Examples of uniformly symmetrizable systems for which the Cauchy problem is ill-posed in the Gevrey class > 2

Tatsuo Nishitani

Osaka University

Yamaguchi University, 2024/01/06

1. Uniformly symmetrizable ($=$ diagonalizable) systems
2. An example
3. Ill-posedness of the Cauchy problem for the example
4. Well-posedness of the Cauchy problem for the example
5. Remarks

Uniformly diagonalizable systems

Consider first-order systems

$$L = D_t - \sum_{j=1}^{n} A_j(t,x) D_{x_j} = D_t - A(t,x,D_x), \ \ (t,x) \in \mathbb{R}^{1+n}$$

L: uniformly diagonalizable$\iff \exists C > 0, \forall (t,x,\xi) \in \mathbb{R}^{1+n} \times S^{n-1}$
$\exists T(t,x,\xi)$ s.t. $\|T(t,x,\xi)\|, \|T^{-1}(t,x,\xi)\| \le C$,
$T(t,x,\xi)A(t,x,\xi)T^{-1}(t,x,\xi) = \text{diagonal (or symmetric)}$

$\iff$ L is uniformly symmetrizable,
$\exists C > 0, \forall (t,x,\xi) \in \mathbb{R}^{1+n} \times S^{n-1}$, $\exists$ symmetric positive definite matrices $S(t,x,\xi)$, such that $\|S(t,x,\xi)\|, \|S^{-1}(t,x,\xi)\| \le C$ and SA is symmetric.

Some related results

Theorem (V.Ivrii and V.Petkov, 1974)

If the Cauchy problem is L^2 well-posed then the system is uniformly symmetrizable.

Theorem (K.Kajitani, 1984)

Assume that $L(t, x, \tau, \xi)$ is uniformly symmetrizable then the Cauchy problem for $L + B$ is well-posed in the Gevrey class of order $1 < s < 2$ for arbitrary lower order term B.

Theorem (G.Métivier, 2014)

If $L(t, x, \tau, \xi)$ is uniformly symmetrizable with a symmetrizer $S(t, x, \xi)$ which is Lipschitz continuous in $(t, x, \xi) \in \mathbb{R}^{1+n} \times S^{n-1}$ then the Cauchy problem for L is L^2 well-posed.

An example

Consider the following 3×3 system proposed by G.Métivier (2014):

$$L_a = \frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} + x \begin{pmatrix} 0 & a & 1 \\ -a & 0 & 0 \\ 1 + a^2 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y}$$
$$= \frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x} + x A_2(a) \frac{\partial}{\partial y} = \frac{\partial}{\partial t} + G_a, \quad a \in \mathbb{C}.$$

L_0 is symmetric hyperbolic system when $a = 0$.

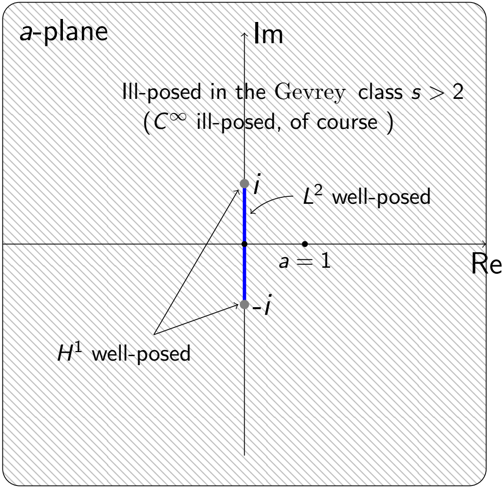
Proposition(G.Métivier) L_a is uniformly symmetrizable for $\forall a \in \mathbb{R}$. If $\mathbb{R} \ni a \neq 0$ there exists no symmetrizer which is continuous at $x = \xi = 0$, $\eta = 1$. When $a = 1$ the Cauchy problem for L_1 is not C^∞ well-posed.

L_a is uniformly diagonalizable for any $a \in \mathbb{C}$

$$P_a(\zeta_0, \zeta) = \zeta_0 I + A_1 \zeta_1 + A_2(a)\zeta_2, \quad \zeta = (\zeta_1, \zeta_2)$$

$\det P_a(\zeta_0, \zeta) = \zeta_0(\zeta_0^2 - \zeta_1^2 - \zeta_2^2)$ is independent of $a \in \mathbb{C}$. The roots of $\det P_a(\zeta_0, \zeta) = 0$ are real distinct $\zeta_0 = 0$, $\zeta_0 = \pm|\zeta|$ then $\exists N(\zeta)$, homogeneous of deg. 0 in ζ, $\|N(\zeta)\|, \|N^{-1}(\zeta)\| \le C$ $(\zeta \neq 0)$ such that $N^{-1} P_a N$ is diagonal. When $\xi^2 + x^2\eta^2 \neq 0$, $N(\xi, x\eta)$ diagonalizes L_a. If $\xi^2 + x^2\eta^2 = 0$ $((\xi, \eta) \neq 0)$ hence $\xi = 0$, $x = 0$ then $L_a = \partial/\partial t$ itself is diagonal.

Ill/Well-posedness of the Cauchy problem for L_a



Special solution to L_a^*

The adjoint of L_a is

$$L_a^* = -\frac{\partial}{\partial t} - A_1^* \frac{\partial}{\partial x} - x\, A_2^* \frac{\partial}{\partial y} = -\frac{\partial}{\partial t} + G_a^*.$$

Look for solutions to $G_a^* V(x,y) = i\beta V(x,y)$ in the form $V = e^{\pm iy} E^{\pm}(x)$ so that the problem is reduced to

$$(A_1^* \partial_x \pm ixA_2^*) E^{\pm}(x) = -i\beta E^{\pm}(x), \quad E^{\pm}(x) = \begin{pmatrix} u^{\pm}(x) \\ v^{\pm}(x) \\ w^{\pm}(x) \end{pmatrix}.$$

If $u^{\pm}(x)$ satisfies $(\partial_x^2 - x^2 + \beta^2 \pm i\bar{a})u^{\pm}(x) = 0$ then with

$$v^{\pm}(x) = \frac{i}{\beta}(\partial_x \pm i\bar{a}x)u^{\pm}(x), \quad w^{\pm}(x) = \mp\frac{x}{\beta} u^{\pm}(x)$$

we have $G_a^* \big(e^{\pm iy} E^{\pm}(x)\big) = i\beta \big(e^{\pm iy} E^{\pm}(x)\big)$.

By homogeneity $G_a^*\big(e^{\pm i\eta^2 y}E^{\pm}(\eta x)\big) = i\eta\beta\big(e^{\pm i\eta^2 y}E^{\pm}(\eta x)\big)$ one has

$$L_a^*\big(e^{i\beta\eta t \pm i\eta^2 y}E^{\pm}(\eta x)\big) = 0.$$

$\beta^2 \pm i\bar{a} = 1 \Longrightarrow (\partial_x^2 - x^2 + 1)e^{-x^2/2} = 0$

Lemma

Assume that $a \notin \{ix \in i\mathbb{R}; -1 \le x \le 1\}$. Then either $\beta^2 + i\bar{a} = 1$ or $\beta^2 - i\bar{a} = 1$ has a root $\beta \in \mathbb{C}$ with $\operatorname{Im}\beta \neq 0$.

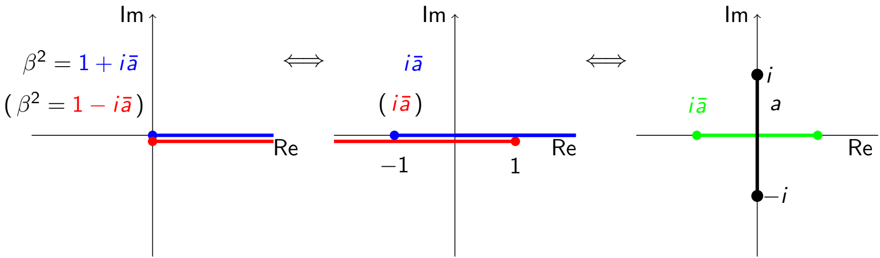
With $u^{\pm}(x) = e^{-x^2/2}$ (we may assume $\operatorname{Im}\beta > 0$)

$$\widetilde{W}_\eta^{\pm}(t,x,y) = \exp\big(i\beta\eta t \pm i\eta^2 y - \frac{1}{2}\eta^2 x^2\big)\big(W_0 + \eta x W_1^{\pm}\big)$$

solves $L_a^*\widetilde{W}_\eta^{\pm} = 0$ where

$$W_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad W_1^{\pm} = \begin{pmatrix} 0 \\ -i(1 \mp i\bar{a})/\beta \\ \mp 1/\beta \end{pmatrix} \text{(constant vectors).}$$

Proof of Lemma



Ill-posedness

Consider the following Cauchy problem

$$\begin{cases} L_a U = 0, \\ U(0, x, y) = \phi(x)\psi(y) W_0 \end{cases} \tag{1}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R})$ are real valued.

Theorem

Assume that $a \notin \{ix \in i\mathbb{R}; -1 \leq x \leq 1\}$. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi \notin \gamma_0^{(2)}(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(0) \neq 0$. Let Ω be any neighborhood of the origin of $\mathbb{R}^3$ such that $\operatorname{supp} \phi(x)\psi(y) \subset \Omega \cap \{t = 0\}$. Then (1) has no $C^1(\Omega)$ solution.

Corollary

Assume that $a \notin \{ix \in i\mathbb{R}; -1 \leq x \leq 1\}$. Then the Cauchy problem for L_a is ill-posed in the Gevrey class of order $s > 2$.

Proof of ill-posedness

Suppose that there were a neighborhood Ω of the origin such that $\mathrm{supp}\,\phi(x)\psi(y) \subset \Omega \cap \{t = 0\}$ and (1) has a solution $U \in C^1(\Omega)$. For $\delta > 0$ we denote

$$D_\delta = \{(t, x, y) \in \mathbb{R}^3 \mid x^2 + y^2 + |t| < \delta\}.$$

Recall

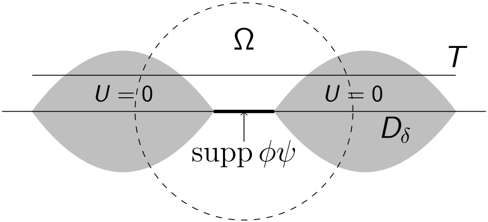
Theorem (Holmgren)

There exists $\delta_0 > 0$ such that if $U(t, x, y) \in C^1(D_\delta)$ with $0 < \delta \leq \delta_0$ verifies

$$\begin{cases} L_a U = 0 \quad in \quad D_\delta, \\ U(0, x, y) = 0 \quad on \quad (x, y) \in D_\delta \cap \{t = 0\} \end{cases}$$

then $U(t, x, y)$ vanishes identically in D_δ.

Holmgren's theorem



From Holmgren's theorem one can choose a $T > 0$ such that $\mathrm{supp}\, U \cap \{0 \le t \le T\} \subset \Omega$. Denoting

$$W_\eta^\pm(t,x,y) = e^{-i\beta\eta T}\widetilde{W}_\eta^\pm(t,x,y)$$
$$= e^{\pm i\eta^2 y - i\beta\eta(T-t)} e^{-\eta^2 x^2/2}\big(W_0 + \eta x W_1^\pm\big)$$

we have obviously $L_a^* W_\eta^\pm = 0$. From

$$0 = \int_0^T (L_a^* W_\eta^\pm, U)dt = \int_0^T (W_\eta^\pm, L_a U)dt$$
$$+(W_\eta^\pm(T), U(T)) - (W_\eta^\pm(0), U(0))$$

it follows that

$$(W_\eta^\pm(T), U(T)) = (W_\eta^\pm(0), U(0)). \tag{2}$$

Proof of ill-posedness:continued

The left-hand side of (2) is

$$(W_\eta^\pm(T), U(T)) = \left(e^{\pm i\eta^2 y - \eta^2 x^2/2}(W_0 + \eta x W_1^\pm), U(T)\right)$$

which is $O(1)$ as $\eta \to \infty$ while the right-hand side is

$$\eta^{-1} e^{-i\beta\eta T} \hat{\psi}(\eta^2) \int e^{-x^2/2} \phi(\eta^{-1} x) dx \qquad (3)$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. From (3) we conclude that there is $C > 0$ such that for large positive η one has

$$|\hat{\psi}(\eta^2)| \leq C\eta e^{(-\operatorname{Im}\beta)\eta T}.$$

Since ψ is even this shows that $|\hat{\psi}(\eta)| \leq C' e^{-c|\eta|^{1/2}}$ with some $c > 0$ and hence $\psi \in \gamma_0^{(2)}(\mathbb{R})$ which is a contradiction.

Well-posedness

Consider

$$\tilde{L}_a U = \frac{\partial}{\partial t} U + A_1 \frac{\partial}{\partial x} U + \phi(x) A_2 \frac{\partial}{\partial y} U = F$$

where $\phi(x)$ is a smooth real valued scalar function with bounded derivatives of all order and $U = {}^t(u, v, w)$ and $F = {}^t(f, g, h)$. We are interested in the case $\phi(x) = x$ in a compact neighborhood of the origin but it is not necessarily assumed.

Theorem

If $a \in \{ix \in i\mathbb{R}; -1 < x < 1\}$ then the Cauchy problem for $\tilde{L}_a$ is L^2 well-posed, and in particular, $\tilde{L}_a$ is strongly hyperbolic.

Theorem

If $a \in \{ix \in i\mathbb{R}; -1 \leq x \leq 1\}$ then the Cauchy problem for $\tilde{L}_a$ is H^1 well-posed.

Proof of well-posedness:symmetrization

Let $a \in \{ix \in i\mathbb{R}; -1 < x < 1\}$ so that $a = i\mu$, $\mu \in \mathbb{R}$, $|\mu| < 1$:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & i\mu & 1 \\ -i\mu & 0 & 0 \\ 1-\mu^2 & 0 & 0 \end{pmatrix}.$$

Denote

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(1-\mu^2) \end{pmatrix}$$

which is symmetric positive definite. It is easy to check that SA_1 and SA_2 are both hermitian, that is A_1 and A_2 are *simultaneously* symmetrizable by S.

Energy estimates

Let $U \in C^1(\mathbb{R}; C_0^\infty(\mathbb{R}^d))$. Then

$$(SA_1\partial U/\partial x, U) + (U, SA_1\partial U/\partial x) = \big((SA_1 - A_1^* S)\partial U/\partial x, U\big) = 0,$$

$$\big(SA_2\phi(x)\frac{\partial U}{\partial y}, U\big) + \big(U, SA_2\phi(x)\frac{\partial U}{\partial y}\big)$$
$$= \big(\phi(x)(SA_2 - A_2^* S)\frac{\partial U}{\partial y}, U\big) = 0.$$

Then one has

$$\frac{d}{dt}(SU, U) = \frac{d}{dt}\|S^{1/2}U\|^2 = 2\mathrm{Re}\,(SU, F) \le 2\|S^{1/2}U\|\|S^{1/2}F\|$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ stands for the $L^2(\mathbb{R}_{x,y})$ inner product and the norm respectively.

Hence we have $\frac{d}{dt}\|S^{1/2}U\| \le \|S^{1/2}F\|$. Integrating this inequality

$$\|S^{1/2}U(t)\| \le \|S^{1/2}U(0)\| + \int_0^t \|S^{1/2}\tilde{L}_a U(s)\| ds.$$

Lemma

For $U \in C^1(\mathbb{R}; C_0^\infty(\mathbb{R}^d))$ one has

$$\|U(t)\| \le \frac{1}{\sqrt{1-\mu^2}}\Big(\|U(0)\| + \int_0^t \|\tilde{L}_a U(s)\| ds\Big). \tag{4}$$

The same estimate holds for $\tilde{L}_a^*$.

Proof: Since $\|V\| \le \|S^{1/2}V\| \le (1-\mu^2)^{-1/2}\|V\|$ the assertion for $\tilde{L}_a$ is immediate. Since S^{-1} symmetrizes A_1^* and A_2^* simultaneously and $\|S^{-1/2}V\| \le \|V\| \le (1-\mu^2)^{-1/2}\|S^{-1/2}V\|$ the same assertion for $\tilde{L}_a^*$ holds.

Proof of well-posedness:continued

Denote by H^s and $\|\cdot\|_s$, $s \in \mathbb{R}$ the usual L^2 based Sobolev space of order s and the norm respectively. Assume that $B = B(t, x, y)$ is smooth with bounded derivatives of all order and fix $T > 0$. For any $s \in \mathbb{R}$ there is $C_s > 0$ such that we have

$$\|U(t)\|_s \leq C\Big(\|U(0)\|_s + \int_0^t \|(\tilde{L}_a + B)^* U(\tau)\|_s d\tau\Big) \tag{5}$$

for any $U \in C^1([0, T]; H^s) \cap C^0([0, T]; H^{s+1})$. Therefore the usual duality arguments proves that for any $U_0 \in H^s$ there exists a solution $U(t) \in C^1([0, T]; H^{s-1}) \cap C^0([0, T]; H^s)$ to the Cauchy problem

$$\begin{cases} (\tilde{L}_a + B)U = 0, \\ U(0) = U_0. \end{cases}$$

The uniqueness of solution follows from (4).

Case $a = \pm i$

$$\tilde{L}_a = \frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \phi(x) \begin{pmatrix} 0 & \pm i & 1 \\ \mp i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y}$$

The third equation of $\tilde{L}_a U = F$ is

$$\frac{\partial w}{\partial t} = h \quad \text{hence} \quad \frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial y}\Big) = \frac{\partial h}{\partial y}$$

from which it follows that

$$\big\|\partial w(t)/\partial y\big\| \leq \big\|\partial w(0)/\partial y\big\| + \int_0^t \big\|\partial h(s)/\partial y\big\| ds. \tag{6}$$

The first two equations of $\tilde{L}_a U = F$ yield a symmetric system for (u, v) assuming $\partial w/\partial y$ to be known.

Case $a = \pm i$: energy estimates

One has

$$\frac{d}{dt}\|U\|^2 = -2\,\mathrm{Re}\,\big(\phi\frac{\partial w}{\partial y}, u\big) + 2\,\mathrm{Re}\,(F, U). \tag{7}$$

The right-hand side is bounded by $C\big(\|\partial w(t)/\partial y\| + \|F(t)\|\big)\|U(t)\|$. From (7) one has

$$\|U(t)\| \le \|U(0)\| + C\int_0^t \Big(\big\|\partial w(s)/\partial y\big\| + \|F(s)\|\Big)\,ds. \tag{8}$$

Inserting (6) into (8) one obtains

$$\begin{aligned}\|U(t)\| \le C'\big(\|U(0)\| + \big\|\partial U(0)/\partial y\big\|\big)\\ +C'\int_0^t \big(\|\tilde{L}_a U(s)\| + \big\|\frac{\partial}{\partial y}\tilde{L}_a U(s)\big\|\big)\,ds\end{aligned} \tag{9}$$

for $0 \le t \le T$.

$\tilde{L}_a^* U = F$

$$\tilde{L}_a^* = -\frac{\partial}{\partial t} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} - \phi(x) \begin{pmatrix} 0 & \mp i & 0 \\ \pm i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y}$$

Since the first two equations of $\tilde{L}_a^* U = F$ split from the third equation and yield a symmetric system then we have with $V = {}^t(u, v)$ and $G = {}^t(f, g)$

$$\begin{aligned} \|V(t)\| + \|\partial V(t)/\partial y\| \le C\big(\|V(0)\| + \|\partial V(0)/\partial y\|\big) \\ + C\int_0^t \big(\|G(s)\| + \|\partial G(s)/\partial y\|\big)\, ds. \end{aligned} \tag{10}$$

From the third equation of $\tilde{L}_a^* U = F$ we have

$$\|w(t)\| \leq \|w(0)\| + C\int_0^t \left(\|h(s)\| + \left\|\partial u(s)/\partial y\right\|\right) ds. \qquad (11)$$

Inserting (10) into (11) one has

$$\|w(t)\| \leq C'\left(\|U(0)\| + \left\|\partial U(0)/\partial y\right\|\right) + C'\int_0^t \left(\|F(s)\| + \left\|\partial F(s)/\partial y\right\|\right) ds.$$

Together with (10) energy estimate (9) holds also for $\tilde{L}_a^*$. Repeating similar arguments one concludes that the Cauchy problem for $\tilde{L}_a$ is C^∞ well-posed (actually H^1 well-posed).

Remarks

The Cauchy problem for

$$L_a^{(0)} = \frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x} + y\, A_2(a) \frac{\partial}{\partial y}, \quad a \in \mathbb{C}$$

is L^2 well-posed, hence C^∞ well-posed for any lower order term.
The Cauchy problem for

$$L_a^{(1)} = \frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x} + t\, A_2(a) \frac{\partial}{\partial y}, \quad a \in \mathbb{C}$$

is C^∞ well-posed for arbitrary lower order term (with regularity loss). These examples can be generalised
$L_a^{(0)} \Longrightarrow$ transversally strictly hyperbolic systems with involutive characteristics (Métivier, N, 2018),
$L_a^{(1)} \Longrightarrow$ transversally strictly hyperbolic systems with symplectic characteristics (N, 2020).