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# THE CAUCHY PROBLEM FOR DIFFERENTIAL OPERATORS WITH DOUBLE CHARACTERISTICS

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ABSTRACT. In this monograph we provide a general picture of the Cauchy problem for differential operators with double characteristics from the viewpoint that the Hamilton map and the geometry of orbits of the Hamilton flow completely characterizes the well/ill-posedness of the Cauchy problem.

## 1. Introduction

Let $P$ be a differential operator of order $m$ defined in a neighborhood of $\bar{x} \in \mathbb{R}^{n+1}$ and let $t = t(x)$ be a real valued smooth function given in a neighborhood of $\bar{x}$ with $t(\bar{x}) = 0$. We assume that $P$ is noncharacteristic with respect to $H = \{t(x) = 0\}$ at $\bar{x}$, that is, $(Pt^m)(\bar{x}) \neq 0$. Let $u_0(x), \dots, u_{m-1}(x)$ be $m$-tuple smooth functions on $H$ defined near $\bar{x}$. Then the Cauchy problem is to find $u$, in a neighborhood of $\bar{x}$, satisfying $Pu = 0$ near $\bar{x}$ and $(\partial/\partial\nu)^j u(x) = u_j(x)$, $j = 0, \dots, m-1$, on $H$, where $\nu$ is the unit normal to $H$. Here $(u_0, \dots, u_{m-1})$ is called the initial data or the Cauchy data. Roughly speaking, the Cauchy problem is said to be $E$ well-posed in the direction $t$ if for any initial data in $E$, which is a function space given beforehand, there exists a unique solution to the Cauchy problem, and the differential operator for which the Cauchy problem is well-posed in the direction $t$ is called hyperbolic in the direction $t$. Our main concern in this monograph is to investigate which operators are hyperbolic in the direction $t$, where $t$ is supposed to be given.

Choosing a system of local coordinates $x = (x_0, x') = (x_0, x_1, \dots, x_n)$ so that $t(x) = x_0$, $\bar{x} = 0$, and dividing $P$ by a nonvanishing function, then we have

$$P = D_0^m + \sum_{|\alpha| \le m, \alpha_0 < m} a_\alpha(x) D^\alpha = \sum_{j=0}^{m} P_j(x, D)$$

in these coordinates, where $P_j(x, D)$ denotes the homogeneous part of $P$ of degree $j$ and $D = (D_0, D') = (D_0, D_1, \dots, D_n)$, $D_j = -\sqrt{-1}\partial/\partial x_j$, $D^\alpha = D_0^{\alpha_0} \cdots D_n^{\alpha_n}$, $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$. The symbol of $P_m(x, D)$,

$$p(x, \xi) = \xi_0^m + \sum_{|\alpha| = m, \alpha_0 < m} a_\alpha(x) \xi^\alpha,$$

is called the principal symbol. We start with giving a very concise definition of the well-posedness of the Cauchy problem which is equivalent to the classical definition

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of the well-posedness that requires the unique existence of a solution with initial data on every $x_0 = \tau$, $|\tau| < \delta$ with a small $\delta > 0$.

**Definition 1.1.** We say that the Cauchy problem for $P$ is $C^\infty$ well-posed near the origin in the $x_0$ direction if one can find a positive constant $\epsilon$ and a neighborhood $\omega$ of the origin such that for any $|\tau| \le \epsilon$ and $f(x) \in C_0^\infty(\omega)$ vanishing in $x_0 < \tau$ there exists a unique $u(x) \in H^\infty(\omega)$ vanishing in $x_0 < \tau$ which satisfies $Pu = f$ in $\omega$. Here $H^\infty(\omega)$ stands for $\bigcap_{k=0}^\infty H^k(\omega)$ with a standard $L^2$ based Sobolev space $H^k(\omega)$.

It follows easily from the definition that if $u \in H^\infty(\omega)$ vanishing in $x_0 < \tau$ satisfies $Pu = 0$ in $x_0 < t$ $(\tau < t < \epsilon)$, then it results $u = 0$ in $x_0 < t$. In this definition we require the causality that the future does not influence the past, which is much weaker, at a glance, than the requirement of the finite propagation speed, but this requirement of the causality consists in the essential part of the hyperbolicity.

**Lemma 1.2.** *Assume that the Cauchy problem for $P$ is $C^\infty$ well-posed near the origin in the $x_0$ direction. Then one can find a positive $\epsilon$ and a neighborhood $\omega$ of the origin such that for any compact set $K \subset \omega$ and for any $p \in \mathbb{N}$ there exist $C > 0$, $q \in \mathbb{N}$ such that the estimate*

$$\|u\|_{H^p(K\cap\{x_0\le t\})} \le C\|Pu\|_{H^q(K\cap\{x_0\le t\})}$$

*holds for every $u \in C_0^\infty(K \cap \{x_0 \ge -\epsilon\})$ and every $|t| < \epsilon$.*

This could be considered to be an expression of the causality by an inequality. Here we recall the definition of strictly hyperbolic operators.

**Definition 1.3.** We say that $P$ is strictly hyperbolic near the origin in the $x_0$ direction if the characteristic roots, that is, the roots of $p(x, \xi_0, \xi') = 0$ with respect to $\xi_0$, are real distinct for any $(x, \xi')$, $\xi' \ne 0$, $x$ in some neighborhood $\Omega$ of the origin.

The Cauchy problem for general higher order strictly hyperbolic systems was first studied by Petrovsky [54], and he derived energy estimates and proved the $C^\infty$ well-posedness for any lower order term. The work was too hard to penetrate, and the first simplification was made by Leray [31], where he derived energy estimates by constructing a symmetrizer and constructed the solution by approximation from the analytic case. Soon afterwards Gårding [12] proved the existence of solutions by functional analysis alone without the approximation process. Shortly afterwards the Fourier analysis approach of Petrovsky reappeared by use of singular integral operators [37, 38]. Nowadays we also have a middle course, reducing higher order equations to first order equations by use of pseudodifferential operators and using energy estimates for first order operators. See [16] for example.

**Theorem 1.4** ([54], [31])**.** *Assume that $P$ is strictly hyperbolic near the origin in the $x_0$ direction. Then for $P + Q$ with any differential operator of order $m - 1$ the Cauchy problem is $C^\infty$ well-posed near the origin in the $x_0$ direction.*

In what follows we often omit "$x_0$ direction" and "$C^\infty$" so that "well-posed" means $C^\infty$ well-posed in the $x_0$ direction.

**Definition 1.5.** $P$ is said to be strongly hyperbolic[1] near the origin if the Cauchy problem for $P+Q$ is $C^\infty$ well-posed near the origin for any $Q$ of order $m-1$.

According to this definition we have the following.

**Corollary 1.6.** *Strictly hyperbolic operators are strongly hyperbolic operators.*

Meanwhile it was proved that the characteristic roots must be real for the Cauchy problem to be well-posed, in [29] for the case of simple characteristics and in [39] in full generality.

**Theorem 1.7** ([29], [39])**.** *Assume that the Cauchy problem for $P$ is well-posed near the origin. Then all the characteristic roots $\xi_0$ are real for any $\xi' \in \mathbb{R}^n$ and any $x \in \omega$ with some neighborhood $\omega$ of the origin.*

After standing about ten years, it was proved that the Levi condition[2] is necessary and sufficient for the well-posedness of the Cauchy problem for differential operators with real characteristics of constant multiplicity of at most two in Mizohata and Ohya [40, 41]. Subsequently the necessity of the Levi condition for the well-posedness was proved in Flaschka and Strang [11] and the sufficiency was proved in Chazarain [8] for differential operators with real characteristics of constant multiplicity of any order. Soon after it was actively studied under which conditions the Cauchy problem is well-posed, assuming a priori the smoothness of characteristic roots. Around the same period the work of Ivrii and Petkov [18] appeared (which introduced the Hamilton map, the linearization of the Hamilton vector field $H_p$ at a multiple characteristic, and clarified some close relations between the well-posedness of the Cauchy problem and the structure of the Hamilton map), which gave an impact to researchers on hyperbolic differential equations. ($p$ fails to be strictly hyperbolic polynomial at singular points of $H_p$.) In this monograph, we try to give an overview on progress of the well-posedness of the Cauchy problem for differential operators with double characteristics obtained in the 1980s along the direction suggested in [18]. For results on the Cauchy problem for differential operators in the 1980s, including an overview of hyperbolic differential operators with constant coefficients, we refer to Gårding [13, 14] and Melrose [34], Ivrii [22].

## 2. Hamilton map and well-posedness of the Cauchy problem

By taking Theorem 1.7 into consideration we assume that the characteristic roots of $p(x,\xi)$ are all real. We start with the definition of characteristics.

**Definition 2.1.** If $p(x,\xi)$ vanishes at $\rho = (x^0,\xi^0) \in \mathbb{R}^{2(n+1)}$, $\xi^0 \neq 0$, of order $r$, that is, $\partial_x^\alpha \partial_\xi^\beta p(\rho) = 0$ for any $|\alpha+\beta| < r$ and $\partial_x^\alpha \partial_\xi^\beta p(\rho) \neq 0$ for some $|\alpha+\beta| = r$, we call $\rho$ a characterisitc of order $r$ of $p$.

By definition strictly hyperbolic operators are those whose characteristics are real and simple. If the Cauchy problem for differential operators with multiple characteristics is well-posed, then the following necessary condition must be verified at such multiple characteristics.

[1] In [18] strongly hyperbolic operators are called regularly hyperbolic operators.

[2] The word Levi condition stems from [32]. One space-dimensional case was also studied in [30].

**Theorem 2.2** ([18])**.** *Assume that the Cauchy problem for $P$ is well-posed near the origin and let $(0,\xi^0)$ be a characteristic of order $r$. Then we have*

$$\partial_x^\alpha \partial_\xi^\beta P_{m-j}(0,\xi^0)=0, \quad |\alpha+\beta|<r-2j \;\; j=0,\dots,[r/2],$$

*where $[r/2]$ stands for the integer part of $r/2$.*

In [18] we find many other necessary conditions for the well-posedness. Here we only cite a necessary condition which is independent of the choice of local coordinates. For differential operators with real simple characteristics we have Theorem 1.4. Then, in what follows, we are concerned with differential operators with double characteristics.

**Definition 2.3.** One calls

$$H_p=\sum_{j=0}^{n}\Big(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j}-\frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big)$$

the Hamilton vector field of $p$. The integral curves of $H_p$, that is, solutions of the Hamilton equation $\dot X=H_p(X)$ on the surface $p=0$, are called the bicharacteristic of $p$.

Multiple characteristics of $p$ are singular (stationary) points of the Hamilton vector field $H_p$. Let $\rho=(x^0,\xi^0)$ be a double characteristic of $p$. We linearize the Hamilton equation $\dot X=H_p(X)$ at $\rho$ where $X=(x,\xi)$; that is, by inserting $X(s)=(x^0,\xi^0)+\epsilon Y(s)$ into the equation, then the term linear in $\epsilon$ in the resulting equation is $\dot Y=2F_p(\rho)Y$, where $F_p(\rho)$ is given by

$$F_p(\rho)=\frac{1}{2}\begin{pmatrix} \dfrac{\partial^2 p}{\partial x\partial \xi}(\rho) & \dfrac{\partial^2 p}{\partial \xi\partial \xi}(\rho) \\ -\dfrac{\partial^2 p}{\partial x\partial x}(\rho) & -\dfrac{\partial^2 p}{\partial \xi\partial x}(\rho)\end{pmatrix}.$$

**Definition 2.4.** We call $F_p(\rho)$ the Hamilton map[3] of $p$ at $\rho$.

A special spectral structure of $F_p(\rho)$ results from the fact that $p(x,\xi_0,\xi')=0$ has only real roots $\xi_0$ for any $(x,\xi')$.

**Lemma 2.5** ([18], [15])**.** *All eigenvalues of the Hamilton map $F_p(\rho)$ are on the imaginary axis, with possibly one exception being a pair of nonzero real eigenvalues $\pm\lambda$, $\lambda>0$.*

**Definition 2.6.** One says that $\rho$ is an effectively hyperbolic characteristic if $F_p(\rho)$ has a nonzero real eigenvalue; we also say that $p(x,\xi)$ is effectively hyperbolic[4] at $\rho$. Otherwise $\rho$ is said to be noneffectively hyperbolic characteristic and $p(x,\xi)$ is called noneffectively hyperbolic at $\rho$.

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[3] We call $F_p$ the Hamilton map after [15]. In [18] they called $F_p$ the fundamental matrix. One of the authors [18] has told me an episode about the origin of the terminology fundamental matrix: When he was a graduate student, they had the following definitions among mathematical students: "Derivative" of the drunken party is the party financed through deposit bottles and in order to be able to get one bottle in the second round one should consume at least 13 in the first. "Fundamental" drunken party is a party with a nonzero second derivative.

[4] The word effective is chosen in [15] and stems from Ivrii's conjecture.

**Definition 2.7.** The positive trace of $F_p(\rho)$ is given by

$$\mathrm{Tr}^+ F_p(\rho) = \frac{1}{2}\sum |\mu_j|,$$

where the sum is taken over all pure imaginary eigenvalues $i\mu_j$ of $F_p(\rho)$ repeated according to their multiplicities.

**Theorem 2.8** ([18])**.** *Assume that $P$ is strongly hyperbolic near the origin. Then there is a neighborhood of the origin where every multiple characteristic of $p$ is at most double and effectively hyperbolic.*

In [18] they conjectured that the converse is also true. This conjecture was affirmatively answered in [19], [42], [23] for special cases, in [24], [35],[5] [45] for general second order operators and in [26], [43] for general higher order operators.

**Theorem 2.9** ([19], [35], [23, 24, 26], [42, 43, 45])**.** *Assume that every multiple characteristic of $p$ is at most double and effectively hyperbolic. Then $P$ is strongly hyperbolic near the origin.*

In [23, 24] the proofs were based on the transformation of the operator $P$ to an operator with "nice" lower order terms by means of integro-pseudodifferential operators and on the energy estimates for the resulting operator, while in [43, 45] the proof was based on weighted energy estimates with pseudodifferential weights of which symbol is a power of (microlocal) time function, after some preliminary transformation by Fourier integral operators. For details we refer to [25], [27]. It is possible to avoid the use of Fourier integral operators in the latter method [49].[6]

In what follows we are concerned with the case where $p$ is noneffectively hyperbolic at double characteristics. The subprincipal symbol $P_{sub}(x,\xi)$ of $P$ is defined by reference to any local coordinates $x$ as follows:[7]

$$P_{sub}(x,\xi) = P_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=0}^{n} \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\xi).$$

**Theorem 2.10** ([18], [15])**.** *Let $\rho = (0,\xi^0)$ be a noneffectively hyperbolic charcteristic of $p$. If the Cauchy problem for $P$ is $C^\infty$ well-posed near the origin, then we have*

$$\mathsf{Im}\, P_{sub}(\rho) = 0, \quad -\mathrm{Tr}^+ F_p(\rho) \le \mathsf{Re}\, P_{sub}(\rho) \le \mathrm{Tr}^+ F_p(\rho). \tag{2.1}$$

This was proved in [18] for one of three cases depending on the properties of $F_p(\rho)$, and the proof for the other two cases was given in [15].

**Definition 2.11.** The condition (2.1) is called the Ivrii-Petkov-Hörmander condition (IPH condition for short). If $\mathrm{Tr}^+ F_p(\rho) = 0$ the IPH condition is reduced to $P_{sub}(\rho) = 0$ and is called the Levi condition.

Let $\rho$ be a double characteristic of $p$. Then we have as $\epsilon \to 0$

$$p(\rho + \epsilon X) = \epsilon^2\big(p_\rho(X) + O(\epsilon)\big), \quad X = (x,\xi) \in \mathbb{R}^{2(n+1)},$$

[5] Estimates proving the regularity of solutions are obtained. But the estimates are not usual ones and are not enough to prove the well-posedness of the Cauchy problem.

[6] If we consider two or more differential operators with the same effectively hyperbolic characteristics, we are forced to treat the problem without Fourier integral operators.

[7] $P_{sub}$ is invariantly defined at double characteristics.

where $p_\rho(X)$ is called the localization of $p$ at $\rho$, which is nothing but the second order term in the Taylor expansion of $p$ at $\rho$, and hence would be considered as the first approximation of $p$ near $\rho$.

**Lemma 2.12** ([18])**.** *$p_\rho(X)$ is a quadratic hyperbolic form in $X = (x, \xi) \in \mathbb{R}^{2(n+1)}$, that is, a quadratic form of signature $(-1, 1, \dots, 1, 0, \dots, 0)$.*

Let $p_\rho(X, Y)$ be the polar form of $p_\rho(X)$. Then it is clear that

$$p_\rho(X, Y) = \sigma(X, F_p(\rho)Y), \quad X, Y \in \mathbb{R}^{2(n+1)},$$

and in particular we have $p_\rho(X) = \sigma(X, F_p(\rho)X)$, where $\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle$ is the standard symplectic 2 form on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $\langle x, y\rangle = \sum_{j=0}^{n} x_j y_j$.

**Lemma 2.13** ([15])**.** *Let $Q(X)$ be a quadratic hyperbolic form on $\mathbb{R}^{2(n+1)}$ and let $F \in M_{2(n+1)}(\mathbb{R})$ be the Hamilton map of $Q$, that is, the map given by the formula*

$$\frac{1}{2}Q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbb{R}^{2(n+1)}.$$

*Then choosing a suitable symplectic basis on $\mathbb{R}^{2(n+1)}$ we see that $Q$ takes one of the following forms:*

(1) $Q = \lambda(x_0^2 - \xi_0^2) + \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{\ell} \xi_j^2$,
(2) $Q = -\xi_0^2 + \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{\ell} \xi_j^2$,
(3) $Q = -\xi_0^2 + 2\xi_0\xi_1 + x_1^2 + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{\ell} \xi_j^2$

*where $\lambda > 0$, $\mu_j > 0$. In the case (1) $F$ has a nonzero real eigenvalue, and in the cases (2) and (3) all eigenvalues of $F$ are on the imaginary axis. In the cases (1) and (2) we have $\operatorname{Ker} F^2 \cap \operatorname{Im} F^2 = \{0\}$, while $\operatorname{Ker} F^2 \cap \operatorname{Im} F^2 \neq \{0\}$ in the case (3).*

Therefore in a suitable symplectic basis, the localization $p_\rho(X)$ of $p$ will be one of (1), (2), (3) in Lemma 2.13. But in studying the well-posedness of the Cauchy problem, not all canonical transformations are allowed,[8] and hence the canonical form of $p_\rho$ in Lemma 2.13 would not always be applicable. This is a main reason why the studies of the Cauchy problem are not so straightforward. In [17], partly motivated by this observation, quadratic hyperbolic operators are intensively studied.

In what follows we restrict ourselves to study the Cauchy problem for differential operators with characteristics at most double which are noneffectively hyperbolic. Since the studies of the Cauchy problem for differential operators with characteristics at most double can be reduced to those for second order operators, differential in $x_0$, and pseudodifferential in $x'$, then in what follows we assume that $P$ takes the form

$$P(x, D) = -D_0^2 + A_1(x, D')D_0 + A_2(x, D')$$

with classical pseudodifferential operators $A_j(x, D')$ of order $j$ so that $A_j(x, \xi') \sim A_{j0} + A_{j1} + \cdots$. Here $A_{jk}$ is of homogeneous of degree $j - k$ in $\xi'$. Then the principal symbol $p(x, \xi)$ is

$$p(x, \xi) = -\xi_0^2 + A_{10}(x, \xi')\xi_0 + A_{20}(x, \xi').$$

[8] One can only use the canonical transformation such that the Fourier integral operator associated with it preserves the causality.

Let us set $P_1(x,\xi) = A_{11}(x,\xi')\xi_0 + A_{21}(x,\xi')$. In this note we always assume that the doubly characteristic set $\Sigma$ of $p(x,\xi)$ verifies the following conditions:[9]

$$\begin{cases} \Sigma \text{ is a } C^\infty \text{ manifold,} \\ \dim T_\rho\Sigma = \dim \operatorname{Ker} F_p(\rho), \quad \rho \in \Sigma, \\ \operatorname{rank}(\sigma|_\Sigma) = \text{constant.} \end{cases} \tag{2.2}$$

By conjugation with a Fourier integral operator with $x_0$ as a parameter, one can assume $p(x,\xi) = -\xi_0^2 + q(x,\xi')$. For this $p$, the conditions (2.2) are equivalent to saying that one can find, at each $\rho \in \Sigma$, $\xi_0 = \phi_0$, $\phi_j(x,\xi')$, $j = 1,\dots,r$, with linearly independent differentials $d\phi_j(\rho)$ such that

$$\begin{cases} p = -\xi_0^2 + \sum_{j=1}^r \phi_j(x,\xi')^2, \quad \Sigma = \{\phi_j = 0, j = 0,\dots,r\}, \\ \operatorname{rank}(\{\phi_i,\phi_j\}(\rho))_{0\le i,j\le r} = \text{constant}, \quad \rho \in \Sigma, \end{cases}$$

holds in a neighborhood of $\rho$, where $\{\phi_i,\phi_j\}$ denotes the Poisson bracket of $\phi_i$ and $\phi_j$:

$$\{\phi_i,\phi_j\} = \sum_{k=0}^n \Big(\frac{\partial\phi_i}{\partial\xi_k}\frac{\partial\phi_j}{\partial x_k} - \frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial\xi_k}\Big) = H_{\phi_i}\phi_j.$$

**Theorem 2.14** ([20], [15]). *Assume that $p$ is noneffectively hyperbolic at every point of $\Sigma$ and*

$$\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 = \{0\} \tag{2.3}$$

*holds on $\Sigma$. If the conditions*

$$\mathsf{Im}\, P_{sub} = 0, \quad \operatorname{Tr}^+ F_p + \mathsf{Re}\, P_{sub} \ge \epsilon \tag{2.4}$$

*hold on $\Sigma$ with some $\epsilon > 0$, then the Cauchy problem for $P$ is well-posed near the origin.*

*Remark* 2.15. If (2.3) and $\operatorname{Tr}^+ F_p = 0$ hold on $\Sigma$, then $\Sigma$ is an involutive manifold, and in this case the Cauchy problem is well-posed if and only if the Levi condition is satisfied on $\Sigma$ ([15]).

We make some comments on the proof of Theorem 2.14 in the next section. In the following we quote condition (2.4) as the strict IPH condition. At this stage the main remaining problem is that in the case that (2.3) fails, and hence the condition

$$\operatorname{Ker} F_p^2 \cap \operatorname{Im} F_p^2 \ne \{0\} \tag{2.5}$$

is assumed on $\Sigma$, is the Cauchy problem still $C^\infty$ well-posed under the strict IPH condition or are other new conditions needed for the well-posedness? There also remains the question as to whether one can take $\epsilon = 0$ in (2.4), that is, whether or not the IPH condition itself is sufficient for the well-posedness.

[9] It seems to be a quite reasonable assumption when one studies a general case.

## 3. Energy estimates and elementary decomposition

In this note pseudodifferential operators are assumed to be Weyl quantized, that is, we define the pseudodifferential operator $p(x', D')$ with symbol $p(x', \xi')$ by

$$p(x', D')u = (2\pi)^{-n} \int e^{i(x'-y')\xi'} p((x'+y')/2, \xi')u(y')dy'd\xi'.$$

We also denote $p(x', D') = \mathrm{Op}(p(x', \xi'))$ and the symbol of $p(x', D')q(x', D')$ by $p(x', \xi')\# q(x', \xi')$. If there is no confusion we denote the symbol $p(x', \xi')$ or the operator $p(x', D')$ by the same $p$. Let us consider

$$P = -M\Lambda + B\Lambda + Q,$$

where $M = D_0 - m(x, D')$, $\Lambda = D_0 - \lambda(x, D')$, and $m(x, \xi')$, $\lambda(x, \xi')$, $B(x, \xi') \in S(\langle\xi'\rangle, g_0)$. We assume $Q(x, \xi') \in S(\langle\xi'\rangle^2, g_0)$, where $g_0 = |dx|^2 + \langle\xi'\rangle^{-2}|d\xi'|^2$, and hence $S(\langle\xi'\rangle^m, g_0) = S^m_{1,0}$. For pseudodifferential operators, we refer to Chapter 18 in [16]. With a large positive parameter $\theta$ we put

$$P_\theta = P(x, D_0 - i\theta, D'), \quad \Lambda_\theta = \Lambda - i\theta, \quad M_\theta = M - i\theta$$

so that $P_\theta = -M_\theta\Lambda_\theta + B\Lambda_\theta + Q$. Denote by $(u, v)$ the $L^2(\mathbb{R}^n)$ inner product. For later use we state an energy inequality in a somewhat more general form than is needed here.

**Proposition 3.1.** *The following inequality holds:*

$$\begin{aligned}2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \geq \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + ((\mathsf{Re}\, Q)u, u) + \theta^2\|u\|^2) + \theta\|\Lambda_\theta u\|^2 \\ +2\theta\mathsf{Re}(Qu, u) + 2((\mathsf{Im}\, B)\Lambda_\theta u, \Lambda_\theta u) + 2((\mathsf{Im}\, m)\Lambda_\theta u, \Lambda_\theta u) \\ +2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\, Q)u) + \mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}\, Q]u, u) \\ +2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\lambda)u) + \theta^3\|u\|^2 + 2\theta^2((\mathsf{Im}\lambda)u, u).\end{aligned}$$

Let $\rho$ be a double characteristic of $p(x, \xi) = -\xi_0^2 + \sum_{j=1}^r \phi_j^2 = -\xi_0^2 + q(x, \xi')$.

**Definition 3.2.** We say that $p(x, \xi)$ admits an elementary decomposition at $\rho$ if we can find classical real valued symbols $\lambda$ , $m$ of degree 1, a nonnegative symbol $Q$ of degree 2 defined in a conic neighborhood $U$ of $\rho$, and a positive constant $C$ such that we have[10]

$$\begin{cases} p(x, \xi) = -(\xi_0 - m)(\xi_0 - \lambda) + Q, \quad \big|\{\xi_0 - \lambda, Q\}\big| \leq CQ, \\ \big|\{\xi_0 - m, \xi_0 - \lambda\}\big| \leq C(\sqrt{Q} + |m - \lambda|) \end{cases} \tag{3.1}$$

in $U$.

Assuming (2.3) it can be seen that $p$ admits a local elementary decomposition, that is, we can find classical symbols $\lambda$, $m$, $Q$ defined in a neighborhood $W$ of the origin such that (3.1) holds in $W$.

**Lemma 3.3.** *Assume* (2.3). *For any $\bar\rho \in \Sigma$ there exist a conic neighborhood $V$ of $\bar\rho$ and a smooth $h(\rho)$ defined in $V \cap \Sigma$ satisfying*

$$h(\rho) \in \mathrm{Ker}\, F_p^2(\rho), \quad p_\rho(h(\rho)) < 0, \quad \sigma(H_{x_0}, F_p(\rho)h(\rho)) = -1. \tag{3.2}$$

*When $h(\rho)$ satisfies* (3.2) *it follows from $\sigma(v, F_p(\rho)h(\rho)) = 0$ and $v \neq 0$ that $p_\rho(v) > 0$.*

[10] In the present case $p = -\xi_0^2 + q$ we have necessarily $m = -\lambda$.

Put $w(\rho) = F_p(\rho)h(\rho)$ with $h(\rho)$ obtained in Lemma 3.3. Since $\operatorname{Im} F_p(\rho)$ is the linear subspace spanned by $\{H_{\xi_0}, H_{\phi_1}, \dots, H_{\phi_r}\}$, one may assume $w(\rho) = H_{\xi_0} - \sum_{j=1}^r \gamma_j H_{\phi_j}$. With

$$\lambda = \sum_{j=1}^{r} \gamma_j(x,\xi')\phi_j(x,\xi')$$

we write

$$p = -(\xi_0+\lambda)(\xi_0-\lambda)+\hat{q}, \quad \hat{q} = \sum_{j=1}^{r}\phi_j^2 - (\sum_{j=1}^{r}\gamma_j\phi_j)^2 = q-\lambda^2.$$

Since $H_{\xi_0-\lambda} \in \operatorname{Ker} F_p$ on $V \cap \Sigma$ we see $\{\xi_0-\lambda, \phi_j\} = 0$, $j = 1, \dots, r$. It follows from Lemma 3.3 that $\sum_{j=1}^r \gamma_j^2 < 1$, and hence

$$|\lambda|^2 \le \delta q \tag{3.3}$$

holds with some $\delta < 1$. This gives an elementary decomposition of $p$ at $\bar{\rho}$. Therefore by a compactness argument we can choose a finite number of $\rho_i' \in \Sigma' \cap \{|\xi'| = 1\}$, $\Sigma' = \{\phi_j = 0, j = 1, \dots, r\}$ and conic neighborhoods $V_i$ of $\rho_i'$ where we have elementary decomposition. Using a partition of unity $\{\chi_i\}$ subordinate to the covering $\{V_i\}$ we define $\lambda = \sum \chi_i\lambda_i$. Since one can take $\delta < 1$ in (3.3), then taking advantage of this space one can show that $p = -(\xi_0+\lambda)(\xi_0-\lambda)+(q-\lambda^2)$ gives a local elementary decomposition.

**Proposition 3.4.** *Assume that* (2.3) *holds on* $\Sigma$. *Then* $p$ *admits a local elementary decomposition.*

We now assume that $p(x,\xi)$ admits a local elementary decomposition near the origin. We first note the following lemma.

**Lemma 3.5.** *Assume that* $p(x,\xi)$ *admits an elementary decomposition at* $\rho$, $p = -(\xi_0-m)(\xi_0-\lambda)+Q$. *Then we have*

$$\operatorname{Tr}^+ F_p(\rho) = \operatorname{Tr}^+ F_{Q_\rho}$$

*where* $Q_\rho$ *is the localization of* $Q$ *at* $\rho$.

By virtue of this lemma the strict IPH condition is reduced to the condition $\operatorname{Tr}^+ F_{Q_\rho} + \mathsf{Re}\, P_{sub}(\rho) \ge \epsilon$, and hence from Melin's inequality [33, 16] there exist $c > 0$, $C > 0$ such that

$$((Q+\mathsf{Re}\, P_{sub})u,u) \ge c\|u\|_{1/2}^2 - C\|u\|^2$$

holds where $\|u\|_s = \|\langle D'\rangle^s u\|$. Noting

$$P = -M\Lambda + Q + \frac{i}{2}\{\xi_0-m,\xi_0-\lambda\}(x,D') + P_{sub} + R, \quad R \in S(1,g_0)$$

we apply Proposition 3.1 with $B = 0$, $\mathsf{Im}\, m = 0$, $\mathsf{Im}\,\lambda = 0$, $\mathsf{Im}\, Q = \{\xi_0-m,\xi_0-\lambda\}/2+\mathsf{Im}\, P_{sub}$, and $\mathsf{Re}\, Q = Q+\mathsf{Re}\, P_{sub}$. Considering the fact that $\{\xi_0-\lambda, Q\}$ is the principal symbol of $-\mathsf{Im}[D_0-\mathsf{Re}\,\lambda, \mathsf{Re}\, Q]$, the energy estimates are easily obtained under the strict IPH condition.

For the question whether one can take $\epsilon = 0$ in Theorem 2.14, there is a counterexample. Let consider the differential operator

$$P = -D_0^2 + \sum_{j=1}^{k}\mu_j(x_j^2D_n^2+D_j^2) + b(x_0)D_n = p(x,D) + b(x_0)D_n. \tag{3.4}$$

In view of Lemma 2.13 the principal symbol $p(x,\xi)$ verifies (2.3). In this case the IPH condition is equivalent to the fact that $b(x_0)$ is real valued and $|b(x_0)| \le \sum_{j=1}^{k}\mu_j$.

**Proposition 3.6** ([47]). *There exists a real valued smooth function $b(x_0)$ defined near the origin which verifies the IPH condition such that the Cauchy problem for $P$ of (3.4) is not $C^\infty$ solvable.*

On the other hand, the question as to which differential operators the IPH condition is sufficient for the well-posedness of the Cauchy problem is closely related to Melin-Hörmander's inequality (Theorem 3.3.1 in [15], Theorem 22.3.2 in [16]). We can find some related results in [53].

## 4. Well-posedness and elementary decomposition

We now ask whether the elementary decomposition might still be possible even under the condition (2.5).

**Lemma 4.1** ([21]). *Assume that $p(x,\xi)$ admits an elementary decomposition at $\rho$. Then there is no bicharacteristic with a limit point in $\Sigma$ (near$\rho$).*

Let $q_i$, $r_i$ be positive constants verifying the condition $\sum_{i=1}^{k} r_i^{-1} = 1$ and consider

$$p(x,\xi) = -\xi_0^2 + \sum_{i=0}^{k-1} q_i(x_i - x_{i+1})^2\xi_n^2 + \sum_{i=1}^{k} r_i\xi_i^2 + \xi_n^{-1}\sum_{i=1}^{k}\epsilon_i\xi_i\xi_k^2, \tag{4.1}$$

where $1 \le k \le n-1$. Note that $\sum_{i=1}^{k} r_i^{-1} = 1$ is equivalent to the condition (2.5). In [44] it was proved that there exists a bicharacteristic of $p(x,\xi)$, with a suitable choice of $\{\epsilon_i\}$, with a limit point in $\Sigma = \{\xi_i = 0, 0 \le i \le k, x_i = x_{i+1}, 0 \le i \le k-1\}$. Hence the elementary decomposition is not possible in general under the condition (2.5). Based on this fact it has been studied under which conditions the elementary decomposition is possible. Before stating the conclusion we prepare a lemma.

**Lemma 4.2.** *Assume that (2.5) holds on $\Sigma$. Then for every $\rho \in \Sigma$ one can find smooth $z_1(\rho)$, $z_2(\rho)$ such that*

$$\begin{aligned}
&z_1(\rho) \in \operatorname{Ker} F_p(\rho) \cap \operatorname{Im} F_p^3(\rho),\\
&z_2(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho),\ \ F_p(\rho)z_2(\rho) \ne 0,\\
&\sigma(w, z_1(\rho)) = 0 \Longrightarrow \sigma(w, F_p(\rho)w) \ge 0
\end{aligned}$$

*hold in a neighborhood of $\rho$ in $\Sigma$.*

Since $F_p(\rho)z_2(\rho)$ is proportional to $z_1(\rho)$, in what follows we may assume that $F_p(\rho)z_2(\rho) = -z_1(\rho)$ without restrictions. Let $S(x,\xi)$ be a smooth function vanishing on $\Sigma$ and satisfying

$$H_S(\rho) \in \operatorname{Ker} F_p^2(\rho) \cap \operatorname{Im} F_p^2(\rho),\ \ F_p(\rho)H_S(\rho) \ne 0, \quad \rho \in \Sigma. \tag{4.2}$$

**Theorem 4.3** ([46], [2]). *Assume that (2.5) is verified on $\Sigma$. Let $S$ be a smooth function satisfying (4.2) and vanishing on $\Sigma$. Then the following two conditions are equivalent:*

(1) *$H_S^3 p = 0$ on $\Sigma$,*
(2) *$p$ admits an elementary decomposition at any point on $\Sigma$.*

This result was proved in [46] with some restrictions and was completed in [2]. We sketch the proof of how to make an elementary decomposition. We first make a preliminary decomposition. In what follows we will work in a conic neighborhood of some $\rho \in \Sigma$ and there will be no special mention of this hereafter. There is a smooth $\Lambda(x,\xi)$ vanishing on $\Sigma$ such that $H_\Lambda$ is proportional to $z_1$. We may assume

$$\Lambda = \xi_0 - \lambda, \quad \lambda = \sum_{j=1}^{r} \gamma_j \phi_j = \langle \gamma, \phi \rangle.$$

Let us write $p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + |\phi|^2 - \langle \gamma, \phi \rangle^2$. It follows from Lemma 4.2 that $\sum_{j=1}^{r} \sigma(w, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j \sigma(w, H_{\phi_j}))^2 \geq 0$ if $\sigma(w, z_1) = 0$. Since the linear subspace $\{(\sigma(w, H_{\phi_j}))_{1\leq j\leq r} \mid \sigma(w, z_1) = 0\}$ coincides with $\mathbb{R}^r$, we conclude that $|\gamma| \leq 1$. On the other hand we see from $0 = \sigma(z_2, F_p z_2) = \sum_{j=1}^{r} \sigma(z_2, H_{\phi_j})^2 - (\sum_{j=1}^{r} \gamma_j \sigma(z_2, H_{\phi_j}))^2$ that $|\gamma| \geq 1$ because $(\sigma(z_2, H_{\phi_j}))_{1\leq j\leq r} \neq 0$. Thus we get $|\gamma| = 1$. We extend $\gamma$ outside $\Sigma$ keeping $|\gamma| = 1$, and we denote such an extended symbol by the same $\gamma$. Let $\psi_1 = \langle \gamma, \phi \rangle$, $\psi_2, \ldots, \psi_r$ be related to $\{\phi_j\}$ by an orthogonal transformation. Switching the notation from $\{\psi_j\}$ to $\{\phi_j\}$ again and renumbering if necessary, one can write

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + |\phi'|^2, \quad \phi' = (\phi_2, \ldots, \phi_r), \tag{4.3}$$

where $\{\xi_0 - \phi_1, \phi_j\} = 0$, $j = 1, \ldots, r$, on $\Sigma$ and $\{\phi_1, \phi_2\} \neq 0$. Although $\{\xi_0 - \phi_1, \phi_j\}$ is a linear combination of $\phi_i$, if one really needs $\phi_1$ in the expression, then one cannot control the term $\{\xi_0 - \phi_1, \phi_j^2\}$ by $|\phi'|^2$. This is the essential difference from the case (2.3).

**Proposition 4.4.** *Assume that $H_S^3 p = 0$ holds on $\Sigma$. Then we can write*

$$p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q,$$

*where $\lambda = \phi_1 + \langle \beta', \phi' \rangle \phi_1 + k\phi_1^3 |\xi'|^{-2}$ and $\xi_0 - \lambda$, $Q$, $\phi_j$ verifies the following:*

$$\begin{aligned} &Q \geq c(|\phi'|^2 + \phi_1^4 |\xi'|^{-2}), \quad |\{\xi_0 - \lambda, Q\}| \leq C(|\phi'|^2 + \phi_1^4 |\xi'|^{-2}), \\ &\{\xi_0 - \lambda, \phi_j\} = O(|\phi|), \;\; j = 1, \ldots, r, \quad \{\phi_1, \phi_2\} \neq 0, \end{aligned} \tag{4.4}$$

*where $c$, $C$ are positive constants, $\beta' = (\beta_2, \ldots, \beta_r)$ is smooth on $\Sigma$ and $k$ is a negative constant. In particular $p$ admits an elementary decomposition at every point on $\Sigma$.*

Making an orthogonal transformation of $\phi_2, \ldots, \phi_r$ we can assume that $\{\phi_1, \phi_j\} = O(|\phi|)$, $j = 3, \ldots, r$. Choosing $k$ large negative it is clear that the first inequality in (4.4) holds. We check that we can choose a smooth $\beta'$ such that Proposition 4.4 holds. With $\{\xi_0 - \phi_1, \phi_j\} = \langle \alpha_j, \phi \rangle$, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jr})$ we have

$$\{\xi_0 - \lambda, Q\} = 2(\langle \alpha'_1, \phi' \rangle + \langle \{\phi', \phi'\}\beta', \phi' \rangle)\phi_1 - 2\phi_1^3 \langle \alpha'_1, \beta' \rangle + O(Q),$$

where $\alpha'_1 = (\alpha_{21}, \ldots, \alpha_{r1})$ and $\{\phi', \phi'\}$ stands for the $(r-1)\times(r-1)$ matrix with $(i,j)$th entry $\{\phi_i, \phi_j\}$. We show that we can take $\beta'$ such that

$$\{\phi', \phi'\}\beta' + \alpha'_1 = 0, \quad \langle \alpha'_1, \beta' \rangle = 0. \tag{4.5}$$

**Lemma 4.5.** *Assume that $H_S^3 p = 0$. Then we have $\langle \alpha'_1, v \rangle = 0$ for any $v$ provided $\{\phi', \phi'\}v = 0$.*

Since the rank of $\{\phi', \phi'\}$ is constant by assumption, there exists a smooth $\beta'$ satisfying the first equation of (4.5) in view of Lemma 4.5. The second identity follows from the first one because $\{\phi', \phi'\}$ is antisymmetric.

Assume (2.5). If $H_S^3 p = 0$ holds on $\Sigma$, then from Theorem 4.3 the elementary decomposition of $p$ is possible at every point on $\Sigma$, and hence under the strict IPH condition one can obtain microlocal energy estimates.[11] Nevertheless since $\phi_1^2$ could not be controlled by $Q$, it is not known whether or not $p$ admits a local elementary decomposition. It also seems to be hard to correct microlocal energy estimates.[12] So we abandon this way and adopt the technique employed in [43]. We fix any $\rho \in \Sigma$. There is a conic neighborhood of $\rho$ where Proposition 4.4 holds. We extend $\phi_j$ by 0 outside such a conic neighborhood of $\rho'$ ($\rho = (0, \rho')$) so that $\phi_j \in S(\langle \xi' \rangle, g_0)$ and define $\lambda$ as in Proposition 4.4. Let $\chi$ be 1 near $\rho'$ with small compact support and let $b(x, \xi')$ be the solution to

$$\{\xi_0 - \lambda, b\} = 0, \quad b(0, x', \xi') = (1 - \chi(x', \xi'))\langle \xi' \rangle. \tag{4.6}$$

Then $\hat{p} = -(\xi_0 + \lambda)(\xi_0 - \lambda) + \hat{Q}$ with $\hat{Q} = q + Mb(x, \xi')^2$ admits a local elementary decomposition provided $M > 0$ is large enough. Take $\chi_1$ of smaller support than $\chi$ and define $b_1(x, \xi')$ just as in (4.6). Then

$$\hat{P} = \hat{p} + P_1 + Mb_1(x, \xi') + P_0$$

satisfies the strict IPH condition. In the following we denote $\hat{P}$, $\hat{Q}$ by $P$, $Q$, respectively, again. Put $\Lambda = D_0 - \lambda$ and set

$$N_s(u) = \|\Lambda u\|_s^2 + \mathsf{Re}\,((Q + \mathsf{Re}\, P_{sub})u, u)_s + \|u\|_{s+1/2}^2.$$

Then we can find $T > 0$ such that with $I = [-T, T]$ we have

$$N_s(u(t)) + \int_{-T}^{t} N_s(u(x_0))dx_0 \le C(s, T) \int_{-T}^{t} \|Pu\|_s^2 dx_0 \tag{4.7}$$

for any $u \in C^2(I; H^\infty(\mathbb{R}^n))$ vanishing in $x_0 < \tau$, $|\tau| < T$ and for any $s \in \mathbb{R}$. For the adjoint operator $P^* = \mathrm{Op}(p + \bar{P}_{sub}) + R$, $R \in S(1, g_0)$ the time reversed estimate (4.7) holds, and then by a standard functional analytic argument we conclude that for any $f \in C^0(I; H^\infty)$ vanishing in $x_0 \le \tau$ there exists $u \in C^2(I; H^\infty)$ vanishing in $x_0 \le \tau$ which verifies $Pu = f$. Let us denote $u = Gf$. Let $\tau$, $\epsilon$, $\nu$ be small positive numbers ($\epsilon < \tau$), and with

$$d_\epsilon(x', \xi'; \kappa') = \{\tilde{\chi}(x' - y')|x' - y'|^2 + |\xi'\langle \xi' \rangle^{-1} - \eta' \langle \eta' \rangle^{-1}|^2 + \epsilon^2\}^{1/2},$$
$$\phi(x, \xi'; \kappa') = x_0 - 2\nu\tau + \nu d_\epsilon(x', \xi'; \kappa'), \quad \kappa' = (y', \eta')$$

we define $\Phi$ by $\Phi = \exp(1/\phi(x, \xi'; \kappa'))$ if $\phi < 0$ and by $\Phi = 0$ if $\phi > 0$, where $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ cuts off a neighborhood of $x' = 0$. Here $\phi(x, \xi'; \kappa')$ is a typical example of a spatial type symbol which was used in estimating wave front sets in [20, 21]. Repeating the arguments in [20] we conclude that there is a $\nu_0 > 0$ such that we have

$$\begin{aligned} &N_s(\Phi u(t)) + \int_{-T}^{t} N_s(\Phi u)dx_0 \\ &\quad \le C(s, T)\Big(N_{s-1/4}(u(t)) + \int_{-T}^{t} \big(\|\Phi P u\|_s^2 + N_{s-1/4}(u)\big)dx_0\Big) \end{aligned}$$

[11] See [2] and Theorem 1.3 in [3].

[12] We can find an explanation about this difficulty in Section 5 in [3].

for $0 < \nu \le \nu_0$. From this estimate it follows that for any open conic sets $\Gamma_i$, $i = 0, 1, 2$, in $\mathbb{R}^{2n} \setminus \{0\}$ with $\Gamma_0 \Subset \Gamma_1 \Subset \Gamma_2$ and for any $h_i(x', \xi') \in S(1, g_0)$, $i = 1, 2$, with $\operatorname{supp} h_1 \subset \Gamma_0$, $\operatorname{supp} h_2 \subset \Gamma_2 \setminus \Gamma_1$, there exists $\delta = \delta(\Gamma_i) > 0$ such that

$$\|D_0^j h_2 G h_1 f(t)\|_p^2 \le C(p, q) \int_{-T}^{t} \|f(x_0)\|_q^2 dx_0, \quad j = 0, 1, \tag{4.8}$$

for any $f \in C^0(I; H^\infty)$ vanishing in $x_0 \le \tau$ and for any $|t| < \delta$, $p$, $q \in \mathbb{R}$. This inequality implies that any perturbation in $\Gamma_0$ does not go out of $\Gamma_1$ within the time interval $\delta$ (modulo the right-hand side term). Then repeating the same arguments in [43] we get the following.

**Theorem 4.6.** *Assume* (2.5) *and that $H_S^3 p = 0$ holds on $\Sigma$. Then the Cauchy problem for $P$ is $C^\infty$ well-posed under the strict IPH condition.*

*Remark* 4.7. Assume (2.5) and that $H_S^3 p = 0$, $\mathrm{Tr}^+ F_p = 0$ hold on $\Sigma$. Then the Cauchy problem is well-posed if and only if the Levi condition is satisfied on $\Sigma$ ([52]).

Now the case that (2.5) holds while $p$ does not admit an elementary decomposition remains to be studied. We study this case in the following sections.

## 5. Geometry of bicharacteristics

In this section we discuss how the geometry of bicharacteristics near the doubly characteristic manifold $\Sigma$ relates to the possibility of elementary decomposition. Recall that bicharacteristics are solutions of the Hamilton equations

$$\dot{x} = \partial p(x, \xi)/\partial \xi, \quad \dot{\xi} = -\partial p(x, \xi)/\partial x \tag{5.1}$$

on which $p(x, \xi)$ vanishes.

**Proposition 5.1** ([44])**.** *Assume that the codimension of $\Sigma$ is* 3*. Then $p$ admits an elementary decomposition if and only if there is no bicharacteristic of $p$ with a limit point in $\Sigma$.*

As for general case we have the following.

**Theorem 5.2** ([1], [48])**.** *Assume* (2.5)*. Let $U$ be a neighborhood of $\rho$ such that $H_S^3 p(\rho) \neq 0$, $\rho \in \Sigma \cap U$. Then one can find a bicharacteristic with a limit point in $U \cap \Sigma$.*

In [1], as in the same way as in [44]; they tried to find a bicharacteristic with a limit point in $\Sigma$ choosing a domain with piecewise smooth boundaries on which $H_p$ points outward except for one piece of the boundary where $H_p$ points inward.[13] In [48] the existence of such bicharacteristics was proved directly. We sketch the proof given in [48]. Assume (2.5). There is an open set $V \subset U$ where $p$ has the form

$$p = -\xi_0^2 + \sum_{j=1}^{\ell} \phi_j^2 + \sum_{j \in I_1} \phi_j^2 + \sum_{j \in I_2} \phi_j^2,$$

where $I_0 = \{0, 1, \dots, \ell\}$, $I_1$, $I_2$ are partitions of the set $\{0, 1, \dots, r\}$ with even $\ell$ $(\ge 2)$ such that $\{\phi_i, \phi_j\} = 0$ if $i$, $j$ belong to different $I_k$, and $\det(\{\phi_i, \phi_j\})_{i,j \in I_1} \neq 0$. Moreover $\{\phi_i, \phi_j\} = 0$ on $V \cap \Sigma$ if $i$, $j \in I_2$ and $\dim \operatorname{Ker}(\{\phi_i, \phi_j\})_{i,j \in I_0} = 1$. Here

[13] In the case of codimension 3 the Hamilton equations are reduced to a dynamical system in the plane and one has an advantage of special aspects of 2-dimensional dynamical systems ([44]).

we have set $\phi_0 = \xi_0$. Since the case $\ell = 2$ is easier than the case $\ell \geq 4$, we assume $\ell \geq 4$. We also assume $I_1 = \emptyset$ so that $p = -\xi_0^2 + \sum_{j=1}^{\ell} \phi_j^2 + \sum_{j=\ell+1}^{r} \phi_j^2$ for simplicity. Note that $\{\phi_i, \phi_j\} = 0$ on $V \cap \Sigma$ if $0 \leq i \leq r$, $\ell + 1 \leq j$. Using the same arguments proving (4.3) one can find a smooth orthogonal transformation of $\{\phi_j\}$, $1 \leq j \leq \ell$ to $\{\tilde{\phi}_j\}$, such that $-\xi_0^2 + \sum_{j=1}^{\ell} \phi_j^2$ is transformed, after switching the notation from $\{\tilde{\phi}_j\}$ to $\{\phi_j\}$ again, to $-(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \sum_{j=2}^{\ell} \phi_j^2$, where $\{\xi_0 - \phi_1, \phi_j\} = 0$ is verified on $V \cap \Sigma$ for every $j$. Choose a system of symplectic coordinates $(X, \Xi)$ such that $X_0 = x_0$, $\Xi_0 = \xi_0 - \phi_1$, and switching the notation from $(X, \Xi)$ to $(x, \xi)$ one can write

$$p = -\xi_0^2 - 2\xi_0\phi_1 + \sum_{j=2}^{\ell} \phi_j^2 + \sum_{j=\ell+1}^{r} \phi_j^2. \tag{5.2}$$

Since $\dim \mathrm{Ker}\,(\{\phi_i, \phi_j\}_{2 \leq i,j \leq \ell}) = 1$ one can choose $c = (c_2, \ldots, c_\ell)$ with $|c| = 1$ which spans $\mathrm{Ker}\,(\{\phi_i, \phi_j\}_{2 \leq i,j \leq \ell})$. We make a smooth orthogonal transformation from $\{\phi_j\}_{2 \leq j \leq \ell}$ to $\{\tilde{\phi}_j\}_{2 \leq j \leq \ell}$ such that $\tilde{\phi}_2 = \sum c_j \phi_j$ and denote $\tilde{\phi}_j$ by $\phi_j$ again. Then we have $\{\phi_2, \phi_j\} = 0$ on $V \cap \Sigma$ unless $j = 1$. We summarize as follows.

**Lemma 5.3.** *Choosing a suitable system of symplectic coordinates, $p$ can be written in the form* (5.2) *in some open set $V$, where*

$$\begin{gathered}\{\xi_0, \phi_j\} = 0,\ \ 1 \leq j \leq r,\ \ \{\phi_2, \phi_j\} = 0,\ \ j \neq 1,\ \ \{\phi_2, \phi_1\} \neq 0,\\ \{\phi_i, \phi_j\} = 0,\ \ 0 \leq i \leq r,\ \ \ell + 1 \leq j \leq r,\ \ \det\left(\{\phi_i, \phi_j\}\right)_{3 \leq i,j \leq \ell} \neq 0\end{gathered}$$

*holds in $V \cap \Sigma$.*

We work in a neighborhood of a fixed $\bar{\rho} \in V$. Since $\{\phi_1, \phi_2\} \neq 0$ we can take $\psi_1, \ldots, \psi_k$ $(r + k = 2n)$ so that $\xi_0$, $x_0$, $\phi_1$, $\phi_2, \ldots, \phi_r$, $\psi_1, \ldots, \psi_k$ to be a system of local coordinates around $\bar{\rho}$ such that

$$\{\xi_0, \psi_j\} = 0,\ \ \{\phi_2, \psi_j\} = 0,\ \ 1 \leq j \leq k,$$

holds on $V \cap \Sigma$. From the Jacobi identity it follows that

$$\{\phi_2, \{\phi_j, \xi_0\}\} = 0,\ \ j = \ell + 1, \ldots, r,$$

holds on $V \cap \Sigma$. Note that a solution $\gamma(s) = (x(s), \xi(s))$ of the Hamilton equations satisfies

$$\frac{d}{ds} f(\gamma(s)) = \{p, f\}(\gamma(s)).$$

With $t = 1/s$ we introduce new unknowns:

$$\begin{cases}\xi_0(s) = t^4 \Xi_0(t),\ x_0(s) = t X_0(t),\\ \phi_1(\gamma(s)) = t^2 \Phi_1(t),\ \phi_2(\gamma(s)) = t^3 \Phi_2(t),\\ \phi_j(\gamma(s)) = t^4 \Phi_j(t),\ 3 \leq j \leq \ell,\\ \phi_j(\gamma(s)) = t^3 \Phi_j,\ \ \ell + 1 \leq j \leq r,\\ \psi_j(\gamma(s)) = t^2 \Psi_j(t),\ 1 \leq j \leq k.\end{cases} \tag{5.3}$$

Then denoting $V = (X_0, \Phi_2, \Xi_0, \Phi_1, \Phi_j, \Psi_j)$, the Hamilton equations (5.1) are reduced to

$$(5.4)\quad \begin{cases} D\Xi_0 = -4\Xi_0 - 2\kappa_2\Phi_1\Phi_2 + tG(t,V), \\ DX_0 = -X_0 + 2\Phi_1 + tG(t,V), \\ D\Phi_1 = -2\Phi_1 + 2\delta\Phi_2 + tG(t,V), \\ D\Phi_2 = -3\Phi_2 - 2\kappa_2\Phi_1^2 + 2\delta\Xi_0 + tG(t,V), \\ tD\Phi_j = -4t\Phi_j - 2\kappa_j\Phi_1^2 - 2\sum_{k=3}^{\ell}\{\phi_k,\phi_j\}(\bar\rho)\Phi_k + tG(t,V),\ 3 \le j \le \ell, \\ D\Phi_j = -3\Phi_j + tG(t,V),\ \ell+1 \le j \le r, \\ D\Psi_j = -2\Psi_j - 2\sum_{k=\ell+1}^{r}\{\phi_k,\psi_j\}(\bar\rho)\Phi_k + tG(t,V),\ 1 \le j \le k, \end{cases}$$

where $D = t(d/dt)$ and $\{\phi_j, \xi_0\} = \sum_{i=1}^r C_i^j\phi_i$, $\kappa_j = C_1^j(\bar\rho)$, $\delta = \{\phi_1,\phi_2\}(\bar\rho)$ and $G(t,V)$ is a smooth function in $(t,V)$ with $G(t,0) = 0$. From Lemma 5.3 we see $F_pH_{\phi_1} = \delta H_{\phi_2}$, $F_pH_{\phi_2} = \delta H_{\xi_0}$, $F_pH_{\xi_0} = 0$, and hence we can take $S = \phi_2$ in (4.2). This proves

$$\kappa_2 = C_1^2 = \frac{\{\phi_2,\{\phi_2,\xi_0\}\}}{\{\phi_2,\phi_1\}} = \frac{H_{\phi_2}^3 p}{\{\phi_2,\phi_1\}} \neq 0$$

on $\Sigma$. Set

$$\mathcal{E} = \{\sum_{0\le j\le i} t^i(\log t)^j V_{ij} \mid V_{ij} \in \mathbb{C}^N\},$$
$$\mathcal{E}^{\#} = \{\sum_{1\le i, 0\le j\le i} t^i(\log t)^j V_{ij} \mid V_{ij} \in \mathbb{C}^N\}.$$

If $V \in \mathcal{E}$ with $\Phi_2(0) \neq 0$ formally satisfies (5.4), then it necessarily follows that $\Phi_2(0) = -1/\kappa_2\delta^2$ and hence $V(0)$ is uniquely determined thanks to

$$\det(\{\phi_i,\phi_j\}(\bar\rho))_{3\le i,j\le\ell} \neq 0.$$

Note that $X_0(0) \neq 0$. Using such uniquely determined $V(0) = \bar V$ we look for a formal solution in the form $\bar V + V$ with $V \in \mathcal{E}^{\#}$. Set

$$V^I = (X_0, \Phi_2, \Xi_0, \Phi_1),\quad V^{II} = (\Phi_3, \dots, \Phi_\ell),$$
$$V^{III} = (\Phi_{\ell+1}, \dots, \Phi_r),\quad V^{IV} = (\Psi_1, \dots, \Psi_k).$$

Then $V = {}^t(V^I, V^{II}, V^{III}, V^{IV})$ satisfies

$$(5.5)\qquad HDV = AV + tF + G(t,V),\quad A = \begin{bmatrix} A_I & O & O & O \\ B_{II} & A_{II} & O & O \\ O & O & -3E & O \\ O & O & B_{IV} & -2E \end{bmatrix}$$

with $H = E \oplus O \oplus E \oplus E$, where $E$ and $O$ are the identity and the zero matrix, respectively, $F$ is a constant vector, and

$$G(t,V) = \sum_{2\le i, 0\le j\le i} G_{ij}t^i(\log t)^j,\quad G_{ij} = G_{ij}(V_{pq} \mid q \le p \le i-1).$$

**Lemma 5.4.** *The eigenvalues of $A_I$ consist of $\{-6,-4,-1,1\}$. $A_{II}$ is the matrix $(\{\phi_i,\phi_j\}(\bar\rho))_{3\le i,j\le\ell}$, and hence $A_{II}$ is diagonalizable with nonzero pure imaginary eigenvalues.*

Applying standard arguments in constructing a formal solution around the regular singular point $t = 0$ we get the following.

**Proposition 5.5.** *There exists a formal solution $V \in \mathcal{E}$ to* (5.5) *with $\Phi_2(0) \neq 0$, $X_0(0) \neq 0$.*

This formal solution is uniquely determined up to a term $at$, $a \in \mathrm{Ker}(H - A)$. We now set $u = V^{II}$, $v = (V^I, V^{III}, V^{IV})$. Since $A_{II}$ is diagonalizable, making a change of unknowns, we may assume that $A_{II}$ is a diagonal matrix. Denoting by $(u_N, v_N)$, which is obtained from a formal solution in Proposition 5.5, by discarding the terms of order greater than $N$, then for any given $m \in \mathbb{N}$ there is $N = N(m)$ such that $(u_N, v_N)$ satisfies (5.4) modulo $O(t^{m+1})$. We look for a solution to (5.4) in the form $(u_N, v_N) + t^m(u, v)$. Then the equation satisfied by $(u, v)$ is (after dividing by $t^m$)

$$
\begin{cases} \big(t^2 d/dt - i\Lambda\big)u = -t(mI - K_1)u + L_1(t)v + tR_1(t, u, v) + tF_1(t), \\ t dv/dt = -(mI - K_2)v + L_2(t)v + tR_2(t, u, v) + tF_2(t), \end{cases} \tag{5.6}
$$

where $R_j(t, u, v)$ are $C^1$ functions defined near $(0, 0, 0) \in \mathbb{R} \times \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ satisfying

$$|R_j(t, u, v)| \leq B_j(|u| + |v|)$$

and $L_j(t)$, $tL_j'(t)$ are continuous and $K_j$ are constant matrices. Moreover, as noted above, $\Lambda$ is assumed to be $\mathrm{diag}(\lambda_1, \ldots, \lambda_{N_1})$, $\lambda_j \in \mathbb{R} \setminus \{0\}$. Although the equation (5.6) is a coupled system of two systems which has $t = 0$ as a singular point of the first kind and the second kind, respectively, taking the fact that $i\Lambda$ is a diagonal matrix with pure imaginary diagonal entries into account we can apply the successive approximation method taking $m$ enough large. Thus conclude the following.

**Proposition 5.6.** *Let $m \in \mathbb{N}$ be enough large. Then there exists a unique solution $(u, v)$ to* (5.6) *with $u(0) = 0$, $v(0) = 0$.*

Thus we see that there is a bicharacteristic with a limit point in $\Sigma$. Noting $X_0(0) \neq 0$ it is clear from (5.3) that this bicharacteristic is tangent to $\Sigma$. From Theorems 4.3 and 5.2 we have the following.

**Theorem 5.7** ([2], [48]). *Assume that* (2.5) *is verified on $\Sigma$. Then $p$ admits an elementary decomposition at every point of $\Sigma$ if and only if there is no bicharacteristic with a limit point in $\Sigma$.*

## 6. Gevrey 5 well-posedness

In this section assuming (2.5) we study the well-posedness of the Cauchy problem for $P$ in the Gevrey classes.

**Definition 6.1.** Let $W$ be an open set in $\mathbb{R}^d$ and let $s \geq 1$. We say $f(x) \in \gamma^{(s)}(W)$ if for any compact set $K \subset W$ one can find $C$, $A > 0$ such that for all $\alpha \in \mathbb{N}^d$ we have

$$|\partial_x^\alpha f(x)| \leq C A^{|\alpha|} |\alpha|!^s, \quad x \in K.$$

We set $\gamma_0^{(s)}(W) = C_0^\infty(W) \cap \gamma^{(s)}(W)$.

In this section the coefficients of $P$ are assumed to be in $\gamma^{(s)}(\Omega)$ with some neighborhood $\Omega$ of the origin, where $s > 1$ is enough close to 1.[14]

[14] We assume that the coefficients are in the Gevrey class also with respect to $x_0$ for simplicity, although this is not necessary.

**Theorem 6.2** ([4]). *Assume that* (2.5) *and the Levi condition* $P_{sub} = 0$ *are satisfied on* $\Sigma$. *Then the Cauchy problem for* $P$ *is well-posed in* $\gamma^{(s)}$ *with* $1 \le s \le 5$ *in a neighborhood of the origin. That is, for any* $f(x) \in C^\infty(\mathbb{R}; \gamma_0^{(s)}(\mathbb{R}^n))$ *vanishing in* $x_0 \le \tau$ *there is a* $u \in C^\infty$ *vanishing in* $x_0 \le \tau$ *and satisfying*

$$Pu = f$$

*in a neighborhood of the origin.*[15]

As stated in Section 4 one can write $p$ in the form of (4.3) in a conic neighborhood of $\rho \in \Sigma$. We may assume $\{\phi_2, \phi_1\}(\rho) > 0$ without restrictions. We extend $\phi_j$ outside a conic neighborhood of $\rho$ to be 0 so that $\phi_j \in S(\langle\xi'\rangle, g_0)$. In the same way as in Section 4 let $b(x, \xi')$ be a solution to (4.6) and put $\phi_{r+1} = Mb(x, \xi')$ with a large positive constant $M$. Then

$$\hat{p} = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \sum_{j=1}^{r+1} \phi_j^2$$

coincides with the original $p$ in a conic neighborhood of $\rho$ and satisfies

$$\{\xi_0 - \phi_1, \phi_j\} = \sum_{k=1}^{r+1} C_{jk}\phi_k, \quad \{\phi_2, \phi_1\} + |\phi_{r+1}| \ge c|\xi'|.$$

In what follows we again denote $\hat{p}$ by $p$. From the Levi condition $P_{sub} = 0$ we can write $P_{sub} = \sum_{j=0}^{r+1} C_j\phi_j$, where $\phi_0 = \xi_0$. Introducing a small parameter $\mu > 0$ and changing the time scale so that $x_0 \to \mu x_0$ we consider

$$\begin{aligned} p(x, \xi, \mu) &= \mu^2 p(\mu x_0, x', \mu^{-1}\xi_0, \xi') \\ &= -(\xi_0 - \phi_1(x, \xi', \mu))(\xi_0 + \phi_1(x, \xi', \mu)) + \sum_{j=2}^{r+1} \phi_j(x, \xi', \mu)^2. \end{aligned}$$

The introduction of the parameter $\mu$, although not necessary, helps much to handle the pseudodifferential calculus.[16] In the following we often omit writing $\mu$ and it is assumed that

$$\kappa = 1/5.$$

We introduce the symbol

$$w(x, \xi', \mu) = \sqrt{\phi_1^2 \langle\mu\xi'\rangle^{-2} + \langle\mu\xi'\rangle^{-4\kappa}}, \quad \Phi = \sqrt{1 - aw},$$

where a constant $a > 0$ is chosen so that $1 - aw \ge c_1 > 0$. Without restrictions we may assume $a = 1$. With

$$g = w(x, \xi', \mu)^{-2}\big(|dx|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2\big), \quad \bar{g} = \langle\mu\xi'\rangle^{4\kappa}\big(|dx|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2\big)$$

we have $w \in S(w, g)$, where $\langle\xi'\rangle_\mu = \mu^{-1}\langle\mu\xi'\rangle$. Note that $g \le \bar{g}$ and for any fixed $\mu > 0$ one has $S(\langle\mu\xi'\rangle^m, \bar{g}) = S^m_{3/5,2/5}$. With $\Phi$ one can write $p(x, \xi, \mu)$ in the form

$$p = -(\xi_0 - \phi_1\Phi)(\xi_0 + \phi_1\Phi) + \sum_{j=2}^{r+1} \phi_j^2 + w\phi_1^2 = -M\Lambda + Q,$$

[15] From the classical result of Bronshtein [7] the Cauchy problem for general second order differential operators $P$ with real characteristics is $\gamma^{(s)}$ well-posed with $1 \le s < 2$ in a neighborhood of the origin.

[16] For example the parameter $\mu$ was efficiently used in [45].

where $\phi_1\Phi = \phi_1 + \psi\phi_1$, $\psi \in S(w, g)$. Since $P = \mathrm{Op}(p + P_{sub}) + R$, $R \in \mu^2 S(1, g)$, taking the Levi condition into account one can write

$$P = -M\Lambda + B_0\Lambda + Q, \ \ Q = \mathrm{Op}(\sum_{j=2}^{r+1} \phi_j^2 + w\phi_1^2 + R + Q_0),$$

where $B_0 \in \mu S(1, g)$, $R = \mathrm{Op}(\sum_{j=1}^{r+1} C_j\phi_j)$, $C_j \in \mu S(1, g)$, $Q_0 \in \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, \bar{g})$.

Although there are so many articles on the composition $e^{\pm\langle\mu D'\rangle^\kappa}$ with pseudodifferential operators of $S_{1,0}$ class,[17] one can hardly find any literature about the composition with pseudodifferential operators of $S_{\rho,\delta}$ class. So we summarize some composition formulas between $e^{\pm\langle\mu D'\rangle^\kappa}$ and pseudodifferential operators which we use in this monograph.

**Definition 6.3.** Let $0 \le \nu < 1$ and let $g_\nu = \langle\mu\xi'\rangle^\nu(|dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2)$. We say $a(x', \xi', \mu) \in \gamma^{(s)}S(m, g_\nu)$ if there is $A > 0$ such that

$$\begin{aligned}|\partial_{x'}^\alpha\partial_{\xi'}^\beta a(x', \xi', \mu)| &\le C_\beta m(x', \xi', \mu)A^{|\alpha|}|\alpha|!^{s/2}\\ &\times(\langle\mu\xi'\rangle^{-\nu/2}\langle\xi'\rangle_\mu)^{-|\beta|}(|\alpha|^{s/2} + \langle\mu\xi'\rangle^{\nu/2})^{|\alpha|}\end{aligned}$$

holds for any $\alpha$, $\beta \in \mathbb{N}^n$.

In particular, if $a(x', \xi', \mu)$ satisfies

$$|\partial_{x'}^\alpha\partial_{\xi'}^\beta a(x', \xi', \mu)| \le C_\beta m(x', \xi', \mu)A^{|\alpha|}|\alpha|!^s\langle\xi'\rangle_\mu^{-|\beta|}, \quad \forall\alpha, \beta \in \mathbb{N}^n, \tag{6.1}$$

then we have clearly $a(x', \xi', \mu) \in \gamma^{(s)}S(m, g_\nu)$.

**Lemma 6.4.** *Let $s > 2$ and assume that $f$ verifies (6.1) with $m(x', \xi', \mu) = 1$. Let $s > 2$ and then we have*

$$\sqrt{f(x', \xi', \mu)^2 + \langle\mu\xi'\rangle^{-\nu}} \in \gamma^{(s)}S(\sqrt{f(x', \xi', \mu)^2 + \langle\mu\xi'\rangle^{-\nu}}, g_\nu).$$

Now assume that $\phi(\xi', \mu) \in S(\langle\mu\xi'\rangle^{1/s}, |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2)$ satisfies

$$\phi(\xi' + \eta', \mu) - \phi(\xi' - \eta', \mu) \le C\langle\mu\eta'\rangle^{1/s}.$$

Then we have the following.

**Proposition 6.5.** *Let $\nu + s^{-1} \le 1$ and let $a(x', \xi', \mu) \in \gamma^{(s)}S(\langle\mu\xi'\rangle^m, g_\nu)$ be independent of $x'$ if $|x'|$ is large enough. With $e^{\phi(D',\mu)}a(x', D', \mu)e^{-\phi(D',\mu)} = b(x', D', \mu)$ we have*

$$\begin{cases} b(x', \xi', \mu) = \sum_{j=0}^{N-1} b_j(x', \xi', \mu) + R_N(x', \xi', \mu),\\ R_N \in \mu^N S(\langle\mu\xi'\rangle^{m-N(1-1/s-\nu/2)+n\nu/2}, g_\nu),\end{cases}$$

*where $b_j \in \mu^j S(\langle\mu\xi'\rangle^{m-j(1-1/s-\nu/2)}, g_\nu)$ is given by*

$$b_j = \sum_{|\alpha|=j}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_{\eta'}^\alpha e^{\phi(\xi'+\eta'/2,\mu)-\phi(\xi'-\eta'/2,\mu)}|_{\eta'=0}\partial_{x'}^\alpha a(x', \xi', \mu).$$

Lemma 6.4 shows that $w \in \gamma^{(s)}S(w, \bar{g})$ $(s > 2)$ if $\phi_1$ satisfies (6.1) with $m = \langle\mu\xi'\rangle$.

[17] For example see [27] and the references therein.

**Proposition 6.6.** *Let $\tau > 0$. We have*

$$e^{(\tau-x_0)\langle\mu D'\rangle^\kappa} P e^{-(\tau-x_0)\langle\mu D'\rangle^\kappa} = -M\Lambda + B_0\Lambda + Q \tag{6.2}$$

*with $M = D_0 - i\langle\mu D'\rangle^\kappa - \mathrm{Op}(m)$, $\Lambda = D_0 - i\langle\mu D'\rangle^\kappa - \mathrm{Op}(\lambda)$, $B_0 \in \mu S(1,\bar{g})$ and $m = -\phi_1\Phi + im_1 + m_2$, $\lambda = \phi_1\Phi + i\lambda_1 + \lambda_2$, where $m_1$, $\lambda_1 \in \mu S(\langle\mu\xi'\rangle^\kappa, g)$ are real valued and $m_2$, $\lambda_2 \in \mu^2 S(\langle\mu\xi'\rangle^{-\kappa},\bar{g})$. Moreover $Q$ verifies*

$$\begin{cases} Q = \mathrm{Op}(q + iq_1 + q_2 + r), \quad q = \sum_{j=2}^{r+1}\phi_j^2 + w\phi_1^2, \\ q_1 = \sum_{j=2}^{r+1} a_j\phi_j + a_1 w\phi_1, \quad q_2 = \sum_{j=1}^{r+1} c_j\phi_j, \end{cases}$$

*where $a_j \in \mu S(\langle\mu\xi'\rangle^\kappa, g)$ are real valued and $c_j \in \mu S(1,\bar{g})$, $r \in \mu^2 S(\langle\mu\xi'\rangle^{2\kappa},\bar{g})$.*

Denote the right-hand side of (6.2) by $P$. Similarly to Proposition 3.1 we have

$$\begin{aligned} 2\mathsf{Im}\,(Pu,\Lambda u) \geq \frac{d}{dx_0}\big(\|\Lambda u\|^2 + ((\mathsf{Re}\,Q)u,u) + \|\langle\mu D'\rangle^\kappa u\|^2\big) + ||\langle\mu D'\rangle^{\kappa/2}\Lambda u|| \\ +2((\mathsf{Im}B_0)\Lambda u,\Lambda u) + 2\mathsf{Re}\,(\langle\mu D'\rangle^\kappa(\mathsf{Re}\,Q)u,u) + 2((\mathsf{Im}\,m)\Lambda u,\Lambda u) \\ +2\mathsf{Re}\,(\Lambda u,(\mathsf{Im}\,Q)u) + \mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u,u) \\ +2\mathsf{Re}\,((\mathsf{Re}\,Q)u,(\mathsf{Im}\,\lambda)u) + (1 - C\mu)\|\langle\mu D'\rangle^{3\kappa/2}u\|^2. \end{aligned}$$

Here we check how the term $\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u,u)$ can be estimated. The main part of the symbol of $-\mathsf{Im}\,[D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]$ is $\{\xi_0 - \phi_1, \phi_j^2\} = 2\{\xi_0 - \phi_1, \phi_j\}\phi_j = \sum_{k=1}^{r+1} C_{jk}\phi_k\phi_j$, $j \geq 2$. If $j,k \geq 2$, then $\mathsf{Re}\,(\mathrm{Op}(C_{jk}\phi_k\phi_j)u,u)$ is easily managed. The main point here is that $C_{j1} \neq 0$ in general because $H_S^3 p = 0$ is not assumed. Thus we must estimate $\mathsf{Re}\,(\mathrm{Op}(C_{j1}\phi_1\phi_j)u,u)$, $j \geq 2$.

**Lemma 6.7.** *Let $a \in \mu S(1,g)$ be real valued. Then we have*

$$\begin{aligned} (\mathrm{Op}(a\phi_1\phi_j)u,u) \leq C\mu\,(\mathrm{Op}(\phi_j^2\langle\mu\xi'\rangle^\kappa)u,u) \\ +C\mu\,(\mathrm{Op}(\phi_1^2 w\langle\mu\xi'\rangle^\kappa)u,u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 \end{aligned}$$

*for $j \neq 1$.*

Write $a\phi_1\phi_j = \mathsf{Re}\,(\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j \# \mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1) + R$, $R \in \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g)$ and note that

$$\begin{aligned} &\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j \# \mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j = \mu\langle\mu\xi'\rangle^\kappa\phi_j^2 + R_1, \\ &\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1 \# \mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1 \\ &\qquad = (\mu^{-1}a^2 w^{-1}\langle\mu\xi'\rangle^{-2\kappa})\phi_1^2 w\langle\mu\xi'\rangle^\kappa + R_2 \end{aligned}$$

with $R_1 \in \mu^3 S(\langle\mu\xi'\rangle^\kappa, g_1)$, $R_2 \in \mu^3 S(\langle\mu\xi'\rangle^{3\kappa}, g)$. Since $w^{-1} \in S(\langle\mu\xi'\rangle^{2\kappa}, g)$ in view of $\mu^{-1}a^2 w^{-1}\langle\mu\xi'\rangle^{-2\kappa} \in \mu S(1,g)$ we get Lemma 6.7. Next check $\mathsf{Re}\,(\Lambda u, (\mathsf{Im}\,Q)u)$. From Proposition 6.6 we see that the main part of $\mathsf{Im}\,Q$ is $q_1$.

**Lemma 6.8.** *We have the following estimates:*

$$\begin{aligned} |(\Lambda u, q_1 u)| \leq C\mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + C\mu\sum_{j=2}^{r+1}(\mathrm{Op}(\langle\mu\xi'\rangle^\kappa\phi_j^2)u,u) \\ +C\mu(\mathrm{Op}(w\langle\mu\xi'\rangle^\kappa\phi_1^2)u,u) + C\mu^3\|\langle\mu D'\rangle^{1/2}u\|^2. \end{aligned}$$

For instance, we sketch how to estimate $(\Lambda u, \mathrm{Op}(a_j\phi_j)u)$. We write $a_j\phi_j = \langle\mu\xi'\rangle^{\kappa/2}\#\langle\mu\xi'\rangle^{-\kappa/2}a_j\phi_j + R$. Then noting $R\in\mu^2S(\langle\mu\xi'\rangle^{2\kappa}, g)$ and writing $R = \langle\mu\xi'\rangle^{\kappa/2}\#\langle\mu\xi'\rangle^{-\kappa/2}R + R_1$, $R_1\in\mu^3S(1,g)$, we have

$$|(\Lambda u, \mathrm{Op}(a_j\phi_j)u)| \le C\mu\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2$$
$$+C\mu\|\mathrm{Op}(\langle\mu\xi'\rangle^{-\kappa/2}a_j\phi_j)u\|^2 + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

Noting $\langle\mu\xi'\rangle^{-\kappa/2}a_j\phi_j = \langle\mu\xi'\rangle^{-\kappa}a_j\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j + R$, $R\in\mu^2S(\langle\mu\xi'\rangle^{1/2}, g)$ we see

$$\mu^{-1}\|\mathrm{Op}(\langle\mu\xi'\rangle^{-\kappa/2}a_j\phi_j)u\|^2 \le C\mu\|\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa/2}\phi_j)u\|^2 + C\mu^3\|\langle\mu D'\rangle^{1/2}u\|^2$$
$$\le C\mu(\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa}\phi_j^2)u,u) + C\mu^3\|\langle\mu D'\rangle^{1/2}u\|^2.$$

Repeating similar arguments we obtain Lemma 6.8. We estimate $\|\langle\mu D'\rangle^{1/2}u\|^2$ in Lemma 6.8. From the assumption one has $\{\phi_1,\phi_2\} + |\phi_{r+1}| \ge c\mu\langle\mu\xi'\rangle$. Estimating the commutator $[\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa/2}\phi_1\sqrt{w}), \mathrm{Op}(\langle\mu\xi'\rangle^{\kappa/2}\phi_2)]$ we see that there is a constant $c>0$ such that

$$c\mu(\mathrm{Op}(\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w})u,u) \le (\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa}w\phi_1^2)u,u)$$
$$+ (\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa}\phi_2^2)u,u) + (\mathrm{Op}(\phi_{r+1}^2)u,u) + C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

Write $C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w} - \mu\langle\mu\xi'\rangle = C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}\big(1 - C^{-1}w^{-1/2}\langle\mu\xi'\rangle^{-\kappa}\big)$, which is $C\mu(\psi\#\psi)+R$, where $\psi\in S(\langle\mu\xi'\rangle^{(1+\kappa)/2}w^{1/4}, g)$, $R\in\mu^3S(\langle\mu\xi'\rangle^{3\kappa}, g)$. Thus we get the following.

**Lemma 6.9.** *We have*

$$\mu\|\langle\mu D'\rangle^{1/2}u\|^2 \le C(\mathrm{Op}(\phi_1^2w\langle\mu\xi'\rangle^{\kappa})u,u)$$
$$+ C(\mathrm{Op}(\phi_2^2\langle\mu\xi'\rangle^{\kappa})u,u) + C(\mathrm{Op}(\phi_{r+1}^2)u,u) + C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

On the other hand $\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}(\mathsf{Re}\,Q)u,u)$ is bounded from below as follows:

$$\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}(\mathsf{Re}\,Q)u,u) \ge \sum_{j=2}^{r+1}(\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa}\phi_j^2)u,u)$$
$$+(\mathrm{Op}(\langle\mu\xi'\rangle^{\kappa}w\phi_1^2)u,u) - C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

From these estimates it follows that $\mathsf{Im}\,([D_0 - \mathsf{Re}\,\lambda, \mathsf{Re}\,Q]u,u)$ is bounded by constant times $\mathsf{Re}\,(\langle\mu D'\rangle^{\kappa}(\mathsf{Re}\,Q)u,u) + \mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2$.

Applying the preceding lemmas we get the following energy estimates.

**Proposition 6.10.** *Denote the right-hand side of* (6.2) *by* $P$. *Then one can find* $\mu_0>0$, $C>0$, $c>0$ *such that for* $0<\mu<\mu_0$ *the inequality*

$$C\int_{-T}^{t}\|\langle\mu D'\rangle^{-\kappa/2}Pu\|^2dx_0 \ge \{\|\Lambda u(t,\cdot)\|^2 + c\|\langle\mu D'\rangle^{\kappa}u(t,\cdot)\|^2\}$$
$$+c\int_{-T}^{t}\{\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 + \|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + \mu\|\langle\mu D'\rangle^{1/2}u\|^2\}dx_0$$

*holds for any* $u\in C^2(I;H^{\infty}(\mathbb{R}^n))$ *vanishing in* $x_0<\tau$, $|\tau|<T$.

By a similar argument as in Section 4 we conclude that there exists a solution operator $G$, $u=Gf$ verifying (4.8). Let us set

$$\hat{G} = e^{-(\tau-x_0)\langle\mu D'\rangle^{\kappa}}Ge^{(\tau-x_0)\langle\mu D'\rangle^{\kappa}}.$$

Then one can take $\delta > 0$ such that for $h_1$, $h_2$ in Section 4 one has the estimate

$$\|e^{(\tau-t)\langle\mu D'\rangle^\kappa} D_0^j h_2 \hat{G} h_1 f(t)\|_p \le C(p,q)\int_{-T}^t \|e^{(\tau-x_0)\langle\mu D'\rangle^\kappa} f(x_0)\|_q dx_0, j=0,1$$

for any $f \in C^0([-T,T];\gamma_0^{(1/\kappa)}(\mathbb{R}^n))$ vanishing in $x_0 \le \tau$ and for any $0 \le t \le \delta$, $p$, $q \in \mathbb{R}$. Then a repetition of the same arguments in [43] proves Theorem 6.2.

## 7. Optimality of Gevrey 5 well-posedness

Consider the differential operator

$$P(x,D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 \tag{7.1}$$

with symbol $p(x,\xi)$.[18] Note that the doubly characteristic manifold of $p$ is given by $\Sigma = \{\xi_0 = 0, x_1 = 0, \xi_1 = 0\}$ $(\xi_2 \neq 0)$. The localization of $p$ at the double characteristic $\rho = (0,(0,0,0,0,1)) \in \Sigma$ is

$$p_\rho = -\xi_0^2 + 2x_1\xi_0 + \xi_1^2,$$

which is case (3) in Lemma 2.13 ($k=1, \ell=1$, where $x_1$ and $\xi_1$ are exchanged[19]) and hence noneffectively hyperbolic at $\rho \in \Sigma$ where $\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\}$. It is also clear that $P_{sub} = 0$ and thus the Levi condition is satisfied. For this $p$ the curve

$$x_1 = -\frac{x_0^2}{4},\ \ x_2 = \frac{x_0^5}{8},\ \ \xi_0 = 0,\ \ \xi_1 = \frac{x_0^3}{8},\ \ \xi_2 = c \tag{7.2}$$

is a bicharacteristic (parametrized by $x_0$) which is tangent to $\Sigma$ as $x_0 \to 0$, where $c \neq 0$ is an arbitrary constant.

**Definition 7.1.** We say that the Cauchy problem for $P$ is locally $\gamma^{(s)}$ solvable at the origin if for any $(u_0,u_1) \in (\gamma^{(s)}(\mathbb{R}^2))^2$ there exist a neighborhood $U$ of the origin and $u(x) \in C^\infty(U)$ satisfying

$$\begin{cases} Pu = 0 \ \text{ in } U, \\ D_0^j u(0,x') = u_j(x'),\ \ x' \in U \cap \{x_0 = 0\},\ \ j = 0,1. \end{cases}$$

**Theorem 7.2** ([4])**.** *The Cauchy problem for $P$ is not locally $\gamma^{(s)}$ solvable at the origin if $s > 5$. In particular, the Cauchy problem for $P$ is not $C^\infty$ well-posed near the origin.*[20]

Making a suitable change of a system of local coordinates leaving the initial plane $x_0 = c$ invariant $P$ can be written

$$P = -D_0^2 + (D_1 + x_0 D_2)^2 + (x_1\sqrt{1+x_1}D_2)^2 = -D_0^2 + A^2 + B^2,$$

where $A^* = A$, $B^* = B$ so that $P$ is of divergence free. After [9], [42] it has been conjectured that for second order differential operators of divergence free with real analytic coefficients

$$-D_0^2 u + \sum_{i,j=1}^n D_{x_i}(a_{ij}(x) D_{x_j} u),\ \ a_{ij}(x) = a_{ji}(x),$$

[18] Exchanging $x_1$ and $\xi_1$, $p(x,\xi)$ turns out to be the case (4.1) with $k=1$.

[19] Since the exchange of $x_1$ and $\xi_1$ is a canonical transformation so that the spectre of $F_p$ and bicharacteristics of $p$ are invariant.

[20] Discussions about the location of zeros of Stokes coefficients given in [4] is insufficient. Here we give a rough sketch on how to modify the arguments there. For more details see [51].

with nonnegative characteristic form $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0$, the Cauchy problem is $C^\infty$ well-posed and has been extensively studied. Theorem 7.2 provides a counterexample for this conjecture.

We give a sketch of the proof. We look for a solution to $PU = 0$ of the form

$$U_\rho(x) = e^{i\rho^5 x_2 + \frac{i}{2}\zeta\rho x_0} w(x_1\rho^2),$$

where $\zeta \in \mathbb{C}$ and $\rho$ is a positive parameter. That is $w$ solves the equation

$$w''(x) = (x^3 + \zeta x - \zeta^2\rho^{-2}/4)w(x).$$

Introducing parameters $\zeta,\ \epsilon \in \mathbb{C}$ with small $|\epsilon|$ let us consider

$$w''(x) = (x^3 + \zeta x + \epsilon)w(x), \quad \zeta,\ \epsilon \in \mathbb{C}, \tag{7.3}$$

so that if $w(x;\zeta,\epsilon)$ verifies (7.3), then

$$U_\rho(x) = e^{i\rho^5 x_2 + \frac{i}{2}\zeta\rho x_0} w(x_1\rho^2;\zeta,\eta), \ \ \eta = -\zeta^2\rho^{-2}/4$$

satisfies $PU_\rho(x) = 0$.

**Theorem 7.3** ([55]). *The differential equation* (7.3) *has a solution $\mathcal{Y}(x;\zeta,\epsilon)$ such that*

(1) *$\mathcal{Y}(x;\zeta,\epsilon)$ is an entire function of $(x,\zeta,\epsilon)$,*
(2) *$\mathcal{Y}(x;\zeta,\epsilon)$ admits an asymptotic representation*

$$\mathcal{Y}(x;\zeta,\epsilon) \sim x^{-3/4}(1 + p(x;\zeta,\epsilon))e^{-(\frac{2}{5}x^{5/2} + \zeta x^{1/2})},$$

*where $p(x;\zeta,\epsilon) \to 0$ uniformly on each compact set in $(\zeta,\epsilon)$ space as $x \to \infty$ in any closed subsector of the open sector $|\arg x| < 3\pi/5$.*

With $\omega = \exp(2\pi i/5)$, $\mathcal{Y}_k(x;\zeta,\epsilon) = \mathcal{Y}(\omega^{-k}x;\omega^{-2k}\zeta,\omega^{-3k}\epsilon)$, $k = 1,2,3,4$, turns out to be a solution to (7.3) and $\mathcal{Y}_k$ is subdominant in $S_k; |\arg x - 2k\pi/5| < \pi/5$ ($\mathcal{Y}_0 = \mathcal{Y}$). The asymptotic representation of $\mathcal{Y}_k$ is obtained from Theorem 7.3 which holds in $|\arg x - 2k\pi/5| < 3\pi/5$. Since $\mathcal{Y}_k$ and $\mathcal{Y}_{k+1}$ are linearly independent one can write

$$\mathcal{Y}_k(x;\zeta,\epsilon) = C_k(\zeta,\epsilon)\mathcal{Y}_{k+1}(x;\zeta,\epsilon) + \tilde{C}_k(\zeta,\epsilon)\mathcal{Y}_{k+2}(x;\zeta,\epsilon).$$

The key to the proof of Theorem 7.2 is the following result about the location of zeros of the Stokes coefficient $C_0(\zeta,0)$.

**Proposition 7.4.** *There is at least one zero of $C_0(\zeta,0)$ with $\mathsf{Im}\,\zeta < 0$.*

Here we summarize several properties of the Stokes coefficients $C_k(\zeta,\epsilon)$, $\tilde{C}_k(\zeta,\epsilon)$ which will be needed in the following.

**Proposition 7.5** ([55]). *The Stokes coefficients have the form $C_0(0,0) = 1+\omega$, $\tilde{C}_k(\zeta,\epsilon) = -\omega$, $C_k(\zeta,\epsilon) = C_0(\omega^{-2k}\zeta,\omega^{-3k}\epsilon)$, and $C_0(\zeta,\epsilon)$ is an entire function of $(\zeta,\epsilon)$ such that*

$$\partial_\zeta C_0(\zeta,\epsilon)|_{(\zeta,\epsilon)=(0,0)} \neq 0. \tag{7.4}$$

*With $c(\zeta) = C_0(\zeta,0)$ we have*

$$c(\zeta) + \omega^2 c(\omega\zeta)c(\omega^4\zeta) - \omega^3 = 0 \quad \forall\zeta \in \mathbb{C}. \tag{7.5}$$

For the proofs we refer to Chapter 5 in Sibuya [55].

**Lemma 7.6** ([56]). *The relation $\overline{C_0(\zeta,\epsilon)} = \bar{\omega}C_0(\bar{\omega}\bar{\zeta},\omega\bar{\epsilon})$ holds. In particular, we have $\overline{c(\zeta)} = \bar{\omega}c(\bar{\omega}\bar{\zeta})$.*

Suppose that $c(\zeta)$ has no zero; then from (7.5) we see that $c(\zeta)$ avoids the values $\omega^3$ and 0. Thus from Picard's Little Theorem it follows that $c(\zeta)$ is a constant which contradicts (7.4). This together with $C_0(0,0) = 1+\omega$ proves the following.

**Lemma 7.7.** *$C_0(\zeta, 0)$ has at least one zero $\zeta_0\,(\neq 0)$.*

**Lemma 7.8** ([51]). *The Stokes coefficient $c(\zeta)$ has no zero in the closed sector $3\pi/5 \le \arg\zeta \le \pi$.*

We turn to the proof of Proposition 7.4. From Lemma 7.7 there exists $\zeta \neq 0$ such that $c(\zeta) = 0$. Lemma 7.8 shows that $-\pi < \arg\zeta < 3\pi/5$. If $-\pi < \arg\zeta < 0$, then this $\zeta$ is a desired zero. If $0 \le \arg\zeta < 3\pi/5$ and hence $-\pi < \arg\bar\omega\bar\zeta \le -2\pi/5$, we see that $\bar\omega\bar\zeta$ is a desired zero by virtue of Lemma 7.6.

From Proposition 7.4 there is a zero $\zeta_0$ of $C_0(\zeta,0)$ with $\mathsf{Im}\,\zeta_0 < 0$. We take $\zeta = \zeta(\epsilon)$ satisfying

$$C_0(\zeta, -\zeta^2\epsilon/4) = 0$$

such that $\zeta(0) = \zeta_0$. Note that $\zeta(\epsilon)$ is given by Puiseux series,

$$\zeta(\epsilon) = \zeta_0 + \sum_{j=0}^{\infty}\zeta_j(\epsilon^{1/p})^j = \tilde\zeta(\epsilon^{1/p}),\ \ \tilde\zeta(0) = \zeta_0,$$

where $p$ is an integer and $\tilde\zeta(z)$ is holomorphic in a neighborhood of $z = 0$. Setting $\eta(\epsilon) = -\tilde\zeta(\epsilon)^2\epsilon^p/4$ we have

$$\mathcal{Y}_0(x;\tilde\zeta(\epsilon),\eta(\epsilon)) = -\omega\mathcal{Y}_2(x;\tilde\zeta(\epsilon),\eta(\epsilon)) \quad \forall x\in\mathbb{C}, \tag{7.6}$$

for enough small $|\epsilon| \ll 1$. Since we have

$$\mathcal{Y}_0(x;\tilde\zeta,\eta) = x^{-3/4}(1 + R(x,\tilde\zeta,\eta))e^{-(\frac{2}{5}x^{5/2}+\tilde\zeta x^{1/2})}$$

in the open sector $|\arg x| < 3\pi/5$, then $\mathcal{Y}_0(x;\tilde\zeta,\eta)$ decays as $\exp(-2x^{5/2}/5)$ when $\mathbb{R}\ni x\to+\infty$. Recall $\mathcal{Y}_0(x;\tilde\zeta(\epsilon),\eta(\epsilon)) = -\omega\mathcal{Y}_0(\omega^{-2}x;\omega^{-4}\tilde\zeta(\epsilon),\omega^{-6}\eta(\epsilon))$ by (7.6). Note that $\omega^{-2}x = e^{\pi i/5}|x|$ if $x < 0$, and hence

$$(\omega^{-2}x)^{5/2} = i|x|^{5/2}, \quad \omega^{-4}\tilde\zeta(\omega^{-2}x)^{1/2} = i\tilde\zeta|x|^{1/2}$$

so that $\mathcal{Y}_0(x;\tilde\zeta,\eta)$ decays as $\exp(\mathsf{Im}\,\tilde\zeta|x|^{1/2})$ when $\mathbb{R}\ni x\to-\infty$. This proves that $\mathcal{Y}_0(x;\tilde\zeta(\epsilon),\eta(\epsilon)) \in \mathcal{S}(\mathbb{R})$. In particular, $\mathcal{Y}_0(x;\tilde\zeta(\epsilon),\eta(\epsilon))$ is bounded uniformly in $x\in\mathbb{R}$ and $|\epsilon|\ll 1$:

$$|\mathcal{Y}_0(x;\tilde\zeta(\epsilon),\eta(\epsilon))| \le B, \ \ x\in\mathbb{R}, \ \ |\epsilon|\ll 1.$$

Take a small $T > 0$ and set

$$U_\rho(x) = \exp\big(-i\rho^5x_2 + \frac{i}{2}\tilde\zeta(\rho^{-2/p})\rho(T-x_0)\big)\mathcal{Y}(x_1\rho^2;\tilde\zeta(\rho^{-2/p}),\eta(\rho^{-2/p}))$$

for $\rho > 0$. It is clear that $PU_\rho = 0$. Take $\phi\in C_0^\infty(\mathbb{R})$ and $\theta\in C_0^\infty(\mathbb{R})$ with small supports near the origin and consider the following Cauchy problem:

$$\begin{cases} Pu = 0, \\ u(0,x') = 0, \ \ D_0u(0,x') = \bar\phi(x_1)\bar\theta(x_2). \end{cases} \tag{7.7}$$

From Holmgren's uniqueness theorem (see Theorem 4.2 in [36] for example; note that $P$ is of polynomial coefficients) we may assume that a solution $u(x)$ to the Cauchy problem (7.7) verifies $u(x) = 0$ in $|x_0| \le T$, $|x'| \ge r$ with small $T > 0$,

$r > 0$. We now integrate the $L^2(\mathbb{R}^2)$ inner product $(PU_\rho, u)$ from 0 to $T$ in $x_0$ to get

$$(D_0U_\rho(T), u(T)) + (U_\rho(T), D_0u(T)) - (2x_1D_2U_\rho(T), u(T)) \\ = (U_\rho(0), D_0u(0)). \tag{7.8}$$

Since $\mathcal{Y}(\rho^2x_1; \tilde{\zeta}, \eta)$ is bounded uniformly in $\rho$ and $x_1$, we see that the left-hand side of (7.8) is $O(\rho^5)$. On the other hand the right-hand side is

$$\int_{\mathbb{R}^2} e^{-i\rho^5x_2+i\tilde{\zeta}\rho T/2}\mathcal{Y}(\rho^2x_1; \tilde{\zeta}, \eta)\phi(x_1)\theta(x_2)dx' \\ = e^{i\tilde{\zeta}\rho T/2}\hat{\theta}(\rho^5)\rho^{-2}\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)\phi(\rho^{-2}x_1)dx_1,$$

where $\hat{\theta}$ stands for the Fourier transform of $\theta$. Noting that $\tilde{\zeta}(\rho^{-2/p}) \to \zeta_0$ as $\rho \to \infty$ one can find $c > 0$ such that $\left|e^{i\tilde{\zeta}\rho T/2}\right| \geq e^{c\rho T}$ for large $\rho$. Therefore we conclude

$$\rho^{-7}e^{c\rho T}|\hat{\theta}(\rho^5)|\left|\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)\phi(\rho^{-2}x_1)dx_1\right| = O(1). \tag{7.9}$$

Note that $\theta \in \gamma_0^{(5)}(\mathbb{R})$ if and only if $|\hat{\theta}(\xi)| \leq Ce^{-L|\xi|^{1/5}}$ with some positive constants $L, C > 0$. Taking this into account we choose an even function $\theta \in C_0^\infty(\mathbb{R})$ satisfying $\theta \notin \gamma_0^{(5)}(\mathbb{R})$. Then it follows that $e^{C\rho}\rho^{-N}|\hat{\theta}(\rho^5)|$ is not bounded when $\rho \to \infty$ for any $N \in \mathbb{N}$ and for any $C > 0$. Let us write

$$\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)\phi(\rho^{-2}x_1)dx_1 = \sum_{k=0}^{2} \frac{\rho^{-2k}}{k!}\phi^{(k)}(0)\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)x_1^k dx_1 + O(\rho^{-6})$$

and note that

$$\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)x_1^k dx_1 \to \int \mathcal{Y}(x_1; \zeta_0, 0)x_1^k dx_1$$

as $\rho \to \infty$.

**Lemma 7.9.** *At least one of*

$$\int \mathcal{Y}(x_1; \zeta_0, 0)x_1^k dx_1, \quad k = 0, 1, 2,$$

*is different from* 0.

It is now clear that, choosing $\phi^{(k)}(0)$, $k = 0, 1, 2$, suitably, (7.9) does not hold. That is, for such initial data the Cauchy problem (7.7) has no $C^\infty$ solution in any neighborhood of the origin.

## 8. Concluding remarks

Denoting $W = \mathrm{Im}F_p^2 \cap \mathrm{Ker}F_p^2$ we can summarize the obtained results on the well-posedness of the Cauchy problem for differential operators with double characteristics in the following table.

<table>
<tr><th>Spectrum of $F_p$</th><th>$W$</th><th>Geometry of bicharacteristics of $p$ near $\Sigma$</th><th>Well-posedness of the Cauchy problem for $P$</th><th>Elementary decomposition</th></tr>
<tr><td>Exists nonzero real eigenvalue</td><td>$W = \{0\}$</td><td>Two bicharacteristics intersect $\Sigma$ transversally[21]</td><td>$C^\infty$ well-posed for any lower order term</td><td>impossible</td></tr>
<tr><td rowspan="3">No nonzero real eigenvalue</td><td>$W = \{0\}$</td><td rowspan="2">No bicharacteristic intersects $\Sigma$</td><td rowspan="2">$C^\infty$ well-posed under the Levi or the strict IPH condition</td><td rowspan="2">possible</td></tr>
<tr><td rowspan="2">$W \neq \{0\}$</td></tr>
<tr><td>Exists a bicharacteristic tangent to $\Sigma$</td><td>Gevrey 5 well-posed under the Levi condition. Under the strict IPH condition?</td><td>impossible</td></tr>
</table>

We will see in this table that the only part which remains to be unclear is what we can assert on the well-posedness of the Cauchy problem if $W \neq \{0\}$, $\mathrm{Tr}^+ F_p > 0$ and there is a bicharacteristic with a limit point in $\Sigma$. A model operator satisfying these conditions is

$$P(x, D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 + a(x_3^2 D_2^2 + D_3^2),$$

where $a > 0$ is a positive constant and hence $\mathrm{Tr}^+ F_p = a$, which coincides with $P$ in (7.1) when $a = 0$. The doubly characteristic manifold is $\Sigma = \{\xi_0 = \xi_1 = \xi_3 = 0, x_1 = x_3 = 0\}$. Since $P_{sub} = 0$ the strict IPH condition is clearly verified. If we define $(x_1, x_2, \xi_0, \xi_1, \xi_2)$ by (7.2) and $(x_3, \xi_3)$ by $x_3 = 0$, $\xi_3 = 0$, then this curve is still a bicharacteristic of $P$ even if $a > 0$. That is, in the viewpoint of "classical mechanics", there exists the singular orbit (7.2) for $P$ with $a \geq 0$. In the case $a = 0$ it seems reasonable to suppose that the existence of this singular orbit causes nonsolvability in $C^\infty$ to the Cauchy problem. From this point it is expected that the Cauchy problem for $P$ with $a > 0$ is not $C^\infty$ well-posed. On the other hand, in the viewpoint of "quantum mechanics" it is forbidden to choose $x_3 = 0$, $\xi_3 = 0$ at the same time by Heisenberg's uncertainty principle. Up to now it is only known that the Cauchy problem for $P$ with $a > 0$ is $\gamma^{(6)}$ well-posed.

[21] One can find more detailed discussions on the behavior of bicharacteristics in [28].

We will see in Theorems 6.2 and 7.2 that the Gevrey 5 class appears very naturally as a function space in which the Cauchy problem is well-posed in the case that there is a bicharacteristic with a limit point in $\Sigma$. This suggests some possible relations between the geometry of bicharacteristics and the Gevrey classes in which the Cauchy problem is well-posed. Indeed, as we explain in the following, there is a close correspondence between them. To formulate this correspondence we introduce the notion of Gevrey strong hyperbolicity (well-posedness) following Definition 1.5. In what follows we assume that the coefficients of $P$ are real analytic or in the Gevrey class $\gamma^{(s)}$ with $s(>1)$ close to 1 in a neighborhood of the origin.

**Definition 8.1.** Let $s > 1$. Then $P$ is said to be Gevrey $s$ strongly hyperbolic at the origin if the Cauchy problem for $P+Q$ with any differential operator of order 1 defined near the origin is locally $\gamma^{(\kappa)}$ solvable at the origin for every $1 \le \kappa < s$.

**Theorem 8.2** ([6])**.** *Assume* (2.5) *and that the codimension of* $\Sigma$ *is* 3*. Then* $P$ *is Gevrey* 3 *strongly hyperbolic at the origin.*

The Gevrey index 3 in the above theorem is optimal in the following sense. Let us consider $P$ of (7.1).

**Theorem 8.3** ([6])**.** *The Cauchy problem for* $P + AD_2$*,* $A \in \mathbb{C} \setminus \mathbb{R}_+$ *is not locally* $\gamma^{(s)}$ *solvable at the origin if* $s > 3$*.*

**Theorem 8.4** ([5])**.** *Assume* (2.5) *and that there is no bicharcateristic with a limit point in* $\Sigma$*. We also assume that the codimension of* $\Sigma$ *is* 3*. Then* $P$ *is Gevrey* 4 *strongly hyperbolic at the origin.*

The Gevrey index 4 in the above theorem is optimal in the following sense. Let us consider the following model operator which verifies the conditions in Theorem 8.4:

$$P = -D_0^2 + 2D_0D_1 + x_1^2D_2^2.$$

**Theorem 8.5** ([17], [50])**.** *The Cauchy problem for* $P + AD_2$*,* $A \in \mathbb{C} \setminus \mathbb{R}_+$ *is not locally* $\gamma^{(s)}$ *solvable at the origin if* $s > 4$*.*

We summarize the preceding results on Gevrey strong hyperbolicity in the following table.

| Spectrum of $F_p$ | W | Geometry of bicharacteristics of $p$ near $\Sigma$ | Gevrey $s$ strong hyperbolicity of $P$ |
|---|---|---|---|
| Exists nonzero real eigenvalue | $W = \{0\}$ | Two bicharacteristics intersect $\Sigma$ transversally | Gevrey $\infty$ strongly hyperbolic[22] |
| No nonzero real eigenvalue | $W \neq \{0\}$ | No bicharacteristic intersects $\Sigma$ | Gevrey 4 strongly hyperbolic |
| | | Exists a bicharacteristic tangent to $\Sigma$ | Gevrey 3 strongly hyperbolic |
| | $W = \{0\}$ | No bicharacteristic intersects $\Sigma$ | Gevrey 2 strongly hyperbolic[23] |

From this table we see that, supposing that the codimension of $\Sigma$ is 3, the threshold of Gevrey strong hyperbolicity occurs only at $s = 2, 3, 4$ and that these

[22] That is, locally $\gamma^{(s)}$ solvable for any $s \ge 1$.

[23] This is a special case of the well known result of [7]. The optimality of the Gevrey index 2 is also well known.

thresholds completely determine the structure of the Hamilton map $F_p$ and the geometry of bicharacteristics near $\Sigma$. The restrictions on the codimension of $\Sigma$ in Theorems 8.2 and 8.4 seem to be technical but not yet removed.

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