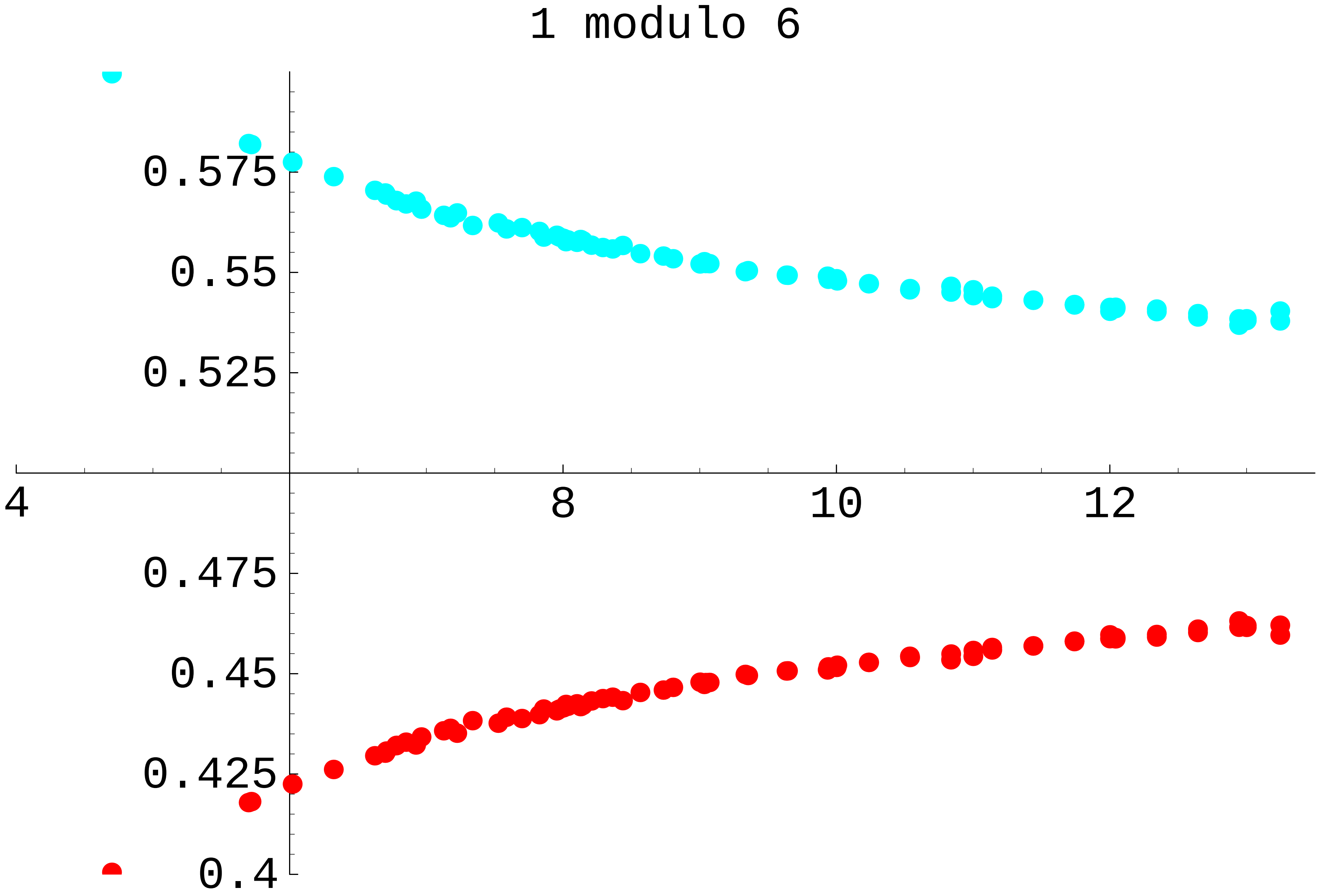
modulo 8

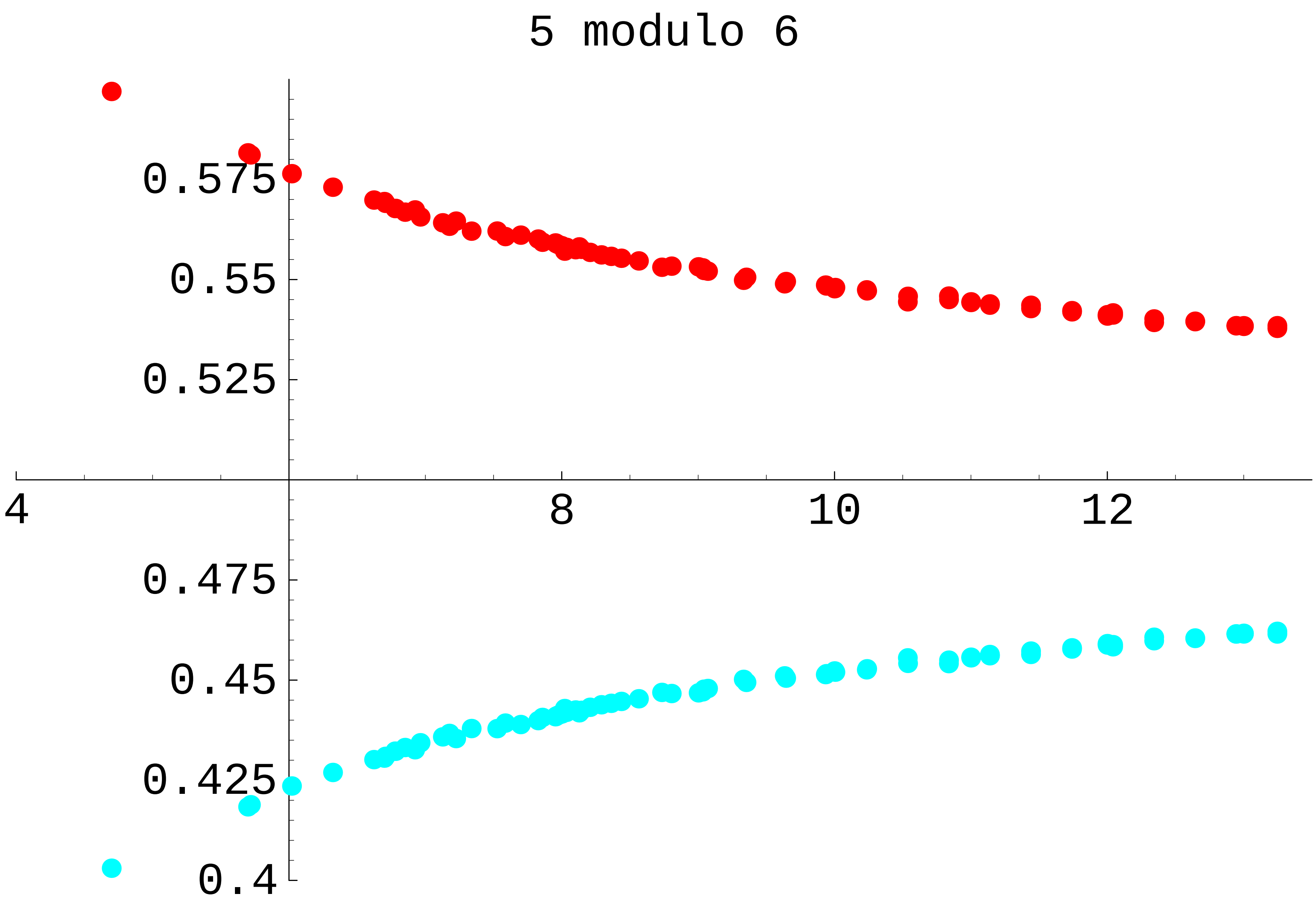


modulo 16



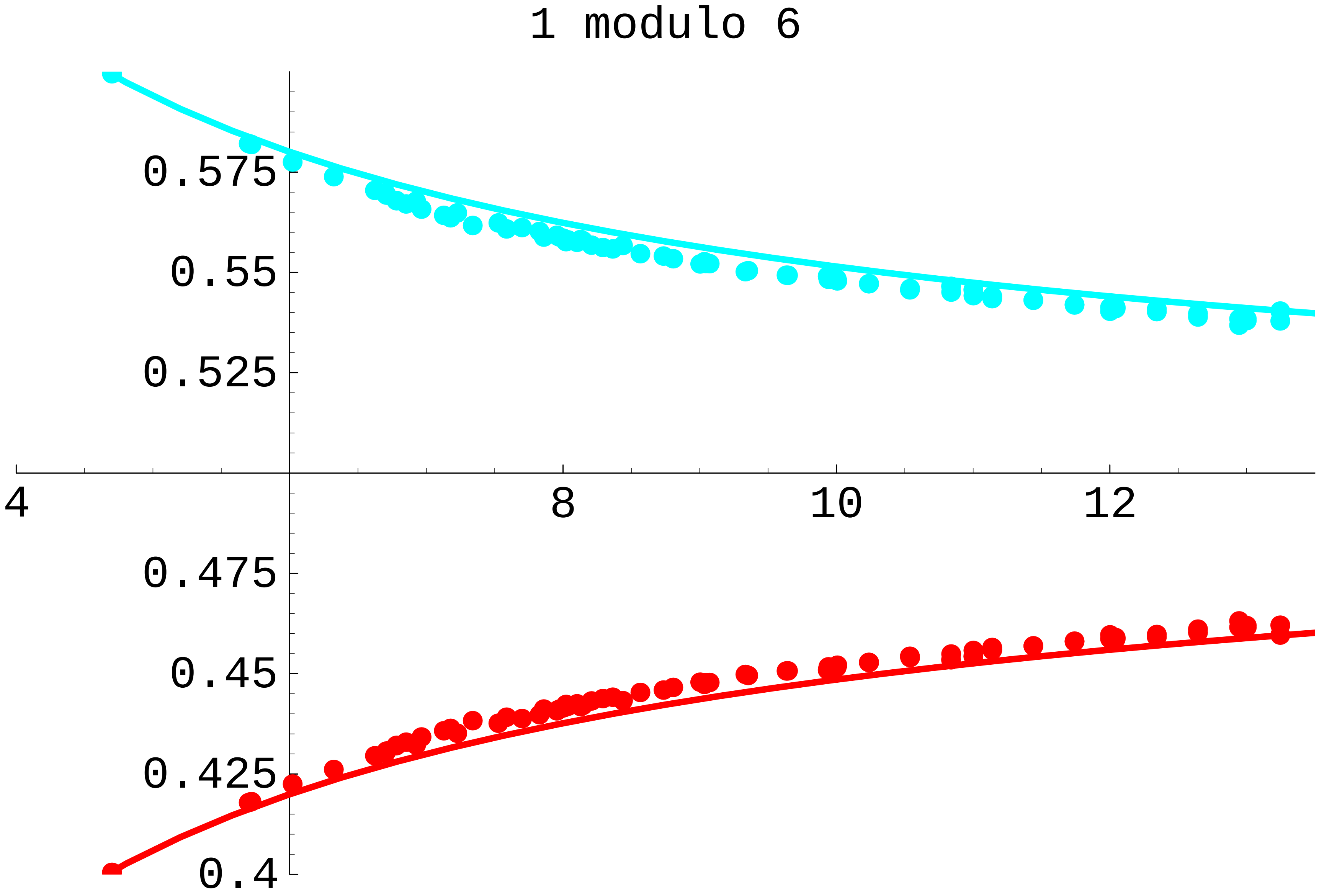
modulo 6

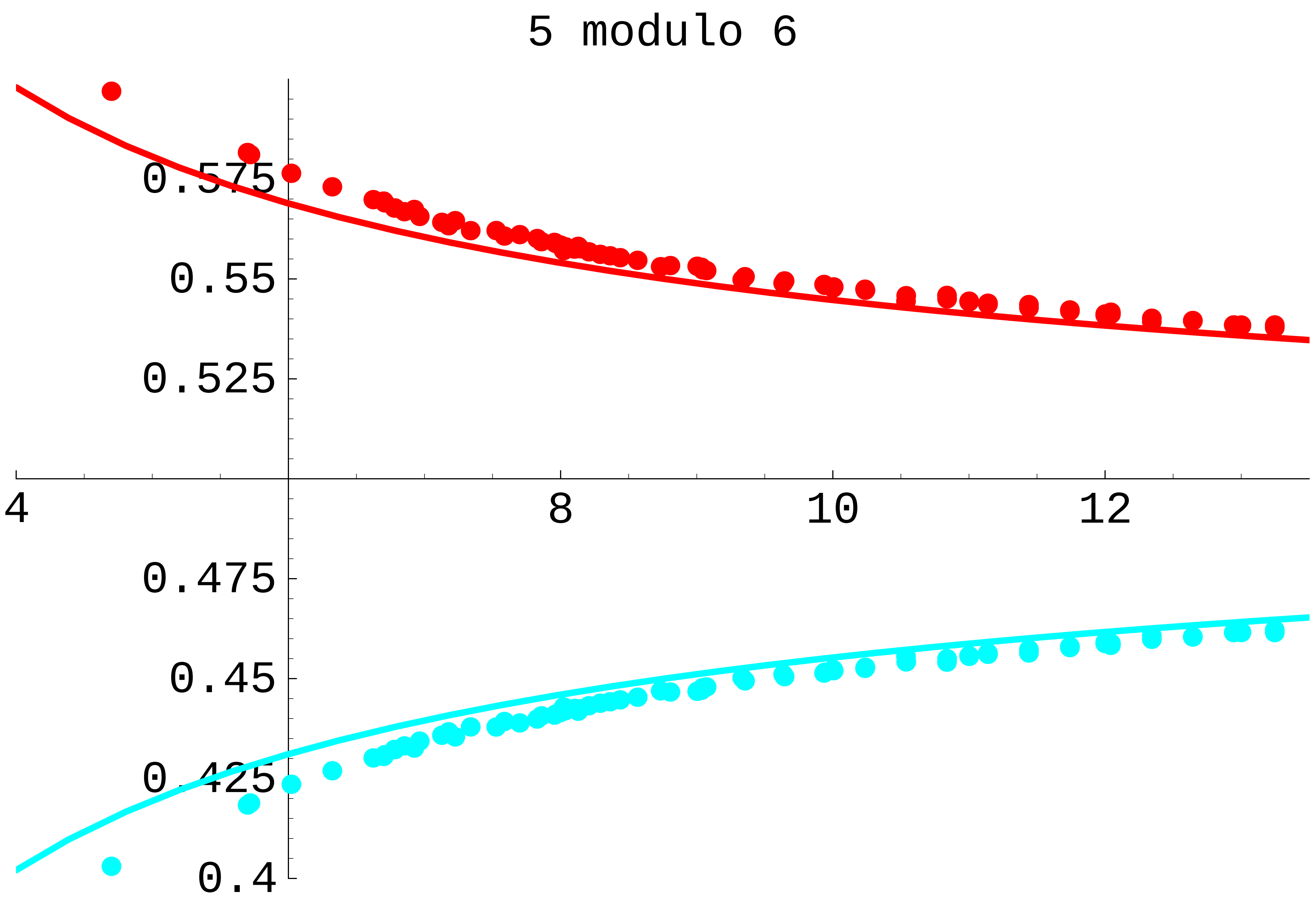




5 modulo 6
0.575
0.55
0.525
0.475
0.45
0.425
0.4
4
8
10
12

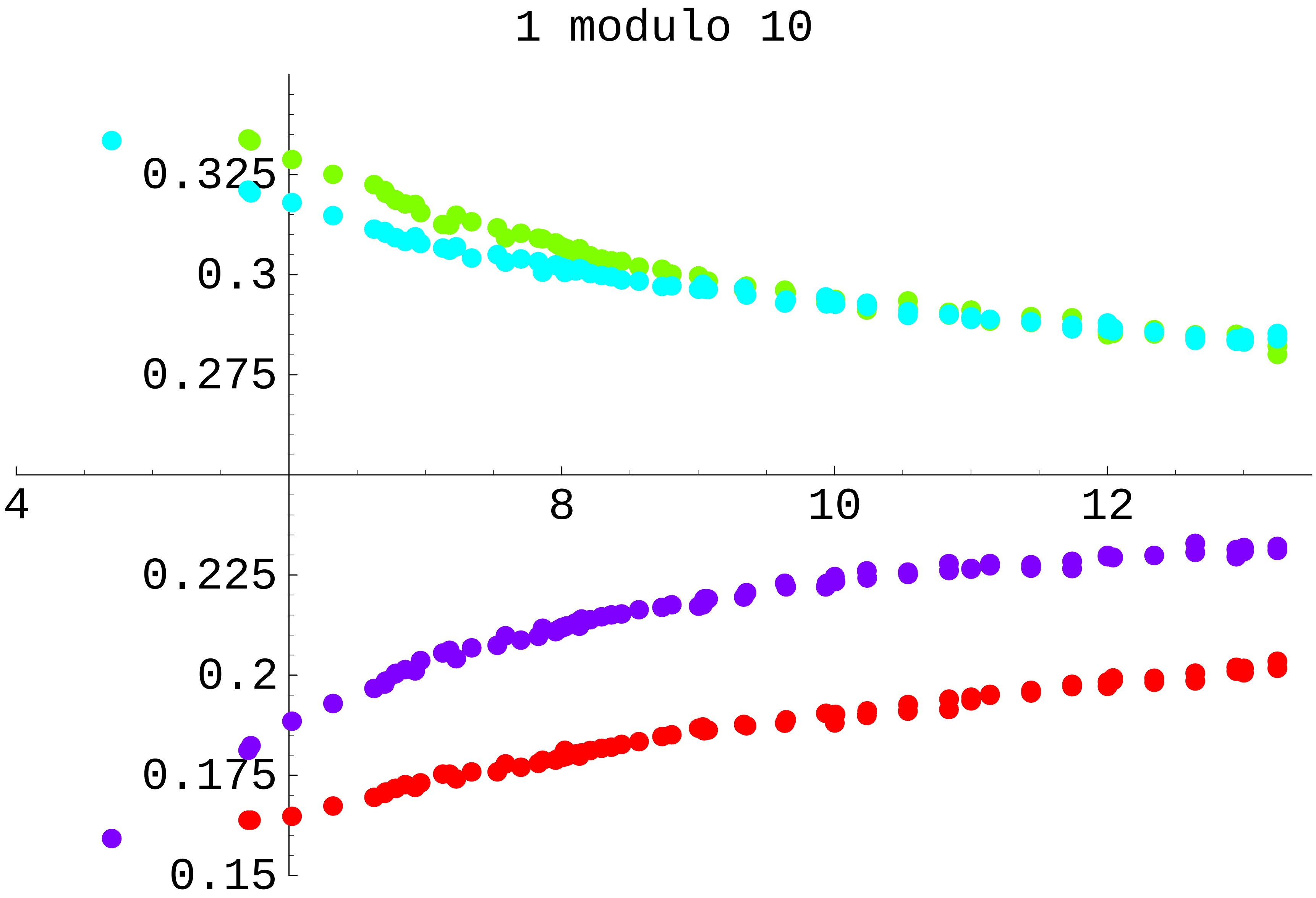
modulo 6

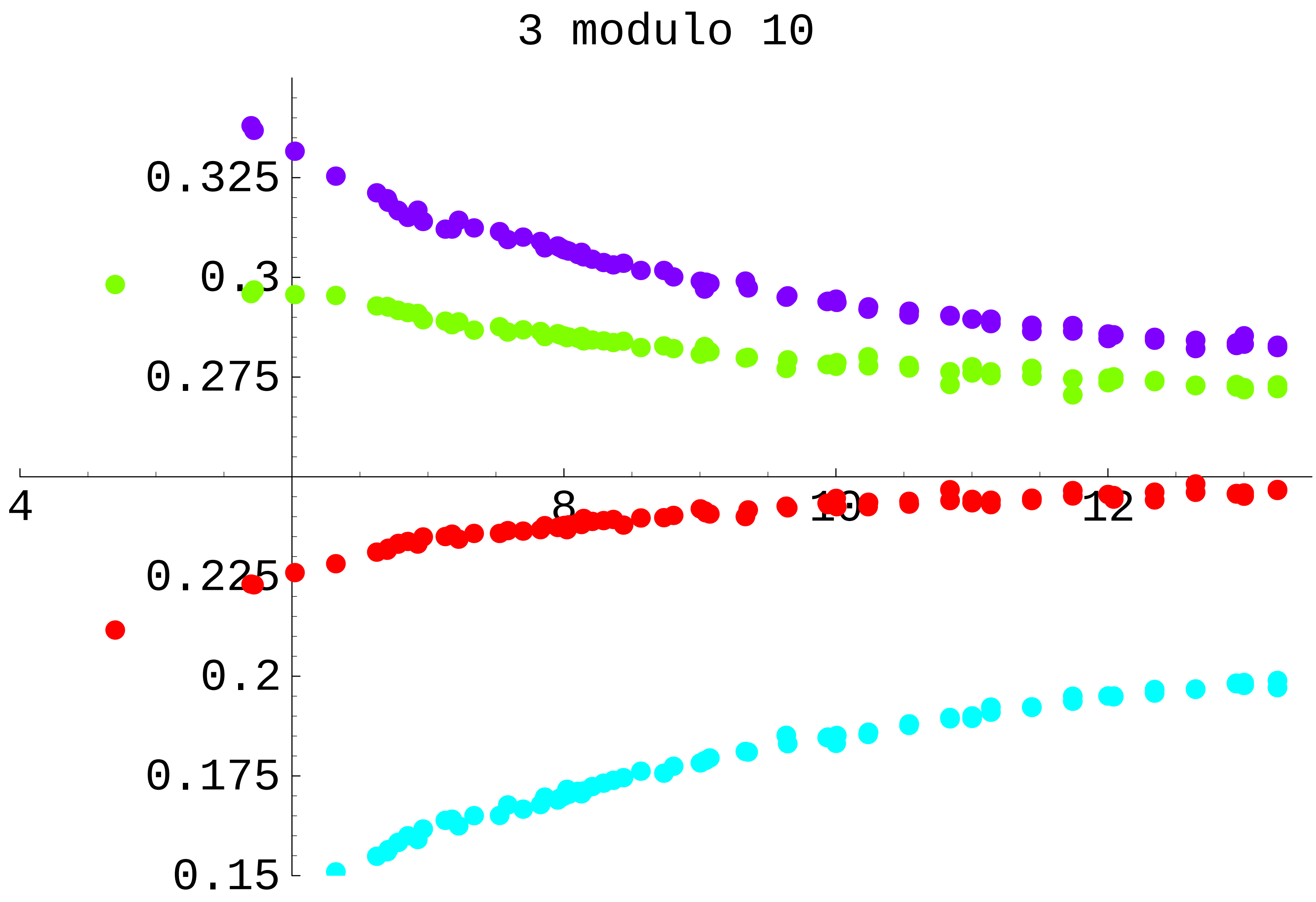


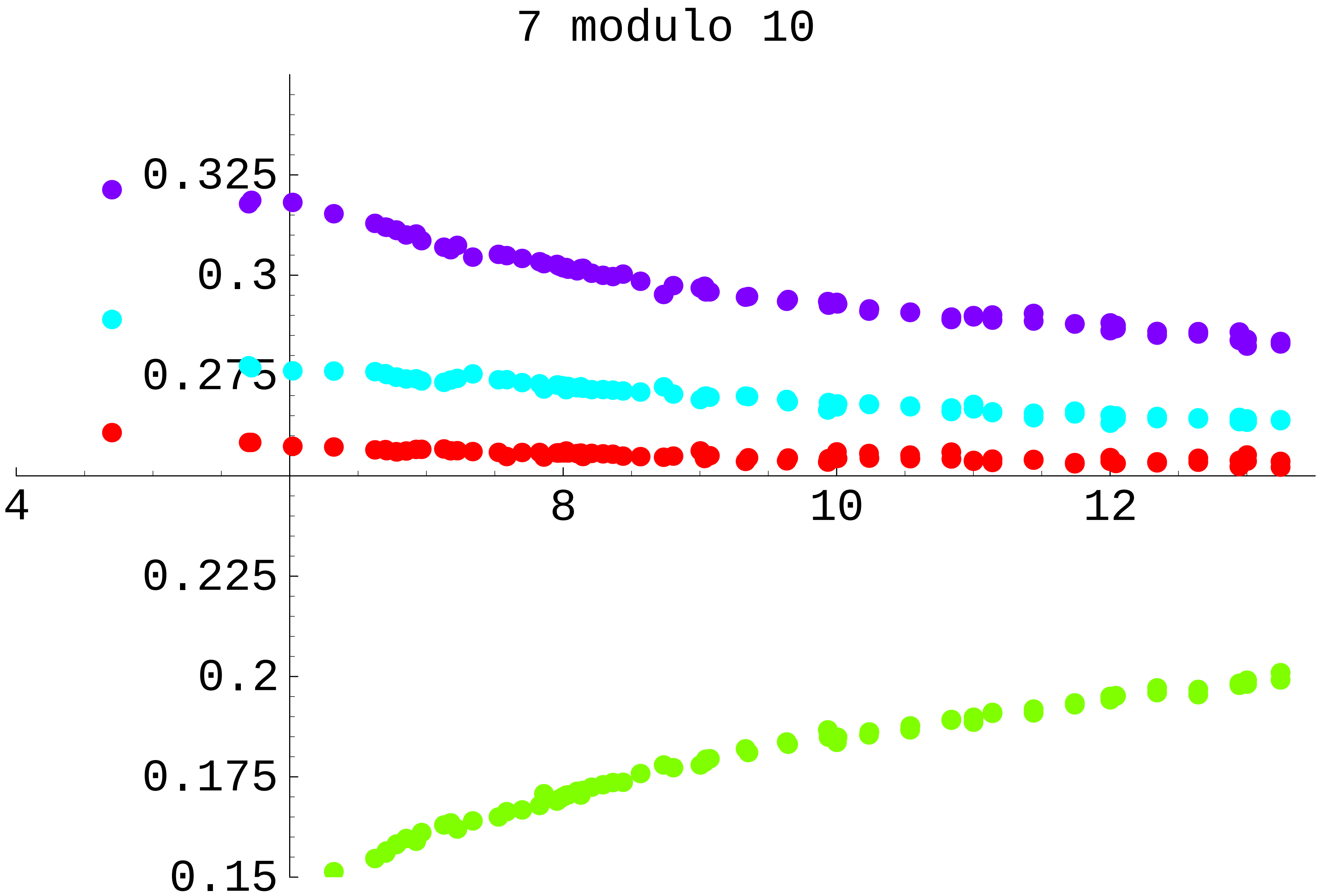


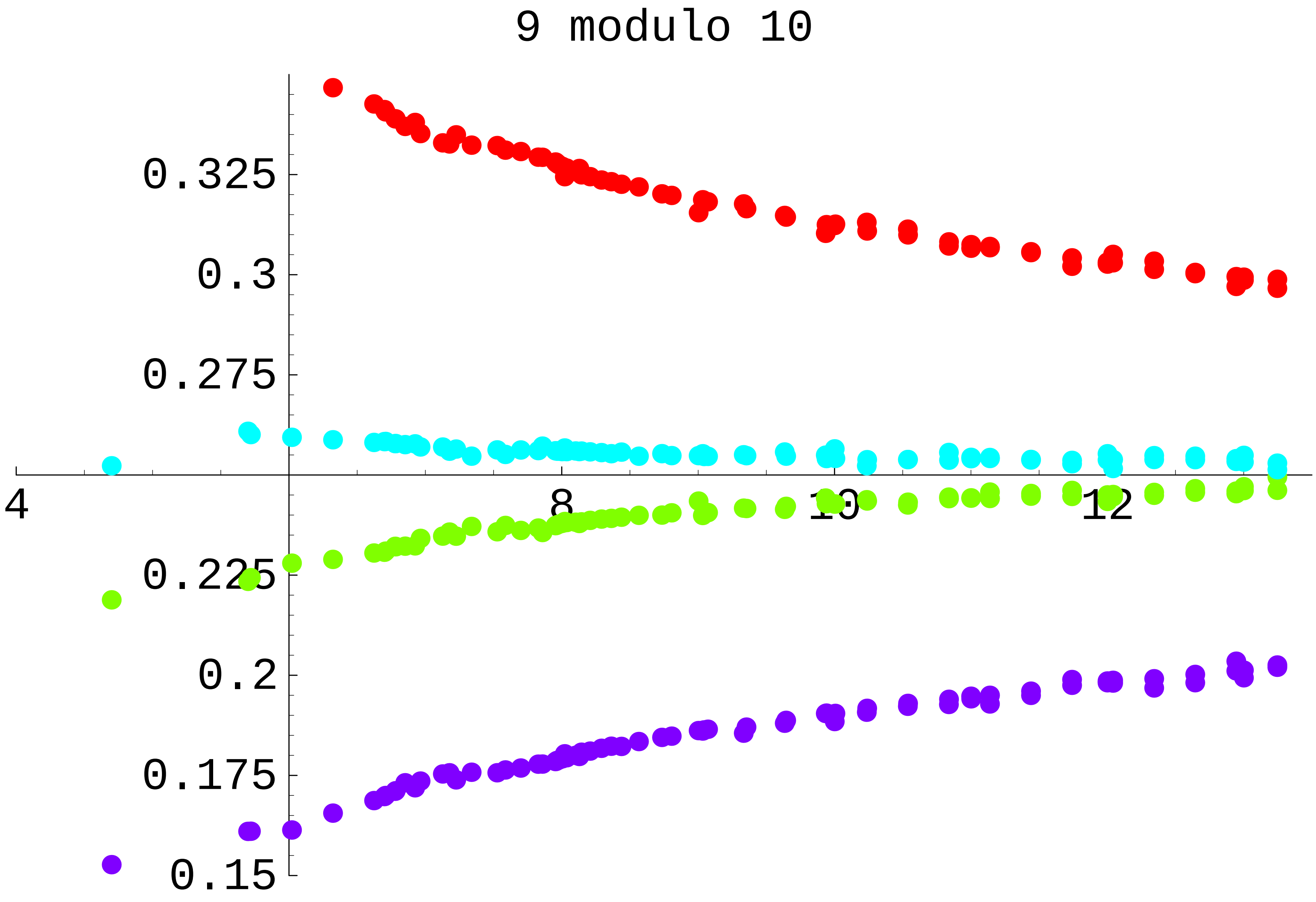
5 modulo 6
0.575
0.55
0.525
0.475
0.45
0.425
0.4
4
8
10
12

modulo 10



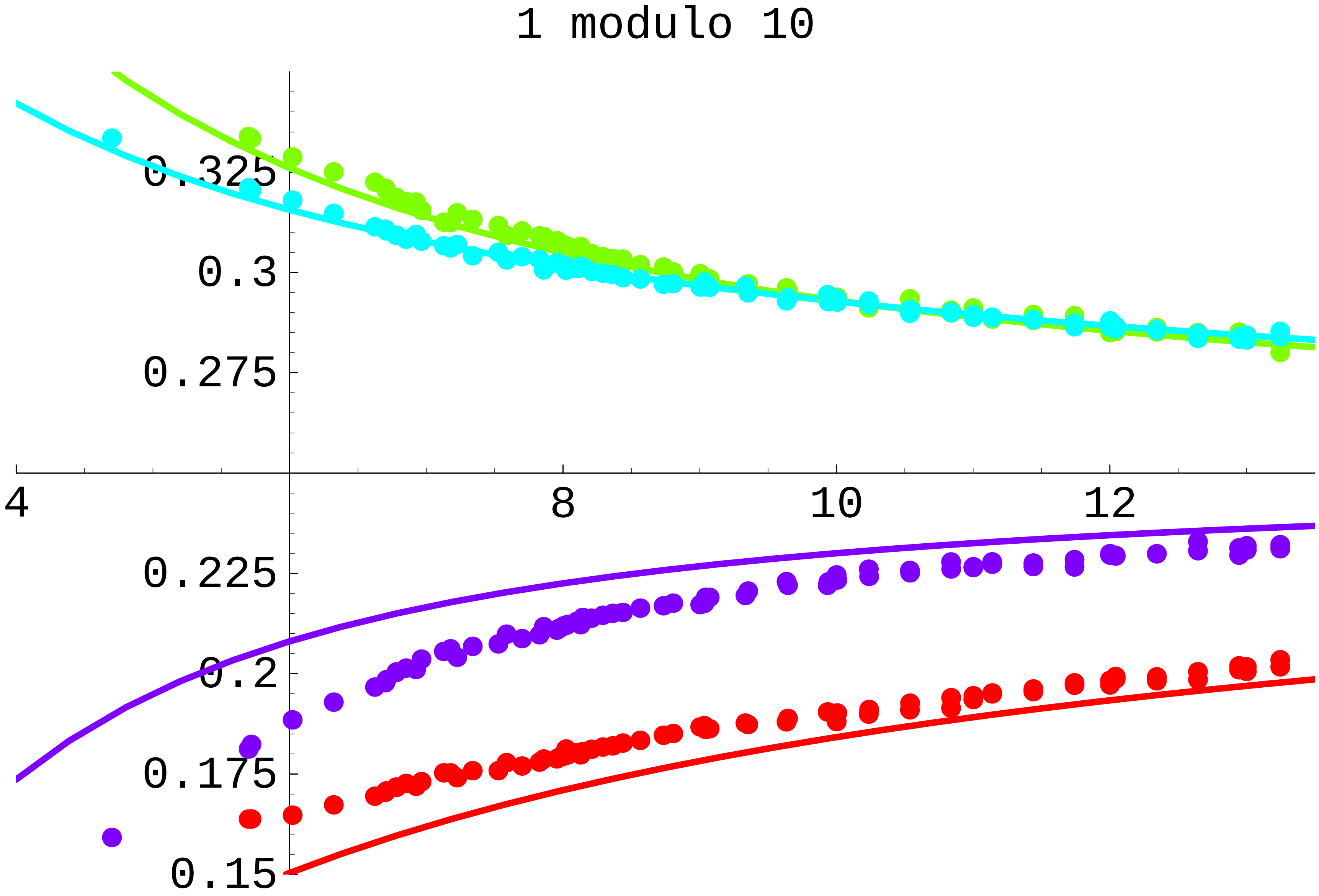

3 modulo 10
0.325
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12

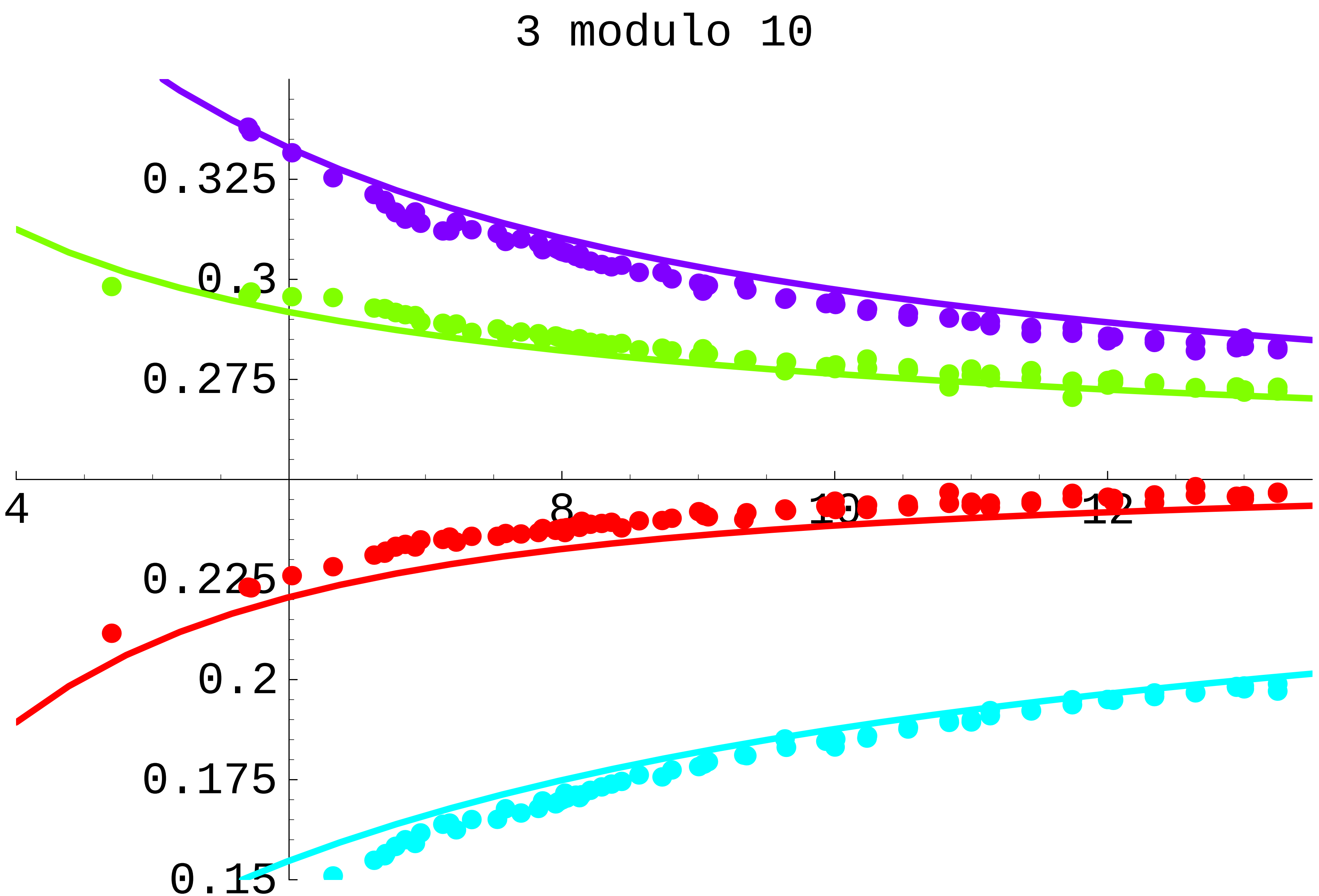

7 modulo 10
0.325
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12



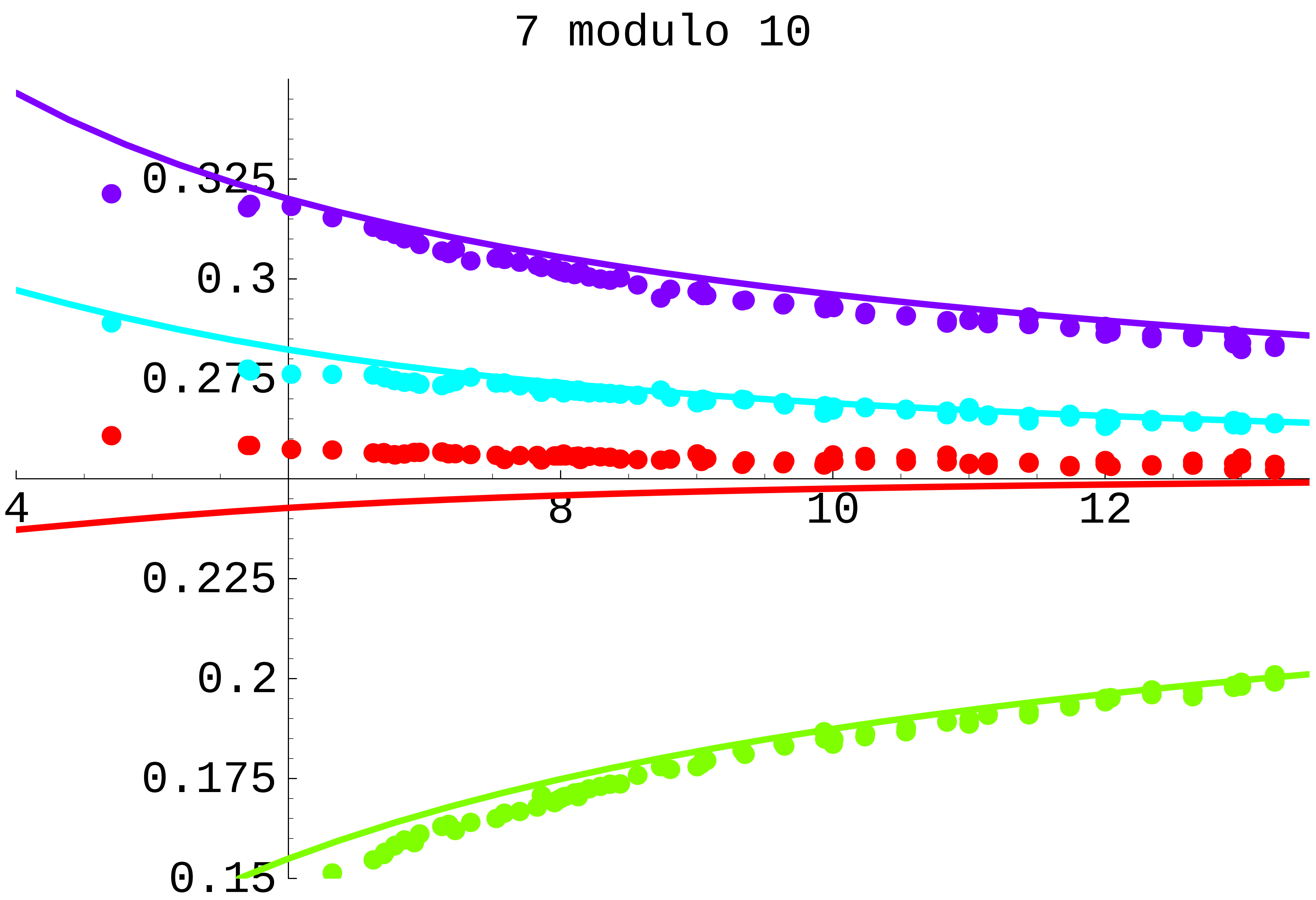
9 modulo 10
0.325
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12

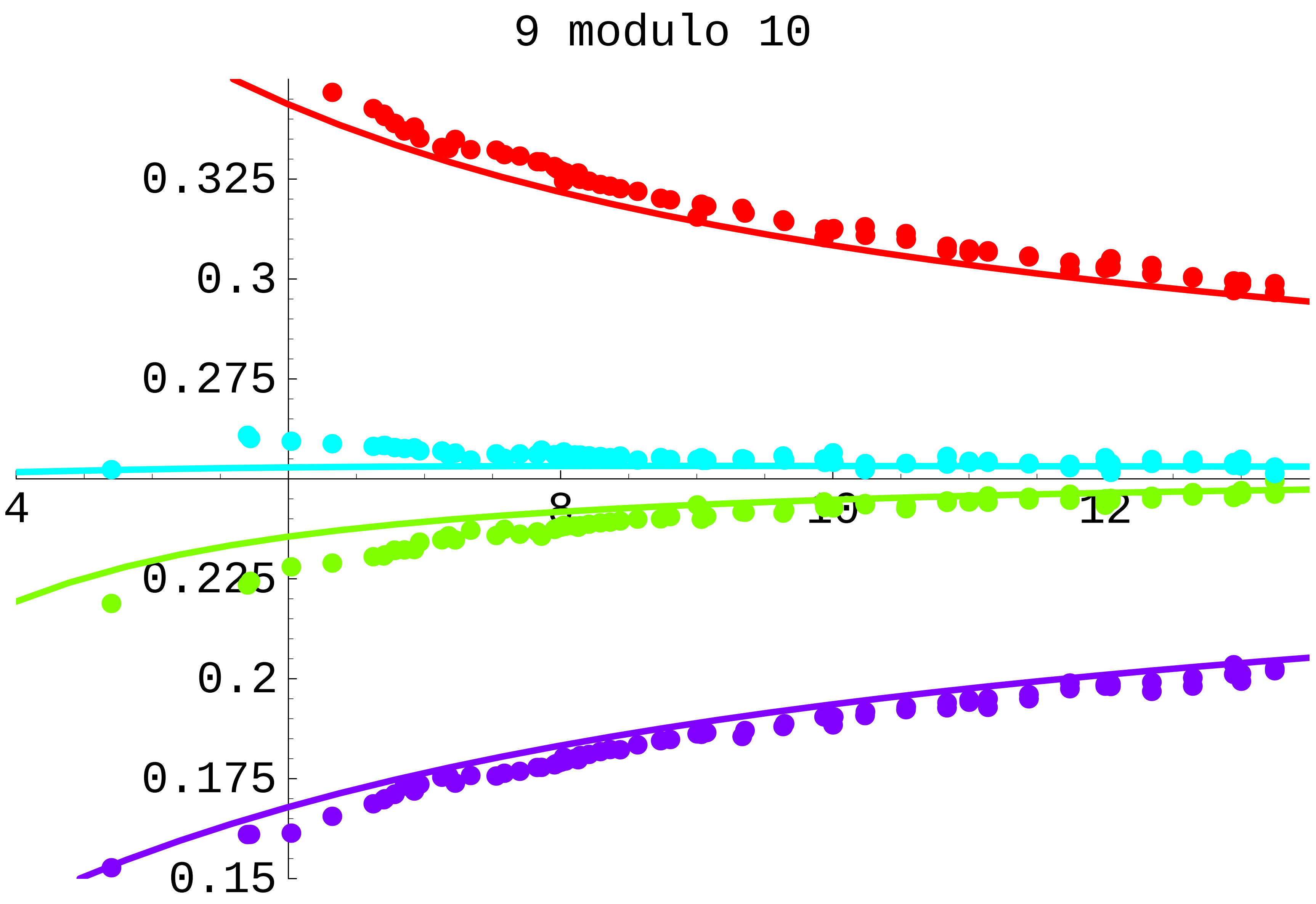
modulo 10





3 modulo 10
0.325
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12


7 modulo 10
0.325
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12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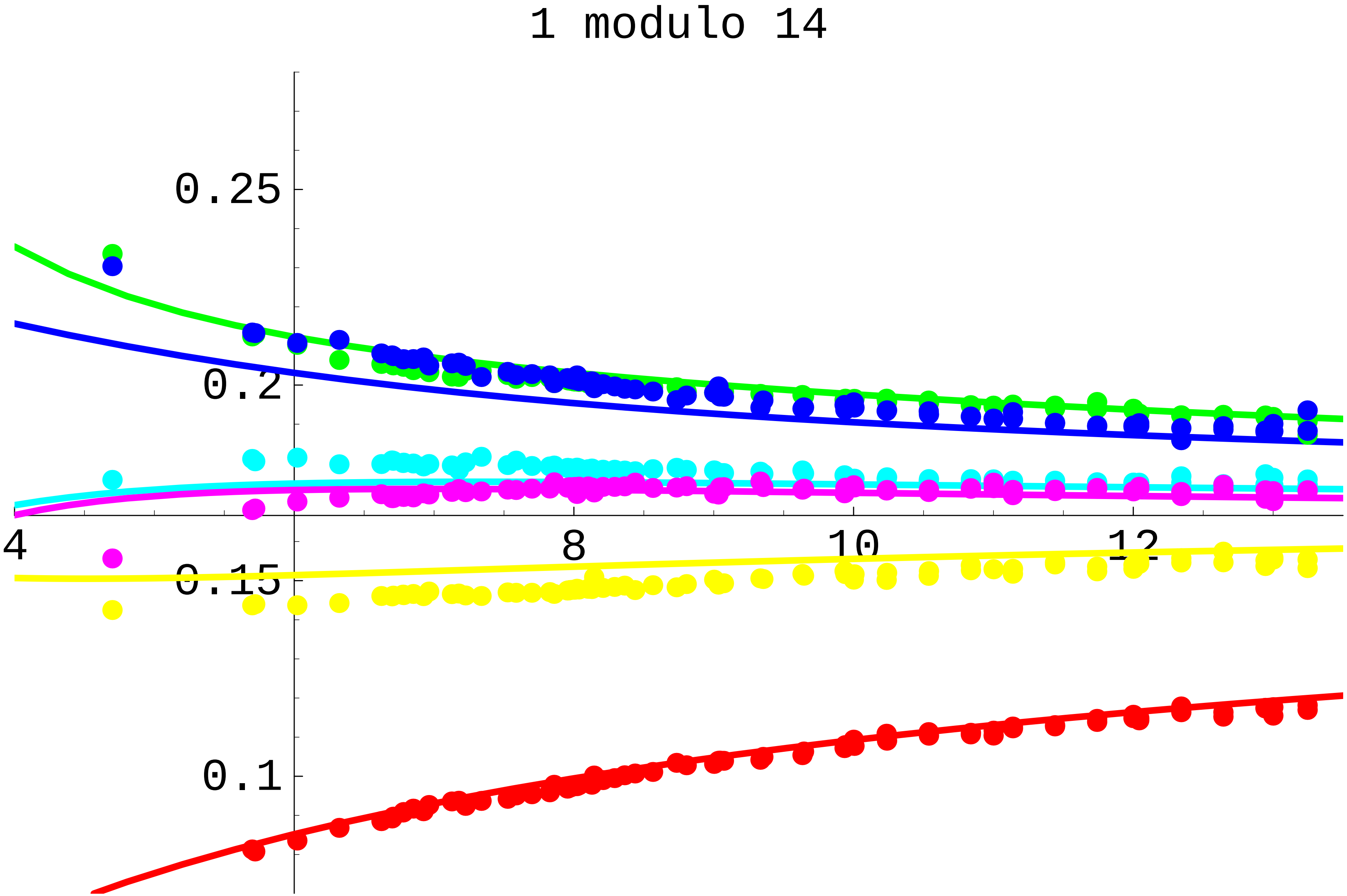
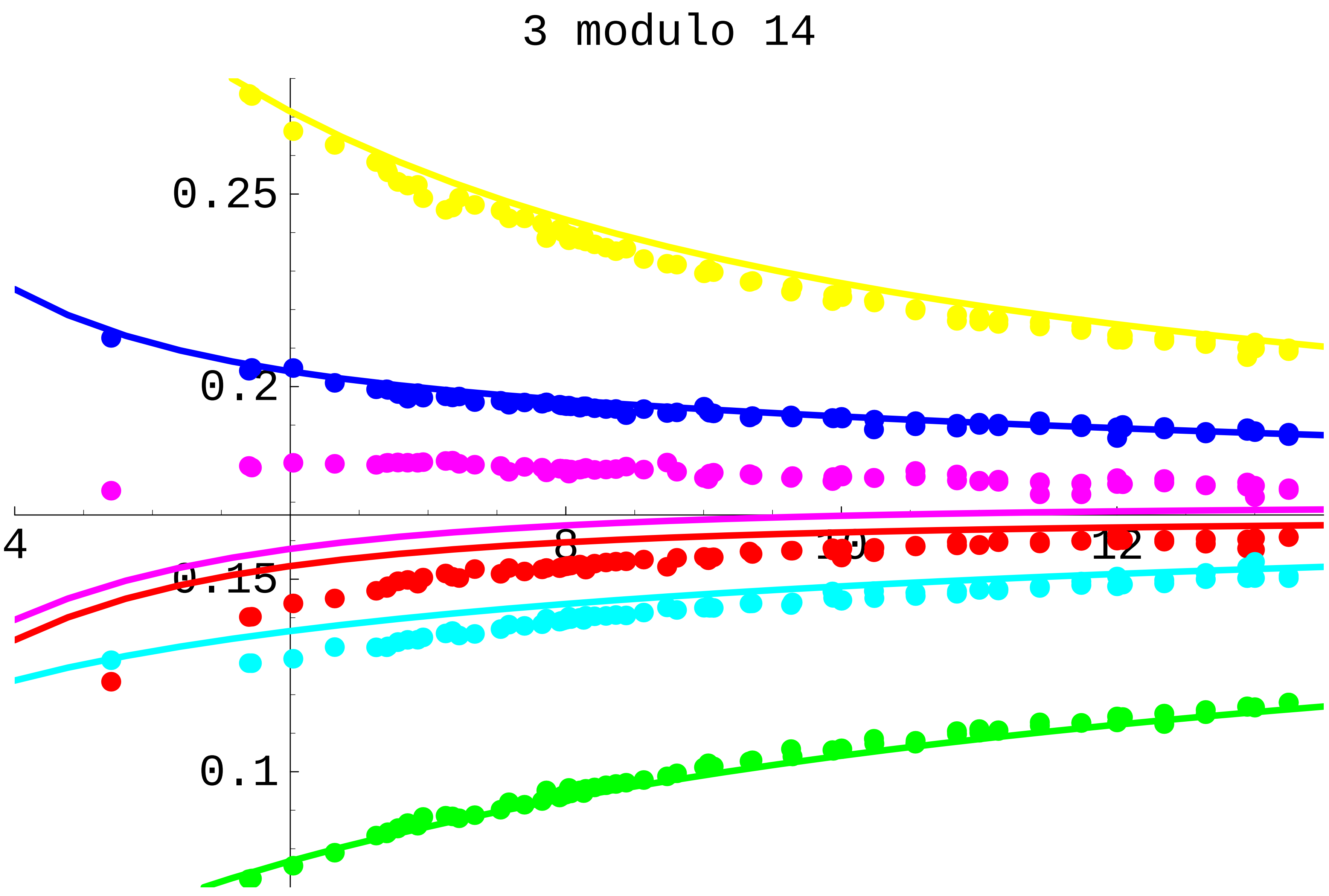
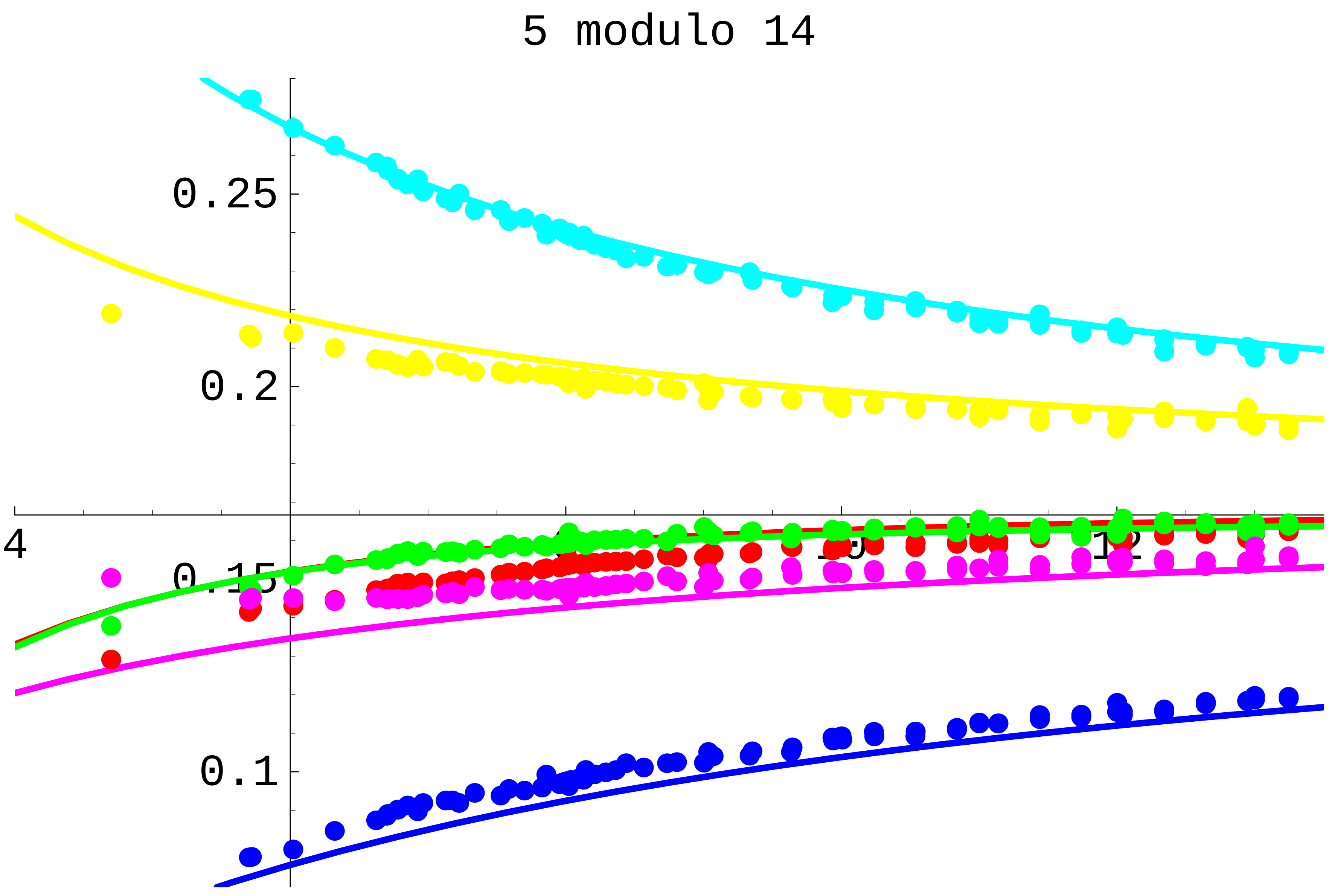
0.3
0.275
0.225
0.2
0.175
0.15
4
8
10
12

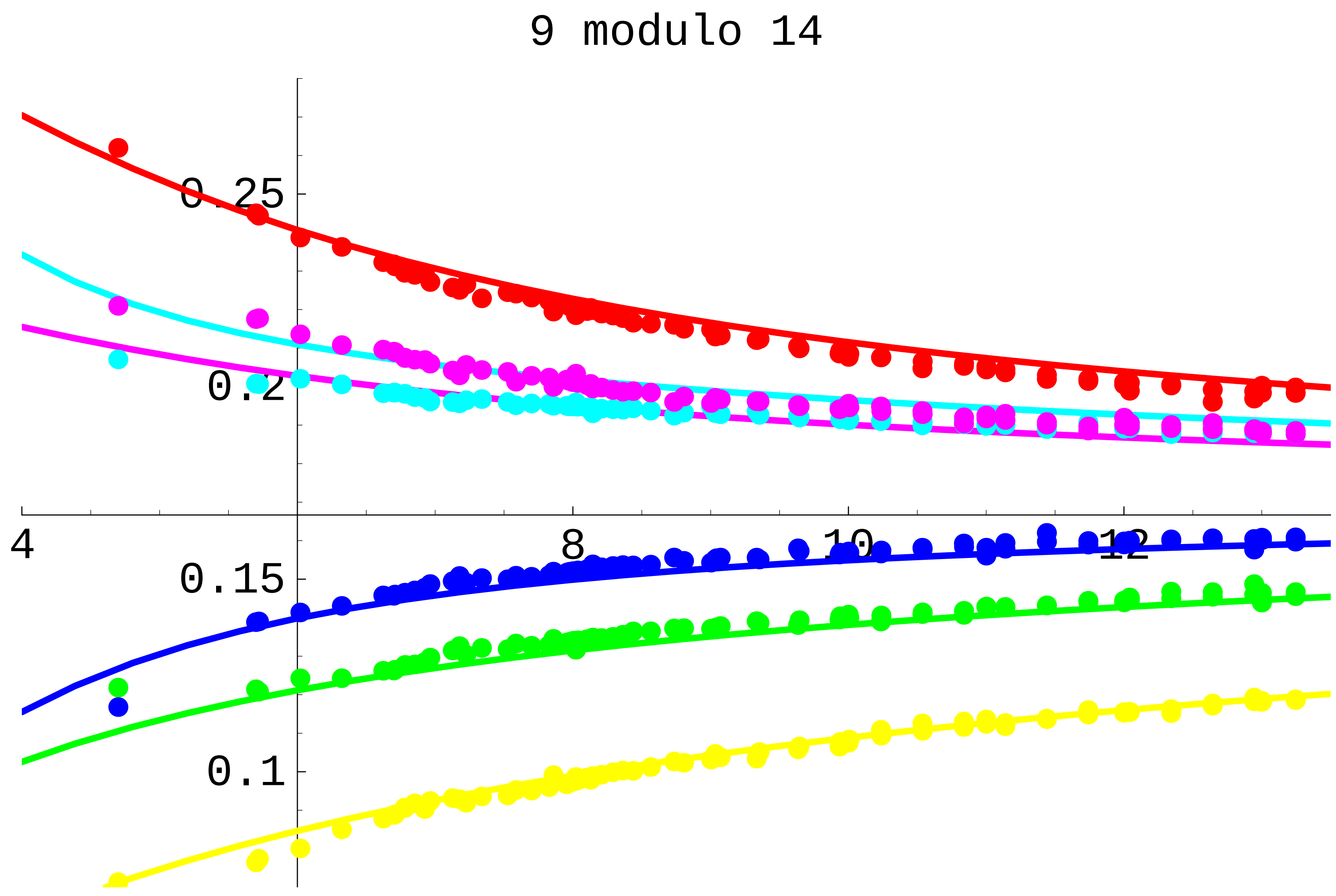
modulo 14



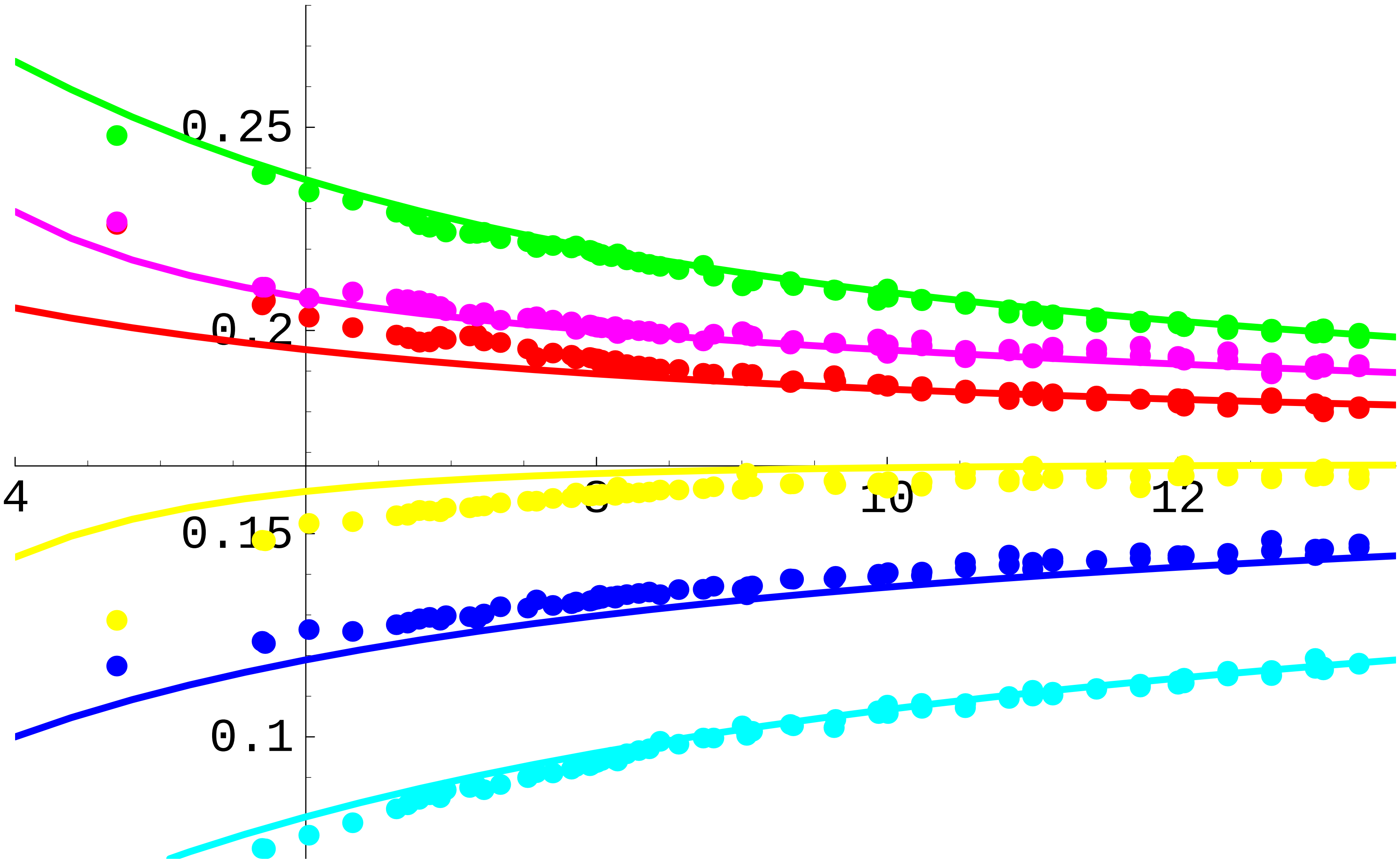

3 modulo 14
0.25
0.2
0.15
0.1
4
8
10
12

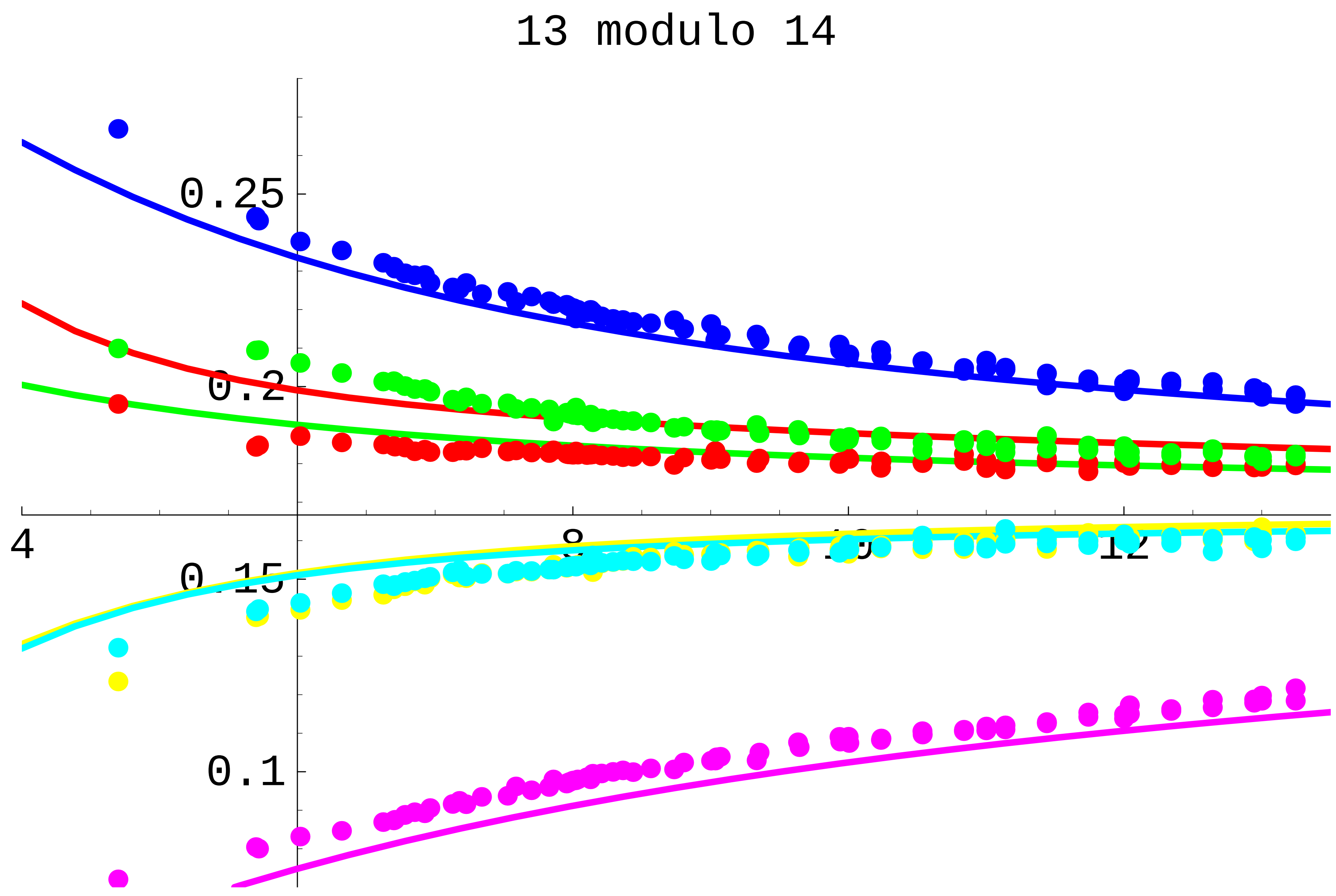


5 modulo 14
0.25
0.2
0.15
0.1
4
12


9 modulo 14
0.25
0.2
0.15
0.1
4
8
10
12

11 modulo 14





13 modulo 14
0.25
0.2
0.15
0.1
4
8
10
12

§6. Next Primes on Quadratic Fields with Class Number 1

K : a quadratic field (*not assume* $\mathrm{ch}(K) = 1$. *assume later*)
$\mathcal{O}_K$: the ring of integers ($= \mathbb{Z}[\omega] = \mathbb{Z} + \mathbb{Z}\omega$)

Next Prime : the *smallest* prime number *greater than* $\cdots$

Next Prime on Number Fields :
Search Path and *Direction*

Search Path $\cdots$ (Random) Walk on Integers
Direction $\cdots\cdots$ Algebraic Polar Coordinate

Algebraic Polar Coordinate (Real Kummer Theory)

$$h_K : \mathbb{P}^1(K) \ni x \mapsto \frac{x-\omega}{x-\omega'} \in \mathbb{P}^1(K) \quad (\omega' : \text{conjugate})$$

$$A_K^\times = h_K^{-1}(K^\times) = \mathbb{P}^1(\mathbb{Q}) \qquad (K^\times = \mathbb{P}^1(K) \setminus \{0, \infty\})$$

$$a \otimes b = \frac{ab - \omega\omega'}{a + b - (\omega + \omega')} \quad (= h_K^{-1}(h_K(a)\, h_K(b)))$$

Prop $(A_K^\times, \otimes, \infty)$: an algebraic group over $\mathbb{Q}$

$$A_K^\times \simeq \ker N_K / \{\pm 1\} \qquad \text{(analog of the unit circle)}$$

Algebraic Polar Coordinate

$$\begin{array}{ccc} K^\times & \longrightarrow & \mathbb{Q}^\times \times A_K^\times \\ \xi & \longmapsto & (N_K(\xi),\, h_K^{-1}(\xi/\xi')) \end{array}$$

$$h_K^{-1}(\xi/\xi') = -\frac{a}{b} \qquad (\xi = a + b\omega \in K,\ a,\ b \in \mathbb{Q})$$

Rem (Real Kummer Theory)

ζ : m-th root of unity (m : odd)

K/k : a quadratic extension s.t. $K = k(\zeta)$, $\zeta + \overline{\zeta} \in k$

$G_k = \mathrm{Gal}(\overline{k}/k)$: the absolute Galois group

$$h_{K/k} : \mathbb{P}^1(K) \ni x \mapsto \frac{x-\zeta}{x-\overline{\zeta}} \in \mathbb{P}^1(K)$$

$$A^{\times}_{K/k} = h^{-1}_{K/k}(\mathbb{P}^1 \setminus \{0, \infty\}) = \mathbb{P}^1 \setminus \{\zeta, \overline{\zeta}\}$$

$A^{\times}_{K/k}$: algebraic group over k

multiplication : $a \otimes b = h^{-1}_{K/k}(h_{K/k}(a)\, h_{K/k}(b))$

identity element : $\infty \in A^{-1}_{K/k} = \mathbb{P}^1(k)$

inverse element : $a \otimes a^{[-1]} = a^{[-1]} \otimes a = \infty$

n-th power : $a^{[n]} = a \otimes a \otimes \cdots \otimes a$ (n times)

Theorem (Komatsu, O. 2003)

$$H^1(\mathrm{Gal}(\overline{k}/k),\, A^{\times}_{K/k}(\overline{k})[m] = \{1\}$$

Rem (*Hilbert Theorem 90*)

$$H^1(\mathrm{Gal}(\overline{K}/K),\, \overline{K}^{\times})[m] = \{1\}$$

Cor $$\begin{array}{rcl} \delta : A^{\times}_{K/k}/(A^{\times}_{K/k})^{[m]} & \simeq & H^1(\mathrm{Gal}(\overline{k}/k),\, A^{\times}_{K/k}[m]) \\ a & \longmapsto & \sigma \mapsto \alpha^{\sigma} \otimes \alpha^{[-1]} \end{array}$$

where $A^{\times}_{K/k}[m] = \langle \eta \rangle \;\; (\eta = h^{-1}_{K/k}(\zeta) \in A^{\times}_{K/k}(k))$

$a = \alpha^{[m]} \;\; (\alpha \in A^{\times}_{K/k}(\overline{k}))$

Walk on $\mathcal{O}_K$ toward $\cdots$

$\mathbb{R} = \mathbb{Q}_\infty$

$A_K^\times = \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R}) \simeq S^1$ (unit circle) $\cdots$ *direction*

α, β : $\mathbb{Z}$-base of $\mathcal{O}_K$ $\cdots$ *steps on Walking*

$$\mathrm{Corn}(\alpha,\, \beta) = \{a\alpha + b\beta \,|\, a,\, b \in \mathbb{N}_0\} \quad (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

$$\mathrm{Dir}(\alpha,\, \beta) = \overline{h_K(\mathrm{Corn}(\alpha,\, \beta))} \subset \mathbb{P}^1(\mathbb{R})$$

$\gamma \in \mathrm{Dir}(\alpha,\, \beta) \subset \mathbb{P}^1(\mathbb{R}) \qquad \xi \in \mathcal{O}_K$

Next Integer of ξ *toward* γ *steps* $\{\alpha,\, \beta\}$

$=$ near γ among $\xi + \alpha$ and $\xi + \beta$.

Walk on $\mathcal{O}_K$ *toward* γ *steps* $\{\alpha,\, \beta\}$

$=$ Sequence of $\mathcal{O}_K$ connented by next integers toward γ steps $\{\alpha,\, \beta\}$

Assume $\mathbf{ch}(K) = 1$

Next Prime of ξ *toward* γ *steps* $\{\alpha, \beta\}$

= First prime in the walk on $\mathcal{O}_K$ starting from ξ toward γ steps $\{\alpha, \beta\}$

Prime Sequence toward γ *steps* $\{\alpha, \beta\}$

= Sequence of $\mathcal{O}_K$ connented by next primes toward γ steps $\{\alpha, \beta\}$

Examples

$K = \mathbb{Q}(\sqrt{-1})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-1}] = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$

$h_K(1) = \infty$, $h_K(\sqrt{-1}) = 0$

$$\mathrm{Dir}(1, \sqrt{-1}) = \overline{h_K(\mathrm{Corn}(1, \sqrt{-1}))} = [-\infty, 0]$$

Walk on $\mathcal{O}_K$ toward $-\sqrt{2}$ steps $\{1, \sqrt{-1}\}$

0	1	$\underline{1+\sqrt{-1}}$	$\underline{2+\sqrt{-1}}$	$2+2\sqrt{-1}$
$\underline{3+2\sqrt{-1}}$	$4+2\sqrt{-1}$	$4+3\sqrt{-1}$	$5+3\sqrt{-1}$	$\underline{5+4\sqrt{-1}}$
$6+4\sqrt{-1}$	$\underline{6+5\sqrt{-1}}$	$7+5\sqrt{-1}$	$\underline{8+5\sqrt{-1}}$	$8+6\sqrt{-1}$
$9+6\sqrt{-1}$	$9+7\sqrt{-1}$	$\underline{10+7\sqrt{-1}}$	$11+7\sqrt{-1}$	$\cdots$

Walk on $\mathcal{O}_K$ toward $-\sqrt{3}$ steps $\{1, \sqrt{-1}\}$

0	1	$\underline{1+\sqrt{-1}}$	$\underline{2+\sqrt{-1}}$	$3+\sqrt{-1}$
$\underline{3+2\sqrt{-1}}$	$4+2\sqrt{-1}$	$4+3\sqrt{-1}$	$5+3\sqrt{-1}$	$6+3\sqrt{-1}$
$6+4\sqrt{-1}$	$7+4\sqrt{-1}$	$8+4\sqrt{-1}$	$\underline{8+5\sqrt{-1}}$	$9+5\sqrt{-1}$
$10+5\sqrt{-1}$	$10+6\sqrt{-1}$	$\underline{11+6\sqrt{-1}}$	$11+7\sqrt{-1}$	$\cdots$

(under line $\cdots$ prime)

§7. Numerical Experiments

$K = \mathbb{Q}(\sqrt{-1})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-1}] = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$

Prime Sequences consisting 10^4 or 10^5 primes

S_2 = Prime Sequence toward $-\sqrt{2}$ steps $\{1, \sqrt{-1}\}$

$1+\sqrt{-1}$	$2+\sqrt{-1}$	$3+2\sqrt{-1}$	$5+4\sqrt{-1}$	$6+5\sqrt{-1}$
$8+5\sqrt{-1}$	$10+7\sqrt{-1}$	$13+10\sqrt{-1}$	$17+12\sqrt{-1}$	$19+14\sqrt{-1}$
$22+15\sqrt{-1}$	$32+23\sqrt{-1}$	$35+24\sqrt{-1}$	$40+29\sqrt{-1}$	$\cdots$

S_3 = Prime Sequence toward $-\sqrt{3}$ steps $\{1, \sqrt{-1}\}$

$1+\sqrt{-1}$	$2+\sqrt{-1}$	$3+2\sqrt{-1}$	$8+5\sqrt{-1}$	$11+6\sqrt{-1}$
$12+7\sqrt{-1}$	$13+8\sqrt{-1}$	$16+9\sqrt{-1}$	$17+10\sqrt{-1}$	$20+11\sqrt{-1}$
$22+13\sqrt{-1}$	$25+14\sqrt{-1}$	$29+16\sqrt{-1}$	$40+23\sqrt{-1}$	$\cdots$

S_5 = Prime Sequence toward $-\sqrt{5}$ steps $\{1, \sqrt{-1}\}$
S_7 = Prime Sequence toward $-\sqrt{7}$ steps $\{1, \sqrt{-1}\}$

S_2 modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	424	605	694	702	2425
2	728	461	595	723	2507
3	636	757	450	649	2492
4	637	684	753	501	2575

%	1	2	3	4
1	17	25	29	29
2	29	18	24	29
3	26	30	18	26
4	25	27	29	19

S_3 modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	532	558	600	809	2499
2	861	474	518	630	2483
3	603	808	520	558	2489
4	503	643	851	531	2528

%	1	2	3	4
1	21	22	24	32
2	35	19	21	25
3	24	32	20	22
4	20	25	34	21

S_5 modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	600	829	666	385	2480
2	574	524	752	637	2487
3	545	586	528	864	2523
4	761	548	578	622	2509

%	1	2	3	4
1	24	33	27	16
2	23	21	30	26
3	22	23	21	34
4	30	22	23	25

S_7 modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	461	821	685	521	2488
2	596	457	730	744	2527
3	625	635	474	772	2506
4	806	613	617	443	2479

%	1	2	3	4
1	19	33	28	21
2	24	18	29	29
3	25	25	19	30
4	33	25	25	18

S_2 modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	99	60	19	17	62	135	190	102	51	1	18	78	832
2	149	93	64	16	7	56	125	166	94	45	3	19	837
3	62	144	72	75	19	7	46	116	169	84	42	3	839
4	18	69	158	85	57	27	4	27	97	194	84	52	872
5	2	14	64	143	87	79	13	3	35	112	178	85	815
6	48	5	15	87	167	95	70	26	5	34	117	197	866
7	83	49	3	24	65	195	81	85	26	7	56	142	816
8	186	99	51	4	25	95	150	100	64	15	6	47	842
9	130	169	76	50	5	30	67	150	81	70	24	6	858
10	35	96	161	95	48	1	8	52	161	74	75	12	818
11	4	35	115	174	92	58	0	13	69	126	89	50	825
12	16	4	41	102	182	88	62	2	6	56	133	86	778

S_3 modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	56	138	108	49	16	21	112	113	116	49	22	17	817
2	65	55	118	94	53	24	24	96	111	116	57	22	835
3	16	60	66	113	126	42	29	28	90	125	125	61	881
4	42	21	57	45	120	107	48	35	24	99	77	144	819
5	79	43	21	53	46	151	112	74	31	35	84	118	847
6	143	71	29	18	55	54	129	105	59	25	23	113	824
7	103	126	75	17	22	57	65	137	109	55	17	24	807
8	92	112	142	57	23	9	43	56	133	108	44	18	837
9	18	97	100	135	44	20	10	50	50	124	125	43	816
10	22	19	112	104	133	69	34	17	47	48	137	111	853
11	62	32	23	98	100	149	67	44	14	57	48	126	820
12	120	61	30	36	109	121	133	82	32	12	61	45	842

S_5 modulo $3+2\sqrt{-1}$ (norm$=13$)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	30	94	94	86	45	47	86	51	102	77	56	54	822
2	55	17	107	91	88	54	30	74	61	100	62	54	793
3	58	50	31	99	88	88	36	54	92	46	121	65	828
4	54	52	51	27	100	81	98	39	52	101	48	120	823
5	80	69	55	58	29	133	94	89	53	38	95	61	854
6	112	70	62	49	42	25	98	109	111	49	38	97	862
7	50	133	64	44	46	48	32	116	106	103	40	29	811
8	81	49	108	64	52	54	52	32	97	89	93	48	819
9	36	91	54	121	71	63	41	64	32	104	95	93	865
10	56	38	90	52	140	85	58	49	51	32	100	93	844
11	97	47	45	95	60	129	50	68	51	58	29	98	827
12	113	82	67	37	93	55	137	74	57	47	50	38	850

S_7 modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	35	98	92	89	59	37	82	42	129	64	55	60	842
2	56	33	102	97	92	58	47	99	46	131	72	52	885
3	75	47	22	76	90	87	52	45	99	35	129	68	825
4	62	70	47	24	97	87	74	49	29	97	45	131	812
5	78	74	67	58	35	104	106	104	59	43	85	44	857
6	105	70	61	44	50	23	102	76	97	56	53	105	842
7	36	133	69	62	66	64	32	104	84	84	72	47	853
8	96	39	128	64	46	65	48	28	76	62	85	43	780
9	38	104	42	115	70	61	72	50	23	76	86	76	813
10	67	52	103	44	96	73	49	65	40	23	93	83	788
11	95	63	52	98	53	132	58	50	66	55	25	105	852
12	99	103	40	41	103	51	131	68	64	62	52	37	851

S_2 modulo $2+\sqrt{-1}$ (norm=5) (10^5 primes)

	1	2	3	4	
1	4962	6294	6830	6734	24820
2	7033	5051	6102	6894	25080
3	6530	7299	4972	6288	25089
4	6296	6436	7185	5093	25010

%	1	2	3	4
1	19	25	28	27
2	28	20	24	27
3	26	29	20	25
4	25	26	29	20

S_3 modulo $2+\sqrt{-1}$ (norm=5) (10^5 primes)

	1	2	3	4	
1	5287	5743	6207	7628	24865
2	7843	5090	5600	6411	24944
3	6381	7694	5299	5711	25085
4	5355	6417	7979	5354	25105

%	1	2	3	4
1	21	23	25	31
2	31	20	22	26
3	25	31	21	23
4	21	26	32	21

S_2 modulo $3+2\sqrt{-1}$ (norm=13) (10^5 primes)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	920	647	163	140	651	1438	1785	869	418	65	329	950	8375
2	1457	869	596	180	167	597	1222	1660	797	419	66	332	8362
3	741	1296	774	660	189	130	473	1124	1612	767	378	65	8209
4	229	706	1353	840	603	186	72	441	1039	1624	844	422	8359
5	37	220	755	1392	891	709	149	82	440	1157	1703	841	8376
6	440	55	218	805	1553	913	686	192	99	460	1200	1782	8403
7	899	430	53	294	924	1759	940	709	168	138	600	1386	8300
8	1749	798	433	80	351	947	1467	884	621	196	143	670	8339
9	1196	1628	762	397	52	300	764	1434	796	663	184	140	8316
10	461	1181	1488	795	440	52	216	680	1424	797	619	172	8325
11	90	439	1157	1584	804	457	38	220	701	1340	778	667	8275
12	156	93	457	1192	1751	915	488	44	201	699	1431	932	8359

S_3 modulo $3+2\sqrt{-1}$ (norm=13) (10^5 primes)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	527	1232	1137	530	276	292	885	1015	1194	695	356	191	8330
2	520	492	1141	1061	557	300	288	888	948	1167	663	348	8373
3	241	559	515	1133	1096	545	361	331	853	904	1146	673	8357
4	491	272	543	506	1180	1130	575	364	279	856	890	1221	8307
5	794	505	259	524	513	1331	1229	644	375	298	840	1057	8369
6	1297	802	420	241	493	546	1217	1221	614	337	279	838	8305
7	1054	1305	691	375	199	534	539	1313	1128	577	262	264	8241
8	850	964	1202	649	371	214	504	574	1145	1082	574	294	8423
9	281	870	937	1144	662	318	233	523	511	1178	1081	571	8309
10	364	304	865	970	1190	675	413	241	513	511	1162	1158	8366
11	683	411	295	859	962	1361	774	485	252	528	511	1178	8299
12	1229	657	352	315	870	1059	1222	824	497	233	535	526	8319

S_2 modulo $3+2\sqrt{-1}$ (norm$=13$) (10^5 primes)

%	1	2	3	4	5	6	7	8	9	10	11	12
1	11	8	2	2	8	17	21	10	5	1	4	11
2	17	10	7	2	2	7	15	20	10	5	1	4
3	9	16	9	8	2	2	6	14	20	9	5	1
4	3	8	16	10	7	2	1	5	12	19	10	5
5	0	3	9	17	11	8	2	1	5	14	20	10
6	5	1	3	10	18	11	8	2	1	5	14	21
7	11	5	1	4	11	21	11	9	2	2	7	17
8	21	10	5	1	4	11	18	11	7	2	2	8
9	14	20	9	5	1	4	9	17	10	8	2	2
10	6	14	18	10	5	1	3	8	17	10	7	2
11	1	5	14	19	10	6	0	3	8	16	9	8
12	2	1	5	14	21	11	6	1	2	8	17	11

S_3 modulo $3+2\sqrt{-1}$ (norm=13) (10^5 primes)

%	1	2	3	4	5	6	7	8	9	10	11	12
1	6	15	14	6	3	4	11	12	14	8	4	2
2	6	6	14	13	7	4	3	11	11	14	8	4
3	3	7	6	14	13	6	4	4	10	11	14	8
4	6	3	7	6	14	14	7	4	3	10	11	15
5	9	6	3	6	6	16	15	8	4	4	10	13
6	16	10	5	3	6	7	15	15	7	4	3	10
7	13	16	8	5	2	6	7	16	14	7	3	3
8	10	11	14	8	4	3	6	7	14	13	7	3
9	3	10	11	14	8	4	3	6	6	14	13	7
10	4	4	10	12	14	8	5	3	6	5	14	14
11	8	5	4	10	12	16	9	6	3	5	6	14
12	15	8	4	4	10	13	15	10	6	3	6	6

S_{11} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	445	765	658	549	2417
2	633	466	700	701	2500
3	602	622	482	795	2501
4	737	646	662	537	2582

%	1	2	3	4
1	18	32	27	23
2	25	19	28	28
3	24	25	19	32
4	29	25	26	20

S_{13} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	520	754	632	573	2479
2	615	507	711	660	2493
3	563	613	515	771	2462
4	782	618	604	562	2566

%	1	2	3	4
1	21	30	25	23
2	25	20	29	26
3	23	25	21	31
4	30	24	24	22

S_{17} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	459	725	710	635	2529
2	682	462	647	694	2485
3	617	695	429	705	2446
4	771	602	661	506	2540

%	1	2	3	4
1	18	29	28	25
2	27	19	26	28
3	25	28	18	29
4	30	24	26	20

S_{19} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	494	727	691	644	2556
2	680	472	683	656	2491
3	634	699	465	689	2487
4	748	593	648	477	2466

%	1	2	3	4
1	19	28	27	25
2	27	19	27	26
3	25	28	19	28
4	30	24	26	19

S_{23} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	473	679	682	627	2461
2	614	466	696	696	2472
3	640	679	495	740	2554
4	734	648	681	450	2513

%	1	2	3	4
1	19	28	27	25
2	25	19	28	28
3	25	27	19	29
4	29	26	27	18

S_{29} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	461	696	700	639	2496
2	673	480	647	700	2500
3	660	663	468	688	2479
4	702	660	665	498	2525

%	1	2	3	4
1	18	29	28	26
2	27	19	26	28
3	27	27	19	28
4	28	26	26	20

S_{31} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	477	695	744	575	2491
2	630	499	676	703	2508
3	664	681	499	708	2552
4	721	632	633	463	2449

%	1	2	3	4
1	19	28	30	23
2	25	20	27	28
3	26	27	20	28
4	29	26	26	19

S_{37} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	521	729	665	632	2547
2	660	484	673	671	2488
3	680	634	523	670	2507
4	686	640	647	485	2458

%	1	2	3	4
1	20	28	26	25
2	27	19	27	27
3	27	25	21	27
4	28	26	26	20

S_{41} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	548	668	685	640	2541
2	642	525	623	686	2476
3	654	633	527	667	2481
4	697	649	646	510	2502

%	1	2	3	4
1	22	26	27	25
2	26	21	25	28
3	26	26	21	27
4	28	26	26	20

S_{43} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	533	709	655	612	2509
2	658	540	627	701	2526
3	664	595	542	646	2447
4	654	681	623	560	2518

%	1	2	3	4
1	21	28	26	24
2	26	21	25	28
3	27	24	22	26
4	26	27	25	22

S_{47} modulo $2+\sqrt{-1}$ (norm=5)

	1	2	3	4	
1	598	631	678	586	2493
2	603	600	609	666	2478
3	597	654	658	662	2571
4	695	592	627	544	2458

%	1	2	3	4
1	24	25	27	24
2	24	24	25	27
3	23	25	26	26
4	28	24	26	22

S_{11} modulo $3 + 2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	51	93	67	89	40	59	87	26	141	50	64	85	852
2	38	56	101	67	108	35	52	115	29	141	41	70	853
3	75	37	80	72	62	118	34	54	94	28	138	40	832
4	63	84	40	37	88	49	92	30	67	88	25	136	799
5	53	78	77	35	82	101	54	116	34	70	107	24	831
6	159	31	66	73	25	58	86	53	87	36	68	94	836
7	25	152	43	69	71	35	51	90	67	113	53	43	812
8	113	17	137	34	60	70	28	49	95	68	98	28	797
9	58	94	18	161	37	67	84	25	67	85	75	94	865
10	47	56	111	18	134	45	59	102	31	63	104	68	838
11	102	35	52	91	23	180	41	87	87	20	66	81	865
12	68	120	41	53	101	19	144	50	65	76	26	57	820

S_{13} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	76	46	77	101	21	78	68	38	128	15	94	53	795
2	39	107	63	92	101	25	81	79	33	149	18	94	881
3	71	55	101	47	78	113	17	85	75	42	161	15	860
4	86	80	36	82	46	88	103	17	57	64	34	155	848
5	23	102	65	32	80	59	101	116	22	79	73	36	788
6	138	12	97	64	45	82	52	90	101	16	75	74	846
7	27	173	19	86	80	32	90	80	83	94	28	82	874
8	75	45	169	10	91	66	39	94	42	93	93	17	834
9	68	67	36	137	13	84	57	44	80	46	75	86	793
10	20	81	79	40	129	26	103	69	38	77	68	69	799
11	93	17	91	73	36	164	20	99	56	48	96	57	850
12	79	97	27	84	68	29	143	23	77	76	35	94	832

S_{17} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	49	47	116	67	67	74	67	76	105	38	108	48	862
2	90	60	39	103	64	51	62	45	70	116	22	97	819
3	42	89	70	27	120	70	68	84	65	75	114	40	864
4	107	39	91	44	37	89	53	62	73	57	63	109	824
5	25	122	41	78	52	41	127	72	64	73	56	76	827
6	110	46	96	33	84	39	45	90	54	71	69	50	787
7	60	95	31	119	35	94	67	31	126	59	66	81	864
8	56	57	92	33	98	35	83	44	39	126	51	59	773
9	93	52	83	116	37	89	50	84	48	38	112	64	866
10	57	78	64	62	98	29	111	32	84	52	46	120	833
11	62	74	80	66	73	117	35	115	42	84	45	53	846
12	111	60	62	76	62	59	96	38	95	44	94	38	835

S_{19} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	39	41	113	43	65	67	71	75	97	51	93	50	805
2	73	37	40	125	46	66	64	79	70	107	40	97	844
3	41	92	52	45	115	50	64	83	58	56	117	53	826
4	105	45	90	39	35	100	50	70	65	64	54	104	821
5	32	116	43	98	53	44	104	61	65	71	60	67	814
6	104	58	107	48	86	41	40	88	48	93	59	61	833
7	58	110	34	100	42	87	47	41	110	62	64	66	821
8	58	69	105	28	82	38	93	49	46	118	56	90	832
9	62	54	64	97	46	124	45	110	47	51	92	59	851
10	74	64	56	68	108	42	96	46	93	48	44	124	863
11	59	92	60	58	70	104	42	102	47	105	44	40	823
12	100	67	62	72	66	70	105	28	104	37	100	56	867

S_{23} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	43	47	108	56	88	46	75	74	114	45	101	56	853
2	78	30	50	108	49	88	56	79	64	114	53	98	867
3	42	107	36	54	98	57	81	68	74	80	109	51	857
4	104	55	93	29	48	98	45	73	51	56	54	111	817
5	64	110	57	102	31	55	92	56	84	53	52	69	825
6	100	58	111	48	93	33	46	110	59	65	51	71	845
7	60	113	56	95	52	99	29	37	105	53	58	59	816
8	66	74	104	50	102	45	67	43	54	99	56	73	833
9	49	58	62	103	50	108	62	101	42	47	112	66	860
10	81	62	53	57	82	53	94	40	75	36	46	105	784
11	59	91	53	69	68	92	55	101	38	84	34	55	799
12	107	62	74	46	64	71	114	51	99	52	73	31	844

S_{29} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	33	69	85	65	80	54	70	70	101	58	110	46	841
2	74	41	52	94	60	92	42	83	46	117	49	103	853
3	46	99	39	61	96	63	80	55	67	61	107	62	836
4	107	65	98	36	47	99	52	88	52	73	60	101	878
5	58	125	53	88	27	56	81	65	75	49	75	59	811
6	116	55	103	54	86	25	57	111	53	66	41	56	823
7	50	101	53	116	53	80	32	65	94	51	75	45	815
8	75	77	102	59	114	53	90	43	42	96	81	69	901
9	45	52	69	114	48	95	58	100	21	56	89	45	792
10	84	46	67	65	85	48	97	52	80	32	51	96	803
11	59	62	42	83	60	104	66	105	69	95	48	67	860
12	94	61	73	43	55	55	90	64	91	49	74	38	787

S_{31} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	29	66	92	67	64	59	78	56	124	65	97	61	858
2	80	35	52	94	63	68	37	73	55	108	49	104	818
3	56	99	37	55	86	49	73	60	79	64	91	45	794
4	103	59	94	44	54	87	54	74	54	90	53	97	863
5	43	104	61	106	33	64	93	61	81	54	73	55	828
6	98	39	83	52	87	30	56	88	62	65	43	87	790
7	58	129	47	90	52	95	32	66	100	70	77	41	857
8	71	62	90	70	97	44	88	33	50	101	57	59	822
9	69	65	61	104	56	87	67	88	34	61	107	64	863
10	91	39	65	50	106	64	109	45	92	31	68	102	862
11	64	61	52	84	61	94	69	117	36	101	33	59	831
12	96	60	60	47	69	49	101	61	96	52	83	40	814

S_{37} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	31	63	91	60	67	41	74	61	108	65	91	57	809
2	91	39	44	103	52	70	41	71	52	101	76	90	830
3	51	111	31	57	79	57	68	53	77	61	114	67	826
4	86	61	96	26	39	95	42	69	53	82	65	98	812
5	54	100	59	100	37	57	84	55	75	51	81	65	818
6	108	66	92	62	95	34	53	84	44	83	40	81	842
7	62	110	74	87	50	98	27	55	112	66	72	50	863
8	74	53	93	56	99	51	83	34	49	88	49	81	810
9	43	74	58	111	63	90	78	92	33	79	83	50	854
10	78	39	73	49	75	78	111	61	93	31	47	123	858
11	47	71	39	62	74	127	69	111	56	96	33	49	834
12	84	43	76	39	88	44	133	64	101	55	84	33	844

S_{41} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	37	58	93	66	66	49	84	55	110	57	106	59	840
2	96	34	54	86	51	88	48	76	52	96	45	86	812
3	69	93	40	61	81	55	61	56	74	57	115	80	842
4	82	66	104	46	49	98	44	61	45	84	47	102	828
5	52	113	54	102	37	48	88	52	75	51	77	65	814
6	124	55	87	52	81	43	83	89	43	78	41	78	854
7	56	119	75	115	62	106	33	55	94	50	78	45	888
8	69	50	99	53	90	51	110	25	49	80	49	67	792
9	51	62	48	91	59	104	66	97	33	57	95	51	814
10	63	42	79	37	107	47	92	53	87	32	57	108	804
11	43	80	39	74	52	110	69	111	68	106	43	63	858
12	98	40	70	45	79	55	110	62	83	56	106	50	854

S_{43} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	46	70	76	66	69	46	81	56	114	58	108	72	862
2	100	40	58	92	45	58	41	77	63	115	57	95	841
3	67	84	50	58	83	52	77	57	78	58	111	65	840
4	97	59	107	31	40	103	46	67	36	79	54	100	819
5	65	115	45	94	38	56	76	37	71	50	81	66	794
6	106	60	101	72	99	35	45	82	40	66	46	72	824
7	45	113	57	86	67	76	35	59	103	44	80	52	817
8	68	56	108	59	87	53	86	32	51	88	35	87	810
9	42	79	52	96	58	103	65	121	39	52	87	56	850
10	71	46	59	49	88	50	105	65	98	43	54	104	832
11	55	71	48	79	51	128	55	95	52	116	47	57	854
12	101	48	79	37	69	64	105	62	104	63	94	31	857

S_{47} modulo $3+2\sqrt{-1}$ (norm=13)

	1	2	3	4	5	6	7	8	9	10	11	12	
1	45	56	97	51	72	44	79	49	108	59	87	83	830
2	96	38	64	109	38	67	48	80	46	93	63	97	839
3	68	112	45	63	85	47	61	47	73	54	108	67	830
4	112	75	93	48	56	95	46	69	39	82	47	102	864
5	52	114	60	97	47	58	111	58	59	37	75	56	824
6	102	63	86	66	99	46	41	99	42	71	53	68	836
7	43	99	59	117	70	105	42	76	97	51	58	52	869
8	81	43	94	50	94	60	105	39	54	92	43	62	817
9	43	54	46	89	63	115	65	87	40	65	97	37	801
10	59	55	75	55	85	58	94	60	90	45	48	102	826
11	45	85	44	79	50	101	64	100	56	103	41	61	829
12	84	45	67	40	65	40	113	53	96	75	109	48	835