A question on the Cauchy problem in the Gevrey classes for weakly hyperbolic equations

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1 Introduction

Consider a differential operator $P(x, D)$ of order m defined in a neighborhood Ω of the origin of $\mathbb{R}^n$ with coordinates $x = (x_1, x_2, \dots, x_n) = (x_1, x')$;

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \;\; D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \;\; D_j = -i\partial/\partial x_j$$

where we assume that $a_\alpha(x)$ are in some Gevrey classes $\gamma^{(s)}(\Omega)$ or $\gamma^{\langle s \rangle}(\Omega)$. By $\gamma^{(s)}(\Omega)$ we mean the set of functions $f \in C^\infty(\Omega)$ such that for any compact set $K \Subset \Omega$ there are constants $C > 0, A > 0$ for which the following inequalities hold:

$$|D^\alpha f(x)| \le C A^{|\alpha|} (|\alpha|!)^s, \;\; x \in K, \;\; \alpha \in \mathbb{N}^n. \tag{1.1}$$

By $\gamma^{\langle s \rangle}(\Omega)$ we mean the set of functions $f \in C^\infty(\Omega)$ such that for any compact set $K \Subset \Omega$ and any $A > 0$ there is a constant $C > 0$ for which (1.1) hold. Evidently $\gamma^{\langle s \rangle} \subset \gamma^{(s)} \subset \gamma^{\langle s' \rangle}$ for $s < s'$. Consider the Cauchy problem

$$\begin{cases} P(x, D) u(x) = 0, \quad (x_1, x') \in \omega \cap \{x_1 > \tau\}, \\ D_1^j u(0, x') = u_j(x'), \;\; j = 0, \dots, m-1, \;\; x' \in \omega \cap \{x_1 = \tau\} \end{cases} \tag{1.2}$$

where $\omega \subset \Omega$ is some open neighborhood of the origin of $\mathbb{R}^n$. We say that the Cauchy problem (1.2) is (uniformly) well-posed in $\gamma^{(s)}$ (resp. in $\gamma^{\langle s \rangle}$) near the origin if there exist ω and $\epsilon > 0$ such that for any $u_j \in \gamma^{(s)}(\mathbb{R}^{n-1})$ (resp. $u_j \in \gamma^{\langle s \rangle}(\mathbb{R}^{n-1})$) and for any $|\tau| < \epsilon$ the Cauchy problem (1.2) has a unique solution $u \in C^m(\omega)$. We say that the Cauchy problem is locally solvable in $\gamma^{(s)}$ at the origin if for any $u_j \in \gamma^{(s)}(\mathbb{R}^{n-1})$ one can find a neighborhood $\omega_{\{u_j\}}$ of the origin such that (1.2) with $\tau = 0$ has a solution $u \in C^m(\omega_{\{u_j\}})$.

Write

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^m = p(x, \xi) + \sum_{j=0}^{m-1} P_j(x, \xi)$$

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where $p(x,\xi)$ is of homogeneous of degree m in ξ, called the principal symbol of P and $P_j(x,\xi)$ denotes the homogeneous part of degree j in ξ. We assume that the hyperplanes $\{x_1 = t\}$ are non-characteristic, that is

$$p(x,\theta) \neq 0, \quad \theta = (1,0,\dots,0) \;\; x \in \Omega, \tag{1.3}$$

which is almost necessary for C^∞ well-posedness of the Cauchy problem ([7]). Then without restrictions one may assume $a_{(m,0,\dots,0)}(x) = 1$. If the Cauchy problem (1.2) is locally solvable in $\gamma^{(s)}$, $s > 1$ at the origin then $p(0,\xi_1,\xi') = 0$ has only real roots for any $\xi' \in \mathbb{R}^{n-1}$ ([8]) so we assume also that

$$p(x,\xi - i\theta) \neq 0, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega, \tag{1.4}$$

that is the equation $p(x,\xi_1,\xi') = 0$ in ξ_1 has only real roots for any $x \in \Omega$, $\xi' \in \mathbb{R}^{n-1}$.

For a given $P(x,\xi)$, which is a polynomial in ξ of degree m, we consider several realizations (quantizations) of $P(x,\xi)$. Let $a(x,\xi) \in S^m_{1,0}$ be a claasical symbol of pseudodifferential operator then we define $\mathrm{op}^t(a)$, $0 \le t \le 1$ by

$$(\mathrm{op}^t(a)u)(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} a((1-t)x + ty, \xi) u(y) dy d\xi.$$

Note that, assuming that $a_\alpha(x)$ are constant outside some compact neighborhood of the origin for simplicity, we see

$$\sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u(x) = \mathrm{op}^0(P)u(x), \quad \sum_{|\alpha| \le m} D^\alpha\big(a_\alpha(x)u(x)\big) = \mathrm{op}^1(P)u(x).$$

When $t = 1/2$ the quantization $\mathrm{op}^{1/2}(a)$ is called Weyl quantization and also denoted by $\mathrm{op}^w(a)$. The next results are implicit in [3] and [4].

Theorem 1.1. *Assume $P(x,\xi) = p(x,\xi) + \sum_{j=0}^r P_j(x,\xi)$ and the coefficients $a_\alpha(x)$ belong to $\gamma^{(m/r)}$* (resp. $\gamma^{\langle m/r \rangle}$)*. Then for any $0 \le t \le 1$ the Cauchy problem for $\mathrm{op}^t(P)$ is well-posed in $\gamma^{(s)}$ near the origin for $1 < s < \min\{m/r, 2\}$* (resp. $\gamma^{\langle s \rangle}$ *for $1 < s \le \min\{m/r, 2\}$*).

Corollary 1.1. *Assume that the coefficients $a_\alpha(x)$ belong to $\gamma^{(2)}$* (resp. $\gamma^{\langle 2 \rangle}$)*. Then for any $0 \le t \le 1$ the Cauchy problem for $\mathrm{op}^t(p)$ is well-posed in $\gamma^{(s)}$ near the origin for $1 < s < 2$* (resp. $\gamma^{\langle s \rangle}$ *for $1 < s \le 2$*).

To confirm the result, first note that under the assumption there is some $M > 0$ such that

$$P(x,\xi + i\tau\theta) \neq 0, \quad |\tau| \ge M(1 + |\xi|)^{r/m}, \quad x,\ \xi \in \mathbb{R}^n \tag{1.5}$$

that is P is m/r -hyperbolic (see [5]). Then Theorem 1.1 was proved for $\mathrm{op}^0(P)$ in [4] and Corollary 1.1 for $\mathrm{op}^0(p)$ is implicit in [3]. Next we recall a formula for change of quantization (e.g. [6]). One can pass from any t -quantization to the t' -quantization by

$$\mathrm{op}^{t'}(a_{t'}) = \mathrm{op}^t(a_t), \quad a_{t'}(x,\xi) = e^{-i(t'-t)D_x D_\xi} a_t(x,\xi). \tag{1.6}$$

In particular, we have

$$\mathrm{op}^t(P(x,\xi)) = \mathrm{op}^0\big(e^{itD_xD_\xi}P(x,\xi)\big).$$

On the other hand, from [3] one has

$$\big|(\partial_x^\beta\partial_\xi^\alpha p)(x,\xi-i\theta)\big| \le C_{\alpha\beta}(1+|\xi|)^{|\beta|}\big|p(x,\xi-i\theta)\big|, \ \ \alpha,\beta\in\mathbb{N}^n.$$

which is sufficient to estimate new terms appear by operating $e^{itD_xD_\xi}$ to $P(x,\xi)$.

2 A question on Theorem 1.1

If P is of constant coefficients, $P(x,\xi)=P(\xi)$, the Cauchy problem for $P(D)$ is $\gamma^{(s)}$ well-posed for $1<s<m/r$ ([5]) if (1.5) holds and it is also clear that the results are optimal considering examples $P(D)=D_1^m+cD_n^r$ with a suitable $c\in\mathbb{C}$. Clearly the case $r=0$ corresponds to the hyperbolicity in the sense of Gårding and the Cauchy problem for $P(D)$ is C^∞ well-posed. In the variable coefficient case, on the other hand, we are restricted to $1<s<\min\{m/r,2\}$ in both Theorem 1.1 and Corollary 1.1. Here we give an example which shows that one can not really exceed 2. Consider

$$P_b(x,\xi)=\xi_1^3-(\xi_2^2+x_2^2\xi_n^2)\xi_1-bx_2^3\xi_n^3 \tag{2.1}$$

with $b\in\mathbb{R}$ which was studied in [2]. Note that (1.4) is equivalent to $b^2\le 4/27$. In [2] it was proved that there is $0<\bar{b}<2/3\sqrt{3}$ such that the Cauchy problem for $\mathrm{op}^0(P_{\bar{b}})$ is not locally solvable at the origin in $\gamma^{(s)}$ for $s>2$. From this we obtain immediately

Proposition 2.1. *Let $m\ge 3$ and $n\ge 3$ and consider*

$$p(x,\xi)=\big(\xi_1^3-(\xi_2^2+x_2^2\xi_n^2)\xi_1-\bar{b}x_2^3\xi_n^3\big)\xi_1^{m-3}$$

which is a homogeneous polynomial in ξ of degree m with polynomial coefficients. For any $0\le t\le 1$ the Cauchy problem for $\mathrm{op}^t(p)$ is not locally solvable at the origin in $\gamma^{(s)}$ for $s>2$, in particular, not well-posed in $\gamma^{(s)}$, $s>2$ near the origin.

Indeed from (1.6) one sees that $\mathrm{op}^{t'}(p)=\mathrm{op}^t(p)$ for any $0\le t',t\le 1$ so that

$$\mathrm{op}^t(p)=\big(D_1^3-(D_2^2+x_2^2D_n^2)D_1-\bar{b}x_2^3D_n^3\big)D_1^3=\mathrm{op}^0(P_{\bar{b}})D_1^{m-3}$$

which proves the assertion.

When $m=2$ we have a result similar to Proposition 2.1:

Proposition 2.2. *Let $n\ge 3$ and consider*

$$P_{mod}(x,\xi)=\xi_1^2-2x_2\xi_1\xi_n-\xi_2^2-x_2^3\xi_n^2$$

which is a homogeneous polynomial in ξ of degree 2 with polynomial coefficients. For any $0\le t\le 1$ the Cauchy problem for $\mathrm{op}^t(P_{mod})$ is not locally solvable at the origin in $\gamma^{(s)}$ for $s>5$, in particular, not well-posed in $\gamma^{(s)}$, $s>5$ near the origin.

This result for $\mathrm{op}^0(P_{mod})$ was proved in [1] (where there is some insufficient part of the proof, see the correction in [10]). Then to conclude Proposition 2.2 it is enough to note $\mathrm{op}^t(P_{mod}) = \mathrm{op}^0(P_{mod})$ for $0 \le t \le 1$.

We would now ask ourselves is there an example of a homogeneous polynomial p in $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $n \ge 3$ of order 2 with real analytic coefficients satisfying (1.3) and (1.4) such that the Cauchy problem for $\mathrm{op}^t(p)$, for any $0 \le t \le 1$, is not well-posed in $\gamma^{(s)}$, $s > 2$, or is there some $2 < \bar{s} \le 5$ such that for any such p the Cauchy problem for $\mathrm{op}^w(p)$ is well-posed in $\gamma^{(s)}$ for $s < \bar{s}$ near the origin.

Note that the case $n = 2$ (and $m = 2$) is quite special.

Proposition 2.3. *Consider*

$$P(x,\xi) = \xi_1^2 - 2a(x)\xi_1\xi_2 + b(x)\xi_2^2 + c(x), \quad x = (x_1, x_2)$$

which is a polynomial in $\xi = (\xi_1, \xi_2)$ with real analytic $a(x), b(x), c(x)$ such that $\Delta(x) = a^2(x) - b(x) \ge 0$ near the origin (a, b are real valued). Then the Cauchy problem for $\mathrm{op}^w(P)$ is C^∞ well-posed near the origin.

In fact if we make a real analytic change of coordinates $y = \kappa(x) = (x_1, \phi(x))$ such that $\phi_{x_1}(x) - a(x)\phi_{x_2}(x) = 0$, $\phi(0, x_2) = x_2$ where $\phi_{x_j} = \partial\phi(x)/\partial x_j$ then we see that

$$\begin{gathered}\mathrm{op}^w(P(x,\xi))(u \circ \kappa) \\ = \Big(\mathrm{op}^0\big(\eta_1^2 - \alpha\tilde{\Delta}\eta_2^2 + \beta_1\tilde{\Delta}_{x_2}\eta_2 + \beta_2\tilde{\Delta}\eta_2 + \beta_3\eta_1 + \beta_4\big)u\Big) \circ \kappa\end{gathered}$$

where $\tilde{\Delta} = \Delta \circ \kappa^{-1}$, $\tilde{\Delta}_{x_2} = \Delta_{x_2} \circ \kappa^{-1}$ and $\alpha = \alpha(y) \ge c > 0$, $\beta_i = \beta_i(y)$ are real analytic near $y = 0$. Noting that $\big|\tilde{\Delta}_{x_2}\big| \le C\big|\tilde{\Delta}_{y_2}\big| \le C'\big|\sqrt{\tilde{\Delta}}\big|$ it suffices to apply [9, Theorem 1.1] to the right-hand side.

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