The hyperbolic Cauchy problem

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(Soeul National University, February, 1992)

1. Review on basic facts

1.1 Hyperbolicity

Let P be a differential operator of order m defined on an open set Ω in $\mathbb{R}^{d+1}$ and let H be a hypersurface in Ω. The Cauchy problem for P with respect to the hypersurface H is:

Find a solution u to the equation $Pu = 0$ of which the first m terms in the Taylor expansion on H coincide with given functions on H?

This is not always possible and hence our main concern is:

For which operators P and hypersurfaces H this problem could be solved?

One almost necessary condition to this problem is that P is non-characteristic with respect to H. That is

Definition 1.1.1: P is said to be *non-characteristic* with respect to H at $\bar{x} \in H$ if

$$\lim_{\lambda \to \infty} \lambda^{-m} e^{-\lambda h(x)} P e^{\lambda h(x)} \neq 0 \text{ at } \bar{x}, \tag{1.1.1}$$

where $h(x)$ is a defining function of H, in the sense that $H = \{h(x) = 0\}$, $dh(x) \neq 0$ on H.

In the analytic category, (1.1.1) is sufficient to assure the solvability of the Cauchy problem (Cauchy-Kowalevsky Theorem). On the other hand, (1.1.1) is far from sufficient to garantee the solvability of the Cauchy problem for general C^∞ data.

Remark: If P is of constant coefficients and H is a hyperplane, this is really necessary ([10]). In the case of variable coefficients, if we assume the existence of a finite dependence domain, this is also necessary ([30], [14], [11]).

Taking the remark in mind, we assume, in what follows, that P is non-characteristic with respect to H.

With a system of local coordinates $x = (x_0, x_1, ..., x_d)$ in Ω, P is expressed as follows:

$$P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha = \sum_{j=0}^{m} P_j(x, D), \tag{1.1.2}$$

where $a_\alpha(x)$ are C^∞ functions on Ω and D is the differential monomial

$$D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_d^{\alpha_d}, D_j = -i\frac{\partial}{\partial x_j}$$

and $\alpha = (\alpha_0, \alpha_1, ..., \alpha_d) \in \mathbb{N}^{d+1}$, $|\alpha| = \sum_{j=0}^{d} \alpha_j$,

$$P_j(x, D) = \sum_{|\alpha|=j} a_\alpha(x) D^\alpha.$$

We choose the local coordinates so that

$$h(x) = x_0, \overline{x} = 0,$$

near $\overline{x}$ and we write P_m in the following form,

$$P_m = \sum_{k=0}^{m} Q_{m-k}(x, D') D_0^k, \tag{1.1.3}$$

where Q_j is a differential operator of order j with respect to $x' = (x_1, ..., x_d)$. In this situation, the condition (1.1.1) yields that

$$Q_0(\overline{x}) \neq 0.$$

Hence, dividing P by $Q_0(x)$, we can assume that the coefficient of D_0^m in (1.1.2) is equal to one near $\overline{x}$.

Here we give an elegant formulation of the Cauchy problem due to [14],

Definition 1.1.2: Let P be a partial differential operator of order m with coefficients in $C^\infty(\Omega)$. Let $t = t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued function. Then the Cauchy problem for P is C^∞ *well posed* at $\overline{x}$ with respect to $t(x)$ if there exist a neighborhood $\omega \subset \Omega$ of $\overline{x}$ and a number $\varepsilon > 0$ such that

$$P : E_\tau = \{v \in C^\infty(\omega) | v = 0 \text{ in } t(x) < t(\overline{x}) + \tau\} \to E_\tau \tag{1.1.4}$$

is an isomorphism if $|\tau| < \varepsilon$.

Our main concern is to characterize differential operators for which the Cauchy problem is C^∞ well posed, that is to characterize *hyperbolic operators*. Another very closely related problem is to characterize strongly hyperbolic operators:

DEFINITION 1.1.3: Let P be a differential operator of order m with $C^\infty(\Omega)$ coefficients and $t(x) \in C^\infty(\Omega)$, be real valued with $dt(x) \neq 0$ in Ω. Then P (or the principal part P_m of P) said to be *strongly hyperbolc* at $\overline{x} \in \Omega$ with respect to $t(x)$ if, for any differential operator Q of order at most $m-1$ with $C^\infty(\Omega)$ coefficients, the Cauchy problem for $P+Q$ is C^∞ well posed at $\overline{x}$ with respect to $t(x)$.

1.2 Operators with constant coefficients

We take $t(x) =< \theta, x >, \theta \in \mathbb{R}^{d+1}$ as a linear function in x so that $dt(x) = \theta$. In this case, the hyperbolicity is completely characterized. Let

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \tag{1.2.1}$$

be a polynomial in $D_0, ..., D_d$. We introduce the following condition; there exists $T > 0$ such that

$$\xi \in \mathbb{R}^{d+1}, \tau \in \mathbb{C}, P(\xi + \tau\theta) = 0 \Longrightarrow |\mathrm{Im}\tau| \leq \mathrm{T}. \tag{1.2.2}$$

Theorem 1.2.1: *Let P have constant coefficients. In order that P to be hyperbolic at $\overline{x}$ w.r.t. θ, it is necessary and sufficient that (1.1.1) and (1.2.2) hold ([8]).*

Here we remark that the hyperbolicity is independent of $\overline{x}$ if $t(x)$ is linear in x. Recall that the principal part of P is given by

$$P_m = P_m(D) = \sum_{|\alpha| = m} a_\alpha D^\alpha. \tag{1.2.3}$$

If P is hyperbolic w.r.t. θ, then P_m is also hyperbolic w.r.t. θ. On the other hand if P is a homogeneous polynomial satisfying (1.1.1) then, for this P, the condition (1.2.2) is equivalent to that

$$\xi \in \mathbb{R}^{d+1}, P(\xi + \tau\theta) = 0 \Longrightarrow \tau \text{ is real}, \tag{1.2.4}$$

The following is also an important characterization of the hyperbolicity.

Theorem 1.2.2: *Suppose that $P(D)$ satisfies (1.1.1) and $P_m(D)$ is hyperbolic w.r.t. θ. In order that P is hyperbolic w.r.t. θ it is necessary and sufficient that we have*

$$|P(\xi)| \leq C \sum_\alpha |D^\alpha P_m(\xi)| \text{ for any } \xi \in \mathbb{R}^{d+1},$$

with some $C > 0$, where the sum is taken over all order derivatives w.r.t. ξ ([48]).

DEFINITION 1.2.1: Let $P(D)$ be given by (1.2.1) , $P(D)$ is said to be *strictly hyperbolic* w.r.t. θ if the roots of the equation $P_m(\xi + \tau\theta) = 0$ in τ are all real and distinct for any $\xi \in \mathbb{R}^{d+1} \setminus \mathbb{R}\theta$.

Theorem 1.2.3: *Let P have constant coefficients. For P to be strongly hyperbolic w.r.t. θ, it is necessary and sufficient that P is strictly hyperbolic w.r.t. θ.*

Let $P(D)$ be hyperbolic w.r.t. $t(x) = x_0$. Then a fundamental solution E of the Cauchy problem for $P(D)$ is a distribution satisfying

$$P(D)E = \delta(x), E = 0 \text{ in } x_0 < 0, \tag{1.2.5}$$

where $\delta(x)$ is the Dirac measure at the origin.

We define $\Gamma(P_m, \theta)$ by

$$\Gamma(P_m, \theta) = \text{the component of } \theta \text{ in } \{\xi | P_m(\xi) \neq 0\},$$

which is a cone with vertex at the origin. Then one can prove that the support of E is contained in $\Gamma^\circ(P_m, \theta)$, which is the dual cone of $\Gamma(P_m, \theta)$:

$$\Gamma^\circ(P_m, \theta) = \{x| < x, y >\geq 0, y \in \Gamma(P_m, \theta)\}.$$

For more detailed studies on the hyperbolicity of operators with constant coefficients, we refer to [8], [1] and [10].

1.3 Strict hyperbolicity

With a system of local coordinates $x = (x_0, ..., x_d)$ in Ω, P is given by

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$

Recall that the *principal part* of P is defined by

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha. \tag{1.3.1}$$

$P_m(x, \xi)$ is invariantly defined as a function on the cotangent bundle $T^*\Omega$.

A first basic result in the characterization of hyperbolicity, in the variable coefficients case, is

Theorem 1.3.1: *Suppose that P is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$. Then there is a neighborhood U of $\overline{x}$ such that $P_m(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ for every $x \in U$, that is $P_m(x, \cdot)$ satisfies (1.2.4) ([25], [31]).*

Definition 1.3.1: We say that a point $z = (x, \xi) \in T^*\Omega \setminus 0$ is a *characteristic of order k* of P_m if

$$d^j P_m(z) = 0, j \leq k - 1, d^k P_m(z) \neq 0. \tag{1.3.2}$$

where $d^j P_m$ is the j-th differential of P_m.

Definition 1.3.2: P is said to be *strictly hyperbolic* at $\overline{x} \in \Omega$ w.r.t. $t(x) \in C^\infty(\Omega)$ if there exists a neighborhood $\omega \subset \Omega$ of $\overline{x}$ such that for any $x \in \omega$, $P_m(x,\cdot)$ is strictly hyperbolic w.r.t. $dt(x)$ in the sense of the definition 1.2.1.

We note that $P_m(x,\cdot)$ is a polynomial on $T_x^*\Omega$ and $dt(x) \in T_x^*\Omega$.

Lemma 1.3.2: *Assume that $P_m(x,\cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\overline{x}$. Then P is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ if and only if there is a neighborhood $\omega \subset \Omega$ of $\overline{x}$ such that every characteristic on $T^*\omega \setminus 0$ of P_m is simple ([14]).*

Theorem 1.3.3: *If P is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ then P is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([46], [23], [9]).*

We assume that P is strictly hyperbolic in Ω w.r.t. $t(x)$ and we define $\Gamma^-(x)$ as follows

$$\cup_{x(s)}\{y \in \mathbb{R}^n | y \in x(s)\}$$

where $x(s)$ varies over all Lipschitz curves such that $(d/ds)x(s)$ belongs to $\Gamma^\circ(p(x(s),\cdot), dt(x(s)))$, $x(0) = x$, x_0 is decreasing along on $x(s)$.

We take $\omega \subset \Omega$, a neighborhood of $\overline{x}$, so that

$$\Gamma^-(x) \cap \{t(x) \geq t(\overline{x})\} \subset\subset \omega \text{ if } x \in \omega^+ = \omega \cap \{t(x) \geq t(\overline{x})\}. \tag{1.3.3}$$

Then we have

Theorem 1.3.4: *Assume that*

$$Pu = f \, in \; \omega^+, \; u = 0 \; in \; t(x) < t(\overline{x}) \; and \; f = 0 \; on \; \Gamma^-(x).$$

Then it follows that

$$u = 0 \; on \; \Gamma^-(x)$$

([23]).

The same conclusion holds for the singularities of the solution of (1.3.4), i.e. if f is C^∞ in a neighborhood of $\Gamma^-(x)$ then so is u. A more refined version of this is the celebrated theorem in the propagation of singularities. For this we need to microlocalize the notion that u is singular at $\overline{x}$, i.e. that u is not C^∞ in some neighborhood of $\overline{x}$ to that of wave front set.

Now we introduce the bicharacteristic of P_m which carries the wave front set of solutions. In the following we assume that P_m is real valued and set

$$P_m(x,\xi) = p(x,\xi)$$

for simplicity. The Hamilton vector field H_p of p is given by

$$H_p = \sum_{j=0}^{d} \frac{\partial p(x,\xi)}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p(x,\xi)}{\partial x_j}\frac{\partial}{\partial \xi_j} \tag{1.3.4}$$

which is a vector field on $T^*\Omega$.

Definition 1.3.3: A *bicharacteristic* of p is an integral curve of H_p on $\{p = 0\}$.

Let γ be a bicharacteristic of p issuing from $z = (\overline{x}, \overline{\xi})$ with $p(z) = 0$, on which x_0 is decreasing.

Theorem 1.3.5: *Assume that P is strictly hyperbolic at $\overline{x}$. If $u \in \mathcal{D}'(\omega)$ satisfies*

$$Pu = f \text{ near } \overline{x} \text{ and } z \notin WF(f),$$

then

$$z \notin WF(u),$$

if $\gamma(-\epsilon) \notin WF(u)$ with a sufficiently small $\epsilon > 0$, where $WF(u)$ denotes the wave front set of u ([11]).

1.4 Operators with constant multiple characteristics

We begin with the following definition.

Definition 1.4.1: Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set. P is said to be of *constant multiple characteristics* if $P_m(x, \xi)$ can be factorized as

$$P_m(x, \xi) = \prod_{j=1}^{k} q_j(x, \xi)^{r_j},$$

where each $q_j(x, \xi)$ is of simple characteristics in Ω and the sets $q_j^{-1}(0)$ are mutually disjoint.

Next, in order to introduce the Levi condition, we define the characteristic function of q_j.

Definition 1.4.2: $\phi(x)$ is a *characteristic function* of q at $\overline{x} \in \Omega$ if there is a neighborhood U of $\overline{x}$ such that

$$q(x, d\phi(x)) = 0, x \in U, d\phi(\overline{x}) \neq 0.$$

Definition 1.4.3: Let P be of constant multiple characteristics. We say that P satisfies the *Leivi condition* at $\overline{x} \in \Omega$ if we have

$$e^{-i\lambda\phi} P(ae^{i\lambda\phi}) = O(\lambda^{m-r_j}), (\lambda \to \infty),$$

for any characteristic function ϕ of q_j and any $a \in C^\infty(\Omega)$ on whose support $d\phi \neq 0$, $j = 1, 2, ..., k$.

Theorem 1.4.1: *Let P be of constant multiple characteristics. If P is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$, then each q_j is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ and P satisfies the Levi condition at $\overline{x}$. Conversely, if each q_j is strictly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ and P satisfies the Levi condition near $\overline{x}$, then P is hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([26], [24], [32], [33], [7], [6]).*

EXAMPLE 1.4.1: We give the simplest example in $\mathbb{R}^2$. Let

$$P(x, D) = D_0^2 + a(x)D_0 + b(x)D_1 + c(x),$$

where $x = (x_0, x_1) \in \mathbb{R}^2$ and $a(x), b(x), c(x)$ are C^∞ functions defined near the origin. Then in order that the Cauchy problem for this P is C^∞ well posed at $x = 0$ it is necessary and sufficient that $b(x) = 0$ near the origin.

2. Hyperbolicity at multiple characteristics

2.1 Effective hyperbolicity

There was a surprising discorvery around 1970, that is there are operators of second order with double characteristics which are strongly hyperbolic. Of course this phenomenon never occur in constant coefficient case.

EXAMPLE 2.1.1: Let

$$P(x, D) = D_0^2 - x_0^2 D_1^2 + a(x)D_0 + b(x)D_1 + c(x)$$

where $x = (x_0, x_1) \in \mathbb{R}^2$. The Cauchy problem for this P is C^∞ well posed at the origin with respect to $t(x) = x_0$ for any $a(x)$, $b(x)$, $c(x) \in C^\infty$ near the origin. On the other hand it is obvious that $z = (0, 0, 0, 1)$ is a double characteristic of P_2. The main feature of this Cauchy problem is that the solution of the Cauchy problem loses the regularity compared with initial data and the loss of derivatives depends on $b(x)$.

Lemma 2.1.1: *Assume that P_m is strongly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$. Then there is a neighborhood U of $\overline{x}$ such that all characteristics of P_m in $T^*U \setminus 0$ are at most double ([14]).*

Let (x, ξ) be a system of symplectic coordinates in $T^*\Omega$. Then the natural symplectic 2-form σ in $T^*\Omega$ is given by

$$\sigma = \sum_{j=0}^{d} d\xi_j \wedge dx_j.$$

Let $h(x, \xi)$ be a smooth function on $T^*\Omega \setminus 0$ and $z = (x, \xi) \in T^*\Omega \setminus 0$ be a double characteristic so that $h(z) = dh(z) = 0$.

Definition 2.1.1: The *Hamilton map* $F_h(z)$ of h at z is defined by

$$\sigma(X, F_h(z)Y) = Q(X, Y), \text{ for any } X, Y \in T_z(T^*\Omega),$$

where Q is the quadratic form corresponding to the Hessian of $h/2$ at z.

Lemma 2.1.2: *Suppose that $P_m(x, \cdot)$ is hyperbolic near $\overline{x}$ w.r.t. $dt(x)$. Let $z \in T^*_{\overline{x}}\Omega \setminus 0$ be a double characteristic of P_m. Then all eigenvalues of $F_{P_m}(z)$ are on the pure imaginary axis possibly with an exception of a pair of $\pm e$, $e \in \mathbb{R}$, $e \neq 0$ ([14], [12]).*

Definition 2.1.2: Suppose that $P_m(x, \cdot)$ is hyperbolic near $\overline{x}$ w.r.t. $dt(x)$. We shall say that P_m is *effectively hyperbolic* at a double characteristic $z \in T^*_{\overline{x}}\Omega \setminus 0$ if $F_{P_m}(z)$ has non-zero real eigenvalue.

Theorem 2.1.3: *In order that P_m is strongly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ it is necessary and sufficient that $P_m(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\overline{x}$ and P_m is effectively hyperbolic at every double characteristic on $T^*_{\overline{x}}\Omega \setminus 0$ ([14], [15], [29], [17], [34]).*

Let $z \in T^*_{\overline{x}}\Omega \setminus 0$ be a double characteristic of p and assume that p is effectively hyperbolic at z.

Definition 2.1.3: Let

$$\gamma : s \mapsto \gamma(s) = (x(s), \xi(s))$$

be a bicharacteristic of p defined in $[s_0, +\infty)$,(resp. $(-\infty, s_0]$) with some s_0. We say that γ is *incoming* (resp. *outgoing*) with respect to z if

$$\gamma(s) \to z \text{ as } s \uparrow +\infty \text{ (resp. as } s \downarrow -\infty).$$

Proposition 2.1.4: *There are exactly two incoming (resp. outgoing) bicharacteristics of p with respect to z. Furthermore one of the incoming (resp. outgoing) bicharacteristics is naturally continued to the other one, and the resulting two curves are C^∞ regular near z as submanifolds of $T^*\Omega$. These two curves are (real) analytic near z whenever p is assumed to be analytic there ([18], [20]).*

2.2 A geometric characterization

We start by the following definition:

DEFINITION 2.2.1: Let z be a multiple characteristic of p. *The localization* $p_z(X)$ of p at z is defined by

$$p_z(X) = d^r p(z; X, ..., X)/r!, X \in T_z(T^*\Omega)$$

which is a homogeneous polynomial of degree r in $X \in T_z(T^*\Omega)$, the tangent space of $T^*\Omega$ at z.

Note that the hyperbolicity of $p_z(X)$ with respect to $\Theta = -H_{x_0}$ follows from the hyperbolicity of $p(x, \cdot)$ with respect to $dt(x) = dx_0$ near $\overline{x}$. Recall that H_ϕ denotes the Hamilton vector field of ϕ defined by

$$\sigma(X, H_\phi(z)) = d\phi(X), X \in T_z(T^*\Omega).$$

Naturally we are led to consider the hyperbolicity cone $\Gamma(p_z, \Theta)$ of p_z. We recall the definition:

$$\Gamma(p_z, \Theta) = \text{ the component of } \Theta \text{ in } \{X \in T_z(T^*\Omega) | p_z(X) \neq 0\}.$$

DEFINITION 2.2.2: *The propagation cone* $\Gamma^\sigma(p_z, \Theta)$ of p_z is defined by

$$\Gamma^\sigma(p_z, \Theta) = \{X \in T_z(T^*\Omega) | \sigma(X, Y) \leq 0, \forall Y \in \Gamma(p_z, \Theta)\}.$$

DEFINITION 2.2.3: Let $t(x, \xi)$ be homogeneous of degree 0 in ξ, C^1 in a conic neighborhood of z. We say that $t(x, \xi)$ is a *time function* at z w.r.t. $\Gamma(p_z, \Theta)$ if $t(z) = 0$ and

$$-H_t(z) \in \Gamma(p_z, \Theta).$$

Note that $t(x, \xi)$ is a time function at z w.r.t. $\Gamma(p_z, \Theta)$ if and only if

$$\Gamma^\sigma(p_z, \Theta) \cap T_z(\{t(x, \xi) = 0\}) = \{0\}.$$

The propagation cone is the *minimal* cone containing the tangents of bicharacterisitcs of p with limit point z. More precisely:

Lemma 2.2.1: *Let $z \in T^*\Omega \setminus 0$ be a characteristic of order r of p. Assume that there are simple characterisitcs z_j and positive numbers λ_j such that*

$$z_j \to z \text{and } \lambda_j p_{z_j}(\Theta) H_p(z_j) \to X(\neq 0) \text{ as } j \to \infty.$$

Then $X \in \Gamma^\sigma(p_z, \Theta)$ ([51]).

Let $q(X)$ be a homogeneous hyperbolic polynomial on $T_z(T^*\Omega)$ with respect to $\Theta \in T_z(T^*\Omega)$. Denote by $\Lambda(q)$ the linearity space of q:

$$\Lambda(q) = \{X \in T_z(T^*\Omega) | q(tX + Y) = q(Y), \forall t \in \mathbb{R}, \forall Y \in T_z(T^*\Omega)\}.$$

Note that $\Lambda(p_z) = \mathrm{Ker} F_p(z)$ if $d^2 p(z) \neq 0$. We now state a geometric characterization of the effective hyperbolicity.

Proposition 2.2.2: *Notations as above. Let $z \in T^*\Omega \backslash 0$ be a double characteristic of p. Then the following conditions are equivalent:*
(a) $\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) = \{0\}$,
(b) $F_p(z)$ *has a non-zero real eigenvalue.*

Let $\theta = (1, 0, ..., 0)$ and assume that the coefficient of D_0^m is equal to 1. Factorizing $p(x, \xi)$ as

$$p(x,\xi) = \prod_{j=1}^{m} q_j(x,\xi)$$

where $q_j(x, \xi) = \xi_0 - \lambda_j(x, \xi')$, we define $h_j(x, \xi)$ as

$$|p(x, \xi - is\theta)|^2 = \sum_{j=0}^{m} s^{2(m-j)} h_j(x,\xi).$$

It is clear that

$$h_k(x,\xi) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} |q_{j_1}(x,\xi)|^2 \cdots |q_{j_k}(x,\xi)|^2, k = 1, 2, ..., m,$$

and $h_0(x, \xi) = 1$, $h_m(x, \xi) = |p(x, \xi)|^2$. We now characterize the effective hyperbolicity in terms of time functions.

Proposition 2.2.3: *Let $z \in T^*\Omega \setminus 0$ be a double characteristic. Assume that p is effectively hyperbolic at z. Then there is a time function $t(x, \xi)$ at z with respect to $\Gamma(p_z, \Theta)$ satisfying*

$$h_{m-1}(x,\xi) \ge ct(x,\xi)^2 |\xi'|^{2(m-1)}$$

near z with a positive constant c. Conversely if the conclusion holds then p is effectively hyperbolic at z.

Definition 2.2.4: $\gamma^+(z)$ (resp. $\gamma^-(z)$) denotes the union of two bicharacteristics of p with the limit point z along which a time function with respect to $\Gamma(p_z, \Theta)$ is increasing (resp. decreasing).

Theorem 2.2.4: *Let $t(x, \xi)$ be a time function with respect to $\Gamma(p_z, \Theta)$. Assume that*

$$WF(u) \cap \{t(x,\xi) = -\epsilon\} \cap \gamma^+(z) = \emptyset$$

and $z \notin WF(Pu)$ with a sufficiently small $\epsilon > 0$. Then $z \notin WF(u)$ ([29], [35]).

2.3 A generalization of effective hyperbolicity

Here we generalize the notion of effective hyperbolicity at characteristics of order exceeding two employing the geometric characterization. We introduce the following assumption:

$(A.i)_z$: there are a conic neighborhood U of z and finite number of time functions $t_l(x,\xi)$, $l = 1, 2, ..., n$ such that

$$h_{m-1}(x,\xi) \geq ct(x,\xi)^2 h_{m-2}(x,\xi)|\xi'|^2, \forall (x,\xi) \in U$$

where $t(x,\xi) = \min_{1 \leq l \leq n} |t_l(x,\xi)|$.

Lemma 2.3.1: *Assume that $(A.i)_z$ holds. Then we have*

$$\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) = \{0\}.$$

QUESTION : Let z be a characteristic of order greater than two. When the conclusion of Lemma 2.3.1 implies $(A.i)_z$?

This is motivated by the geometric characterization given in Proposition 2.2.3. Let us denote by P_j the homogeneous part of degree j so that P is the sum of P_j, $j = 0, 1, ..., m-1$ and $p = P_m$. A general necessary condition for hyperbolicity at a multiple characterisitc is:

Theorem 2.3.2: *Let $z = (\overline{x}, \overline{\xi}) \in T^*\Omega \setminus 0$ be a characteristic of order r of p. Suppose that P is hyperbolic at $\overline{x}$, that is the Cauchy problem for P is C^∞ well posed at $\overline{x}$ w.r.t. $t(x) = x_0$. Then P_j vanishes at least of order $r - 2(m-j)$ at z whenever $r - 2(m-j) > 0$ ([14]).*

We assume the following:

$(A.ii)_z$: there are a conic neighborhood U of z and $C > 0$ such that

$$|P_j(x,\xi)| \leq C|h_{2j-m}(x,\xi)|^{1/2}|\xi'|^{m/2}, \forall (x,\xi) \in U$$

for $[m/2] + 1 \leq j \leq m-1$.

It should be noted that $h_{2j-m}(x,\xi) \neq 0$ near z if $2j - m \leq m - r$, i.e. $j \leq (2m-r)/2$ when z is a characteristic of order r because there are $m - r$ of q_j which do not vanish at z and hence $(A.ii)_z$ gives no restriction on P_j near z if $j \leq (2m-r)/2$.

Now we have

Theorem 2.3.3: *Assume that the conditions* $(A.i)_z$ *and* $(A.ii)_z$ *are satisfied for every multiple characteristic* $z \in T^*_{\overline{x}}\Omega \setminus 0$*. Then the Cauchy problem for* P *is* C^∞ *well posed at* $\overline{x}$ *w.r.t.* $t(x) = x_0$ *([21]).*

QUESTION : Let z be a characteristic of order greater than two. Suppose that the Cauchy problem for P is C^∞ well posed at $\overline{x}$ w.r.t. $t(x) = x_0$ for every lower order term P_j verifying $(A.ii)_z$. Can we conclude that

$$\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) = \{0\}?$$

In the next theorem we follow the notations in subsection 1.3 and assume the condition (1.3.3).

Theorem 2.3.4: *Assume that the conditions* $(A.i)_z$ *and* $(A.ii)_z$ *are fulfilled at every multiple characteristic* $z \in T^*\Omega \setminus 0$*. Suppose that*

$$Pu = f \, in \, \omega^+, \; u = 0 \; in \; t(x) < t(\overline{x}) \; and \; f = 0 \; on \; \Gamma^-(x).$$

Then it follows that

$$u = 0 \; on \; \Gamma^-(x)$$

([21]).

EXAMPLE 2.3.1: Here we give a simple example to elucidate the geometric meanings of the conditions $(A.i)_z$ and $(A.ii)_z$. Let $p(x, \xi)$ be factorized as

$$p(x, \xi) = e(x, \xi) \prod_{j=1}^{r} q_j(x, \xi), q_j(z) = 0$$

in a conic neighborhood U of z where $e(x, \xi)$, $q_j(x, \xi)$ are smooth near z, homogeneous of degree $m - r$ and 1 respectively and $e(z) \neq 0$, $dq_j(z) \neq 0$ and $q_j(\overline{x}, \theta) > 0$. Assume, for simplicity, that dq_j are linearly independent at z. Recall that the cone generated by the Hamilton vector fields $H_{q_j}(z)$ of q_j forms the propagation cone $\Gamma^\sigma(p_z, \Theta)$ of the localization $p_z(X)$. Then the condition $(A.i)_z$ is fulfilled if and only if $\Gamma^\sigma(p_z, \Theta)$ is transversal to the tangent space at z of each intersection of any two hypersurfaces $\{q_k = 0\}$, $\{q_l = 0\}$:

$$\Gamma^\sigma(p_z, \Theta) \cap T_z\{q_k = 0, q_l = 0\} = \{0\}, \forall k \neq l.$$

On the other hand, the condition $(A.ii)_z$ is satisfied if and only if $P_j(x, \xi)$ vanishes of order $r - 2(m - j)$ on each intersection of any two hypersurfaces $\{q_k = 0\}$, $\{q_l = 0\}$ near z whenever $r - 2(m - j) > 0$.

EXAMPLE 2.3.2: Here we give an example verifying the conditions $(A.i)_z$ and $(A.ii)_z$ which is not necessarily factorized smoothly. Denote by Σ the set of characteristics of order r of p:

$$\Sigma = \{(x,\xi) \in T^*\Omega \setminus 0 | p(x,\xi) = dp(x,\xi) = \cdots = d^{r-1}p(x,\xi) = 0\}.$$

We assume that

(i) Σ is a C^∞ manifold near $z = (\overline{x}, \overline{\xi})$.

It then follows that

$$p_z(X + tY) = p_z(X) \; \forall t \in \mathbb{R}, \forall Y \in T_z\Sigma, \forall X \in T_z(T^*\Omega)$$

so that $T_z\Sigma = \Lambda(p_z)$ and we may regard $p_z(X)$ as a polynomial on $N_\Sigma(T^*\Omega)_z$ which is defined by $T_z(T^*\Omega)/T_z\Sigma$. Denoting by $[X]$ the equivalence class of $X \in T_z(T^*\Omega)$ we assume that

(ii) $p_z([X])$ is strictly hyperbolic with respect to $[\Theta] \in N_\Sigma(T^*\Omega)_z$

and that $\Gamma^\sigma(p_z, \Theta)$ is transversal to Σ at z:

$$\Gamma^\sigma(p_z, \Theta) \cap T_z\Sigma = \{0\}.$$

We also assume that

(iii) $P_j(x,\xi)$ vanishes of order $r - 2(m-j)$ on Σ near z when $r - 2(m-j) > 0$.

Then the conditions $(A.i)_z$ and $(A.ii)_z$ are fulfilled for p.

2.4 Non effective hyperbolicity

The necessity of effective hyperbolicity in Theorem 2.1.3 is a special case of a more general condition for hyperbolicity. At any double characteristic $z \in T^*\Omega \setminus 0$ of p, the subprincipal symbol of P is well defined by reference to any local coordinates x:

$$P^s(x,\xi) = P_{m-1}(x,\xi) + \frac{i}{2}\sum_{j=0}^{d} \frac{\partial^2}{\partial x_j \partial \xi_j} p(x,\xi).$$

DEFINITION 2.4.1: We define *the positive trace* Tr^+F_p of p at z as

$$Tr^+F_p(z) = \sum i\mu_j$$

where $i\mu_j$ are the eigenvalues of $F_p(z)$ on the imaginary axis, repeated according to their multiplicities.

Theorem 2.4.1: *Let $z = (\overline{x}, \overline{\xi}) \in T^*_{\overline{x}}\Omega \setminus 0$ be a double characteristic. Assume that P is hyperbolic at $\overline{x}$ w.r.t. $t(x) = x_0$. Then we have*

$$\text{Im } P^s(z) = 0, \ |\text{Re } P^s(z)| \le Tr^+ F_p(z)$$

([14], [12]).

For the converse of Theorem 2.4.1 we refer to [16], [12]. When the multiplicity of z exceeds 2, according to Proposition 2.2.2, it would be natural to call that P is not of effective type at z if

$$\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) \neq \{0\}.$$

In what follows, in this subsection, we study operators of non effective type. We recall that the localization $p_z(X)$ is a well defined hyperbolic polynomial on $T^*(\Omega)/\Lambda(p_z)$ w.r.t. $[\Theta]$, where $[X]$ denotes the equivalence class of X as in Example 2.3.2. It is easy to check that if z is a double characteristic then $p_z(X)$ is strictly hyperbolic on $T^*_z(\Omega)/\Lambda(p_z)$ w.r.t. $[\Theta]$. It is then natural to assume, as an ideal case, that $p_z(X)$ is strictly hyperbolic w.r.t. $[\Theta]$ even when z is a characteristic of order greater than 2. When z is a triple characteristic with

$$\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z)$$

we refer to a recent work [2].

As for the case

$$\Gamma^\sigma(p_z, \Theta) \not\subset \Lambda(p_z)$$

we state a typical necessary condition in order that P is hyperbolic at $\overline{x}$ w.r.t. $t(x) = x_0$ when p has a triple characteristic $z \in T^*_{\overline{x}}\Omega \setminus 0$ (see also [3]). We list up the assumptions we make:

The localization $p_z(X)$ of p at z satisfies the following conditions:

(i) $p_z(X) = L(X)Q(X)$ where $L(X)$ is a linear form and $Q(X)$ is a real quadratic form such that

$$\text{Ker } F_Q^2 \cap \text{Im } F_Q^2 = \{0\}.$$

(ii) $H_L \in \Lambda(p_z)$.

Theorem 2.4.2: *In order that P is hyperbolic at $\overline{x}$ w.r.t. $t(x) = x_0$ the followings are necessary:*

(L1) $P^s(z) = 0$

(L2) $\text{Im } H_{P^s}(z) = 0, \ Tr^+ F_Q \ H_L \pm \text{Re } H_{P^s}(z) \in \Gamma^\sigma(p_z, \Theta)$

([4]).

Question : What conditions are necessary when we drop the assumption (ii) in Theorem 2.4.2?

Remark: Assuming that P is not effective type at a multiple characteristic z, we could expect, in general, neither $\Gamma^{\sigma}(p_z, \Theta) \subset \Lambda(p_z)$ nor p_z is factorized. In such general cases, few facts are known concerning with both necessity and sufficiency of C^{∞} well posedness of the Cauchy problem. However see [40]. An interesting approach to this problem is found in [41].

3. First order systems

3.1 Preliminaries

Let L be a differential operator of first order on $C^{\infty}(\Omega, \mathbb{C}^N)$. Let (x, ξ) be a system of local coordinates on $T^*\Omega$ and $e_1, ..., e_N$ be a frame in $\mathbb{C}^N$. With these coordinates and frame, the principal symbol of L is given by

$$L_1(x, \xi) = \sum_{j=0}^{d} L_j(x)\xi_j. \tag{3.1.1}$$

We set

$$h(x, \xi) = \det\, L_1(x, \xi),$$

which is invariantly defined as a function on $T^*\Omega$.

Definition 3.1.1: Let $t(x) \in C^{\infty}(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. We say that L is *non-characteristic* w.r.t. $H = \{t(x) = 0\}$ at $\overline{x} \in H$ if

$$\lim_{\lambda \to \infty} \lambda^{-1} e^{-\lambda t(x)} L(e^{\lambda t(x)}),$$

is a surjection on $\mathbb{C}^N$ at $\overline{x}$.

As in subsection 1.1 we are assuming that L is non-characteristic w.r.t. H at the reference point $\overline{x}$.

Definition 3.1.2: Let L be a differential operator of first order on $C^{\infty}(\Omega, \mathbb{C}^N)$ and $t(x) \in C^{\infty}(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. Then L is said to be *hyperbolic* w.r.t. $t(x)$ at $\overline{x} \in \Omega$ if there are a neighborhood $\omega \subset \Omega$ of $\overline{x}$ and $\epsilon > 0$ such that

$$L : E_{\tau} = \{U \in C^{\infty}(\omega, \mathbb{C}^N) | U = 0 \text{ on } t(x) < t(\overline{x}) + \tau\} \to E_{\tau}$$

is an isomorphism if $|\tau| < \epsilon$.

Definition 3.1.3: Let L be a differential operator of first order on $C^{\infty}(\Omega, \mathbb{C}^N)$ and $t(x) \in C^{\infty}(\Omega)$ be real valued. Then L_1 is said to be *strongly hyperbolic* at $\overline{x}$ w.r.t. $t(x)$ if, for any $Q \in C^{\infty}(\Omega, M(N, \mathbb{C}))$, $L + Q$ is hyperbolic at $\overline{x}$ w.r.t. $t(x)$.

3.2 Systems with constant coefficients

Let

$$L(D) = \sum_{j=0}^{d} A_j D_j + B,$$

where A_j, B are constant square matrices of order N. We take $t(x) =< \theta, x >$ as a linear function in x.

Theorem 3.2.1: *Assume that L is of constant coefficients. For that L to be hyperbolic at $\overline{x}$ w.r.t. θ it is necessary and sufficient that* $\det L(D)$ *is hyperbolic at $\overline{x}$ w.r.t. θ ([1]).*

For $L(D)$ to be strongly hyperbolic the strict hyperbolicity of $\det L(D)$ is sufficient but not necessary:

Theorem 3.2.2: *Assume that L is of constant coefficients. In order that L is strongly hyperbolic w.r.t. θ it is necessary and sufficient that the following condition holds for every $\xi \in \mathbb{R}^{d+1} \setminus \mathbb{R}\theta$,*

$$|L_1(\xi + \tau\theta)^{-1}| \le C(\mathrm{Re}\ \tau)^{-1} \text{ for } \mathrm{Re}\ \tau > 0,$$

([22], [47]).

Theorem 3.2.3: *If h is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ then L is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$. This statement also holds in the variable coefficient case.*

Recall that symmetric or symmetrizable systems are always strongly hyperbolic. It happens that the converse is also true. We first recall that L_1 is said to be symmetrizable if there is a positive definite Hermitian symmetric matrix $S \in M(N, \mathbb{C})$ such that $SL_1(\xi)$ becomes to be Hermitian symmetric for every $\xi \in \mathbb{R}^{d+1}$.

Proposition 3.2.4: *Every 2×2 strongly hyperbolic system is symmetrizable ([47]).*

Without restrictions we may assume that $A_0 = I$, the identity matrix of order N. Let us set

$$d(L_1) = \dim\ \mathrm{span}\{I, A_1, ..., A_d\}$$

which is called *the reduced dimension* of L_1.

Proposition 3.2.5: *Assume that A_j are real and*

$$d(L_1) \ge \frac{N(N+1)}{2} - 1.$$

If L_1 is strongly hyperbolic then L_1 is symmetrizable ([36]).

For another related results we refer to [49].

Question : Let $N > 2$. For what k can one find a strongly hyperbolic system L_1 with $d(L_1) = k$ which is not symmetrizable? Recently a complete classification of 3×3 strongly hyperbolic systems with real constant coefficients is given in [42], [43].

3.3 Systems with constant multiple characterisitcs

Definition 3.3.1: L is said to be of *constant multiple characteristics* if $h(x, \xi)$ $= \det L_1(x, \xi)$ satisfies the conditions in the definition 1.4.1.

If L is of constant multiple characteristics then $h(x, \xi)$ can be factorized as

$$h(x, \xi) = \prod_{j=1}^{k} q_j(x, \xi)^{r_j}.$$

We introduce the following hypothesis.

For every characteristic function ϕ of q_j at $\overline{x} \in \Omega$, we have

$$\text{rank } L_1(\overline{x}, d\phi(\overline{x})) = N - 1 \text{ for any } j. \tag{3.3.1}$$

If we assume that (3.3.1) holds near $\overline{x} \in \Omega$ then we can find $N_j \in C^\infty(\omega, \mathbb{C}^N)$ such that

$$L_1(x, d\phi(x))N_j(x, d\phi(x)) = 0, \ 1 \leq j \leq k,$$

where ω is a neighborhood of $\overline{x}$. Using N_j we introduce the Levi condition.

Definition 3.3.2: We shall say that L satisfies the Levi condition at $\overline{x}$ if, for every characteristic function ϕ of q_j at $\overline{x}$ and for every $a \in C_0^\infty(\Omega)$ on whose support $d\phi \neq 0$, there exists $V_i^{(j)}(x; \phi, a)$ belonging to $C^\infty(\Omega, \mathbb{C}^N)$ such that

$$e^{-i\lambda\phi} L\{e^{i\lambda\phi}(aN_j + \sum_{i=1}^{r_j - 1} \lambda^{-i} V_i^{(j)})\} = O(\lambda^{1-r_j}),$$

for $j = 1, 2, ..., k$.

Theorem 3.3.1: *Assume that L is of constant multiple characteristics and the hypothesis (3.3.1) is realized near $\overline{x}$. If L is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ then each q_j is strictly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ and L satisfies the Levi condition at $\overline{x}$. Conversely if each q_j is strictly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ and L satisfies the Levi condition near $\overline{x}$, then L is hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([44], [52]).*

Theorem 3.3.2: *Assume that L is of constant multiple characteristics. In order that L is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ it is necessary and sufficient that*

$$\dim \text{Ker} L_1(x, \xi) = r_j, \ \forall (x, \xi) \text{ with } q_j(x, \xi) = 0, \ x \text{ near } \overline{x}$$

for $j = 1, 2, ..., k$ ([19]).

For studies on hyperbolicity of systems with constant multiple characteristics without the condition (3.3.1), we refer to recent works [50] and [27].

3.4 Systems with double characteristics

With a system of local coordinates (x, ξ) in $T^*\Omega$ and a frame in $\mathbb{C}^N$, the full symbol of $L(x, D)$ is expressed as follows

$$L(x, \xi) = L_1(x, \xi) + L_0(x).$$

We define $\mathcal{L}(x, \xi)$ by

$$\mathcal{L}(x, \xi) = L^s(x, \xi) L_1^{co}(x, \xi) - \frac{i}{2}\{L_1, L_1^{co}\}(x, \xi),$$

where

$$L^s(x, \xi) = L_0(x) + \frac{i}{2} \sum_{j=0}^{d} \frac{\partial^2}{\partial x_j \partial \xi_j} L_1(x, \xi),$$

$$\{L_1, L_1^{co}\} = \sum_{j=0}^{d} \frac{\partial L_1}{\partial \xi_j} \frac{\partial L_1^{co}}{\partial x_j} - \frac{\partial L_1}{\partial x_j} \frac{\partial L_1^{co}}{\partial \xi_j}$$

and $L_1^{co}(x, \xi)$ is the *cofactor matrix* of $L_1(x, \xi)$. Note that $\mathcal{L}(x, \xi)$ is invariantly defined at a multiple characteristic z in Hom $(\mathbb{C}^N, \mathbb{C}^N)/L_1(z)$Hom $(\mathbb{C}^N, \mathbb{C}^N)$.

Theorem 3.4.1: *Assume that L is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ and $h = \det L_1$ is not effectively hyperbolic and the rank of L_1 is $N-1$ at the multiple characteristic $z \in T_{\overline{x}}^*\Omega \setminus 0$. Then there is a real number $\alpha, |\alpha| \leq 1$ such that*

$$\mathcal{L}(z) + \alpha Tr^+ h(z) I = O,$$

in Hom $(\mathbb{C}^N, \mathbb{C}^N)/L_1(z)$Hom $(\mathbb{C}^N, \mathbb{C}^N)$ *([37]).*

Corollary 3.4.2: *Assume that L_1 is strongly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ and $z \in T_{\overline{x}}^*\Omega \setminus 0$. Then h is effectively hyperbolic at z or the rank of $L_1(z)$ is less than or equal to $N-2$.*

Theorem 3.4.3: *Suppose that $h(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\overline{x} \in \Omega$ and h is effectively hyperbolic at every multiple characteristic in $T_{\overline{x}}^*\Omega \setminus 0$. Then L_1 is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([38]).*

In the following we assume that all characteristics are at most double and we denote by Σ the doubly characteristic set:

$$\Sigma = \{z | h(z) = dh(z) = 0\}.$$

We introduce the following hypotheses concerning the doubly characteristic set.

(i) Σ is a C^∞ manifold,
(ii) rank Hess h = codim Σ.

Theorem 3.4.4: *Assume that (i) and (ii) hold and $h(x,\cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\overline{x}$ and one of the following conditions is verified at every point $z \in T^*_{\overline{x}} \cap \Sigma$,*

(a) h *is effectively hyperbolic at* z,
(b) $\text{rank } L_1 \le N-2$, *near* z *on* Σ.

Then L *is strongly hyperbolic ([38], [37], [5]).*

We present some interesting facts which are valid at double characteristics. Let us denote by Hess $h(z)$ the Hessian of h at z.

Lemma 3.4.5: *Let* $z \in T^*_{\overline{x}}\Omega \setminus 0$ *be a double characteristic. Then we have*

$$\text{rank Hess}h(z) \le 4.$$

If all $L_j(x)$ *are real valued then we have*

$$\text{rank Hess}h(z) \le 3.$$

DEFINITION 3.4.1: We say that a double characteristic z is *non degenerate* if

$$\text{rank Hess}h(z) = 4$$

(resp. rank Hess$h(z) = 3$ if all $L_j(x)$ are real valued).

Proposition 3.4.6: *Let* $\overline{z}$ *be a non degenerate double characteristic.*

(i) The doubly characteristic set $\Sigma = \{z | h(z) = dh(z) = 0\}$ *is a smooth manifold near* $\overline{z}$ *of codimension 4 (resp. 3 if* $L_j(x)$ *are real).*
(ii) There is a smooth symmetrizer of $L_1(x,\xi)$ *near* $\overline{z}$, *that is there is a positive definite Hermitian symmetric matrix* $S(x,\xi')$, *smoothly depending on* (x,ξ') *satisfying*

$$S(x,\xi')L_1(x,\xi) = L_1(x,\xi)^* S(x,\xi')$$

([37], [5]).

We now turn to the stability of non degenerate double characteristics. Let

$$\tilde{L}_1(x,\xi) = \sum_{j=0}^{d} \tilde{L}_j(x)\xi_j$$

be another system and set $\tilde{h}(x,\xi) = \det \tilde{L}_1(x,\xi)$. We assume that $\tilde{h}(x,\cdot)$ is hyperbolic w.r.t. x_0.

Proposition 3.4.7: *Suppose that $\tilde{L}_j(x)$ are sufficiently close to $L_j(x)$ in C^3 near $\overline{x}$. Then $\tilde{h}$ has a non degenerate double characteristic near $\overline{z} = (\overline{x}, \overline{\xi})$ ([13]).*

This shows that non degenerate double characteristics are very stable and we can not remove them by small perturbations.

We now introduce the notion of localization of L_1 at a multiple characteristic z following [49].

DEFINITION 3.4.2: Let z be a characteristic of order r with

$$\dim \text{Ker} L_1(z) = r.$$

Let Ker $L_1(z)$ = span $\{u_1, ..., u_r\}$ and Ker ${}^tL_1(z)$ = span $\{v_1, ..., v_r\}$. We set $U = (u_1, ..., u_r)$ and $V = (v_1, ..., v_r)$ and define *the localization* of L_1 at z as

$$L_{loc}(U,V)(X) = dL(U,V)(z; X)$$

where $L(U,V) = {}^t V L_1(x,\xi) U$.

Lemma 3.4.8: *Let $\tilde{U}$, $\tilde{V}$ be another pair of basis for* Ker $L_1(z)$ *and* Ker ${}^tL_1(z)$. *Then with some non singular M_i we have*

$$L_{loc}(\tilde{U}, \tilde{V})(X) = M_1 L_{loc}(U,V)(X) M_2.$$

DEFINITION 3.4.3: Let z be a characteristic of order r with dim Ker$L_1(z) = r$. We say that z is *non degenerate* if

$$d(L_{loc}(U,V)) \geq r(r+1)/2.$$

QUESTION : Let $L_1(\overline{z}) = O$ and suppose that $\overline{z}$ is a non degenerate characteristic of order N. Is $L_1(x,\xi)$ diagonalizable at every point (x,ξ) near $\overline{z}$?

QUESTION : Assume that dim Ker$L_1(z) = r(z)$, the multiplicity of z, for every multiple characteristic near $\overline{z}$. Suppose that $\overline{z}$ is non degenerate. Let $\tilde{L}_1$ be sufficiently close to L_1 in C^∞. Is there a characteristic of order $r = r(\overline{z})$ of $\tilde{h}$ near $\overline{z}$?

3.5 Systems with multiple characteristics

In this subsection we state some recent necessary conditions for strong hyperbolicity of first order systems at characteristics of order exceeding two. We adopt the following definitions.

DEFINITION 3.5.1: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. Then L is said to be *hyperbolic* w.r.t. $t(x)$ both *future and past* at $\overline{x} \in \Omega$ if there are a neighborhood $\omega \subset \Omega$ of $\overline{x}$ and $\epsilon > 0$ such that both

$$L : E_\tau^\pm = \{U \in C^\infty(\omega, \mathbb{C}^N) | U = 0 \text{ on } \pm (t(x) - t(\overline{x})) < \tau\} \to E_\tau^\pm$$

are isomorphisms if $|\tau| < \epsilon$.

DEFINITION 3.5.2: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$ be real valued. Then L_1 is said to be *strongly hyperbolic* at $\overline{x}$ w.r.t. $t(x)$ if, for any $Q \in C^\infty(\Omega, M(N, \mathbb{C}))$, $L + Q$ is hyperbolic at $\overline{x}$ both future and past w.r.t. $t(x)$.

Let us denote by M the cofactor matrix $L_1^{co}(x, \xi)$ of $L_1(x, \xi)$. As before we set $h(x, \xi) = \det L_1(x, \xi)$. Recall that

$$L_1(x, \xi) = \sum_{j=0}^{d} L_j(x) \xi_j.$$

Theorem 3.5.1: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \backslash 0$ be a characteristic of order r of $h(x, \xi)$. Then if L is strongly hyperbolic at the origin w.r.t. $t(x) = x_0$, it follows that*

$$d^j M(z) = O, j < r - 2 \text{ i.e. } \partial_\xi^\alpha \partial_x^\beta M(z) = O, |\alpha + \beta| < r - 2.$$

Moreover every element of $d^{r-2} M(z; X) = d^{r-2} M(z; X, ..., X)/(r-2)!$ is divisible by $\prod g_j(X)^{r_j - 1}$ where $\prod g_j(X)^{r_j}$ is an irreducible factorization of $p_z(X)$ ([39]).

Corollary 3.5.2: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a multiple characteristic of $h(x, \xi)$ and V_0 be the generalized eigenspace for $L_1(z)$ associated to the zero eigenvalue. Then if L is strongly hyperbolic at the origin w.r.t. x_0 we have*

$$(L_1(z)|_{V_0})^2 = O,$$

where $L_1|_{V_0}$ is the restriction of $L_1(z)$ to V_0.

This corollary clearly corresponds to Lemma 2.1.1.

Theorem 3.5.3: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a characteristic of order r of $h(x,\xi)$. Suppose that*

$$\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z).$$

Then if L is strongly hyperbolic at the origin w.r.t. $t(x) = x_0$ we have

$$d^j M(z) = O, j < r - 1,$$

([39]).

Corollary 3.5.4: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a characteristic of order r of $h(x,\xi)$ with $\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z)$. If L is strongly hyperbolic at the origin w.r.t. x_0 then we have*

$$\dim \text{ Ker } L_1(z) = r.$$

For another approach to systems with multiple characteristics, we refer to [28], [53].

Question : In Theorems 3.5.1 and 3.5.3 can we drop the assumption of analyticity?

Finally we state a basic question.

Question: Let z be a characteristic of order r. Assume that L_1 is strongly hyperbolic and $\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) \neq \{0\}$. Then $\dim \text{ Ker} L_1(z) = r$ is necessary?

If this is affirmative, combining Theorem 3.5.3, we could conclude that; if L is strongly hyperbolic and z is a characteristic of order r then we have either

$$\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) = \{0\}$$

or

$$\dim \text{ Ker} L_1(z) = r.$$

Clearly the first case corresponds to a generalization of effective hyperbolicity and the second case means the symmetrizability of L_1 at z.

Question : Let z be a characteristic of order r with $\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z)$. Assume that L_1 is strongly hyperbolic . Then the localization $L_{loc}(U, V)(X)$ is diagonalizable for every X?

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