Hyperbolic Equations with Double Characteristics

Contents

Preface

This text is a note from lectures for a Ph.D. course I gave at the University of Pisa in the spring of 2000. It is intended to provide basic facts on hyperbolic polynomials and to be a quick introduction to hyperbolic operators with double characteristics.

The text is divided in three parts. The first part collects several facts including the proofs, with a few exceptions, which are commonly used in studies on hyperbolic problems. In spite of their elementary aspects and common use, since they are scattered to different issues, it is hard to find in a book with proofs. I also added some rather recent results which concern the approximation problem of hyperbolic systems by strictly hyperbolic ones.

The second part deals with several general necessary conditions for the well-posedness of the Cauchy problem, that is necessary conditions that are available without restrictions on multiplicities or the geometric aspects of the characteristics. Two of which I gave proofs. Some techniques of the proof are a little bit refined compared to the previous ones.

In the third part, this part should be considered as an essential part of the text, after stating basic notions and results on hyperbolic operators with double characteristics I gave proof of Ivrii-Petkov-Hörmander theorem. At a double characteristic point, the Taylor expansion starts with a quadratic form. The representative of this quadratic form with respect to the canonical 2-form on the cotangent bundle over an open set we are working on is called the Hamilton map. The theorem states that in order for the Cauchy problem to be well-posed the subprincipal symbol has to be real and bounded, in modulus, by the sum of the modulus of pure imaginary eigenvalues of the Hamilton map.

I mentioned nothing about the effects of the possible presence of zero eigenvalues of the Hamilton map. In fact, one of the most missing parts in the theory of hyperbolic operators with double characteristics is studies on operators with characteristics where the Hamilton map admits non-semi-simple zero eigenvalues.

In the appendix, I added two lemmas which will be useful when we prove the Ivrii-Petkov-Hörmander theorem in full generality.

At the end of each section, a few references are given. I hope that the reader will find more suitable papers among the references of the references given here. At the end of the text, I also listed a few references for the convenience of catching a general picture of recent developments on the theory of linear hyperbolic equations.

I am very grateful to Ferruccio Colombini and Sergio Spagnolo for their invitation to give this series of lectures and very kind hospitality to that I owe my so pleasant stay in Pisa.

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Chapter 1

Hyperbolic polynomials

1.1 Hyperbolic polynomials

Definition 1.1.1. Let $P(\zeta)$ be a (monic) polynomial in ζ of degree m. Then we say that $P(\zeta)$ is a hyperbolic polynomial iff the all zeros of $P(\zeta)$ are real.

We note that the coefficients of $P(\zeta)$ are real.

Lemma 1.1.1. *Let $P(\zeta)$ be a hyperbolic polynomial. Then*

$$\left(\frac{\partial}{\partial\zeta}\right)^j P(\zeta), \quad 0 \le j \le m-1$$

are also hyperbolic polynomials.

To prove this lemma, we apply the following Lucas theorem.

Theorem 1.1.1. (Lucas) *Assume that the zeros of $P(\zeta)$ are contained in a half plane (with the boundary). Then the zeros of $P'(\zeta)$ are contained in the same half plane.*

Proof. Let us write $P(\zeta) = k(\zeta-\alpha_1)(\zeta-\alpha_2)\cdots(\zeta-\alpha_m)$. Then it is clear that

$$\frac{P'(\zeta)}{P(\zeta)} = \frac{1}{\zeta-\alpha_1} + \frac{1}{\zeta-\alpha_2} + \cdots + \frac{1}{\zeta-\alpha_m}.$$

Assume that the half plane is given by $H : \mathrm{Im}(\zeta-a)/b \le 0$. Let $\zeta \notin H$. Then we have

$$\mathrm{Im}\,\frac{\zeta-\alpha_k}{b} = \mathrm{Im}\,\frac{\zeta-a}{b} + \mathrm{Im}\,\frac{a-\alpha_k}{b} > 0.$$

This implies that $\mathrm{Im}\, b/(\zeta-\alpha_k) < 0$ and hence

$$\mathrm{Im}\, b\frac{P'(\zeta)}{P(\zeta)} = \sum_{i=1}^{m} \mathrm{Im}\,\frac{b}{\zeta-\alpha_i} < 0$$

where we have $P'(\zeta) \neq 0$. Thus we have proved the assertion. □

Proof of Lemma 1.1.1: Let us denote by $H^{\pm}$ the upper (lower) half-plane with the real line, then by the assumption the zeros of $P(\zeta)$ are contained in both H^+ and H^-. Hence so are the zeros of $P'(\zeta)$. This is the desired assertion. □

Assume again that the zeros of $P(\zeta)$ are contained in $H = \{\mathrm{Im}(\zeta - a)/b \le 0\}$. Let $s \in \mathbb{R}$ and study the zeros of $P(\zeta) + sP'(\zeta)$. Noting that

$$\mathrm{Im}\, b \frac{P(\zeta) + sP'(\zeta)}{P(\zeta)} = \mathrm{Im}\, b + s \sum_{i=1}^{m} \mathrm{Im}\, \frac{b}{\zeta - \alpha_i}$$

we conclude that if $\mathrm{Im}\, b = 0$ and $\zeta \notin H$ then the left-hand side is equal to

$$s \sum_{i=1}^{m} \mathrm{Im}\, \frac{b}{\zeta - \alpha_i}$$

which is > 0 and < 0 according to $s > 0$ and $s < 0$. This proves the following

Lemma 1.1.2. *Assume that the zeros of $P(\zeta)$ are contained in $H = \{\mathrm{Im}\, \zeta \le a\}$ (resp.$\{\mathrm{Im}\, \zeta \ge \mathrm{a}\}$). Then for any $s \in \mathbb{R}$ the zeros of $P(\zeta) + sP'(\zeta)$ are contained in the same H.*

Corollary 1.1.1. *Let $P(\zeta)$ be a hyperbolic polynomial. Then $P(\zeta) + sP'(\zeta)$ $(s \in \mathbb{R})$ is also a hyperbolic polynomial.*

We make more precise studies on the zeros of $P(\zeta) + sP'(\zeta)$. Let us write

$$P(\zeta) = \prod_{j=1}^{s} (\zeta - \lambda_j)^{r_j}, \quad \lambda_j \in \mathbb{R}, \quad \sum_{j=1}^{s} r_j = m$$

where λ_j are different from each other.

Lemma 1.1.3. *We have*

(i) *Let $s > 0$. Then $P(\zeta) + sP'(\zeta)$ has λ_j as a zero of order $r_j - 1$ and in each $(\lambda_j, \lambda_{j+1})$, $j = 0, 1, \ldots, k-1$ $(\lambda_0 = -\infty)$, $P(\zeta) + sP'(\zeta)$ has exactly one simple zero.*

(ii) *Let $s < 0$. Then $P(\zeta) + sP'(\zeta)$ has λ_j as a zero of order $r_j - 1$ and in each $(\lambda_j, \lambda_{j+1})$, $j = 1, \ldots, k$ $(\lambda_{k+1} = +\infty)$, $P(\zeta) + sP'(\zeta)$ has exactly one simple zero.*

Proof. We prove the case (i). The proof for the case (ii) is similar. Let λ be one of λ_j and let us write $P(\zeta) = (\zeta - \lambda)^l Q(\zeta)$ where $l \ge 1$ and $Q(\lambda) \ne 0$. Note that

$$\frac{P(\zeta) + sP'(\zeta)}{P(\zeta)} = 1 + s\frac{l}{\zeta - \lambda} + s\frac{Q'(\zeta)}{Q(\zeta)}.$$

When $\zeta \uparrow \lambda$ the right-hand side goes to $-\infty$ and if $\zeta \downarrow \lambda$ the right-hand side grows to $+\infty$. This shows that when $\zeta \uparrow \lambda$, $P(\zeta)$ and $P(\zeta) + sP'(\zeta)$ have the same sign and have the opposite sign when $\zeta \downarrow \lambda$. Let λ_k and λ_{k+1} be successive zeros of $P(\zeta)$. Since $P(\zeta)$ is different from zero in $(\lambda_k, \lambda_{k+1})$, assuming $P(\zeta) > 0$

in this open interval without restrictions, we see that $P(\zeta) + sP'(\zeta) > 0$ (< 0) when $\zeta \downarrow \lambda_k$ $(\zeta \uparrow \lambda_{k+1})$. This shows that $P(\zeta) + sP'(\zeta)$ has at least one zero in $(\lambda_k, \lambda_{k+1})$. When $\zeta \uparrow \lambda_1$, $P(\zeta)$ and $P(\zeta)+sP'(\zeta)$ have the opposite sign and if ζ goes to $-\infty$ we see that they have the same sign. This shows that $P(\zeta)+sP'(\zeta)$ has at least one zero in $(-\infty, \lambda_1)$.

On the other hand it is clear that λ_j is a zero of $P(\zeta)+sP'(\zeta)$ of order $r_j - 1$. Hence taking the number of zeros into account we get the desired assertion. □

Let us put $(T_s P)(\zeta) = P(\zeta) + sP'(\zeta)$ and $Q(\zeta; s) = (T_s^m P)(\zeta)$. Then

Lemma 1.1.4. *For $s \in \mathbb{R}$, $s \neq 0$, $Q(\zeta; s)$ has m different real zeros.*

Proof. Let $s \neq 0$. Then from Lemma 1.1.3, it follows that every zero of $(T_s P)(\zeta)$ is at most of order $m - 1$. Repeating this arguments we get the result. □

References

[1] L.Ahlfors: Complex Analysis. MacGraw-Hill, 1979.

1.2 A theorem of Nuij

Let $P(\xi_0, \xi_1, \ldots, \xi_n)$ be a homogeneous polynomial of degree m which is hyperbolic with respect to ξ_0, that is, $P(\xi_0, \xi_1, \ldots, \xi_n) = 0$ has only real roots for any $\xi' = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. We define $T_{k,s}$ and F_s by

$$(T_{k,s}P)(\xi) = P(\xi) + s\xi_k \frac{\partial}{\partial \xi_0} P(\xi), \quad s \in \mathbb{R}, \quad k = 1, \ldots, n$$

and $(F_s P)(\xi) = (T_{1,s}^m \cdots T_{n,s}^m P)(\xi)$. Then we have

Lemma 1.2.1. *Denote $Q(\xi; s) = (F_s P)(\xi)$. Then $Q(\xi_0, \xi_1, \ldots, \xi_n; s) = 0$ has m different real roots with respect to ξ_0 for any $s \in \mathbb{R}$, $s \neq 0$, $\xi' \neq 0$.*

Proof. Fix $\xi' \neq 0$. Without restrictions we may suppose that $\xi_k \neq 0$. By Lemma 1.1.4, $(T_{k,s}^m [T_{k+1,s}^m \cdots T_{n,s}^m P])(\xi) = 0$ has m different real roots because $[T_{k+1,s}^m \cdots T_{n,s}^m P](\xi)$ is a hyperbolic polynomial in ξ_0. Hence

$$[T_{1,s}^m \cdots T_{k-1,s}^m] T_{k,s}^m [T_{k+1,s}^m \cdots T_{n,s}^m P])(\xi) = 0$$

has m different real roots. This finishes the proof. □

Remark 1.2.1. $Q(\xi; s)$ is a polynomial in s of degree at most mn. If P is monic in ξ_0 then so is $Q(\xi; s)$.

Definition 1.2.1. Let $P(\xi_0, \xi_1, \ldots, \xi_n)$ be a homogeneous polynomial which is hyperbolic with respect to ξ_0. We say that $P(\xi)$ is strictly hyperbolic polynomial with respect to ξ_0 if $P(\xi) = 0$ has m different real roots for any $\xi' = (\xi_1, \ldots, \xi_n) \neq 0$.

Theorem 1.2.1. (Nuij) *Every hyperbolic polynomial $P(\xi)$ is a limit of strictly hyperbolic polynomials.*

Exercise: Introduce a topology in the space of polynomials and make precise the statement of Theorem 1.2.1.

In the case of systems, the situation is completely different. Let us take $m \times m$ matrix $P(x)$ depending smoothly in $x = (x_0, x_1, \ldots, x_n)$, defined near the origin of $\mathbb{R}^{n+1}$.

Definition 1.2.2. We say that $P(x)$ is a hyperbolic matrix with respect to $\theta \in \mathbb{R}^{n+1}$ if $P(\theta)$ is positive definite and

$$\det P(\lambda\theta + x) = 0$$

has only real roots for any x near the origin.

We give a typical example.

Example 1. Let us take $P(x) = x_0 I + A(x')$ where I denotes the identity matrix of order m and $x' = (x_1, \ldots, x_n)$. Then if $A(x')$ is symmetric then $P(x)$ is a hyperbolic matrix with respect to $\theta = (1, 0, \ldots, 0)$.

Definition 1.2.3. Let $P(x)$ be a $m \times m$ matrix valued function defined near the origin. We say that $x = 0$ is a non degenerate characteristic for P if the following conditions are verified:

(i) $P(0) = O$,

(ii)

the dimension of the space spanned by $\{\frac{\partial P}{\partial x_j}(0) \mid j = 0, 1, \ldots, n\} = \frac{m(m+1)}{2}$,

(iii)

$$P_0(x) = \sum_{j=0}^{m} \frac{\partial P}{\partial x_j}(0) x_j$$

is diagonalizable for every x.

Example 2. Let us take

$$P(x) = x_0 I + \sum_{j=1}^{n} A_j x_j$$

where the matrices I, $A_1, \ldots, A_n$ span the space of all real symmetric matrices.

Then we have

Theorem 1.2.2. *Assume that $P(x)$ is a $m \times m$ real valued hyperbolic matrix, real analytic near the origin. Assume that $x = 0$ is a non degenerate characteristic of order m for P. If $\tilde{P}(x)$ is another real valued hyperbolic matrix, real analytic near the origin which is sufficiently close to $P(x)$ in C^2, then $\tilde{P}(x)$ has a non degenerate characteristic of order m close to $x = 0$.*

Corollary 1.2.1. *Assume that $P(x)$ verifies the same assumptions as in Theorem 1.2.2. Then near $P(x)$ there is no hyperbolic matrix which has m different real eigenvalues.*

Here we apply Theorem 1.2.2 to

$$L(\xi) = \xi_0 I + \sum_{j=1}^{n} A_j \xi_j = \xi_0 + A(\xi')$$

where A_j are real $m \times m$ constant matrices. We assume that $A(\xi')$ has only real eigenvalues and is diagonalizable for every ξ' and moreover the matrices I, $A_1, \ldots, A_n$ span a $m(m+1)/2$ dimensional space. We now assume

$$A(\bar{\xi}') \text{ has } m \text{ fold eigenvalue } \bar{\lambda}$$

at some $\bar{\xi}' \neq 0$. Note that $L(-\bar{\lambda}, \bar{\xi}') = O$ because $A(\bar{\xi}')$ is diagonalizable. Let us set $P(\xi) = L((-\bar{\lambda}, \bar{\xi}') + \xi)$ so that $P(\xi)$ is real hyperbolic, real analytic and $P(0) = O$. Note that

$$\frac{\partial P}{\partial \xi_0} = I, \quad \frac{\partial P}{\partial \xi_j}(0) = A_j, \quad j = 1, \ldots, n$$

because $P(\xi) = L(\xi)$. Let

$$\tilde{L}(\xi) = \xi_0 I + \sum_{j=1}^{n} \tilde{A}_j \xi_j = \xi_0 + \tilde{A}(\xi')$$

be a hyperbolic matrix which is close to $L(\xi)$. With $\tilde{P}(\xi) = \tilde{L}((-\bar{\lambda}, \bar{\xi}') + \xi)$, $\tilde{P}(\xi)$ is a hyperbolic matrix and close to $P(\xi)$. Then from Theorem 1.2.2 there is a non degenerate characteristic η of order m for $\tilde{P}$ close to 0. In particular

$$O = \tilde{P}(\eta) = \tilde{L}((-\bar{\lambda}, \bar{\xi}') + \eta)$$

and this shows that $\tilde{A}(\bar{\xi}' + \eta')$ has a m fold eigenvalue $\bar{\lambda} - \eta_0$. EXERCISE: Let

$$L(\xi) = \xi_0 I + \sum_{j-1}^{n} A_j \xi_j = \xi_0 I + A(\xi')$$

be a 2×2 hyperbolic system, that is $A(\xi')$ are 2×2 constant matrices and $A(\xi')$ has only real eigenvalues for any ξ'. Let $n \geq 3$ and give an example which can not be approximated by strictly hyperbolic systems. On the other hand, prove that if $n = 1$ then $L(\xi)$ can be approximated by strictly hyperbolic systems.

References

[1] L.Ahlfors: Complex Analysis. MacGraw-Hill, 1979.
[2] W.Nuij: A note on hyperbolic polynomials. Math. Scand. **23** (1968) 69-72.

[3] T.Nishitani: Symmetrization of hyperbolic systems with non degenerate characteristics. J. Func. Anal. **132** (1995) 251-272.
[4] L.Hörmander: Hyperbolic systems with double characteristics. Comm. Pure Appl. Math. **46** (1993) 261-301.
[5] F.John: Algebraic conditions for hyperbolicity of systems of partial differential equations. Comm. Pure Appl. Math. **31** (1978) 787-793.

1.3 Polynomial whose roots separate the roots of another polynomial, I

We start with the following definition.

Definition 1.3.1. For a polynomial in $(\zeta, \bar{\zeta})$

$$h(\zeta, \bar{\zeta}) = \sum_{i,j=0}^{m-1} h_{ij} \zeta^i \bar{\zeta}^j, \quad h_{ij} \in \mathbb{C}$$

we associate a quadratic form in $z = (z_0, z_1, \ldots, z_{m-1}) \in \mathbb{C}^m$ in such a way

$$\hat{h}(z, \bar{z}) = \sum_{i,j=0}^{m-1} h_{ij} z_i \bar{z}_j.$$

For a polynomial in ζ

$$p(\zeta) = \sum_{j=0}^{m} a_j \zeta^j, \quad a_j \in \mathbb{C}$$

we associate a linear function in $z = (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1}$ by

$$\hat{p}(z) = \sum_{j=0}^{m} a_j z_j.$$

Let

$$D_t = \frac{1}{i} \frac{d}{dt}$$

and $\hat{h}(z, \bar{z})$ is coming from $h(\zeta, \bar{\zeta})$. We study a differential quadratic form

$$\hat{h}(Du, \overline{Du}) = \sum_{i,j=0}^{m-1} h_{i,j} D_t^i u \cdot \overline{D_t^j u}$$

where $Du = (u, D_t u, \ldots, D_t^{m-1} u)$. It is easy to see that

$$D_t \hat{h}(Du, \overline{Du}) = \sum_{i,j=0}^{m-1} h_{i,j} (D_t^{i+1} u \cdot \overline{D_t^j u} - D_t^i u \cdot \overline{D_t^{j+1} u}) = \hat{g}(Du, \overline{Du})$$

where $\hat{g}(z, \bar{z})$ comes from

$$g(\zeta, \bar{\zeta}) = (\zeta - \bar{\zeta})h(\zeta, \bar{\zeta}). \tag{1.1}$$

We want to estimate the right-hand side of

$$\frac{d}{dt}\hat{h}(Du, \overline{Du}) = i\hat{g}(Du, \overline{Du})$$

in terms of a given differential operator

$$p(D_t)u = \sum_{j=0}^{m} a_j D_t^j u.$$

Lemma 1.3.1. *If $h(\zeta, \bar{\zeta}) = p(\zeta)q(\bar{\zeta})$ where p and q are polynomials then we have*

$$\hat{h}(z, \bar{z}) = \hat{p}(z)\hat{q}(\bar{z}).$$

Proof. Clear. □

Remark 1.3.1. Note that $\hat{h}(z, \bar{z}) = \hat{p}(z, \bar{z})\hat{q}(z, \bar{z})$ is not true any more even if $h(\zeta, \bar{\zeta}) = p(\zeta, \bar{\zeta})q(\zeta, \bar{\zeta})$.

We remark that

$$\hat{p}(Du) = \sum_{j=0}^{m} a_j D_t^j u = p(D_t)u.$$

We *suppose* that $q(\zeta, \bar{\zeta})$ can be written as

$$q(\zeta, \bar{\zeta}) = p(\zeta)q(\bar{\zeta}) + p(\bar{\zeta})r(\zeta) \tag{1.2}$$

with some real polynomials p and r. Then by Lemma 1.3.1 we get

$$\begin{aligned}\hat{g}(Du, \overline{Du}) &= \hat{p}(Du)\hat{q}(\overline{Du}) + \hat{p}(\overline{Du})\hat{r}(Du) \\ &= p(D_t)u \cdot \overline{q(D_t)u} + \overline{p(D_t)u} \cdot r(D_t)u.\end{aligned}$$

This implies that

$$\frac{d}{dt}\hat{h}(Du, \overline{Du}) = i\left[p(D_t)u \cdot \overline{q(D_t)u} + \overline{p(D_t)u} \cdot r(D_t)u\right].$$

From (1.1) and (1.2) we get

$$(\zeta - \bar{\zeta})h(\zeta, \bar{\zeta}) = p(\zeta)q(\bar{\zeta}) + p(\bar{\zeta})r(\zeta).$$

Taking $\zeta = \bar{\zeta}$ we see hat $r(\zeta) = -q(\zeta)$ and hence

$$h(\zeta, \bar{\zeta}) = \frac{p(\zeta)q(\bar{\zeta}) - p(\bar{\zeta})q(\zeta)}{\zeta - \bar{\zeta}}.$$

Definition 1.3.2. Let $p(\zeta)$ and $q(\zeta)$ be polynomials in ζ. We call

$$h_{p,q}(\zeta,\bar{\zeta}) = \frac{p(\zeta)\overline{q(\zeta)} - \overline{p(\zeta)}q(\zeta)}{\zeta - \bar{\zeta}} \tag{1.3}$$

the Bezout form of p and q.

Exercise: Assume that p is a real polynomial. Prove that $h_{p,q}(\zeta,\bar{\zeta})$ is a polynomial in $(\zeta,\bar{\zeta})$ iff $q(\zeta)$ is real.

Let $p(\zeta)$ be a monic hyperbolic polynomial of degree m so that

$$p(\zeta) = \prod_{j=1}^{m}(\zeta - \lambda_k), \quad \lambda_k \in \mathbb{R}.$$

We introduce the following polynomial in $(\zeta,\bar{\zeta})$

$$h_p(\zeta,\bar{\zeta}) = \frac{1}{m}\sum_{k=1}^{m}\prod_{j\neq k}^{m}|\zeta - \lambda_k|^2. \tag{1.4}$$

We assume that λ_k are different from each other. Let $q(\zeta)$ be a real polynomial of degree at most $m-1$. By the Lagrange interpolation formula, one can write

$$q(\zeta) = \sum_{k=1}^{m}\frac{q(\lambda_k)p(\zeta)}{p'(\lambda_k)(\zeta - \lambda_k)}.$$

With

$$p_k(\zeta) = \prod_{j\neq k}^{m}(\zeta - \lambda_j), \quad \alpha_k = \frac{q(\lambda_k)}{p_k(\lambda_k)} \tag{1.5}$$

we see

$$q(\zeta) = \sum_{k=1}^{m}\alpha_k p_k(\zeta).$$

Then we have

$$\begin{aligned} p(\zeta)q(\bar{\zeta}) - p(\bar{\zeta})q(\zeta) &= \sum_{k=1}^{m}\alpha_k p_k(\zeta)p_k(\bar{\zeta})(\zeta - \lambda_k) - \sum_{k=1}^{m}\alpha_k p_k(\bar{\zeta})p_k(\zeta)(\bar{\zeta} - \lambda_k) \\ &= (\zeta - \bar{\zeta})\sum_{k-1}^{m}\alpha_k p_k(\zeta)p_k(\bar{\zeta}) = (\zeta - \bar{\zeta})\sum_{k-1}^{m}\alpha_k|p_k(\zeta)|^2 \end{aligned}$$

which gives

$$h_{p,q}(\zeta,\bar{\zeta}) = \sum_{k=1}^{m}\alpha_k|p_k(\zeta)|^2, \quad h_p(\zeta,\bar{\zeta}) = \frac{1}{m}\sum_{k=1}^{m}|p_k(\zeta)|^2. \tag{1.6}$$

Lemma 1.3.2. *Assume that p is a strictly hyperbolic polynomial and q is a polynomial of degree at most $m-1$. If there is a $c>0$ such that*

$$h_{p,q}(\zeta,\bar{\zeta}) \geq c\sum_{k=1}^{m}|p_k(\zeta)|^2 = cmh_p(\zeta,\bar{\zeta}) \tag{1.7}$$

then we have

(i) *$q(\zeta)$ is a hyperbolic polynomial of degree $m-1$ and the coefficient of ζ^{m-1} is positive.*

(ii) *$q(\zeta)$ separates $p(\zeta)$, that is the zeros $\{\mu_k\}$ of $q(\zeta)$ verify*

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{m-1} < \mu_{m-1} < \lambda_m.$$

Conversely if (i) and (ii) are satisfied then (1.7) *holds with some $c>0$.*

Proof. Assume (1.7). From (1.6) it follows that $\alpha_k > 0$ because $h_{p,q}(\lambda_s,\lambda_s) = \alpha_s|p_k(\lambda_s)|^2$. Then it is clear that $q(\zeta)$ is of degree $m-1$ and the coefficient of ζ^{m-1} is positive. Since $\alpha_k = q(\lambda_k)/p'(\lambda_k)$ it follows that

$$q(\lambda_k)p'(\lambda_k) > 0, \quad q(\lambda_{k+1})p'(\lambda_{k+1}) > 0.$$

On the other hand we get

$$p'(\lambda_k)p'(\lambda_{k+1}) < 0 \tag{1.8}$$

because $p(\lambda_k)$ is strictly hyperbolic. These show that $q(\lambda_k)q(\lambda_{k+1}) < 0$ and hence $q(\zeta)$ has a zero in $(\lambda_k,\lambda_{k+1})$. Since $q(\zeta)$ is of degree $m-1$ this proves the assertion (ii).
Conversely we assume (i) and (ii). This shows that $q(\lambda_k)q(\lambda_{k+1}) < 0$. In virtue of (1.8) we see that $q(\lambda_k)p'(\lambda_k)$ and $q(\lambda_{k+1})p'(\lambda_{k+1})$ have the same sign. $q(\lambda_1)p'(\lambda_1)$ has the same sign as that of $q(\zeta)p'(\zeta)$ when $\zeta \downarrow -\infty$. On the other hand $q(\zeta)p'(\zeta)$ is positive when $\zeta \downarrow -\infty$ because of (ii). Thus we get

$$q(\lambda_k)p'(\lambda_k) > 0, \quad k = 1,2,\ldots,m$$

and hence $\alpha_k > 0$ which proves the result. □

Lemma 1.3.3. *Assume that $p(\zeta)$ is a strictly hyperbolic polynomial and $q(\zeta)$ is a hyperbolic polynomial of degree $m-1$. Assume that $q(\zeta)$ separates $p(\zeta)$ and the coefficient of ζ^{m-1} is positive. Let $r(\zeta)$ be a polynomial of degree $m-1$. Then there is a $C>0$ such that*

$$C\hat{h}_{p,q}(z,\bar{z}) \geq |\hat{r}(z)|^2.$$

Proof. Since one can express

$$r(\zeta) = \sum_{k=1}^{m} c_k p_k(\zeta)$$

we gets

$$|\hat{r}(z)|^2 \le \left(\sum_{k=1}^{m} |c_k|^2 \alpha_k^{-1}\right) \left(\sum_{k=1}^{m} \alpha_k |\hat{p}_k(z)|^2\right).$$

This proves the assertion because

$$h_{p,q}(z,\bar{z}) = \sum_{k=1}^{m} \alpha_k |p_k(z)|^2$$

which follows from (1.6). □

1.4 Polynomial whose roots separate the roots of another polynomial, II

We study more general hyperbolic polynomials. Let

$$p(\zeta) = \prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j}, \quad \lambda_{(j)} \in \mathbb{R}, \quad \sum_{j=1}^{s} r_j = m$$

where $\lambda_{(j)}$ are different from each other. We also write the same $p(\zeta)$ as

$$p(\zeta) = \prod_{j=1}^{m} (\zeta - \lambda_j)$$

so that $\{\lambda_1, \ldots, \lambda_m\} = \{\lambda_{(1)}, \ldots, \lambda_{(1)}, \lambda_{(2)}, \ldots, \lambda_{(2)}, \ldots\}$.

Definition 1.4.1. Let $q(\zeta)$ be a hyperbolic polynomial of degree $m-1$. Then we say that $q(\zeta)$ separates $p(\zeta)$ if q has the form

$$q(\zeta) = c \prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j - 1} \prod_{j=1}^{s-1} (\zeta - \mu_j), \quad (c > 0)$$

where $\lambda_{(1)} < \mu_1 < \lambda_{(2)} < \cdots < \mu_{s-1} < \lambda_{(s)}$.

Lemma 1.4.1. *Assume that $q(\zeta)$ separates $p(\zeta)$ and the coefficient of ζ^{m-1} in $q(\zeta)$ is positive. Then there is a $c > 0$ such that*

$$h_{p,q}(\zeta, \bar{\zeta}) \ge c h_p(\zeta, \bar{\zeta})$$

holds. The converse is also true.

Proof. If $s = 1$ then the assertion is clear. Let $s \ge 2$. Let us put

$$a(\zeta) = \prod_{j=1}^{s} (\zeta - \lambda_{(j)}), \quad b(\zeta) = \prod_{j=1}^{s-1} (\zeta - \mu_j)$$

so that

$$b(\zeta)\{\prod_{j=1}^{s}(\zeta-\lambda_{(j)})^{r_j-1}\}=q(\zeta).$$

Set

$$a_k(\zeta)=\prod_{j\neq k}^{s}(\zeta-\lambda_{(j)}),\quad \alpha_k=\frac{b(\lambda_{(k)})}{a_k(\lambda_{(k)})}>0. \tag{1.9}$$

The same argument as before gives

$$a(\zeta)b(\bar{\zeta})-a(\bar{\zeta})b(\zeta)=(\zeta-\bar{\zeta})\sum_{k=1}^{s}\alpha_k a_k(\zeta)a_k(\bar{\zeta}).$$

Now we have

$$\frac{p(\zeta)q(\bar{\zeta})-p(\bar{\zeta})q(\zeta)}{\zeta-\bar{\zeta}}=\frac{\left|\prod_{j=1}^{s}(\zeta-\lambda_{(j)})^{r_j-1}\right|^2(a(\zeta)b(\bar{\zeta})-a(\bar{\zeta})b(\zeta))}{\zeta-\bar{\zeta}}$$
$$=\sum_{k=1}^{s}\alpha_k\left|\prod_{j=1}^{s}(\zeta-\lambda_{(j)})^{r_j-1}\right|^2|a_k(\zeta)|^2=\sum_{k=1}^{s}\alpha_k\left|\prod_{j=1}^{s}(\zeta-\lambda_{(j)})^{r_j-\delta_{kj}}\right|^2$$

where δ_{kj} is the Kronecker's delta. This proves that

$$h_{p,q}(\zeta,\bar{\zeta})=\sum_{k=1}^{s}\alpha_k\prod_{j=1}^{s}|\zeta-\lambda_{(j)}|^{2(r_j-\delta_{kj})}. \tag{1.10}$$

Since

$$h_p(\zeta,\bar{\zeta})=\frac{1}{m}\sum_{k=1}^{m}\prod_{j\neq k}^{m}|\zeta-\lambda_j|^2=\frac{1}{m}\sum_{k=1}^{s}r_k\prod_{j=1}^{s}|\zeta-\lambda_{(j)}|^{2(r_j-\delta_{kj})}$$

we get the desired inequality.
We turn to the proof of the converse. Since

$$\frac{\partial p}{\partial\zeta}(\zeta)q(\zeta)-p(\zeta)\frac{\partial q}{\partial\zeta}(\zeta)=h_{p,q}(\zeta,\zeta),\quad \zeta\in\mathbb{R} \tag{1.11}$$

it is clear from the assumption that the zeros of q other than $\{\lambda_{(j)}\}$ are simple. It is also clear from (1.12) that the coefficient of ζ^{m-1} in q is positive. We examine that q has no zero in $(-\infty,\lambda_{(1)})$. If there were, we denote the minimal one by μ. Then we see that

$$\frac{\partial q}{\partial\zeta}(\mu)>0\ (<0)$$

if m is even (odd). On the other hand $p(\mu)$ has the sign $(-1)^m$ it follows that

$$-p(\mu)\frac{\partial q}{\partial\zeta}(\mu)<0$$

and hence $h_{p,q}(\mu,\mu) < 0$, contradicting the assumption. We then examine that q has no zero in $\zeta > \lambda_{(s)}$. This can be checked by a similar way. We next show that q has at most one zero in each $(\lambda_{(k)}, \lambda_{(k+1)})$. If not there are two successive simple zeros $\mu_i \in (\lambda_{(k)}, \lambda_{(k+1)})$, $i = 1, 2$ and hence $p(\zeta) \cdot \partial q(\zeta)/\partial\zeta$ has different signs at μ_1 and μ_2 and hence a contradiction. Thus we can conclude that either $q(\zeta)$ separates $p(\zeta)$ or some $\lambda_{(j)}$ is a zero of $q(\zeta)$ of order greater than $r_j - 1$. Assume that this is the case. Then one can write

$$q(\zeta) = (\zeta - \lambda_{(j)})^l r(\zeta), \quad l \geq r_j.$$

Taking $\zeta = \lambda_{(j)} + \xi$ we see that the right-hand side of (1.12) is $O(|\xi|^{l+r_j-1})$. On the other hand it is clear that

$$|h_{p,q}(\zeta,\zeta)| \geq c|\xi|^{2(r_j-1)}$$

with some $c > 0$. This contradicts the assumption. □

Corollary 1.4.1. *Assume that $q(\zeta)$ separates $p(\zeta)$. Then we have*

$$\hat{h}(z,\bar{z}) \geq c\hat{h}_p(z,\bar{z}), \quad z \in \mathbb{C}^m$$

with some $c > 0$.

Proof. Let us set

$$\phi_k(\zeta) = \prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j - \delta_{kj}}.$$

Then from (1.10) we see

$$h_{p,q}(\zeta,\bar{\zeta}) = \sum_{k=1}^{s} \alpha_k \phi_k(\zeta)\phi_k(\bar{\zeta})$$

and hence

$$\hat{h}(z,\bar{z}) = \sum_{k=1}^{s} \alpha_k \hat{\phi}_k(z)\hat{\phi}_k(\bar{z}).$$

On the other hand since $p_k(\zeta)$ coincides with some $\phi_l(\zeta)$ this proves the assertion. □

Lemma 1.4.2. *Let $p(\zeta)$ be a hyperbolic polynomial. Then $p'(\zeta)$ separates $p(\zeta)$.*

Proof. Let

$$p(\zeta) = \prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j}$$

then it is clear that one can write

$$p'(\zeta) = \{\prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j - 1}\} b(\zeta), \; p(\zeta) = \{\prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j - 1}\} a(\zeta).$$

Then

$$\frac{p'(\zeta)}{p(\zeta)} = \sum_{k=1}^{s} \frac{r_k}{\zeta - \lambda_{(k)}} = \frac{b(\zeta)}{a(\zeta)}.$$

It is obvious that $b(\zeta)/a(\zeta) \downarrow -\infty$ as $\zeta \uparrow \lambda_{(k)}$ and $b(\zeta)/a(\zeta) \uparrow +\infty$ as $\zeta \downarrow \lambda_{(k)}$. Noting that $a(\zeta)$ is different from zero in $(\lambda_{(k)}, \lambda_{(k+1)})$ we see that

$$b(\lambda_{(k)})b(\lambda_{(k+1)}) < 0.$$

This proves that $b(\zeta)$ has a zero in $(\lambda_{(k)}, \lambda_{k+1)})$. Since $b(\zeta)$ is of degree $s-1$, $b(\zeta)$ separates $a(\zeta)$ and hence the result. □

Let us set

$$q(\zeta) = \frac{1}{m}\frac{\partial p}{\partial \zeta}(\zeta). \tag{1.12}$$

Then as shown above $q(\zeta)$ separates $p(\zeta)$.

Lemma 1.4.3. *We have*

$$C\hat{h}_p(z, \bar{z}) \geq |\hat{q}(z)|^2$$

with some $C > 0$.

Proof. Since

$$\frac{\partial p}{\partial \zeta}(\zeta) = \sum_{k=1}^{m} \prod_{j \neq k}^{m} (\zeta - \lambda_j) = \sum_{k=1}^{s} r_k \prod_{j=1}^{s} (\zeta - \lambda_{(j)})^{r_j - \delta_{kj}} = \sum_{k=1}^{s} r_k \phi_k(\zeta)$$

one can write

$$q(\zeta) = \frac{1}{m} \sum_{k=1}^{s} r_k \phi_k(\zeta).$$

From this we get

$$|\hat{q}(z)|^2 \leq \frac{1}{m^2} \left(\sum_{k=1}^{s} r_k^2 \alpha_k^{-1} \right) \left(\sum_{k=1}^{s} \alpha_k |\hat{\phi}_k(z)|^2 \right) = \frac{1}{m^2} \left(\sum_{k=1}^{s} r_k^2 \alpha_k^{-1} \right) \hat{h}_{p,q}(z, \bar{z}).$$

Then the assertion follows from the next lemma. □

Lemma 1.4.4. *Let* $q(\zeta)$ *be defined by* (1.12)*. Then we have*

$$\hat{h}_{p,q}(z, \bar{z}) = \hat{h}_p(z, \bar{z}).$$

Proof. Note that

$$p(\zeta)q(\bar{\zeta}) = \frac{1}{m}\frac{\partial}{\partial \bar{\zeta}}[p(\zeta)p(\bar{\zeta})], \quad \frac{1}{m}\frac{\partial}{\partial \zeta}[p(\zeta)p(\bar{\zeta})].$$

This shows that

$$h_{p,q}(\zeta, \bar{\zeta}) = \frac{1}{m} \left(\frac{\partial}{\partial \bar{\zeta}} \prod_{j=1}^{m} (\zeta - \lambda_j)(\bar{\zeta} - \lambda_j) - \frac{\partial}{\partial \zeta} \prod_{j=1}^{m} (\zeta - \lambda_j)(\bar{\zeta} - \lambda_j) \right) / (\zeta - \bar{\zeta}).$$

Here we remark that

$$[\frac{\partial}{\partial\bar{\zeta}} - \frac{\partial}{\partial\zeta}](\zeta - \lambda_j)(\bar{\zeta} - \lambda_j) = \zeta - \bar{\zeta}$$

and this shows

$$h_{p,q}(\zeta, \bar{\zeta}) = \frac{1}{m}\sum_{k=1}^{m}\prod_{j\neq k}^{m}|\zeta - \lambda_j|^2 = h_p(\zeta, \bar{\zeta}).$$

□

References

[1] V.Ja.Ivrii: Correctness of the Cauchy problem in Gevrey classes for non strictly hyperbolic operators. Math. USSR. Sbornik **25** (1975) 365-387.
[2] J.Leray: Hyperbolic Differential Equations. Inst. Adv. Study Princeton 1953.
[3] L.Gårding: Solution directe du problème de Cauchy pour les équations hyperboliques. Colloque International C.N.R.S. Asterisque Soc. Math. France **2-3** (1973) 71-90.

1.5 A theorem of Hermite

Assume that $p(\zeta)$ is a monic polynomial of degree m and verifies

$$p(\zeta) = 0 \Longrightarrow \operatorname{Im}\zeta > 0. \tag{1.13}$$

Let us write

$$p(\zeta) = p_0(\zeta) - ip_1(\zeta) \tag{1.14}$$

where $p_i(\zeta)$ are real polynomials.

Theorem 1.5.1. *Assume* (1.13)*. Then $p_0(\zeta)$ and $p_1(\zeta)$ are strictly hyperbolic polynomials of degree m and $m-1$ respectively and $p_1(\zeta)$ separates $p_0(\zeta)$.*

Proof. It is clear that $p_0(\zeta)$ is of degree m. Let us write

$$p(\zeta) = \prod_{j=1}^{m}(\zeta - \lambda_j) = \zeta^m + a_1\zeta^{m-1} + \cdots + a_m$$

so that $\operatorname{Im}\lambda_j > 0$. Since $\operatorname{Im} a_1 = -\sum \operatorname{Im}\lambda_j < 0$ and hence $\operatorname{Im} a_1 \neq 0$ we see that $p_1(\zeta)$ is of degree $m-1$. Let us set

$$\theta(\zeta) = \arg p(\zeta).$$

Let $\zeta \in \mathbb{R}$ vary from $-\infty$ to $+\infty$. Note that

$$\arg p(\zeta) = \sum_{j=1}^{m}\arg(\zeta - \lambda_j).$$

It is clear that $\pi < \arg(\zeta - \lambda_j) < 2\pi$ and $\arg(\zeta - \lambda_j)$ is strictly increasing as ζ varies from $-\infty$ to $+\infty$. This shows that $\theta(\zeta) = \arg p(\zeta)$ is strictly increasing from $m\pi$ to $2m\pi$ when ζ increases from $-\infty$ to $+\infty$. Since $\theta(\zeta)$ is strictly increasing, $\cos\theta(\zeta)$ has exactly m zeros in $(m\pi, 2m\pi)$ which are different from each other. Since

$$\cos\theta(\zeta) = \frac{p_0(\zeta)}{\sqrt{p_0(\zeta)^2 + p_1(\zeta)^2}}$$

this proves that $p_0(\zeta)$ has m different zeros. We next observe $\sin\theta(\zeta)$. By similar arguments we see that $p_1(\zeta)$ has exactly $m-1$ zeros and $p_1(\zeta)$ separates $p_0(\zeta)$. □

We note that

$$\begin{aligned} p(\zeta)\overline{p(\zeta)} &= (p_0(\zeta) - ip_1(\zeta))(p_0(\zeta) + ip_1(\zeta)) \\ &= |p_0(\zeta)|^2 + |p_1(\zeta)|^2 + i[p_0(\zeta)p_1(\bar\zeta) - p_0(\bar\zeta)p_1(\zeta)] \\ &= |p_0(\zeta)|^2 + |p_1(\zeta)|^2 + i(\zeta - \bar\zeta)h_{p_0,p_1}(\zeta,\bar\zeta). \end{aligned}$$

Lemma 1.5.1. *Assume* (1.13). *Then we have*

$$|p(\zeta)|^2 = |p_0(\zeta)|^2 + |p_1(\zeta)|^2 + i(\zeta - \bar\zeta)h_{p_0,p_1}(\zeta,\bar\zeta).$$

Recall that

$$\frac{d}{dt}\hat h_{p_0,p_1}(Du, \overline{Du}) = \hat g(Du, \overline{Du})$$

where $g(\zeta,\bar\zeta) = i(\zeta - \bar\zeta)h_{p_0,p_1}(\zeta,\bar\zeta)$. From Lemma 1.5.1 it follows that

$$g(\zeta,\bar\zeta) = |p(\zeta)|^2 - |p_0(\zeta)|^2 - |p_1(\zeta)|^2.$$

This gives

$$\frac{d}{dt}\hat h_{p_0,p_1}(Du, \overline{Du}) = |p(D_t)u|^2 - |p_0(D_t)u|^2 - |p_1(D_t)u|^2.$$

1.6 Hyperbolic polynomials with parameters

Let us study

$$f(t,s) = t^r + f_1(s)t^{r-1} + \cdots + f_r(s) \tag{1.15}$$

where $f_i(s) \in C^\infty(J)$ and J is an open interval containing the origin. We assume that $f(t,s) = 0$ has only real roots with respect to t for any $s \in J$. We also assume that

$$f_i(0) = 0, i = 1, 2, \ldots, r \tag{1.16}$$

that is $t = 0$ is a r fold root of $f(t,0)$.

Lemma 1.6.1. *Assume* (1.16). *Then we have*

$$f_i(s) = O(s^i) \text{ as } s \to 0, \quad i = 1, 2, \ldots, r$$

and one can write

$$f(t, s) = f_{(0,0)}(t, s) + O(|t| + |s|)^{r+1}$$

where $f_{(0,0)}(t, s)$ *is of homogeneous of degree* r *and hyperbolic with respect to* t *for all* $s \in \mathbb{R}$.

Remark 1.6.1. Note that $f_{(0,0)}(t, s)$ is given by

$$f(\mu t, \mu s) = \mu^r\{f_{(0,0)}(t, s) + O(\mu)\}, \quad \mu \to 0.$$

Proof. Take $\sigma_j \in \mathbb{N}$ such that $f_j(s) = O(s^{\sigma_j})$ (if $f_j(s) = O(s^k)$ for any k then we take σ_j sufficiently large). Put

$$\min_{1 \le j \le r} \frac{\sigma_j}{j} = \lambda = \frac{q}{p} > 0$$

where p, q are relatively prime. We first prove $f_i(s) = O(s^i)$. It is enough to prove $\lambda \ge 1$. We suppose $0 < \lambda < 1$ and derive a contradiction. Plug $t = w|s|^\lambda$ into $f(t, s) = 0$ which yields

$$0 = \sum_{j=0}^{r} w^j |s|^{\lambda j} f_{r-j}(s), \; f_0(s) = 1.$$

Multiplying $|s|^{-\lambda r}$ we get

$$0 = \sum_{j=0}^{r} w^j f_{r-j}(s)|s|^{-\lambda(r-j)}.$$

Let $s \to \pm 0$ then we have

$$0 = \sum_{j=0}^{r} w^j f^{\pm}_{r-j} = 0, \quad f^{\pm}_{r-j} = \lim_{s \to \pm 0} |s|^{-\lambda(r-j)} f_{r-j}(s). \tag{1.17}$$

By the assumption there is at least one $0 \le j \le r - 1$ such that $f^{\pm}_{r-j} \ne 0$. We first note that the equation (1.17) has r real roots. Otherwise since $f^{\pm}_0 = f_0(s) = 1$, by Rouché's theorem, $f(t, s) = 0$ would have a non real root for small s which contradicts the assumption. We first treat the case $q > 2$. If $f^{\pm}_j \ne 0$ then $\sigma_j q = pj$ and hence $j = nq$ with some $n \in \mathbb{N}$. Then (1.17) with $+$ sign is reduced to

$$w^r + a_1 w^{r-q} + \cdots + a_l w^{r-lq} = 0.$$

(1.17) with $-$ sign is reduced to a similar equation. One can express

$$w^r \left(1 + a_1(\frac{1}{w})^q + \cdots + a_l(\frac{1}{w})^{lq}\right) = 0, \; (a_l \ne 0).$$

With $W = (1/w)^q$ this turns out to be

$$a_l W^l + \cdots + a_1 W + 1 = 0. \tag{1.18}$$

Noting that (1.18) has a non zero root, W, we get a non real root w from $w^q = 1/W$ because $q > 2$ and hence a contradiction. We turn to the case $q = 2$ and hence $p = 1$. From the same arguments (1.17) is reduced to

$$w^r + a_1^{\pm} w^{r-2} + \cdots + a_l^{\pm} w^{r-2l} = 0.$$

Since $f_{2k}(s) = s^k(a_{2k} + O(s))$, $s \to 0$ we see that $a_k^+ = a_k^-$ if k is even and $a_k^+ = -a_k^-$ if k is odd. As before we are led to

$$w^r \left(1 + a_1^{\pm}(\frac{1}{w})^2 + \cdots + a_l^{\pm}(\frac{1}{w})^{2l}\right) = 0.$$

With $W = (1/w)^2$ we have

$$a_l^{\pm} W^l + \cdots + a_1^{\pm} W + 1 = 0. \tag{1.19}$$

As observed above, W and $-W$ are the roots of (1.19) at the same time, and hence from $w^2 = 1/W$ we get a non-real root and a contradiction. Thus we have proved that $\lambda \geq 1$ and hence the result.
We turn to the second assertion. Set $t = ws$ and plug this into $f(t,s) = 0$. Then we have

$$\begin{aligned} s^{-r} f(t,s) = w^r + a_1 w^{r-1} + \cdots + a_r + sg(w,s) \\ = f_{(0,0)}(w,1) + sg(w,s). \end{aligned}$$

From this we see that $f_{(0,0)}(w,1) = 0$ has only real roots. Since

$$f_{(0,0)}(t,s) = s^r f_{(0,0)}(\frac{t}{s}, 1)$$

we get the desired assertion. □

Definition 1.6.1. Let $P(\zeta)$ be a polynomial and assume that $P(\eta) = 0$. We define $P_\eta(\zeta)$ by

$$P(\eta + \mu\zeta) = \mu^r\{P_\eta(\zeta) + O(\mu)\}, \ \mu \to 0, \ P_\eta(\zeta) \not\equiv 0.$$

We call $P_\eta(\zeta)$ the localization of $P(\zeta)$ at η.

This is nothing but

$$P_\eta(\zeta) = \sum_{|\alpha|=r} \frac{1}{\alpha!}(\frac{\partial}{\partial\zeta})^\alpha P(\eta)\zeta^\alpha$$

and hence of homogeneous of degree r. We study now hyperbolic polynomial with parameter $x \in \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be a neighborhood of the origin of $\mathbb{R}^n$ and

$$P(t,s) = t^r + f_1(x)t^{r-1} + \cdots + f_r(x), \quad f_i(0) = 0$$

which is hyperbolic with respect to t for all $x \in U$.

Lemma 1.6.2. *Under the above assumption we have*

$$(\frac{\partial}{\partial x})^\alpha f_j(0) = 0,\ |\alpha| \le j-1$$

and the localization $P_{(0,0)}(t,x)$ at $(0,0)$ is hyperbolic with respect to t for all $x \in U$.

Proof. Fix $\eta \in \mathbb{R}^n$, $\eta \neq 0$ and consider

$$P(t, s\eta) = f(t, s; \eta) = t^r + f_1(s\eta)t^{r-1} + \cdots + f_r(s\eta).$$

For $|s| < \delta$, $f(t,s;\eta)$ is hyperbolic with respect to t. From Lemma 1.6.1 it follows that

$$(\frac{d}{ds})^k f_j(s\eta)|_{s=0} = \langle \eta, \frac{\partial}{\partial x}\rangle^k f_j(0) = 0, \quad k \le j-1.$$

Since η is arbitrary this shows the assertion. On the other hand

$$P(\mu t, \mu x) = f(\mu t, \mu; x) = \mu^r\{f_{(0,0)}(t,1;x) + O(\mu)\}$$

where $f_{(0,0)}(t,1;x)$ is hyperbolic with respect to t by Lemma 1.6.1. This shows that $P_{(0,0)}(t,x) = f_{(0,0)}(t,1;x)$ and hence the result. □

We finally study the general case:

$$P(t,x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x)$$

which is hyperbolic w.r.t. t for any $x \in U$. Assume that

$$(\frac{\partial}{\partial t})^k P(\hat{t},\hat{x}) = 0,\ k = 0,1,\ldots,r-1,\ \ (\frac{\partial}{\partial t})^r P(\hat{t},\hat{x}) \neq 0. \tag{1.20}$$

Then we have

Corollary 1.6.1. *Assume* (1.20). *Then*

$$(\frac{\partial}{\partial t})^j(\frac{\partial}{\partial x})^\alpha P(\hat{t},\hat{x}) = 0, \quad j + |\alpha| \le r-1$$

and the localization $P_{(\hat{t},\hat{x})}(t,x)$ is hyperbolic w.r.t. t for any $x \in \mathbb{R}^n$.

Proof. We first note that there is a neighborhood V of $\hat{x}$ such that one can write

$$P(t,x) = Q(t,x)R(t,x), \quad x \in V$$

where Q and R are hyperbolic polynomials in t of degree r and $m-r$ respectively and

$$(\frac{\partial}{\partial t})^k Q(\hat{t},\hat{x}) = 0,\ k = 0,\ldots,r-1,\ \ R(\hat{t},\hat{x}) \neq 0.$$

Applying Corollary 1.6.2 to $Q(\hat{t}+t, \hat{x}+x)$ to get

$$Q(\hat{t}+t, \hat{x}+x) = Q_{(\hat{t},\hat{x})}(t,x) + O(|t|+|x|)^{r+1}.$$

Since $R(\hat{t}, \hat{x}) \neq 0$ we get

$$P(\hat{t}+t, \hat{x}+x) = R(\hat{t}, \hat{x}) Q_{(\hat{t},\hat{x})}(t,x) + O(|t|+|x|)^{r+1}.$$

This proves the assertion. □

References

[1] L.Hörmander: The Cauchy problem for differential equations with double characteristics. J.Analyse Math. **32** (1977) 118-196.
[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.

1.7 A theorem of Rellich

Consider

$$P(t,s) = t^m + a_1(s)t^{m-1} + \cdots + a_m(s)$$

which is hyperbolic w.r.t. t for any $s \in J$, an open interval containing the origin.

Lemma 1.7.1. *Assume $a_j(s) \in C^\omega(J)$. Then there are $\delta > 0$ and m functions $\lambda_j(s) \in C^\omega(\{|s| < \delta\})$, $j = 1, 2, \ldots, m$ such that*

$$P(t,s) = \prod_{j=1}^{m} (t - \lambda_j(s)).$$

Proof. Recall that the roots of $P(t,s) = 0$ can be expanded by a Puiseux series:

$$t(s) = \sum_{j=0} A_j s^{j/h}, \quad 0 < |s| < r.$$

We show that $A_j s^{j/h}$ are real in $0 < |t| < r$, $t \in \mathbb{R}$. Suppose otherwise. Let $A_k s^{k/h}$ is the first term which does not verify the above assertion. Then we see

$$\arg(A_k s^{k/h}) = \arg A_k + \frac{kq\pi}{h} \neq p\pi \text{ for any } p \in \mathbb{N}$$

where we have set $\arg s = q\pi$. Then

$$t(s) = \sum_{j=1}^{k-1} A_j s^{j/h} + A_k s^{k/h}(1 + o(1)), \; s \to 0$$

and hence

$$\operatorname{Im} t(s) = \{\operatorname{Im} A_k s^{k/h}\}(1 + o(1)), \; s \to 0$$

which contradicts the assumption. Since

$$\arg A_j + \frac{j}{h}2n\pi = p\pi, \quad \arg A_j + \frac{j}{h}(2n+1)\pi = p'\pi$$

we get $j/h = p' - p \in \mathbb{Z}$ and hence $\arg A_j = p\pi - (p'-p)2n\pi \in \pi\mathbb{Z}$. Thus we get $A_j s^{j/h} = A_j s^{p'-p}$. Therefore $t(s)$ is a power series of s and hence analytic. □

We turn to the case $a_j(s) \in C^\infty(J)$. Let us write

$$P(t,0) = \prod_{j=1}^{s}(t-\tau_j)^{r_j}, \quad \sum_{j=1}^{s} r_j = m$$

where τ_j are different from each other. Recall that one can write

$$P(t,s) = P_1(t,s)\cdots P_s(t,s)$$

where

$$P_j(t,s) = t^{r_j} + a_{j,1}(s)t^{r_j-1} + \cdots + a_{j,r_j}(s)$$

and $t = \tau_j$ is a root of $P_j(t,0)$ of multiplicity r_j and P_j is hyperbolic with respect to t. It is enough to study $P_j(\tau_j + t, s)$ and then we study $P(t,s)$ which is of degree m and $P(t,0) = t^m$. Note that

$$P(st,s) = s^m[P_{(0,0)}(t,1) + sR(t,s)].$$

From Lemma 1.6.1, $P_{(0,0)}(t,1)$ is a hyperbolic polynomial and hence one can write

$$P_{(0,0)}(t,1) = \prod_{j=1}^{s}(t-\lambda_j)^{r_j}$$

where λ_j are different from each other. Take circles $C(\lambda_j;\epsilon)$, centered at λ_j with radius $\epsilon > 0$. Take $\epsilon > 0$ so small so that each circle lies outside of the other. We have

$$|P_{(0,0)}(t,1)| \geq c\epsilon^{r_j}$$

with some $c > 0$ on $C_j(\lambda_j;\epsilon)$ and then one can take $s_0(\epsilon)$ so that

$$|s^{m+1}R(t,s)| < |s^m P_{(0,0)}(t,1)| \text{ on } C(\lambda_j;\epsilon),\ j = 1,\ldots,s,\ \ |s| \leq s_0(\epsilon).$$

Then by Rouché's theorem $P(st,s) = 0$ has exactly r_j roots in $C(\lambda_j;\epsilon)$. Let us name these roots

$$\lambda_1^j(s) \leq \lambda_2^j(s) \leq \cdots \leq \lambda_{r_j}^j(s)$$

and note that $\lambda_k^j(s) \to \lambda_j$, $s \to 0$. With $t_k^j(s) = s\lambda_k^j(s)$, $j = 1,\ldots,s$, $k = 1,\ldots,r_j$ we have $P(t_k^j(s),s) = 0$ and

$$\frac{t_k^j(s) - t_k^j(0)}{s} = \lambda_k^j(s) \to \lambda_j \text{ as } s \to 0.$$

This proves that $t_k^j(s)$ is differentiable at $s = 0$ and moreover

$$\{\frac{d}{ds} t_k^j(0)\} = \{\text{the roots of } P_{(0,0)}(t,1) = 0\}.$$

We apply this arguments to each $P_j(\tau_j + t, s)$ to get

Lemma 1.7.2. *Assume $a_j(s) \in C^\infty(J)$. Then there are $\delta > 0$ and m continuous functions $t_j(s) \in C^0(\{|s| < \delta\})$ which are differentiable at $s = 0$ such that*

$$P(t, s) = \prod_{j=1}^m (t - t_j(s)).$$

Moreover

$$\{\frac{d}{ds} t_j(0) \mid j = 1, ..., m\} = \{\text{the roots of } P_{(0,0)}(t,1) = 0\}.$$

We next study the case

$$P(t, x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x)$$

where $x = (x_1, \ldots, x_n) \in U \subset \mathbb{R}^n$ and $P(t, x)$ is hyperbolic w.r.t. t for $x \in U$.

Corollary 1.7.1. *Assume $a_j(x) \in C^\omega(U)$. Let e_μ be the unit vector with μ-th component 1. Then there are $\delta > 0$ and m real analytic functions $\lambda_j(s; e_\mu) \in C^\omega(\{|s| < \delta\})$ such that*

$$P(t, \hat{x} + se_\mu) = \prod_{j=1}^m (t - \lambda_j(s; e_\mu)).$$

Corollary 1.7.2. *Assume $a_j(x) \in C^\infty(U)$. Let e_μ be the unit vector with μ-th component 1. Then there are $\delta > 0$ and m continuous functions $\lambda_j(s; e_\mu) \in C^0(\{|s| < \delta\})$ which are differentiable at $s = 0$ such that*

$$P(t, \hat{x} + se_\mu) = \prod_{j=1}^m (t - \lambda_j(s; e_\mu)).$$

We end this section with a special result. Let us consider a polynomial of degree 2:

$$P(z, t, s) = z^2 + a(t, s)z + b(t, s)$$

where $a(t, s)$, $b(t, s) \in C^\omega(U)$, U is a neighborhood of the origin of $\mathbb{R}^2$ and $P(z, t, s)$ is hyperbolic with respect to z for $(t, s) \in U$. Then we have

Lemma 1.7.3. *There are a neighborhood V of the origin and two functions $\mu(t, s)$, $\lambda(t, s) \in C^0(V) \cap C^\omega(V \setminus (0, 0))$ such that*

$$P(z, t, s) = (z - \lambda(t, s))(z - \mu(t, s))$$

and the first derivatives of $\lambda(t, s)$ and $\mu(t, s)$ are bounded in V.

References

[1] T.Kato: Perturbation Theory. Springer, 1984.
[2] M.D.Bronshtein: Smoothness of roots of polynomials depending on parameters. Sib. Mat. Zh. **20** (1979) 493-501.
[3] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[4] T.Nishitani: The Cauchy problem for weakly hyperbolic equations of second order. Comm. P.D.E. **5** (1980) 1273-1296.

1.8 Malgrange's preparation theorem

We first state the Weierstrass' preparation theorem.

Theorem 1.8.1. *Let $g(t,z)$ be holomorphic in a neighborhood of the origin in $\mathbb{C}^{1+n}$ and verify*

$$g = \frac{\partial}{\partial t} g = \cdots = (\frac{\partial}{\partial t})^{k-1} g = 0,\ (\frac{\partial}{\partial t})^k g \neq 0 \quad \text{at } (0,0). \tag{1.21}$$

Then g can be uniquely factorized:

$$g(t,z) = c(t,z)\{t^k + a_1(z)t^{k-1} + \cdots + a_k(z)\}$$

where $c(t,z)$, $a_j(z)$ are holomorphic at $(0,0)$, 0 respectively and $c(0,0) \neq 0$, $a_j(0) = 0$.

Theorem 1.8.2. *Assume that $g(t,x,\xi)$ is defined in $|t| < 2r$, $|x| < \delta$, $\xi \in U$, U is an open set in $\mathbb{R}^k$, verifying*

$$\partial_x^\alpha \partial_\xi^\beta g(t,x,\xi) \in C^0(\{|t| < 2r\} \times \{|x| < \delta\} \times U),\ |\alpha| \le M,\ |\beta| \le L$$

and holomorphic in t, $|t| < 2r$. Moreover we assume that

$$(\frac{\partial}{\partial t})^j g(0,0,\xi) = 0,\ 0 \le j \le k-1,\ \ (\frac{\partial}{\partial t})^k g(0,0,\xi) \neq 0,\ \ \xi \in U.$$

Let $K \subset\subset U$ be a compact. Then there are $\bar{r} > 0$, $\bar{\delta} > 0$ and open set $(K \subset)W$ such that

$$g(t,x,\xi) = c(t,x,\xi)\{t^k + a_1(x,\xi)t^{k-1} + \cdots + a_k(x,\xi)\}$$

holds in $|t| < \beta r$, $|x| < \bar{\delta}$, $\xi \in W$ where $c(0,0,\xi) \neq 0$, $a_j(0,\xi) = 0$, $\xi \in W$ and

$$\partial_x^\alpha \partial_\xi^\beta c(t,x,\xi) \in C^0(\{|t| < \bar{r}\} \times \{|x| < \bar{\delta}\} \times W),$$
$$\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi) \in C^0(\{|x| < \bar{\delta}\} \times W)$$

for $|\alpha| \le M$, $|\beta| \le L$, and holomorphic in t.

We turn to Malgrange's preparation theorem.

Theorem 1.8.3. *Assume that* $g(t,x,\xi,\eta)$ *is defined in the set* $|t|\le r$, $|x|\le\delta$, $(\xi,\eta)\in U\times V$, *where* U *and* V *are some open sets in* $\mathbb{R}^p$ *and* $\mathbb{C}^q$, *and verifies the followings:*

$$(\frac{\partial}{\partial t})^l g(t,x,\xi,\eta)\in C^0(\{|t|\le r\}\times\{|x|\le\delta\}\times U\times V),\ 0\le l\le 2k+3$$

which are holomorphic in η *and*

$$(\frac{\partial}{\partial t})^j g(0,0,\xi,\eta)=0,\ 0\le j\le k-1,\ (\frac{\partial}{\partial t})^k g(0,0,\xi,\eta)\ne 0,\ (\xi,\eta)\in U\times V.$$

Then for any compacts $K\subset\subset U$, $L\subset\subset V$ *there are* $\bar r>0$, $\bar\delta>0$, $K\times L\subset W$ *and* $c(t,x,\xi,\eta)$, $a_j(t,x,\xi,\eta)$ *such that*

$$g(t,x,\xi,\eta)=c(t,x,\xi,\eta)\{t^k+a_1(x,\xi,\eta)t^{k-1}+\cdots+a_k(x,\xi,\eta)\}$$

holds where $c(0,0,\xi,\eta)\ne 0$, $a_j(0,\xi,\eta)=0$, $(\xi,\eta)\in W$ *and*

$$(\frac{\partial}{\partial t})^l c(t,x,\xi,\eta)\in C^0(\{|t|\le\bar r\}\times\{|x|\le\bar\delta\}\times W),\ 0\le l\le k,$$
$$a_j(t,x,\xi,\eta)\in C^0(\{|x|\le\bar\delta\}\times W)$$

which are holomorphic in η.

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, I. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.
[2] L.Nirenberg: A proof of the Malgrange preparation theorem. Lecture Notes in Maths. **192**, Springer 1971.

1.9 Hyperbolic cone

Let

$$P(t,x)=t^m+a_1(x)t^{m-1}+\cdots+a_m(x)$$

be hyperbolic for $|x|<\delta$, $x\in\mathbb{R}^n$. Assume that $t=0$ is a zero of order r of $P(t,0)$. Then we can write

$$P(t,x)=c(t,x)\{t^r+\tilde a_1(x)t^{r-1}+\cdots+\tilde a_r(x)\}$$

where $\tilde a_i(0)=0$ and $c(t,x)$ is a polynomial of degree $m-r$ in t such that $c(0,0)\ne 0$. Hence we may consider (changing notations)

$$P(t,x)=t^r+a_1(x)t^{r-1}+\cdots+a_r(x),\quad a_i(0)=0$$

which is hyperbolic for $|x|<\delta$. Let $P(t_0,x_0)=0$. Recall that the localization $P_{(t_0,x_0)}(\tau,\xi)$ is defined by

$$P(t_0+\mu,x_0+\mu\xi)=\mu^\nu\{P_{(t_0,x_0)}(\tau,\xi)+O(\mu)\},\quad \mu\to 0.$$

From Corollary 1.6.2, $P_{(t_0,x_0)}(\tau,\xi)$ is hyperbolic with respect to τ.

Definition 1.9.1. We define $\Gamma_{(t_0,x_0)}$ as the connected component of $(1,0,\ldots,0)$ of the set
$$\{(\tau,\xi) \mid P_{(t_0,x_0)}(\tau,\xi) \neq 0\}$$
and this is called the hyperbolic cone of $P_{(t_0,x_0)}(\tau,\xi)$.

The next proposition is well known.

Proposition 1.9.1. *$\Gamma(t_0,x_0)$ is a convex cone.*

Proof. Let us denote $P(\tau,\xi) = P_{(t_0,x_0)}(\tau,\xi)$ and $\Gamma = \Gamma_{(t_0,x_0)}$. Let $\theta = (1,0,\ldots,0)$ and

$$\Delta = \{(\tau,\xi) \mid P((\tau,\xi)+t\theta) = 0, t \in \mathbb{C} \Rightarrow t < 0\}$$

and show that $\Gamma = \Delta$. Take $(\bar{z},\bar{\xi}) \in \Gamma$. Then there is a curve $(\tau(s),\xi(s))$, $0 \leq s \leq 1$ which connects θ and $(\bar{\tau},\bar{\xi})$ in Γ. We study

$$P((\tau(s),\xi(s))+t\theta)) = 0.$$

From the hyperbolicity, the roots $\{t_j(s)\}$ are real and hence

$$t(s) = \max_j t_j(s)$$

is a continuous function in s. Note that $t(0) < 0$. We show that $t(1) < 0$. Otherwise there were $\hat{s}$ such that $t(\hat{s}) = 0$. Then

$$P(\tau(\hat{s}),\xi(\hat{s})) = 0$$

which is a contradiction. Hence we have proved $\Gamma \subset \Delta$.
Conversely, take $(\bar{\tau},\bar{\xi}) \in \Delta$. Then

$$P(s(\bar{\tau},\bar{\xi})+(1-s)\theta) = s^\nu P((\bar{\tau},\bar{\xi})+(1-s)s^{-1}\theta) \neq 0$$

for $0 < s \leq 1$. This shows that $(\bar{\tau},\bar{\xi}) \in \Gamma$ and hence $\Delta \subset \Gamma$. We have actually proved

Lemma 1.9.1. *We have*

$$\Gamma_{(t_0,x_0)} = \{(\tau,\xi) \mid P_{(t_0,x_0)}((\tau,\xi)+t\theta) = 0 \Rightarrow t < 0\}.$$

Corollary 1.9.1. *Let*

$$P_{(t_0,x_0)}(\tau,\xi) = \prod_{j=0}^{\nu}(\tau-\lambda_j(\xi)).$$

Then we have
$$\Gamma_{(t_0,x_0)} = \{(\tau,\xi) \mid \tau > \max_j \lambda_j(\xi)\}.$$

Exercise: Prove Corollary 1.9.1.

Remark 1.9.1. Let us denote by $\lambda_j(\xi; t, x)$ the roots of $P_{(t,x)}(\tau, \xi)$. Note that the number of the roots depends on (t, x). We see

$$\max_j \lambda_j(\xi; t_1, x_1) \le \max_j \lambda_j(\xi; t_0, x_0)$$

if $\Gamma_{(t_0,x_0)} \subset \Gamma_{(t_1,x_1)}$.

We now show that Δ is convex. Take $(\hat\tau, \hat\xi) \in \Delta$ and fix a small $\epsilon > 0$. Set

$$E_{(\hat\tau,\hat\xi)} = \{(\tau, \xi) \mid P((\tau, \xi) + i\epsilon\theta + it(\hat\tau, \hat\xi)) = 0,\ t \in \mathbb{C} \Rightarrow \operatorname{Re} t < 0\}.$$

Note that $E_{(\hat\tau,\hat\xi)}$ is open because

$$P((\tau, \xi) + i\epsilon\theta + it(\hat\tau, \hat\xi)) \neq 0 \text{ if } \operatorname{Re} t = 0.$$

Note that $(0, 0) \in E_{(\hat\tau,\hat\xi)}$ because

$$P(i\epsilon\theta + it(\hat\tau, \hat\xi)) = (it)^\nu P(\epsilon t^{-1}\theta + (\hat\tau, \hat\xi)) = 0 \Rightarrow t < 0.$$

For any fixed $(\tau, \xi) \in \mathbb{R}^{n+1}$, we consider

$$P(s(\tau, \xi) + i\epsilon\theta + it(\hat\tau, \hat\xi)) = 0.$$

Then $t(s) = \max_j \operatorname{Re} t_j(s)$, $0 \le s \le 1$ is continuous and $t(0) < 0$. Show that $t(1) < 0$. Otherwise there were $\hat s$ so that $t(\hat s) = 0$, that is

$$P(\hat s(\tau, \xi) + i\epsilon\theta + i\hat t(\hat\tau, \hat\xi)) = 0, \quad \operatorname{Re} \hat t = 0$$

which contradicts the hyperbolicity. This proves that

$$E_{(\hat\tau,\hat\xi)} = \mathbb{R}^{n+1}.$$

Note that this shows

$$P((\tau, \xi) + i(\epsilon\theta + (\hat\tau, \hat\xi))) \neq 0 \text{ for } (\hat\tau, \hat\xi) \in \Delta,\ (\tau, \xi) \in \mathbb{R}^{n+1}.$$

Since for any $(\hat\tau, \hat\xi) \in \Delta$ we can take $\epsilon > 0$ so that $-\epsilon\theta + (\hat\tau, \hat\xi) \in \Delta$, we conclude that

$$P((\tau, \xi) + i(\hat\tau, \hat\xi)) \neq 0 \text{ for } (\tau, \xi) \in \mathbb{R}^{n+1},\ (\hat\tau, \hat\xi) \in \Delta.$$

Thus

$$P((\tau, \xi) + t(\hat\tau, \hat\xi)) = P((\tau, \xi) + \operatorname{Re} t(\hat\tau, \hat\xi) + i \operatorname{Im} t(\hat\tau, \hat\xi)) = 0$$

has only real roots t. We summarize

Lemma 1.9.2. *Let $(\hat\tau, \hat\xi) \in \Gamma_{(t_0,x_0)}$. Then $P_{(t_0,x_0)}(\tau, \xi)$ is hyperbolic with respect to $(\hat\tau, \hat\xi) \in \Gamma_{(t_0,x_0)}$.*

Repeating the same arguments as before we conclude that

$$\Delta = \{(\tau,\xi) \mid P((\tau,\xi)+t(\hat{\tau},\hat{\xi}))=0 \Rightarrow t<0\}.$$

This implies that $P(s(\tau,\xi)+(1-s)(\hat{\tau},\hat{\xi})+t(\hat{\tau},\hat{\xi}))=0$ shows that $1-s+t<0$ and hence $t<-(1-s)\le 0$. That is $s(\tau,\xi)+(1-s)(\hat{\tau},\hat{\xi})\in\Delta$. Hence Δ is convex. □

EXERCISE: Carry out the proof of the fact that $E_{(\hat{\tau},\hat{\xi})}$ is open.

References

[1] L.Gårding: Linear hyperbolic partial differential equations with constant coefficients. Acta Math. **85** (1951) 1-62.
[2] L.Hörmander: The Analysis of Linear Partial Differential Operators, II. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

1.10 Semi-continuity of hyperbolic cones

We want to prove

Theorem 1.10.1. *Let $P(t,x)=t^m+a_1(x)t^{m-1}+\cdots+a_m(x)$ be a hyperbolic polynomial and $a_j(x)\in C^{2r+3}(\{|x|<\delta\})$. Assume that $t=0$ is a root of $P(t,0)=0$ of multiplicity r. Let a compact $K\subset\Gamma_{(0,0)}$ be given. Then there is a $\delta_0>0$ such that we have*

$$K\subset\Gamma_{(t,x)} \quad \text{for} \quad |t|<\delta_0,\ |x|<\delta_0.$$

We first explain the idea of the proof. Take the convex hull of K and denote it by $\hat{K}$ and show that $\hat{K}\subset\Gamma_{(t,x)}$. Suppose that $\hat{K}\not\subset\Gamma_{(t,x)}$. Take $(\hat{\tau},\hat{\xi})\in\hat{K}$ and $(\hat{\tau},\hat{\xi})\notin\Gamma_{(t,x)}$ and consider

$$P_{(t,x)}((1-s)\theta+s(\hat{\tau},\hat{\xi})) \tag{1.22}$$

which is positive at $s=0$. This can not be positive in $0<s\le 1$ by assumption, then there is a $\hat{s}$, $0<\hat{s}\le 1$ such that (1.22) becomes zero. We denote by $(\hat{\tau},\hat{\xi})$ the point $(1-\hat{s})\theta+\hat{s}(\hat{\tau},\hat{\xi})$ again. Recall that

$$\begin{aligned} P(t+\lambda\tau,x+\lambda\xi) &= \lambda^\nu\{P_{(t,x)}(\tau,\xi)+O(\lambda)\} \\ &= \sum_{l=0}^r\Big(\sum_{j+|\alpha|=l}\tfrac{1}{j!\alpha!}\partial_t^j\partial_\xi^\alpha P(t,x)\tau^j\xi^\alpha\Big)\lambda^l+O(\lambda^{r+1}) \\ &= \sum_{l=0}^r\lambda^l p_l(t,x;\tau,\xi)+O(\lambda^{r+1}). \end{aligned}$$

This shows that $p_l(t,x;\tau,\xi)=0$, $l=0,1,\ldots,\nu-1$ and $p_\nu(t,x;\tau,\xi)=P_{(t,x)}(\tau,\xi)$. Take $\xi=\hat{\xi}$ and formally we replace λ by $i\sigma$ in the above equality to get

$$\begin{gathered} P(t+i\sigma\tau,x+i\sigma\hat{\xi}) \\ =(i\sigma)^\nu\{P_{(t,x)}(\tau,\hat{\xi})+\sum_{l=\nu+1}^r(i\sigma)^{l-\nu}p_l(t,x;\tau,\hat{\xi})\}+O(|\sigma|^{r+1}). \end{gathered}$$

Since $P_{(t,x)}(\hat{\tau},\hat{\xi})=0$, there is $\tau(\sigma)$ with $\tau(0)=\hat{\tau}$ which annulates the right-hand side. On the other hand *if* $P(t+\lambda\tau, x+\lambda\xi)$ *is hyperbolic with respect to* $(\hat{\tau},\hat{\xi})$ one has

$$|P((t,x)+i\sigma(\hat{\tau},\hat{\xi}))|\geq c|\sigma|^r$$

with some $c>0$ which is a contradiction.
We go to the proof. Recall that we may assume that we are working with

$$P(t,x)=t^r+a_1(x)t^{r-1}+\cdots+a_r(x),\quad a_j(0)=0$$

with $a_j(x)\in C^{2r+3}(\{|x|<\delta\})$. Let us set

$$g(t,x,\tau,\xi,\lambda)=P(t+\lambda\tau,x+\lambda\xi)$$

which is defined in $|\lambda|<\bar{\lambda}$, $|t|<\bar{t}$, $|x|<\bar{x}$, $|\tau|,|\xi|<M$, where we note that we can take M as large as we please taking $\bar{\lambda}$ small. Let a compact $L\subset\Gamma_{(0,0)}$ be given. We take $\bar{\lambda}$, M so that L is contained in $\{(t,\xi)\mid |\tau|,|\xi|\leq M\}$. Note that

$$g(0,0,\tau,\xi,\lambda)=\lambda^r P_{(0,0)}(\tau,\xi)+O(\lambda^{r+1}),\ \lambda\to 0$$

and $P_{(0,0)}(\tau,\xi)\neq 0$ for $(\tau,\xi)\in L$. Then it follows

$$\partial_\lambda^j g(0,0,\tau,\xi,0)=0,\ 0\leq j\leq r-1,\ \ \partial_\lambda^r g(0,0,\tau,\xi,0)\neq 0$$

and hence one can apply Malgrange's preparation theorem. Therefore we have

$$g(t,x,\tau,\xi,\lambda)=c(t,x,\tau,\xi,\lambda)p(t,x,\tau,\xi,\lambda)$$

holds in $|t|\leq\delta_0$, $|x|\leq\delta_0$ and $(\tau,\xi)\in W$ where $L\subset W$ and

$$p(t,x,\tau,\xi,\lambda)=\lambda^r+a_1(t,x,\tau,\xi)\lambda^{r-1}+\cdots+a_r(t,x,\tau,\xi)$$

and $c(0,0,\tau,\xi,0)\neq 0$, $a_j(0,0,\tau,\xi)=0$, $(\tau,\xi)\in W$. Moreover c and a_j are holomorphic in (t,τ). We now prove

Lemma 1.10.1. *We have with some* $\delta_1>0$

$$p(t,x,\tau,\xi,\lambda)\neq 0\quad\text{if}\quad \operatorname{Im}t\leq 0,\ (\tau,\xi)\in L,\ \operatorname{Im}\lambda<0 \tag{1.23}$$

for $|t|\leq\delta_1$, $|x|\leq\delta_1$.

An immediate corollary is

Corollary 1.10.1. *We have*

$$|p(t,x,\tau,\xi,i\sigma)|\geq c|\sigma|^r$$

for $\sigma<0$, $|t|\leq\delta_0$, $|x|\leq\delta_0$, $(\tau,\xi)\in L$.

Proof. Since one can write

$$p(t,x,\tau,\xi,\lambda)=\prod_{j=1}^{r}(\lambda-\lambda_j(t,x,\tau,\xi))$$

with $\operatorname{Im}\lambda_j(t,x,\tau,\xi)\geq 0$ by Lemma 1.10.1, the assertion follows. □

Proof of Lemma 1.10.1 Let us introduce

$$Q(t,x,\tau,\xi,\lambda;z)=\sum_{j=0}^{r}\frac{1}{j!}\partial_\lambda^j c(t,x,\tau,\xi,\lambda)z^j,\quad z\in\mathbb{C}$$

and put

$$G(t,x,\tau,\xi,\lambda;i\sigma)=Q(t,x,\tau,\xi,\lambda;i\sigma)p(t,x,\tau,\xi,\lambda+i\sigma).$$

We first show

$$|p(t,x,1,0,\lambda+i\sigma)|\geq c|\sigma|^r,\ \operatorname{Im}t\leq 0,\ \sigma<0. \tag{1.24}$$

Note that

$$P(t+\lambda+i\sigma,x)=\sum_{j=0}^{r}\frac{1}{j!}\partial_\lambda^j P(t+\lambda,x)(i\sigma)^j+O(|\sigma|^{r+1})$$

and

$$\partial_\lambda^j P(t+\lambda,x)=\partial_\lambda^j[c(t,x,1,0,\lambda)p(t,x,1,0,\lambda)]$$
$$=(\frac{1}{i}\frac{\partial}{\partial\sigma})^j[Q(t,x,1,0,\lambda;i\sigma)p(t,x,1,0,\lambda+i\sigma)]|_{\sigma=0}$$

for $0\leq j\leq r$ and hence

$$P(t+\lambda+i\sigma,x)=G(t,x,1,0,\lambda;i\sigma)+O(|\sigma|^{r+1}). \tag{1.25}$$

On the other hand it is clear that

$$|P(t+\lambda+i\sigma,x)|\geq|\operatorname{Im}t+\sigma|^r$$

for $\operatorname{Im}t\leq 0$, $\sigma<0$. Then form (1.25) one gets $|G(t,x,1,0,\lambda;i\sigma)|\geq c|\sigma|^r$ with some $c>0$ and hence (1.24) since $Q(t,x,1,0,\lambda;0)=c(t,x,1,0,\lambda)$.
Now suppose that (1.23) were not true so that there is $(\hat\tau,\hat\xi)\in L$ such that $p(t,x,\hat\tau,\hat\xi,\lambda)=0$, $\operatorname{Im}t\leq 0$ has a root $\operatorname{Im}\lambda<0$. Moving t little bit, we may suppose that $p(t,x,\hat\tau,\hat\xi,\lambda)=0$, $\operatorname{Im}t<0$ has a root $\operatorname{Im}\lambda<0$. Let $(\tau(s),\xi(s))$ be a curve connecting θ and $(\hat\tau,\hat\xi)$. With $\Lambda(s)=\min_j\operatorname{Im}\lambda_j(s)$ where $\lambda_j(s)$ are the roots of $p(t,x,\tau(s),\xi(s),\lambda)=0$, it is clear that $\Lambda(s)$ is continuous and $\Lambda(0)\geq 0$ and $\Lambda(1)<0$ by hypothesis. Then there is a $\hat s$ such that

$$p(t,x,\tau(\hat s),\xi(\hat s),\hat\lambda)=0,\ \operatorname{Im}\hat\lambda=0.$$

Since $\operatorname{Im}t<0$ this contradicts the hyperbolicity of P. This ends the proof. □

We next prove

Lemma 1.10.2. *We have*

$$\sum_{l=0}^{r}(i\sigma)^l p_l(t,x,\tau,\xi) = G(t-\sigma\,\mathrm{Im}\,\tau, x, \mathrm{Re}\,\tau, \xi; i\sigma) + O(|\sigma|^{r+1}). \tag{1.26}$$

Proof. Recall that

$$\partial_\lambda^j G(t,x,\tau,\xi,0;0) = \sum_{k+|\alpha|=j}\frac{j!}{k!\alpha!}\partial_t^k\partial_x^\alpha P(t,x)\tau^k\xi^\alpha,\ 0\le j\le r.$$

This gives that

$$\sum_{\mu+\nu=l}\frac{1}{\mu!\nu!}\partial_t^\nu\partial_\lambda^\mu G(t,x,\tau,\xi,0;0)\zeta^\nu = \sum_{j+|\alpha|=l}\frac{1}{j!\alpha!}\partial_t^j\partial_x^\alpha P(t,x)(\tau+\zeta)^j\xi^\alpha.$$

The left-hand side is equal to

$$\frac{1}{l!}\left(\frac{1}{i}\frac{\partial}{\partial\sigma}\right)^l G(t+i\sigma\zeta, x,\tau,\xi,0;i\sigma)|_{\sigma=0}.$$

After multiplying $(i\sigma)^l$ to both sides we sum up over $l=0,1,\ldots,r$ to get

$$\sum_{l=0}^{r}(i\sigma)^l p_l(t,x,\tau+\zeta,\xi) = G(t+i\sigma\zeta,x,\tau,\xi;i\sigma) + O(|\sigma|^{r+1}).$$

Plugging $i\,\mathrm{Im}\,\tau$, $\mathrm{Re}\,\tau$ into ζ and τ respectively we get the desired assertion. □

Proof of Theorem 1.10.1. As noted before we can take $\tau(\sigma)$ so that the left-hand side of (1.26) is $O(|\sigma|^{r+1})$ with $\xi=\hat{\xi}$. On the other hand from Corollary 1.10.1 we see that the right-hand side is bounded from below by positive constant times $|\sigma|^r$ which is a contradiction. □

References

[1] M.D.Bronshtein: Smoothness of roots of polynomials depending on parameters. Sib. Math. Zh. **20** (1979) 493-501.
[2] S.Wakabayashi: Singularities of solutions of the Cauchy problem for symmetric hyperbolic systems. Comm. P.D. E. **9** (1984) 1147-1177.

1.11 Smoothness of the roots of hyperbolic polynomials

Let us consider a hyperbolic polynomial in t

$$P(t,x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x).$$

Assume that $P(t,0)=t^m$. For any y, $|y|=1$, we study $P(t,x+sy)$ which is defined in $|x|<\delta$, $|s|<\delta(x)$. From Corollary 1.7.2 one can write

$$P(t,x+sy)=\prod_{j-1}^{m}(t-t_j(s;x))$$

where $t_j(s;x)$ are differentiable at $s=0$.

Theorem 1.11.1. *There are $C>0$ and $\delta_1>0$ such that*

$$\left|\frac{d}{ds}t_j(0;x)\right|\leq C \quad \textit{for} \quad |x|<\delta_1.$$

Proof. Let $T_k(x)$ be different roots of $P(t,x)=0$, $k=1,\ldots,j(x)$. As we have seen that

$$\{\frac{d}{dt}t_j(0;x)\mid t_j(0,x)=T_k(x)\}=\{\text{roots of } P_{(T_k(x),x)}(t,y)=0\}. \tag{1.27}$$

This can be seen in another way: let us write

$$P(T_k(x)+st,x+sy)=s^{r_k(x)}(P_{(T_k(x),x)}(t,y)+O(s))$$

where r_k is the multiplicity of $T_k(x)$. The left-hand side is

$$\prod_{t_j(0,x)\neq T_k(x)}\{st-(t_j(s;x)-T_k(x))\}\prod_{t_j(0,x)=T_k(x)}\{st-(t_j(s;x)-T_k(x))\}.$$

Dividing s^{r_k} and letting $s\to 0$ we get

$$\prod_{t_j(0,x)\neq T_k(x)}(t_j(0,x)-T_k(x))\prod_{t_j(0,x)=T_k(x)}(t-\frac{d}{dt}t_j(0,x))$$

which is $P_{(T_k(x),x)}(t,y)$ and hence (1.27).
It is clear that $K=\{(t,x)\mid |t-1|\leq\epsilon,|x|\leq\epsilon\}\subset\Gamma_{(0,0)}$ taking $\epsilon>0$ small because $\theta\in\Gamma_{(0,0)}$. Since $T_k(x)$ is continuous in x we may suppose that if $|x|<\delta_1$ then $|T_k(x)|\leq\delta_0$. Then we have

$$K\subset\Gamma_{(T_k(x),x)}$$

by Theorem 1.10.1. Since $\Gamma_{(T_k(x),x)}$ is a cone we have $(t,y)\in\Gamma_{(T_k(x),x)}$ if $t>0$ and $|y/t|\leq\epsilon$, that is

$$t>0,\quad P_{(T_k(x),x)}(t,y)=0\Longrightarrow |y|>\epsilon t.$$

It $t<0$ and $P_{(T_k(x),x)}(t,y)=0$ then $P_{(T_k(x),x)}(-t,-y)=0$ and hence $|y|>\epsilon|t|$. This proves the assertion. □

References

[1] M.D.Bronshtein: Smoothness of roots of polynomials depending on parameters. Sib. Math. Zh. **20** (1979) 493-501.
[2] S.Wakabayashi: Singularities of solutions of the Cauchy problem for symmetric hyperbolic systems. Comm. P.D. E. **9** (1984) 1147-1177.

Chapter 2

Necessary conditions for well-posedness

2.1 Well-posedness of the Cauchy problem

Let $P(x, D)$ be a differential operator defined in a neighborhood Ω of the origin of $\mathbb{R}^{n+1}$ with coordinates $x = (x_0, x_1, \ldots, x_n) = (x_0, x')$:

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \tag{2.1}$$

where $D^\alpha = D_0^{\alpha_0} \cdots D_1^{\alpha_n}$, $D_j = -i\partial/\partial x_j$ and $a_\alpha(x) \in C^\infty(\Omega)$. We assume that hyperplanes $x_0 =$ const. are non characteristic for P. We may assume $a_{(m,0,\ldots,0)}(x) = 1$.

Definition 2.1.1. We say that the Cauchy problem for P is (C^∞) well posed near the origin if there are $\epsilon > 0$ and a neighborhood ω of the origin such that: for any $|\tau| \le \epsilon$ and for any $f(x) \in C_0^\infty(\omega)$ vanishing in $x_0 < \tau$ there is a unique solution $u(x) \in H^\infty(\omega)$ to $Pu = f$ in ω vanishing in $x_0 < \tau$, where $H^\infty(\omega) = \cap_{p=0}^\infty H^p(\omega)$ and $H^p(\omega)$ denotes the usual Sobolev space of order p.

Assume that $u \in H^\infty(\omega)$ vanishes in $x_0 < \tau$ with $|\tau| < \epsilon$. If $Pu = 0$ in $x_0 < t$ ($|t| < \epsilon$) then we conclude that $u = 0$ in $x_0 < t$. To see this, note that the equation $Pw = Pu$ has a solution $w \in H^\infty(\omega)$ vanishing in $x_0 < \tau$. Since $w - u = 0$ in $x_0 < \min(\tau, t)$, and $P(w - u) = 0$, by the uniqueness we get $w = u$ and hence $u = 0$ in $x_0 < t$.

Lemma 2.1.1. *Assume that the Cauchy problem for P is well posed near the origin. Then for any $U \Subset \omega$, the following classical Cauchy problem has a unique solution $u \in H^\infty$*

$$\begin{cases} Pu = f \quad in \quad \omega \cap \{x_0 > \tau\} \\ D_0^j u(\tau, x') = u_j(x'), \quad j = 0, 1, \ldots, m-1 \end{cases} \tag{2.2}$$

for any given $f(x) \in C_0^\infty(\omega)$ and $u_j(x') \in C_0^\infty(\omega \cap \{x_0 = \tau\})$.

Proof. Since $x_0 = \tau$ is non characteristic, we compute $u_j(x') = D_0^j u(\tau, x')$ for $j = m, m+1, \ldots$ from $u_j(x')$, $j = 0, \ldots, m-1$ and the equation $Pu = f$. By a Borel's lemma, we can take $\hat{u} \in C_0^\infty(\omega)$ so that $D_0^j \hat{u}(\tau, x') = u_j(x')$ for all $j \in \mathbb{N}$. Clearly we have $D_0^j(P\hat{u} - f) = 0$ on $\{x_0 = \tau\}$ for all $j \in \mathbb{N}$. The function g, defined by $g = P\hat{u} - f$ in $x_0 > \tau$ and zero in $x_0 < \tau$ is in $C_0^\infty(\omega)$. By assumption there is $v \in H^\infty(\omega)$ such that $Pv = g$ in ω and $v = 0$ in $x_0 < \tau$. This shows that

$$\begin{cases} P(\hat{u} - v) = f \quad \text{in} \quad \omega \cap \{x_0 > \tau\} \\ D_0^j(\hat{u} - v) = u_j(x') \quad \text{on} \quad \omega \cap \{x_0 = \tau\} \end{cases}$$

so that $\hat{u} - v \in H^\infty(\omega)$ is a desired solution to (2.2). □

Exercise: In Lemma 2.1.1, prove the uniqueness.

For the operator (2.1) we write

$$P(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha = P_m(x,\xi) + P_{m-1}(x,\xi) + \cdots$$

where $P_j(x,\xi)$ denotes the homogeneous part of degree j with respect to ξ. Recall that $P_m(x,\xi)$ is called the principal symbol.

Definition 2.1.2. Let P be as above. We say that P or P_m is strongly hyperbolic near the origin if for any differential operator of order less than m defined in Ω, the Cauchy problem for $P + Q$ is well posed near the origin.

Remark 2.1.1. In this definition, the open set ω may depend on Q. In conclusion, at least in the scalar case, ω is independent of Q.

For differential operators with constant coefficients we can give a characterization for which operators the Cauchy problem is well posed. Let

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha \tag{2.3}$$

where $a_{(m,0,\ldots,0)} = 1$.

Theorem 2.1.1. (Gårding) *Let P be as in (2.4). Then the Cauchy problem for P is well posed iff $P(\xi)$ verifies with some $T > 0$*

$$\xi \in \mathbb{R}^{n+1},\ \tau \in \mathbb{C},\ P(\xi + \tau\theta) = 0 \Longrightarrow |\operatorname{Im}\tau| < T \tag{2.4}$$

where $\theta = (1, 0, \ldots, 0)$.

Theorem 2.1.2. *Let P be a differential operator with constant coefficients. Then P is strongly hyperbolic iff $P_m(\xi)$ is a strictly hyperbolic polynomial with respect to ξ_0.*

We turn to differential operators with variable coefficients.

Theorem 2.1.3. (Lax-Mizohata) *Assume that the Cauchy problem for P is well posed near the origin. Then there is a neighborhood U of the origin such that for any $x \in U$, the polynomial $P_m(x,\xi)$ is hyperbolic polynomial with respect to ξ_0.*

Definition 2.1.3. A point $(\bar{x},\bar{\xi})$ is called a characteristic (point) of order r if

$$\partial_\xi^\alpha \partial_x^\beta P_m(\bar{x},\bar{\xi}) = 0,\ |\alpha+\beta| < r,\ \partial_\xi^\alpha \partial_x^\beta P_m(\bar{x},\bar{\xi}) \neq 0 \text{ some } |\alpha+\beta| = r.$$

Lemma 2.1.2. *Assume that $P_m(x,\xi)$ is a hyperbolic polynomial with respect to ξ_0 and*

$$\left(\frac{\partial}{\partial\xi_0}\right)^j P_m(\bar{x},\bar{\xi}) = 0,\ 0 \le j \le r-1, \quad \left(\frac{\partial}{\partial\xi_0}\right)^r P_m(\bar{x},\bar{\xi}) \neq 0.$$

Then $(\bar{x},\bar{\xi})$ is a characteristic of order r.

Proof. By Corollary 1.6.3.5. □

Theorem 2.1.4. (Ivrii-Petkov) *Assume that the Cauchy problem for P is well posed near the origin and $(0,\bar{\xi})$ is a characteristic of order r. Then we have*

$$\partial_x^\beta \partial_\xi^\alpha P_{m-j}(0,\bar{\xi}) = 0, \quad |\alpha+\beta| < r-2j.$$

Corollary 2.1.1. *Assume that P is strongly hyperbolic near the origin. Then every multiple characteristic is at most double.*

Let $(0,\bar{\xi})$ be a characteristic of order r. Then from Theorem 2.1.4 one can define

$$P_{m-j,(0,\bar{\xi})}(x,\xi) = \sum_{|\alpha+\beta|=r-2j} \frac{1}{\alpha!\beta!} \partial_x^\beta \partial_\xi^\alpha P_{m-j}(0,\bar{\xi}) x^\beta \xi^\alpha$$

Then we define $P_{(0,\bar{\xi})}(x,\xi)$ by

$$P_{(0,\bar{\xi})}(x,\xi) = \sum_{j=0} P_{m-j,(0,\bar{\xi})}(x,\xi).$$

Note that $P_{(0,\bar{\xi})}(x,\xi)$ is not homogeneous.

Theorem 2.1.5. *Assume that the Cauchy problem for P is well posed near the origin and $(0,\bar{\xi})$ is a characteristic of order r. Then $P_{(0,\bar{\xi})}(x,\xi)$ is a hyperbolic polynomial in the sense of Gårding, that is verifies* (2.4).

References

[1] L.Gårding: Linear hyperbolic partial differential equations with constant coefficients. Acta Math. **85** (1951) 1-62.
[2] L.Hörmander: The Analysis of Linear Partial Differential Operators, II. Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[3] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[4] P.D.Lax: Asymptotic solutions of oscillatory initial value problems. Duke Math. J. **24** (1957) 627-646.
[5] S.Mizohata: Some remarks on the Cauchy problem. J.Math. Kyoto Univ. **1** (1961) 109-127.
[6] T.Nishitani: Hyperbolicity of localizations. Ark. Mat. **31** (1993) 377-393.

2.2 Well-posedness of the Cauchy problem for first order systems

In this section, we study a first order system $P(x,D) = L(x,D) + B(x)$ where

$$L(x,D) = D_0 I + \sum_{j=1}^{n} A_j(x) D_j = D_0 I + A(x, D')$$

and $A_j(x)$ are $m \times m$ matrix valued C^∞ functions defined in a neighborhood Ω of the origin of $\mathbb{R}^{n+1}$. We first study P with constant coefficients.

Theorem 2.2.1. *Assume that P is of constant coefficients. Then the Cauchy problem for P is well posed iff the Cauchy problem for $\det P(D)$ is well posed. Here $\det P(\xi)$ denotes the determinant of $P(\xi)$.*

Definition 2.2.1. We say that $L(x,\xi)$ is uniformly diagonalizable if there is $C > 0$ such that for any (x,ξ), x near the origin, $\xi \in \mathbb{R}^{n+1}$ one can find a matrix $T(x,\xi)$ with $\|T(x,\xi)^{-1}\|$, $\|T(x,\xi)\| \le C$ and

$$T(x,\xi)^{-1} L(x,\xi) T(x,\xi)$$

is diagonal.

Theorem 2.2.2. (Kasahara-Yamaguti) *Let P be of constant coefficients. Then P is strongly hyperbolic iff P is uniformly diagonalizable.*

We turn to P with variable coefficients. Let us denote

$$h(x,\xi) = \det L(x,\xi).$$

Theorem 2.2.3. (Lax-Mizohata) *Assume that the Cauchy problem for $P(x,D)$ is well posed. Then every eigenvalue of $A(x,\xi')$ is real for any x near the origin and any $\xi' \in \mathbb{R}^n$.*

Theorem 2.2.4. *Assume that $A_j(x)$ are real analytic. Assume that P is strongly hyperbolic near the origin and let $(0,\bar{\xi})$ be a characteristic of order r of $h(x,\xi)$. Then every $(m-1)$-th minor of $L(x,\xi)$ vanishes of order $r-2$ at $(0,\bar{\xi})$.*

Corollary 2.2.1. *Assume that $A_j(x)$ are real analytic and L is strongly hyperbolic near the origin. Let λ be a multiple eigenvalue of $A(0,\bar{\xi}')$. Then the size of Jordan blocks corresponding to the eigenvalue λ is at most two.*

References

[1] L.Gårding: Linear hyperbolic partial differential equations with constant coefficients. Acta Math. **85** (1951) 1-62.
[2] K.Kasahara, M.Yamaguti: Strongly hyperbolic systems of linear partial differential equations with constant coefficients. Mem. Coll. Sci. Univ. Kyoto Ser A. **33** (1960) 1-23.
[3] T.Nishitani: Necessary conditions for strong hyperbolicity of first order systems. J. Analyse Math. **61** (1993) 181-229.

2.3 Implications of well-posedness

We shall prove the next lemma.

Lemma 2.3.1. *Assume that the Cauchy problem for P is well posed near the origin. Then there are open neighborhood ω and $\epsilon > 0$ such that: for any compact $K \Subset \omega$ and $p \in \mathbb{N}$ there are $C > 0$, $q \in \mathbb{N}$ such that*

$$\|u\|_{H^p(K^t)} \le C\|Pu\|_{H^q(K^t)}$$

for any $u \in C_0^\infty(K_{-\epsilon})$ and $|t| < \epsilon$ where $K^t = \{x \in K \mid x_0 \le t\}$ and similarly $K_t = \{x \in K \mid x_0 \ge t\}$

Proof. Take an open set V so that $K \Subset V \Subset \omega$. Let us define F_M, $M = 1, 2, \ldots$ by

$$F_M = \{f \in C_0^\infty(\overline{V_{-\epsilon}}) \mid \exists u \in H^p(\omega) \text{ such that}$$
$$Pu = f \text{ in } \omega, \|u\|_{H^p(\omega)} \le M, u = 0 \text{ in } x_0 \le -\epsilon\}.$$

By the well-posedness assumption, it is clear that

$$\bigcup_{M=1}^{\infty} F_M = C_0^\infty(\overline{V_{-\epsilon}}).$$

It is also clear that F_M is symmetric and convex. Let $F_M \ni f_j \to f$ in $C_0^\infty(\overline{V_{-\epsilon}})$. Then there exist u_j such that $Pu_j = f_j$ and $\|u_j\|_{H^p(\omega)} \le M$, taking a subsequence, we may suppose that

$$u_j \to u \text{ in } H^{p-1}_{loc}(\omega) \text{ and } u_j \to u \text{ weak in } H^p(\omega),\ u \in H^p(\omega).$$

It is clear that $Pu = f$ in ω and $u = 0$ in $x_0 \le -\epsilon$. This shows that F_M is closed. Then from the Baire's theorem some F_M contains a neighborhood of 0 in $C_0^\infty(\overline{V_{-\epsilon}})$. That is, there is $q \ge 0$ such that

$$f \in C_0^\infty(\overline{V_{-\epsilon}}),\ \ \|f\|_{H^q(V)} \le \delta \Longrightarrow f \in F_M.$$

For any $f \in C_0^\infty(\overline{V_{-\epsilon}})$, taking $\delta f/\|f\|_{H^q(V)}$, we get

$$\|u\|_{H^p(\omega)} \le M\delta^{-1}\|f\|_{H^q(V)}. \tag{2.5}$$

We summarize: for any $f \in C_0^\infty(\overline{V_{-\epsilon}})$, the solution u to $Pu = f$ in ω vanishing in $x_0 \le -\epsilon$ satisfies (2.5).
Let $u \in C_0^\infty(K_{-\epsilon})$. Take $\chi \in C_0^\infty(V)$ so that $\chi = 1$ on K. Take $g \in \mathcal{S}(\mathbb{R}^{n+1})$ so that $Pu = g$ in $x_0 < t$. Then the solution v, vanishing in $x_0 \le -\epsilon$, to $Pv = \chi g$ coincides with u in $x_0 < t$ as we remarked after Definition 2.1.1. Hence

$$\begin{aligned}\|v\|_{H^p(V^t)} = \|u\|_{H^p(V^t)} &\le C_0\|\chi g\|_{H^q(V)}\\ &\le C_0'\|g\|_{H^q(V)} \le C_0''\|g\|_{H^q(\mathbb{R}^{n+1})}.\end{aligned}$$

Since this holds for any $g \in \mathcal{S}(\mathbb{R}^{n+1})$ provided $g = Pu$ in $x_0 < t$, this shows that

$$\|u\|_{H^p(V^t)} \le C_0''\|Pu\|_{H^q(\{x_0<t\})} = c_0''\|Pu\|_{H^q(K^t)}$$

and hence the result. □

Corollary 2.3.1. *Assume that the Cauchy problem for P is well-posed near the origin. Then there are a neighborhood ω of the origin and $\epsilon > 0$ such that for any compact $K \Subset \omega$ one can find $C > 0$ and $p \in \mathbb{N}$ such that*

$$|u|_{C^0(K^t)} \le C|Pu|_{C^p(K^t)}$$

for any $u \in C_0^\infty(K_{-\epsilon})$, $|t| < \epsilon$ where $|u|_{C^p(K)} = \sup_{x\in K, |\alpha|\le p} |\partial_x^\alpha u(x)|$.

Proof. By a Sobolev embedding theorem. □

Let $P(x,\xi)$ be the full symbol of $P(x,D)$. Let us set

$$P_\lambda(x,\xi) = P(y(\lambda) + \lambda^{-\sigma}x, \lambda^\kappa\eta(\lambda) + \lambda^\sigma\xi)$$

where $\sigma = (\sigma_0,\dots,\sigma_n)$, $\lambda^{-\sigma}x = (\lambda^{-\sigma_0}x_0,\dots,\lambda^{-\sigma_n}x_n)$ and $y(\lambda)$, $\eta(\lambda)$ are $\mathbb{R}^{n+1}$ valued continuous functions defined near $\lambda = \infty$.

Lemma 2.3.2. *Assume that $0 \in \Omega$ and $y(\infty) = 0$ and the Cauchy problem for P is well posed near the origin. Then for every compact $W \subset \mathbb{R}^{n+1}$ and every positive $T > 0$ there are $C > 0$, $\bar\lambda > 0$ and $p \in \mathbb{N}$ such that*

$$|u|_{C^0(W^t)} \le C\lambda^{(\kappa+\bar\sigma)p}|P_\lambda u|_{C^p(W^t)}$$

for any $u \in C_0^\infty(W)$, $\lambda \ge \bar\lambda$, $|t| < T$ where $\bar\sigma = \max_j \sigma_j$.

Proof. Let $u \in C_0^\infty(K)$ and put $v(x) = e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle}u(x) \in C_0^\infty(K)$. Then from Corollary 2.3.1 we get

$$\begin{aligned}|v|_{C^0(K^t)} = |u|_{C^0(K^t)} &\le C|Pv|_{C^p(K^t)}\\ = C|e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle}\tilde P u|_{C^p(K^t)} &\le C'\lambda^{\kappa p}|\tilde P u|_{C^p(K^t)}\end{aligned} \tag{2.6}$$

where $\tilde{P}(x,D) = e^{-i\lambda^\kappa\langle\eta(\lambda),x\rangle}P(x,D)e^{i\lambda^\kappa\langle\eta(\lambda),x\rangle} = P(x,\lambda^\kappa\eta(\lambda)+D)$. Take a compact K so that K contains the origin and $\inf\{x_0 \mid x \in K\} > -\epsilon$. Let $W \subset \mathbb{R}^{n+1}$ be a given compact. Then there is λ_1 such that for any $u \in C_0^\infty(W)$ we have

$$u(\lambda^\sigma(x-y(\lambda))) \in C_0^\infty(K) \quad \text{if} \quad \lambda \geq \lambda_1$$

remarking $y(\infty)=0$. Set $v(x) = u(\lambda^\sigma(x-y(\lambda)))$ and $t(\lambda) = \lambda^{-\sigma_0}s + y_0(\lambda)$. Take λ_2 so that $\lambda \geq \lambda_2$ and $|s| < T$ implies $|t(\lambda)| < \epsilon$. Then from (2.6) it follows that

$$|v|_{C^0(K^{t(\lambda)})} \leq C\lambda^{\kappa p}|\tilde{P}v|_{C^0(K^{t(\lambda)})}.$$

This shows that, by change of coordinates $z = \lambda^\sigma(x-y(\lambda))$,

$$|u|_{C^0(W^s)} \leq C'\lambda^{\kappa p+\bar{\sigma}p}|P_\lambda u|_{C^p(W^s)}$$

and hence the result. □

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[3] T.Nishitani: Hyperbolicity of localizations. Ark. Mat. **31** (1993) 377-393.

2.4 Proof of Theorem 2.1.3

Let us write $p(x,\xi) = P_m(x,\xi)$. Assume that with some $\bar{\xi}' \in \mathbb{R}^n$

$$p(0,\xi_0,\bar{\xi}') = 0$$

has a root $\bar{\xi}_0$ with $\mathrm{Im}\,\bar{\xi}_0 \neq 0$. Note that $\bar{\xi}' \neq 0$. Since $p(x,\xi)$ is homogeneous of degree m, we may assume that

$$\mathrm{Im}\,\bar{\xi}_0 = -1$$

taking $a\bar{\xi}'$, $a \in \mathbb{R}$ in place of $\bar{\xi}'$. Moreover we may suppose $\bar{\xi}'_\mu \neq 0$. Then with a new system of local coordinates

$$y_0 = x_0,\ y_1 = \langle x',\bar{\xi}'\rangle + x_0\,\mathrm{Re}\,\bar{\xi}_0,\ y_j = x_j (j \neq 1,\mu),\ y_\mu = x_1$$

we can assume that

$$\bar{\xi}' = (1,0,\ldots,0),\ \bar{\xi}_0 = -i$$

because the corresponding contragredient change of the dual variables is:

$$\xi_0 = \eta_0 + (\mathrm{Re}\,\bar{\xi}_0)\eta_1,\ \xi_1 = \bar{\xi}_1\eta_1 + \eta_\mu,\ \xi_j = \bar{\xi}_j\eta_1 + \eta_j\ (j \neq \mu),\ \xi_\mu = \bar{\xi}_\mu.$$

Hence one can write

$$p(0,\xi_0,\xi_1,0,\dots,0)=(\xi_0+i\xi_1)^r Q(\xi_0,\xi_1),\ Q(-i,1)\neq 0. \tag{2.7}$$

Take $\nu(>r)$ sufficiently large and take a new system of coordinates depending on a large parameter λ

$$y_j=\lambda^{2\nu}x_j\ (j=0,1),\ \ y_j=\lambda^{\nu}x_j\ (j\geq 2)$$

and consider

$$P_\lambda(y,D)=P(\lambda^{-\tilde\nu}y,\lambda^{\tilde\nu}D)$$

where $\tilde\nu=\nu(2,2,1,\dots,1)$. It is clear that

$$\lambda^{-2m\nu}(P_\lambda(y,D)-p(0,\lambda^{\tilde\nu}D))=O(\lambda^{-\nu})$$

where $O(\lambda^{\nu})$ denotes a differential operator of order m such that the coefficients are $O(\lambda^{-\nu})$. Note that

$$p(0,\lambda^{\tilde\nu}D)=\lambda^{2m\nu}(D_0+iD_1)^rQ(D_0,D_1)+O(\lambda^{2m\nu-\nu})$$

because $p(0,\xi)-p(0,\xi_0,\xi_1,0,\dots,0)=\sum_{j=2}^n\xi_jp_j(\xi)$ where $p_j(\xi)$ are homogeneous of degree $m-1$. This shows that

$$\lambda^{-2m\nu}P_\lambda(y,D)=(D_0+iD_1)^rQ(D_0,D_1)+O(\lambda^{-\nu}). \tag{2.8}$$

Here we remark that

$$D_0+iD_1=\frac{1}{i}\left(\frac{\partial}{\partial y_0}+i\frac{\partial}{\partial y_1}\right)$$

is the Cauchy-Riemann operator in $z=y_0+iy_1$ plane. Take

$$\psi=-i(y_0+iy_1)-i(y_0+iy_1)^2\ \ (=-iz-iz^2)$$

which is holomorphic in z verifying

$$\psi(0)=0,\ \operatorname{Im}\psi=-y_0-y_0^2+y_1^2\geq 0$$

for $y_0\leq 0$ and $|z|$ small. We now set

$$\phi(y)=\psi(y_0+iy_1)+i(y_2^2+\cdots+y_n^2)$$

then we have $(D_0+iD_1)\phi=0$ and

$$\operatorname{Im}\phi(y)\geq c|y|^2\quad\text{when}\quad y_0\leq 0,\ \ |y|\leq\delta$$

with some $c>0$ and $\delta>0$.

In what follows we look for an asymptotic null solution to $P_\lambda u\sim 0$ in the form

$$u=\sum_{j=0}^{\infty}e^{i\lambda\phi(y)}v_j(y)\lambda^{-j}. \tag{2.9}$$

Notice that

$$\lambda^{-2m\nu}e^{-i\lambda\phi}P_\lambda e^{i\lambda\phi}v = \lambda^{m-r}a(D_0+iD_1)^r v + O(\lambda^{m-r-1})v$$

where $a = Q(\partial\phi/\partial y_0, \partial\phi/\partial y_1)$ because

$$e^{-i\lambda\phi}(D_0+iD_1)e^{i\lambda\phi} = D_0+iD_1,$$
$$e^{-i\lambda\phi}Q(D_0,D_1)e^{i\lambda\phi} = \lambda^{m-r}Q\left(\frac{\partial\phi}{\partial y_0},\frac{\partial\phi}{\partial y_1}\right) + O(\lambda^{m-r-1}).$$

Therefore we get

$$\lambda^{-2m\nu}e^{-i\lambda\phi}P_\lambda e^{i\lambda\phi}\sum_{j=0} v_j\lambda^{-j}$$
$$= \lambda^{m-r}\sum_{j=0}\{a(D_0+D_1)^r v_j - F_j(v_0,\ldots,v_{j-1})\}\lambda^{-j}$$

where $F_0 = 0$. Thus we are led to

$$a(D_0+iD_1)^r v_0 = 0,\ a(D_0+iD_1)^r v_j = F_j(v_0,\ldots,v_{j-1}),\ j\geq 1. \tag{2.10}$$

We take $v_0 \in C_0^\infty(\mathbb{R}^{n+1})$ so that $v_0 = 1$ near the origin. We note that $a \neq 0$ near the origin for $Q(\partial\phi/\partial y_0, \partial\phi/\partial y_1) = (-i,1)$ at the origin. Then to solve the second equation in (2.10) we remark that for $\phi(z) \in C_0^\infty(\mathbb{C})$ the function

$$\Phi(z) = \frac{1}{2\pi i}\int \frac{\phi(\zeta)}{\zeta - z}d\zeta\wedge\bar{\zeta}$$

solves $\partial\Phi(z)/\partial\bar{z} = \phi(z)$. Then we can find smooth $v_j(y)$ which verifies (2.10) near the origin. Here we remark that it is enough to solve (2.10) just near the origin. For any N, set

$$u_\lambda^{(N)} = \sum_{j=0}^{L(N)} e^{i\lambda\phi}v_j(y)\lambda^{-j}$$

where $L(N)$ is chosen suitably so that

$$e^{-i\lambda\phi}P_\lambda u_\lambda^{(N)} = O(\lambda^{-N}). \tag{2.11}$$

Take $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ which is 1 near the origin so that (2.11) holds on the support of χ. Then we see that

$$P_\lambda\chi u_\lambda^{(N)} = \chi P_\lambda u_\lambda^{(N)} + [P_\lambda,\chi]u_\lambda^{(N)}.$$

Since $-\operatorname{Im}\phi \leq -\epsilon$ if $[P_\lambda,\chi]u_\lambda^{(N)} \neq 0$, $y_0 \leq 0$, this proves that with $K = \operatorname{supp}\chi$

$$|P_\lambda\chi u_\lambda^{(N)}|_{C^p(K^0)} \leq C_N\lambda^{-N+p}.$$

On the other hand we have

$$|\chi u_\lambda^{(N)}|_{C^0(K^0)} \geq c > 0$$

with some $c > 0$ because $\phi(0) = 0$. Taking N large enough we get a contradiction to the inequality in Lemma 2.3.2. □

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.

2.5 Proof of Theorem 2.1.4

Recall that we are interested in a differential operator of order m

$$P(x,D) = P_m(x,D) + P_{m-1}(x,D) + \cdots .$$

Assume that $(\bar{x}, \bar{\xi})$ is a characteristic of order r. Taking a new system of local coordinates, we may assume, without restrictions, that $(\bar{x}, \bar{\xi}) = (0, e_n)$ where $e_n = (0, \dots, 0, 1)$.

EXERCISE: Give this new system of local coordinates.

Let us denote $z = (0, e_n)$ and put

$$P_{m-j}(z + \mu(x,\xi)) = \mu^{s_j}\{P_{m-j,z}(x,\xi) + O(\mu)\}$$

where $P_{m-j,z}(x,\xi)$ is the localization of P_{m-j} at z (Definition 1.6.1). Assume that Theorem 2.1.4 does not hold, that is there is $j \geq 1$ such that $s_j < r - 2j$. Let us define

$$\theta_0 = \min_{j, s_j < r-2j} \frac{j}{r - s_j}$$

then $\theta_0 < 1/2$ by assumption. Put

$$\hat{P}(x,\xi) = \sum_{m - r\theta_0 = m - j - s_j\theta_0, j \geq 0} P_{m-j,z}(x,\xi).$$

Lemma 2.5.1. *There is $(\bar{x}, \bar{\xi}')$ near $(0,0)$ such that $\hat{P}(\bar{x}, \xi_0, \bar{\xi}') = 0$ has a non real root.*

Proof. Let us write

$$q(x,\xi) = \sum_{m - r\theta_0 = m - j - s_j\theta_0, j \geq 1} P_{m-j,z}(x,\xi).$$

We first assume $q(0, \xi_0, 0) \not\equiv 0$. Then one can write

$$\hat{P}(0, \xi_0, 0) = \xi_0^r + \sum_{r\theta_0 = j + s_j\theta_0} a_j \xi_0^{s_j}.$$

Put $\theta_0 = p/q$ where p and q are relatively prime. Since $\theta_0 < 1/2$ we get $q \geq 3$. Then $(r - s_j)\theta_0 = j$ implies $r - s_j = nq$ with some $n \in \mathbb{N}$. Hence we can express

$$\hat{P}(0, \xi_0, 0) = \xi_0^r \left(1 + \sum \tilde{a}_l (\frac{1}{\xi_0})^{lq}\right).$$

Now to prove that $\hat{P}(0,\xi_0,0)=0$ has a non real root it is enough to repeat the same arguments as in the proof of Lemma 1.6.1.
We now assume that $q(0,\xi_0,0)\equiv 0$. It is clear that there is $(\bar{x},\bar{\xi}')$ such that $q(s\bar{x},\xi_0,s\bar{\xi}')$ is not identically zero in (ξ_0,s) by the assumption. Consider

$$f(\xi_0,s)=\hat{P}(s\bar{x},\xi_0,s\bar{\xi}')=\xi_0^r+\sum_{j\geq 1}f_j(s)\xi_0^{r-j}$$

which verifies $f(\xi_0,0)=\xi_0^r$. From Lemma 1.6.1 it follows that $f_j(s)=O(s^j)$ if $f(\xi_0,s)=0$ has only real roots for small s. But this is not the case by the assumption again. This ends the proof. □

Exercise: Show that there is a non real root ξ_0 with $\operatorname{Im}\xi_0<0$.

Put $\sigma_0=1-2\theta_0$ and study

$$\begin{aligned}
P(\lambda^{-\theta_0}x,\lambda e_n+\lambda^{\theta_0}D)&=\sum_{j=0}^m P_{m-j}(\lambda^{-\theta_0}x,\lambda e_n+\lambda^{\theta_0}D)\\
&=\sum_{j=0}^m \lambda^{m-j}P_{m-j}(\lambda^{-\theta_0}x,e_n+\lambda^{-\theta_0-\sigma_0}D)\\
&=\sum_{j=0}^m\ \sum_{k\geq 0,m-j-s_j\theta_0-k\theta_0>-M}\lambda^{m-j-s_j\theta_0-k\theta_0}\\
\times\sum_{|\alpha+\beta|=s_j+k}&\frac{1}{\alpha!\beta!}P_{(\beta)}^{(\alpha)}(0,e_n)x^\beta(\lambda^{-\sigma_0}D)^\alpha+O(\lambda^{-M})
\end{aligned}$$

where M is a sufficiently large integer and by $O(\lambda^{-M})$, as before, we denote a differential operator whose coefficients are bounded by λ^{-M} on any preassigned open set U in $\mathbb{R}^{n+1}$ and $P_{(\beta)}^{(\alpha)}(x,\xi)=\partial_x^\beta\partial_\xi^\alpha P(x,\xi)$. Let us set

$$\begin{aligned}
G^{(0)}(x,\xi;\lambda)=\sum_{j=0}^m\ \sum_{m-j-s_j\theta_0-k\theta_0>-M}&\lambda^{-j+(r-s_j)\theta_0-k\theta_0}\\
\times\sum_{|\alpha+\beta|=s_j+k}&\frac{1}{\alpha!\beta!}P_{(\beta)}^{(\alpha)}(0,e_n)x^\beta\xi^\alpha
\end{aligned}$$

so that

$$P_\lambda(x,D)=\lambda^{m-r\theta_0}G^{(0)}(x,\lambda^{-\sigma_0}D;\lambda)+O(\lambda^{-M}).$$

It is useful to rewrite $G^{(0)}(x,\xi;\lambda)$ in the following way

$$G^{(0)}(x,\xi;\lambda)=\sum_{j\geq 0}\lambda^{\delta_j(G^{(0)})}G_j^{(0)}(x,\xi)$$

where $0=\delta_0(G^{(0)})<\delta_1(G^{(0)})\cdots$. It is clear that

$$G_0^{(0)}(x,\xi)=\sum_{j-(r-s_j)\theta_0=0}P_{m-j,z}(x,\xi).\tag{2.12}$$

Definition 2.5.1. We say that a differential operator $P(x, D; \lambda)$ with a parameter λ is in $\mathcal{R}(U)$ if there are $\kappa \in \mathbb{Q}_+$ and differential operators $P_j(x, D)$ with coefficients in $C^\infty(U)$ such that

$$P(x, D; \lambda) = \sum_{j=0} \lambda^{-\kappa j} P_j(x, D)$$

where it is understood that the sum is finite.

Lemma 2.5.2. *Let $G(x, D)$ be a differential operator with coefficients in $C^\infty(U)$ and let σ, $\theta \in \mathbb{Q}_+$ be such that $\sigma \geq \theta > 0$ and let $\phi \in C^\infty(U)$. Then*
(i) *we have*

$$e^{-i\lambda^\theta \phi} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi} = G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D)) + \lambda^{-\theta} r(x, \lambda^{-\theta} D; \lambda)$$

with $r(x, \xi; \lambda) \in \mathcal{R}(U)$.
(ii) *If $G(x, \xi) = O(|\xi|^q)$ as $\xi \to 0$ uniformly with respect to $x \in U$, then*

$$\begin{aligned} e^{-i\lambda^\theta \phi} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi} &= G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D)) \\ &\quad + \lambda^{-(\sigma-\theta)q-\theta} r(x, \lambda^{-\theta} D; \lambda) \end{aligned}$$

with $r(x, \xi; \lambda) \in \mathcal{R}(U)$.

Remark 2.5.1. It is important to remark that in the notation above the quantity $G(x, \lambda^{-(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D))$ does not contain the terms in which the derivatives land on $\phi_x(x)$, as will be clear from the proof, these terms are pushed into the "error" terms r and thus $G(x, \lambda^{(\sigma-\theta)}(\phi_x + \lambda^{-\theta} D))$ is to be thought as a commutative expression.

Proof. Denote by $\psi(x, y) = \phi(x) - \phi(y) - \langle y - x, \phi_x(x) \rangle$. Then if $u(x)$ is smooth

$$\begin{aligned} & e^{-i\lambda^\theta \phi(x)} G(x, \lambda^{-\sigma} D) e^{i\lambda^\theta \phi(x)} u(x) \\ = \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \phi_x(x)) (\lambda^{-\sigma} D_y)^\alpha & \left[e^{\lambda^\theta \psi(x,y)} u(y) \right]_{y=x} \\ = \sum_{\alpha \geq 0} \frac{1}{\alpha!} & G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \phi_x(x)) (\lambda^{-\sigma} D_x)^\alpha u(x) \\ & + \sum_{\alpha \geq 0} \frac{1}{\alpha!} G^{(\alpha)}(x, \lambda^{-\sigma+\theta} \phi_x(x)) \\ \times \sum_{2 \leq |\beta| \leq |\alpha|} \binom{\alpha}{\beta} & \left[(\lambda^{-\sigma} D_y)^\beta e^{i\lambda^\theta \psi(x,y)} (\lambda^{-\sigma} D_y)^{\alpha-\beta} u(y) \right]_{y=x}. \end{aligned}$$

The first term is what has been called $G(x, \lambda^{-(\sigma-\theta)}(\phi_x(x) + \lambda^{-\sigma} D))$. Let us take a closer look at the second term. Due to the vanishing of $\psi(x, y)|_{y=x}$ and $\nabla \psi(x, y)|_{y=x}$ the quantity $D^\beta e^{i\lambda^\theta \psi(x,y)}|_{y=x}$ is a polynomial in the variable λ^θ

of degree less than or equal to $[|\beta|/2]$. Factoring out $\lambda^{\theta|\beta|/2}$ we obtain a polynomial of the same degree in the variable λ^θ. Thus the second sum above can be written as

$$\begin{aligned}
&\sum_{\alpha\ge 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}\\
&\times\sum_{2\le|\beta|\le|\alpha|}\lambda^{-\theta|\alpha|+\theta|\beta|/2}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})D^{\alpha-\beta}u(x)\\
&=\sum_{\alpha\ge 0}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))\lambda^{-(\sigma-\theta)|\alpha|}\\
&\times\sum_{\nu=1}^{[|\alpha|/2]}\lambda^{-\theta\nu}\sum_{|\beta|=2\nu}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}u(x).
\end{aligned}$$

With

$$\begin{aligned}
b_{k\nu}(x,D;\lambda)=\lambda^{-(\sigma-\theta)k}&\sum_{|\alpha|=k}\frac{1}{\alpha!}G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x(x))\\
&\times\sum_{|\beta|=2\nu}P_{\alpha,\beta,\phi}(x;\lambda^{-\theta})(\lambda^{-\theta}D)^{\alpha-\beta}
\end{aligned}$$

this can be rewritten as

$$\lambda^{-\theta}r(x,\lambda^{-\theta}D;\lambda),\quad r(x,D;\lambda)=\sum_{k=0}\sum_{\nu=1}^{[k/2]}b_{k\nu}(x,D;\lambda).$$

It is clear that $r(x,\xi;\lambda)\in\mathcal{R}(U)$.

We turn to the second assertion. It is obvious that nothing is changed in the first term, so that all we have to do is just look at the second term. If $k\ge q$ then

$$\lambda^{-(\sigma-\theta)k}\le\lambda^{-(\sigma-\theta)q}.$$

If $k<q$ then our assumption implies that

$$G^{(\alpha)}(x,\xi)=O(|\xi|^{q-k})$$

and hence $G^{(\alpha)}(x,\lambda^{-(\sigma-\theta)}\phi_x)=O(\lambda^{-(\sigma-\theta)(q-k)})$. This proves the assertion. □

From the assumption we may start off assuming that there is an analytic function $\tau_0(x,\xi')$ with $\operatorname{Im}\tau_0\ne 0$ such that

$$\det G_0^{(0)}(x,\xi)=(\xi_0-\tau_0(x,\xi'))^{q_0}\Delta_0(x,\xi),\ \Delta_0(x,\tau_0(x,\xi'),\xi')\ne 0$$

in some open set $U\times V$ in $\mathbb{R}^{n+1}\times\mathbb{R}^n$. In the sequel, U and V stands for an open set in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ respectively which may differ from line to line but *the subsequent one is contained in the preceding one.* Denote by $\phi^{(0)}(x)$ a real analytic function in U such that

$$\partial_{x_0}\phi^{(0)}(x) = \tau_0(x, \partial_{x'}\phi^{(0)}(x)).$$

Then we have

$$\begin{aligned} &e^{-i\lambda^{\sigma_0}\phi^{(0)}(x)}G^{(0)}(x, \lambda^{-\sigma_0}D; \lambda)e^{i\lambda^{\sigma_0}\phi^{(0)}(x)} \\ &= G^{(0)}(x, \phi_x^{(0)}(x) + \lambda^{-\sigma_0}D; \lambda) + \lambda^{-\sigma_0}R^{(0)}(x, \lambda^{-\sigma_0}D; \lambda) \end{aligned}$$

with $R^{(0)}(x, \xi; \lambda) \in \mathcal{R}(U)$ by Lemma 2.5.2.
We prepare the following lemma for our induction.

Lemma 2.5.3. *Consider a differential operator*

$$G^{(p)}(x, \phi_x^{(p)}(x) + \lambda^{-\sigma_p}D; \lambda) + \lambda^{-\sigma_p}R^{(p)}(x, \lambda^{-\sigma_p}D; \lambda) \tag{2.13}$$

where $\sigma_p = \sigma_{p-1} - \theta_p$, $\theta_p \in \mathbb{Q}_+$ *with* $\sigma_{-1} = 1 - \theta_0$, $R^{(p)}(x, \xi; \lambda) \in \mathcal{R}(U)$, *which is verifying*

$$(i)_p \qquad G^{(p)}(x, \xi; \lambda) = \sum_{j=0} \lambda^{-\delta_j(G^{(p)})}G_j^{(p)}(x, \xi),\ 0 = \delta_0(G^{(p)}) < \delta_1(G^{(p)}) < \cdots$$

the sum being finite and $G_j^{(p)}(x, D)$ *denoting a differential operator with real analytic coefficients and* $\phi^{(p)}(x)$ *is a real analytic function in* U *such that* $\phi_x^{(p)}(x)$ *is a root of* $G^{(p)}(x, \xi) = 0$ *with uniform multiplicity* q_p, *that is*
$(ii)_p$

$$\begin{cases} \partial_{x_0}\phi^{(p)}(x) = \tau_p(x, \partial_{x'}\phi^{(p)}(x)) \quad in \quad U \\ G_0^{(p)}(x, \xi) = (\xi_0 - \tau_p(x, \xi'))^{q_p}\Delta_p(x, \xi), \quad \Delta_p(x, \tau_p(x, \xi'), \xi') \neq 0 \ in\ U \times V \end{cases}$$

where $\tau_p(x, \xi')$ *is a real analytic in* $U \times V$ *and there is* $k(p) \in \mathbb{N}$ *such that*

$$(iii)_p \qquad \sigma_p,\ \theta_p,\ \delta_j(G^{(p)})\ (j \geq 1) \in \mathbb{N}/k(p).$$

Then we can find $\theta_{p+1} \in \mathbb{Q}_+$ *and a real analytic* $\phi^{(p+1)}(x)$ *in* U *such that with* $\sigma_{p+1} = \sigma_p - \theta_{p+1}$ *that*

$$e^{-i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)}[G^{(p)}(x, \phi_x^{(p)}(x) + \lambda^{-\sigma_p}D; \lambda) \tag{2.14}$$

$$+\lambda^{-\sigma_p}R^{(p)}(x, \lambda^{-\sigma_p}D; \lambda)]e^{i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)} \tag{2.15}$$

$$\begin{aligned} &= \lambda^{-\theta_{p+1}q_p}[G^{(p+1)}(x, \phi_x^{(p+1)}(x) + \lambda^{-\sigma_{p+1}}D; \lambda) \\ &\qquad +\lambda^{-\sigma_{p+1}}R^{(p+1)}(x, \lambda^{-\sigma_{p+1}}D; \lambda)] \end{aligned} \tag{2.16}$$

where

$$(i)_{p+1} \qquad \begin{cases} G^{(p+1)}(x, \xi; \lambda) = \sum_{j=0} \lambda^{-\delta_j(G^{(p+1)})}G_j^{(p+1)}(x, \xi), \\ 0 = \delta_0(G^{(p+1)}) < \delta_1(G^{(p+1)}) < \cdots \end{cases}$$

the sum being finite and $G_j^{(p+1)}(x,D)$ denoting a differential operator with analytic coefficients and $\phi^{(p+1)}(x)$ is a real analytic function such that $\phi_x^{(p+1)}$ is a root of $G_0^{(p+1)}(x,\xi)=0$ with uniform multiplicity q_{p+1} that is

$$(ii)_{p+1} \qquad \begin{cases} \partial_{x_0}\phi^{(p+1)}(x) = \tau_{p+1}(x,\partial_{x'}\phi^{(p+1)}(x)) \quad in \quad U \\ G_0^{(p+1)}(x,\xi) = (\xi_0 - \tau_{p+1}(x,\xi'))^{q_{p+1}}\Delta_{p+1}(x,\xi), \\ \Delta_{p+1}(x,\tau_{p+1}(x,\xi'),\xi') \neq 0 \ in \ U\times V \end{cases}$$

where $\tau_{p+1}(x,\xi')$ is real analytic in $U\times V$ and there is $k(p+1)\in\mathbb{N}$ such that

$$(iii)_{p+1} \qquad \sigma_{p+1},\ \theta_{p+1},\ \delta_j(G^{(p+1)})\ (j\geq 1)\in \mathbb{N}/k(p+1).$$

Proof. Set

$$\tilde{G}^{(p)}(x,\xi;\lambda) = G^{(p)}(x,\phi_x^{(p)}(x)+\xi;\lambda) + \lambda^{-\sigma_p}R^{(p)}(x,\xi;\lambda)$$

then $\tilde{G}^{(p)}(x,\xi;\lambda)$ can be written as

$$\tilde{G}^{(p)}(x,\xi;\lambda) = \sum_{j=0}\lambda^{-\delta_j(\tilde{G}^{(p)})}\tilde{G}_j^{(p)}(x,\xi)$$

where $\delta_0(\tilde{G}^{(p)}) = \delta_0(G^{(p)}) = 0$ and

$$\tilde{G}_0^{(p)}(x,\xi) = G_0^{(p)}(x,\phi_x^{(p)}(x)+\xi)$$

and hence $\tilde{G}_0^{(p)}(x,0)=0$. Thus one can write

$$\tilde{G}_j^{(p)}(x,\lambda^{-\theta}\xi) = \lambda^{-\theta s_j^p}\left[\hat{G}_j^{(p)}(x,\xi) + O(\lambda^{-\theta})\right] \tag{2.17}$$

for any $\theta>0$. From $(ii)_p$ it is clear that $s_0^p = q_p$ and

$$\tilde{G}_0^{(p)}(x,\lambda^{-\theta}\xi) = \hat{G}_0^{(p)}(x,\phi_x^{(p)}(x)+\lambda^{-\theta}\xi)$$
$$= \lambda^{-\theta q_p}\left[\sum_{|\alpha|\leq q_p}\frac{1}{\alpha!}(G_0^{(p)})^{(\alpha)}(x,\phi_x^{(p)}(x))\xi^\alpha + O(\lambda^{-\theta})\right]$$

hence

$$\hat{G}_0^{(p)}(x,\xi) = \sum_{|\alpha|=q_p}\frac{1}{\alpha!}(G_0^{(p)})^{(\alpha)}(x,\phi_x^{(p)}(x))\xi^\alpha. \tag{2.18}$$

We now define

$$\theta_{p+1} = \min_{j\geq 1, s_j^p\leq s_0^p}\left(\frac{\delta_j(\tilde{G}^{(p)})}{s_0^p - s_j^p},\theta_p\right)$$

so that, in particular, $\theta_{p+1}\leq\theta_p$. Let

$$\sigma_{p+1} = \sigma_p - \theta_{p+1}.$$

For our present purpose we shall assume that $\sigma_{p+1} > 0$. If $\sigma_{p+1} \le 0$ we make a different argument in the following.
Let now $\phi^{(p+1)}(x)$ be a real analytic function in U, in the following we shall precise this function. Applying Lemma 2.5.2 we compute

$$\begin{aligned} &e^{-i\lambda^{\sigma_{p+1}}\phi^{(p+1)}(x)}\tilde{G}^{(p)}(x,\lambda^{-\sigma_p}D;\lambda)e^{i\lambda^{\sigma_{p+1}}\phi^{(p+1)}} \\ &= \sum_{j\ge 0}\lambda^{-\delta_j(\tilde{G}^{(p)})}\tilde{G}_j^{(p)}(x,\lambda^{-\theta_{p+1}}(\phi_x^{(p+1)}(x)+\lambda^{-\sigma_{p+1}}D)) \\ &\quad + \sum_{j\ge 0}\lambda^{-\delta_j(\tilde{G}^{(p)})-\theta_{p+1}s_j^p-\sigma_{p+1}}\tilde{R}_j^{(p+1)}(x,\lambda^{-\sigma_{p+1}}D;\lambda) \end{aligned} \tag{2.19}$$

where $\tilde{R}_j^{(p+1)}(x,\xi;\lambda) \in \mathcal{R}(U)$. Define $G^{(p+1)}(x,\xi;\lambda)$ and $R^{(p+1)}(x,\xi;\lambda)$ by

$$\begin{aligned} \tilde{G}^{(p)}(x,\lambda^{-\theta_{p+1}}\xi;\lambda) &= \lambda^{-\theta_{p+1}s_0^p}G^{(p+1)}(x,\xi;\lambda) \\ &= \lambda^{-\theta_{p+1}s_0^p}\sum_{j\ge 0}\lambda^{-\delta_j(G^{(p+1)})}G_j^{(p+1)}(x,\xi) \end{aligned}$$

and

$$\begin{aligned} &\sum_{j\ge 0}\lambda^{-\delta_j(\tilde{G}^{(p)})-\theta_{p+1}s_j^p}\tilde{R}_j^{(p+1)}(x,\xi;\lambda) \\ &\qquad = \lambda^{-\theta_{p+1}s_0^p}R^{(p+1)}(x,\xi;\lambda). \end{aligned}$$

(Note that $\delta_j(\tilde{G}^{(p)}) + \theta_{p+1}s_j^p \ge \theta_{p+1}s_0^p$). Then this proves (2.17). From (2.17) we obtain

$$G_0^{(p+1)}(x,\xi) = \sum_{\theta_{p+1}s_0^p=\theta_{p+1}s_j^p+\delta_j(\tilde{G}^{(p)})}\hat{G}_j^{(p)}(x,\xi) \tag{2.20}$$

where $\hat{G}_j^{(p)}(x,\xi)$ being homogeneous of degree s_j^p with respect to ξ. Then it is clear that $\hat{G}_0^{(p+1)}(x,\xi)$ is a polynomial in ξ of degree $q_0 = s_0^p$ and the coefficient of $\xi_0^{q_0}$ is different from zero. Then one can find some open sets U and V and real analytic $\tau_{p+1}(x,\xi')$ defined in $U \times V$, and real analytic $\phi^{(p+1)}(x)$ in U such that $(ii)_{p+1}$ holds. This proves the lemma. □

Lemma 2.5.4. *Assume that there exists a* $\bar{p} \in \mathbb{N}$ *such that*

$$q_{\bar{p}} = q_{\bar{p}+1} = \cdots = q.$$

Then there exists a $k = k(\bar{p})$ *such that for all* $p \ge \bar{p}$

$$\sigma_p,\ \theta_p,\ \delta_j(G^{(p)}),\ j \ge 1 \text{ belong to } \mathbb{N}/k.$$

Proof. The fact that $q_p = q_{p+1}$ implies that there is no roots of $G_0^{(p+1)}(x,\xi) = 0$ with respect to ξ_0 with uniform multiplicity less than q_p. Two cases may occur: either the sum in (2.20) has $\hat{G}_0^{(p)}$ as the only summand or there are also other summands. In the former case, we have

$$\theta_{p+1}s_0^p < \theta_{p+1}s_j^p + \delta_j(\tilde{G}^{(p)}) \text{ for every } j \geq 0$$

which implies that

$$\theta_{p+1} < \frac{\delta_j(\tilde{G}^{(p)})}{s_0^p - s_j^p}$$

that is, $\theta_{p+1} = \theta_p$. Assume now that there are terms other than $\hat{G}_0^{(p)}$, corresponding to $j > 0$. Then the condition defining the sum implies that there is $j \geq 1$ such that

$$\delta_j(\tilde{G}^{(p)}) = \theta_{p+1}$$

because of the following lemma.

Lemma 2.5.5. *Let*

$$f(\tau) = \sum_{j=0}^{s} a_j \tau^{q_j}$$

where $0 = q_0 < q_1 < \cdots < q_s$ *and* $a_j \neq 0$. *Then the roots of* $f(\tau) = 0$ *have multiplicity at most* s.

Exercise: Prove Lemma 2.5.5.

In both cases we conclude that either $\theta_{p+1} = \theta_p$ or $\theta_{p+1} = \delta_j(\tilde{G}^{(p)})$ holds. In particular this implies $\theta_{p+1} \in \mathbb{N}/k(p)$ and hence $k(p+1) = k(p)$ since $\delta_j(G^{(p+1)})$ are obtained summing and multiplying rational numbers whose denominator is $k(p)$. □

From Lemma 2.5.4 the above iteration procedure occurs only a finite number of times before reaching a point where

$$\sigma_{p+1} = \sigma_0 - \sum_{i=1}^{\bar{p}} \theta_i \leq 0$$

for a suitable integer $\bar{p}$. We may assume for a certain $t > 0$

$$\sigma_t > 0, \quad \sigma_{t+1} = \sigma_t - \theta_{t+1} \leq 0$$

that is

$$\theta_{t+1} = \min\left(\frac{\delta_j(\tilde{G}^{(t)})}{s_0^t - s_j^t}, \theta_t\right) \geq \sigma_t.$$

Our purpose is to construct an asymptotic null solution to the operator

$$\tilde{G}^{(t)}(x, \lambda^{-\sigma_t}D; \lambda) = G^{(t)}(x, \phi_x^{(t)}(x) + \lambda^{-\sigma_t}D; \lambda) + \lambda^{-\sigma_t}R^{(t)}(x, \lambda^{-\sigma_t}D; \lambda)$$

where $R^{(t)}(x, \xi; \lambda) \in \mathcal{R}(U)$. With

$$\tilde{G}^{(t)}(x, \xi; \lambda) = \sum_{j \geq 0} \lambda^{-\delta_j(\tilde{G}^{(t)})} \tilde{G}_j^{(t)}(x, \xi)$$

repeating the same arguments as in the proof of Lemma 2.5.3 one can write

$$\tilde{G}^{(t)}(x,\lambda^{-\sigma_t}D;\lambda) = \sum_{j\geq 0}\lambda^{-\delta_j(\tilde{G}^{(t)})-s_j^t\sigma_t}\left[\hat{G}_j^{(t)}(x,D)+O(\lambda^{-\sigma_t})\right]$$

$$= \lambda^{-\sigma_t s_0^t}\left[\sum_{\sigma_t s_0^t=\sigma_t s_j^t+\delta_j(\tilde{G}^{(t)})}\hat{G}_j^{(t)}(x,D)+\sum_{j\geq 1}\lambda^{-\bar{\delta}_j}K_j(x,D)\right]$$

because $\delta_j(\tilde{G}^{(p)})+\sigma_t s_j^t \geq \sigma_t s_j^t$ where $0<\bar{\delta}_1<\bar{\delta}_2<\cdots$. Since $\hat{G}_0^{(t)}(x,D)$ is a differential operator of order s_0^t which is non characteristic with respect to x_0 =const, disposing of the power λ in front of the operator in square brackets, we are left with the operator

$$P(x,D)+\sum_{j\geq 1}\lambda^{-j/k}P_j(x,D) \tag{2.21}$$

where $P(x,D)$ has the principal part $\hat{G}^{(t)}(x,D)$ and $P_j(x,D)$ are differential operators. One can then seek an asymptotic null solution to (2.21) in the form

$$\sum_{j\geq 0}\lambda^{-j/k}u_j(x).$$

By the Cauchy-Kowalevski theorem, we solve $u_j(x)$ successively with $u_0(x)\neq 0$. Note that we may assume that

$$\operatorname{Im}\tau_0(x,\xi')\leq -c \quad \text{in} \quad U\times V$$

with some $c>0$ where $(\hat{x},\hat{\xi}')\in U\times V$. We solve $\phi^{(0)}(x)$ under the condition

$$\phi^{(0)}(\hat{x}_0,x') = i|x'-\hat{x}'|^2+\langle x',\hat{\xi}'\rangle.$$

Then it is easy to see that $\phi^{(0)}(x)$ verifies

$$\operatorname{Im}\phi^{(0)}(x)\geq c\{\hat{x}_0-x_0+|x'-\hat{x}'|^2\},\ \ x_0\leq\hat{x}_0$$

near $\hat{x}$ with some $c>0$. The rest of the proof is just a repetition of that of Theorem 2.1.2. □

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[3] A.Bove and T.Nishitani: Necessary conditions for the well-posedness of the Cauchy problem for hyperbolic systems. preprint 1999.

Chapter 3

Hyperbolic operators with double characteristics

3.1 Basic notions and results

Let us denote by $T^*\Omega$ the cotangent bundle over Ω with a system of local coordinates $x = (x_0, x_1, \ldots, x_n)$. Let (x, ξ) be a system of canonical coordinates on $T^*\Omega$, then the canonical 2-form σ on $T^*\Omega$ is given by

$$\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$$

in these coordinates. This 2-form gives a symplectic structure on $T^*\Omega$.

Remark 3.1.1. σ can be defined in coordinates free. Let π be the projection $\pi : T^*\Omega \to \Omega$. We define the canonical 1-form ω by

$$\langle \omega_\gamma, t \rangle = \langle \pi_* t, \gamma \rangle, \quad \gamma \in T_\gamma(T^*\Omega).$$

Then σ is given by $d\omega$.

EXERCISE: Check this (note that with $t = \alpha\partial/\partial x + \beta\partial/\partial\xi$ we have $\pi_* t = \alpha\partial/\partial x$).

Let $f \in C^\infty(T^*\Omega)$. Then we define the Hamilton vector field H_f of f by

$$-df(\cdot) = \sigma(H_f, \cdot). \tag{3.1}$$

In canonical coordinates (x, ξ), with $X = \alpha\partial/\partial x + \beta\partial/\partial\xi$, $H_f = a\partial/\partial x + b\partial/\partial\xi$ we have

$$-df(X) = -\alpha\frac{\partial f}{\partial x} - \beta\frac{\partial f}{\partial \xi} = d\xi \wedge dx(H_f, \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial \xi}) = \langle b, \alpha \rangle - \langle a, \beta \rangle.$$

That is, $b = -\partial f/\partial x$, $a = \partial f/\partial \xi$ and hence

$$H_f = \frac{\partial f}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial \xi}.$$

Let $P(x, D)$ be a differential operator of order m on Ω and let

$$P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \cdots .$$

Then the principal symbol $P_m(x, \xi) \in C^\infty(T^*\Omega)$, that is if $y = (y_0, y_1, \ldots, y_n)$ is a second system of local coordinates and (y, η) is canonical coordinates on $T^*\Omega$ then

$$P_m(x, \xi) = P_m(y, \eta) \quad \text{if} \quad \eta = {}^t\left(\frac{\partial x}{\partial y}\right)\xi.$$

Definition 3.1.1. Let $p(x, \xi) \in C^\infty(T^*\Omega)$. Then bicharacteristics of p is an integral curve of H_p lying on $\{p = 0\}$.

In what follows we assume that $P_m(x, \xi)$ is hyperbolic and monic with respect to ξ_0 and we write $p(x, \xi) = P_m(x, \xi)$. By definition, bicharacteristics of p is an integral curve of the following Hamilton system

$$\begin{cases} \dot{x} = \frac{\partial p}{\partial \xi}(x, \xi) \\ \dot{\xi} = -\frac{\partial p}{\partial x}(x, \xi) \end{cases} \tag{3.2}$$

The general picture is *energy propagates along bicharacteristics*. Let $z = (\bar{x}, \bar{\xi})$ be a multiple characteristic of $p(x, \xi)$. Then z is a singular (stationary) point of the Hamilton system (3.2).

Definition 3.1.2. The propagation cone C_z of the localization $p_z(x, \xi)$ is given by

$$C_z = \{X \in T_z(T^*\Omega) \mid \sigma(X, Y) \le 0, \forall Y \in \Gamma_z\}.$$

When z is a simple characteristic then p_z is a linear function in $X = (x, \xi)$ and $p_z(X) = dp(z; X)$. Then it is clear that $\Gamma_z = \{X \mid dp(z; X) > 0\}$ and $C_z = H_p(z) \cdot \mathbb{R}^+$. One can prove that C_z is the *minimal* cone including every bicharacteristic which has z as a limit point.

Lemma 3.1.1. *Let $z \in T^*\Omega \setminus \{0\}$ be a multiple characteristic of p. Assume that there are simple characteristics z_j and positive numbers λ_j such that*

$$z_j \to z \quad \text{and} \quad \lambda_j H_p(z_j) \to X (\neq 0), \quad j \to \infty.$$

Then $X \in C_z$.

Proof. Let K be any compact in Γ_z. Then from Theorem 1.10.1 it follows that for sufficiently large j we have $K \subset \Gamma_{z_j}$. Since

$$dp(z_j; Y) = \sigma(Y, H_p(z_j)) > 0, \quad \forall Y \in K$$

it follows $X \in C_z$. □

Definition 3.1.3. Let $t(x,\xi)$ be homogeneous of degree 0 in ξ, C^∞ near z. We say that t is a time function for p near z if $t(z)=0$ and

$$-H_t(z) \in \Gamma_z.$$

Note that $t(x,\xi)$ is a time function near z for p if and only if

$$C_z \cap T_z(\{t(x,\xi)=0\}) = \{0\}.$$

By Theorem 2.1.4, if $p(x,D)$ is strongly hyperbolic then every multiple characteristic is at most double. Moreover if z is a double characteristic, then the localization p_z at z is a "quadratic" form in (x,ξ).
To make a closer look at behaviors of bicharacteristics near double characteristics, we linearize the Hamilton system at the reference double characteristic $z=(\bar{x},\bar{\xi})$. Let

$$x(t)=\bar{x}+\epsilon y(t),\ \xi(t)=\bar{\xi}+\epsilon\eta(t)$$

and plug this into (3.2). Then the linear term in ϵ gives

$$\frac{d}{dt}\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 p}{\partial\xi\partial x} & \frac{\partial^2 p}{\partial\xi\partial\xi} \\ -\frac{\partial^2 p}{\partial x\partial x} & -\frac{\partial^2 p}{\partial x\partial\xi} \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix}.$$

Definition 3.1.4. We call

$$F_p(z) = \begin{pmatrix} \frac{\partial^2 p}{\partial\xi\partial x}(z) & \frac{\partial^2 p}{\partial\xi\partial\xi}(z) \\ -\frac{\partial^2 p}{\partial x\partial x}(z) & -\frac{\partial^2 p}{\partial x\partial\xi}(z) \end{pmatrix}$$

the Hamilton map of p at z.

Let $Q(X,Y)$ be the quadratic form (polar form) corresponding to p_z, then $F_p(z)$ is given by

$$Q(X,Y)=\sigma(X,F_pY), \quad \forall X,Y\in T_z(T^*\Omega). \tag{3.3}$$

Note that this definition is coordinates-free.

EXERCISE: Check (3.3). Show that F_p is well defined using local coordinates.

Remark 3.1.2. We define the map V^* by

$$V^* : T_z^*(T^*\Omega) \ni d\phi \mapsto d(H_p\phi) \in T_z^*(T^*\Omega)$$

(here $\phi\in C^2(T^*\Omega)$ with $\phi(z)=0$). Then the dual map
$V: T_z(T^*\Omega)\to T_z(T^*\Omega)$ of V^* is the Hamilton map $H_p(z)$.

Lemma 3.1.2. *Let $z\in T^*\Omega\setminus\{0\}$ be a double characteristic of p. Then all eigenvalues of $F_p(z)$ are on the pure imaginary axis possibly with one exception of a pair of $\pm e$, $e\in\mathbb{R}$. More precisely the eigenvalues of $F_p(z)$ consists of*

$$\pm e,\ \pm i\lambda_1,\ldots,\pm i\lambda_r, \quad e,\ \lambda_j\in\mathbb{R}_+.$$

Definition 3.1.5. We say that p is effectively hyperbolic at z if F_p has non zero real eigenvalues.

Definition 3.1.6. We define the linearity of p_z by

$$\Lambda_z = \{X \in T_z(T^*\Omega) \mid p_z(tX + Y) = p_z(Y),\ \forall t \in \mathbb{R}, \forall Y \in T_z(T^*\Omega)\}.$$

When z is a simple characteristic then

$$p_z(tX + Y) = dp(z; tX + Y) = tdp(z; X) + dp(z; Y)$$

hence it is clear thar $\Lambda_z = \{X \in T_z(T^*\Omega) \mid dp(z; X) = 0\}$.

EXERCISE: Show that $\Lambda_z = \mathrm{Ker} F_p(z)$ when z is a double characteristic.

Lemma 3.1.3. *The following conditions are equivalent.*

(i) F_p *has real non zero eigenvalues*

(ii) $(\mathrm{Ker} F_p)^\sigma \cap \Gamma_z \neq \emptyset$

(iii) $C_z \cap \Lambda_z = \{0\}$
where $(\mathrm{Ker} F_p)^\sigma = \{X \in T_z(T^*\Omega) \mid \sigma(X, Y) = 0,\ \forall Y \in \mathrm{Ker} F_p\}$.

Let $m = 2$ and we give a useful characterization of effective hyperbolicity. Write p as

$$-p(x, \xi) = -(\xi_0 - a(x, \xi'))^2 + q(x, \xi')$$

where $\xi' = (\xi_1, \ldots, \xi_n)$ and $q(x, \xi') \geq 0$ by Theorem 2.1.4. Then we have

Lemma 3.1.4. *Assume that p is effectively hyperbolic at z. Then there is a time function $t(x, \xi')$ near z for p verifying*

$$q(x, \xi') \geq ct(x, \xi')^2 |\xi'|^2 \quad \text{near} \quad z \tag{3.4}$$

with some $c > 0$. Conversely if (3.8) holds with some time function $t(x, \xi')$ then p is effectively hyperbolic at z.

Theorem 3.1.1. *In order that $p(x, D)$ is strongly hyperbolic it is necessary and sufficient that $p(x, \xi)$ is effectively hyperbolic at every double characteristic.*

Definition 3.1.7. Let $z \in T^*\Omega \setminus \{0\}$ be a double characteristic. We define $\mathrm{Tr}^+ F_p$ by

$$\mathrm{Tr}^+ F_p(z) = \sum \lambda_j$$

where $i\lambda_j$ are the eigenvalues of $F_p(z)$ on the positive imaginary axis repeated according to their multiplicities.

Definition 3.1.8. Let $z \in T^*\Omega \setminus \{0\}$ be a double characteristic. Then the subprincipal symbol of $P(x, D)$ is defined by

$$P_{\mathrm{sub}}(x, \xi) = P_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=0}^{n} \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x, \xi).$$

EXERCISE: Show that $P_{\text{sub}}(x,\xi)$ is well defined on double characteristics, that is if y is a second system of local coordintes near z and (y,η) is canonical coordinates on $T^*\Omega$. Let $(\bar{y},\bar{\eta})$ be the coordinates of z. Then show $P_{\text{sub}}(\bar{x},\bar{\xi}) = P_{\text{sub}}(\bar{y},\bar{\eta})$.

Theorem 3.1.2. (Ivrii-Petkov, Hörmander) *Assume that the Cauchy problem for P is well posed near the origin. Let $z \in T_0^*\Omega \setminus \{0\}$ be a multiple characteristic and p is not effectively hyperbolic at z. Then we have*

$$\text{Im}P_{\text{sub}}(z) = 0, \quad |P_{\text{sub}}(z)| \leq \frac{1}{2}\text{Tr}^+ F_p(z).$$

We end this section, with a conjecture proposed by Melrose. Let Σ_e be the subset in $T^*\Omega \setminus \{0\}$ on which p is effectively hyperbolic. Denote by $e(z)$ the positive eigenvalue of $F_p(z)$ when $z \in \Sigma_e$. Set

$$s(z) = |\operatorname{Im} P_{\text{sub}}(z)| + \inf\{|\operatorname{Re} P_{\text{sub}}(z) - s| \mid |s| \leq \frac{1}{2}\text{Tr}^+ F_p(z)\}.$$

Then CONJECTURE: Assume that the Cauchy problem for P is well posed

near the origin. Then there is a neighborhood U of the origin such that $s(z)/e(z)$ is uniformly bounded in $\Sigma \cap (T^*U \setminus \{0\})$.

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
[2] V.Ja.Ivrii: Linear hyperbolic equations. Partial Differential Equations IV, Encyclopaedia Math. Sci. **33**, Springer, 1988 pp.149-235.
[3] L.Gårding: Some recent results for hyperbolic differential equations. Proceeding of the nineteenth Nordic congress of mathematicians. Reykavik, 1985 pp.50-59.
[4] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[5] L.Hörmander: The Cauchy problem for differential equations with doube characteristics. J.Analyse Math. **32** (1977) 118-196.
[6] V.Ja. Ivrii: Sufficient conditions for regular and completely regular hyperbolicity. Trans. Moscow Math. Soc. **1** (1978) 1-65.
[7] V.Ja.Ivrii: The well-posedness of the Cauchy problem for non strictly hyperbolic operators III, the energy integral. Trans. Moscow Math. Soc. **34** (1978) 149-168.
[8] N.Iwasaki: The Cauchy problem for effectively hyperbolic equations (general case). J. Math. Kyoto Univ. **25** (1985) 727-743.
[9] N.Iwasaki: The Cauchy problem for effectively hyperbolic equations (standard type). Pbul. RIMS Kyoto Univ. **20** (1984) 551-592.
[10] T.Nishitani: Local energy integrals for effectively hyperbolic operators I, II. J. Math. Kyoto Univ. **24** (1984) 623-658, 659-666.

[11] T.Nishitani: Microlocal energy estimates for hyperbolic operators with double characteristics. Hyperbolic Equations and Related Topics, Kinokuniya, Tokyo, 1986, pp.235-255.
[12] T.Nishitani: The Cauchy problem for effectively hyperbolic operators. Non-Linear Variational Problems, Pitman, 1985 pp.9-25.
[13] T.Nishitani: Note on Ivrii-Petkov-Hörmander condition of hyperbolicity. Sci. Rep. College Gen. Ed. Osaka Univ. **39** (1990) 7-9.
[14] K.Kajitani and T.Nishitani: The Hyperbolic Cauchy Problem. Lecture Notes in Math. **1505**, Springer, 1991.
[15] R.Melrose: The Cauchy problem and propagation of singularities. Seminar on Nonlinear Partial Differential Equations, Springer, 1984 pp.185-201.
[16] E.Bernardi, A.Bove and C.Parenti: Geometric results for a class of hyperbolic operators with double characteristics, II. J. Funct. Analysis. **116** (1993) 62-82.
[17] E.Bernardi and A.Bove: Geometric Transition for a Class of Hyperbolic Operators with Double Characteristics. Japanese J.Math. **23** (1997) 1-87.

3.2 Systems with double characteristics

In this section, we give a few results on hyperbolic systems with double characteristics. Let

$$P(x,D) = L(x,D) + B(x),\ L(x,D) = D_0 I_m + \sum_{j=1}^{n} A_j(x) D_j$$

where $A_j(x)$ are $m \times m$ matrix valued smooth function on Ω. We introduce a matrix symbol $\mathcal{L}(x,\xi)$ which plays an important role treating systems.

Definition 3.2.1. We define $\mathcal{L}(x,\xi)$ by

$$\mathcal{L}(x,\xi) = P_{\rm sub}(x,\xi) L^{\rm co}(x,\xi) - \frac{i}{2}\{L, L^{\rm co}\}(x,\xi)$$

where

$$P_{\rm sub}(x,\xi) = B(x) + \frac{i}{2}\sum_{j=0}^{n} \frac{\partial^2}{\partial x_j \partial \xi_j} L(x,\xi),$$

$$\{L, L^{\rm co}\} = \sum_{j=0}^{n} \frac{\partial L}{\partial \xi_j}\frac{\partial L^{\rm co}}{\partial x_j} - \frac{\partial L}{\partial x_j}\frac{\partial L^{\rm co}}{\partial \xi_j}$$

and $L^{\rm co}(x,\xi)$ is the cofactor matrix of $L(x,\xi)$. This $\mathcal{L}(x,\xi)$ is invariantly defined at multiple characteristics z up to the multiplication by $L(z)$ to the left.

Theorem 3.2.1. *Assume that the Cauchy problem for P is well posed near the origin and* $\det L(x,\xi)$ *is not effectively hyperbolic and the rank of* $L(x,\xi)$ *is*

$m-1$ at a multiple characteristic $z \in T_0^*\Omega \setminus \{0\}$. Then there is a real number α with $|\alpha| \le 1$ such that

$$\mathcal{L}(z) + \alpha \mathrm{Tr}^+ \det L(z) I = L(z) K$$

with some matrix K.

Corollary 3.2.1. *Assume that P is strongly hyperbolic near the origin and let $z \in T_0^*\Omega \setminus \{0\}$ be a multiple characteristic. Then either $\det L(x,\xi)$ is effectively hyperbolic at z or the rank of $L(z)$ is less than or equal to $m-2$.*

In the following we assume that all characteristics are at most double and we denote by Σ the doubly characteristic set:

$$\Sigma = \{z = (x,\xi) \mid \det L(x,\xi) = 0,\ d\det L(x,\xi) = 0\}.$$

We introduce the following hypothesis on the doubly characteristic set.

(i) Σ is a C^∞ manifold

(ii) rank Hess $\det L$ = codim Σ

Then we have

Theorem 3.2.2. *Assume that (i) and (ii) hold and one of the following conditions is verified at every point $z \in T_0^*\Omega \cap \Sigma$*

(a) $\det L(x,\xi)$ *is effectively hyperbolic at z*

(b) rank$L \le m-2$ *near z on Σ.*

Then L is strongly hyperbolic near the origin.

We close this section with a conjecture. Before mentioning a conjecture we explain the background.

Theorem 3.2.3. *Assume that $A_j(x)$ are real analytic in Ω. Let $z \in T_0^*\Omega \setminus \{0\}$ be a characteristic of order r of $\det L(x,\xi)$ verifying $C_z \subset \Lambda_z$. If P is strongly hyperbolic near the origin then we have*

$$\dim\ \mathrm{Ker} L(z) = r.$$

We state a conjecture. CONJECTURE: Assume that P is strongly hyperbolic and z is a multiple characteristic of $\det L(x,\xi)$. Then we have

$$\text{either} \quad C_z \cap \Lambda_z = \{0\} \quad \text{or} \quad \dim\ \mathrm{Ker} L(z) = r$$

where of course C_z and Λ_z are defined from $\det L(x,\xi)$.

Taking Lemma 3.2.2 into account, the condition $C_z \cap \Lambda_z = \{0\}$ should read as a generalization of effective hyperbolicity of $\det L(x,\xi)$ and the condition $\dim\ \mathrm{Ker} L(z) = r$ means trivially that $L(z)$ is diagonalizable.

References

[1] T.Nishitani: Une condition nécessaire pour systèmes hyperboliques. Osaka J. Math. **26** (1989) 71-88.

[2] E.Bernardi and T.Nishitani: Remarks on symmetrization of 2×2 systems and the characteristic manifold. Osaka J.Math. **29** (1992) 129-134.

3.3 Quadratic forms on a symplectic vector space

Definition 3.3.1. Let S be a finite dimensional vector space over $\mathbb{R}$ ($\mathbb{C}$) and let σ be a non degenerate anti-symmetric bilinear form on S. Then we call S a (finite dimensional) real (complex) symplectic vector space. Let S_i ($i = 1, 2$) be two symplectic vector spaces with symplectic forms σ_i. If a linear bijection

$$T : S_1 \to S_2$$

verifies $T^*\sigma_2 = \sigma_1$, then T is called a symplectic isomorphism.

Remark 3.3.1. σ is said to be non degenerate if

$$\sigma(\gamma, \gamma') = 0,\ \forall \gamma' \in S \Longrightarrow \gamma = 0.$$

EXAMPLE: Let $T^*\mathbb{R}^n = \{(x, \xi) \mid x, \xi \in \mathbb{R}^n\}$ with

$$\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle.$$

Proposition 3.3.1. *Let S be a finite dimensional real symplectic vector space. Then the dimension of S is even and there is a symplectic isomorphism*

$$T : S \to T^*\mathbb{R}^n$$

with some n.

Proof. Let e_j, f_j be the unit vector along x_j, ξ_j axis in $T^*\mathbb{R}^n$ respectively. It is clear that

$$\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0,\ \ \sigma(f_j, e_k) = \delta_{jk} \tag{3.5}$$

where δ_{jk} is the Kronecker's delta. To prove this proposition it is enough to show that there exists a basis of S verifying (3.5). Take $f_1 \in S$, $f_1 \neq 0$. Since σ is non degenerate one can take $e_1 \in S$ so that $\sigma(f_1, e_1) = 1$. Note that f_1 and e_1 are linearly independent. Let $S_0 = \text{span}\{f_1, e_1\}$ and

$$S_1 = S_0^\sigma = \{\gamma \in S \mid \sigma(\gamma, S_0) = 0\}.$$

Then we have $S = S_1 \oplus S_0$ for if $\gamma \in S_1 \cap S_0$ then writing $\gamma = af_1 + be_1$ one gets

$$\sigma(\gamma, f_1) = -b = 0,\ \sigma(\gamma, e_1) = a = 0$$

and hence $\gamma = 0$. We now show that S_1 is a symplectic vector space with the symplectic form σ. It is enough to check that σ is non degenerate on S_1. Suppose $\sigma(\gamma, S_1) = 0$, $\gamma \in S_1$. By definition we see $\sigma(\gamma, S_0) = 0$ hence $\sigma(\gamma, S) = 0$ which gives $\gamma = 0$. The rest of the proof is carried out by induction. □

Definition 3.3.2. Let S be a symplectic vector space of dimension $2n$ with the symplectic form σ. A basis $\{f_j, e_j\}_{j=1}^n$ verifying (3.5) is called a symplectic basis.

Proposition 3.3.2. *Let S be a symplectic vector space of dimension $2n$ with the symplectic form σ. Let A, B be subsets of $J = \{1, 2, \dots, n\}$. Assume that $\{e_j\}_{j\in A}$, $\{f_k\}_{k\in B}$ are linearly independent and verify* (3.5)*. Then one can choose $\{e_j\}_{j\in J\setminus A}$, $\{f_k\}_{k\in J\setminus B}$ so that $\{e_j\}_{j\in J}$ and $\{f_k\}_{k\in J}$ become a full symplectic basis.*

Proof. Assume $B \setminus A \neq \emptyset$. Take $l \in B \setminus A$. Then there exists $g \in S$ such that $\sigma(g, f_l) = -1$. With $V = \mathrm{span}\{e_j, f_k \mid j \in A, k \in B\}$ we have $g \notin V$ because $\sigma(V, f_l) = 0$ by assumption. Choosing α_i, β_i, $i \in A \cap B$ suitably one can assume that

$$e_l = g - \sum_{i\in A\cap B} \alpha_i e_i - \sum_{i\in A\cap B} \beta_i f_i$$

verifies

$$\sigma(e_l, e_j) = 0,\ j \in A,\ \ \sigma(e_l, f_k) = -\delta_{lk},\ k \in B.$$

Repeating this argument we may assume that $B \subset A$. Applying the same arguments to $A \setminus B$ we may assume $A = B$. If $A = B \neq J$ then with

$$S_0 = \mathrm{span}\{e_j, f_k \mid j \in A, k \in B\}$$

we consider $S_1 = S_0^{\sigma}$. Since S_1 is a symplectic vector space, then by Proposition 3.3.1 there is a symplectic basis for S_1 and hence it is enough to add this basis to $\{e_j, f_j\}_{j\in A=B}$. □

Let Q be a quadratic form on S. We define F by

$$Q(X, Y) = \sigma(X, FY), \quad \forall X, Y \in S \tag{3.6}$$

where $Q(X, Y)$ denotes the polar form of $Q(X)$, that is

$$2Q(X, Y) = Q(X + Y) - Q(X) - Q(Y).$$

Since Q is symmetric we have $\sigma(FX, Y) = -\sigma(X, FY)$ and hence F is skew symmetric with respect to σ. Let $S_{\mathbb{C}}$ be the complexification of S and V_λ, $\lambda \in \mathbb{C}$ be the generalized eigenspace associated with the eigenvalue λ of F.

Lemma 3.3.1. *If $\lambda + \mu \neq 0$ then $Q(V_\lambda, V_\mu) = 0$ and $\sigma(V_\lambda, V_\mu) = 0$. In particular $Q(V_\lambda, V_\lambda) = 0$, $\sigma(V_\lambda, V_\lambda) = 0$ if $\lambda \neq 0$.*

Proof. Since $\lambda + \mu \neq 0$ then $F + \mu$ is bijective on V_λ. From

$$\sigma((F + \mu)^N V_\lambda, V_\mu) = \sigma(V_\lambda, (-F + \mu)^N V_\mu) = 0$$

for large N and hence $\sigma(V_\lambda, V_\mu) = 0$. Noticing that

$$\begin{aligned} Q(V_\lambda, V_\mu) &= \sigma(V_\lambda, FV_\mu) = \sigma(V_\lambda, (F - \mu)V_\mu) + \mu\sigma(V_\lambda, V_\mu) \\ &= \sigma(V_\lambda, (F - \mu)V_\lambda) = \cdots = \sigma(V_\lambda, (F - \mu)^N V_\mu) = 0 \end{aligned}$$

we get $Q(V_\lambda, V_\mu) = 0$. □

Remark 3.3.2. Since F is a real map, we see that

$$V_{\bar{\lambda}} = \overline{V_\lambda}.$$

If $\lambda + \mu \neq 0$, $\bar{\lambda} + \mu \neq 0$ then

$$Q(V_\lambda, V_\mu) = Q(\overline{V_\lambda}, V_\mu) = Q(\overline{V_\lambda}, \overline{V_\mu}) = Q(V_\lambda, \overline{V_\mu}) = 0.$$

This shows that any two of $\mathrm{Re}\, V_\lambda$, $\mathrm{Im}\, V_\lambda$, $\mathrm{Re}\, V_\mu$ and $\mathrm{Im}\, V_\mu$ are Q orthogonal. Similar arguments prove that any pair of these spaces are also σ orthogonal.

Remark 3.3.3. Note that

$$\dim_{\mathbb{C}} V_\lambda \leq \dim_{\mathbb{R}} \mathrm{Re}\, V_\lambda.$$

To see this let $V_\lambda = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_s\}$. Suppose $\mathrm{Re}\, V_\lambda = \mathrm{span}_{\mathbb{R}}\{f_1, \ldots, f_r\}$ with $r < s$. Since $\mathrm{Re}\, e_i$, $\mathrm{Im}\, e_i \in \mathrm{Re}\, V_\lambda$ and hence $e_i \in \mathrm{span}_{\mathbb{C}}\{f_1, \ldots, f_r\}$ which is a contradiction.

Let us denote

$$\mathrm{RadQ} = \{X \mid Q(X, Y) = 0,\ \forall Y \in S\} = \mathrm{Ker} F.$$

In what follows we assume that Q is non negative definite or hyperbolic, that is Q has the signature $(q, 1)$.

Lemma 3.3.2. *Let $V \subset S$ be a subspace. Assume that $V \cap \mathrm{Rad}\ Q = \{0\}$ and $Q(V) \leq 0$. Then $\dim V \leq 1$.*

Proof. Let us write $S = \mathrm{Rad}\ Q \oplus S_0$ and let V_0 be the projection of V into S_0 along $\mathrm{Rad}\ Q$. Since $\mathrm{Rad}\ Q \cap V = \{0\}$ we have $\dim V_0 = \dim V$. Note that $Q(V_0, V_0) \leq 0$ and Q is non degenerate on S_0 and hence positive definite or has the Lorenz signature on S_0. Then V_0 is a proper cone in S_0 and hence $\dim V_0 \leq 0$. □

Proof of Lemma 3.1.2: We first show that if λ is an eigenvalue of F with $\mathrm{Re}\, \lambda \neq 0$ then λ is real and $\dim V_\lambda = 1$. In particular all eigenvalues of F are either real or pure imaginary. To see this note

$$Q(V_\lambda + \overline{V_\lambda}, V_\lambda + \overline{V_\lambda}) = Q(V_{\bar{\lambda}}, V_\lambda) + Q(V_\lambda, V_{\bar{\lambda}}) = 0$$

because $\lambda + \bar{\lambda} \neq 0$ (see Remark above). This shows $Q(\mathrm{Re}\, V_\lambda) = 0$. Hence by Lemma 3.3.2 we have $\dim \mathrm{Re}\, V_\lambda \leq 1$. On the other hand the Remark above shows that $\dim_{\mathbb{C}} V_\lambda \leq \dim_{\mathbb{R}} \mathrm{Re}\, V_\lambda \leq 1$ and hence $\dim_{\mathbb{C}} V_\lambda = \dim_{\mathbb{R}} \mathrm{Re}\, V_\lambda = 1$. Let us put $\mathrm{Re}\, V_\lambda = \mathrm{span}_{\mathbb{R}}\{f\}$ and $V_\lambda = \mathrm{span}_{\mathbb{C}}\{e\}$. Then it is clear that $e = \alpha f$ with some $\alpha \in \mathbb{C}$. Since $Fe = \lambda e$ and hence $Ff = \lambda f$ this shows that λ is real. We next show that if λ, λ' are non zero real eigenvalues of F then $\lambda = \pm\lambda'$. Suppose $\lambda + \lambda' \neq 0$. Let $V_\lambda = \mathrm{span}_{\mathbb{C}}\{e\}$ and $V_{\lambda'} = \mathrm{span}_{\mathbb{C}}\{f\}$ where e, $f \in S$. Then we have

$$Q(\alpha e + \beta f, \alpha e + \beta f) = 2\alpha\beta Q(e, f) = 0.$$

From Lemma 3.3.2 it follows that $\dim(V_\lambda + V_{\lambda'}) \le 1$ and hence e and f are linearly dependent. Since the decomposition into the sum of eigenspaces is unique one conclude that $\lambda = \lambda'$. □

In the rest of this section, we make more detailed studies on Q. We have thus proved

$$S_{\mathbb{C}} = \sum_{\mu>0} \oplus(V_{i\mu} + V_{-i\mu}) \oplus (V_\lambda + V_{-\lambda}) \oplus V_0, \quad \lambda \in \mathbb{R},\ \lambda \neq 0.$$

Recall that the sum is Q and σ orthogonal. We first study $V_{\pm\lambda}$. Let $V_\lambda = \mathrm{span}\{e\}$, $V_{-\lambda} = \mathrm{span}\{f\}$, $e,\ f \in S$. Then we have $\sigma(e,f) \neq 0$ otherwise we would have $\sigma(e, S_{\mathbb{C}}) = 0$ and a contradiction. Thus we may assume that $\sigma(f,e) = 1$ and hence

$$V_\lambda + V_{-\lambda} = \mathrm{span}\{f,e\}, \quad Q(xe + \xi f) = \lambda x \xi. \tag{3.7}$$

We turn to pure imaginary eigenvalues.

Lemma 3.3.3. *Let λ be a pure imaginary eigenvalue of F. Then V_λ consists of simple eigenvectors and*

$$Q(v, \bar{v}) > 0 \quad \text{if} \quad v \in V_\lambda,\ v \neq 0.$$

Proof. Let us fix $v \in V_{i\mu}$ ($\mu \neq 0$). With $v = v_1 + iv_2$, $v_i \in S$ we have $\bar{v} = v_1 - iv_2 \in V_{-i\mu}$ and

$$Q(v + \bar{v}, v + \bar{v}) = 2Q(v, \bar{v}) = 4Q(v_1, v_1) = 4Q(v_2, v_2) \tag{3.8}$$

because $Q(v,v) = Q(\bar{v}, \bar{v}) = 0$. Suppose $Q(v, \bar{v}) \le 0$ and hence $Q(v_1, v_1) \le 0$, $Q(v_2, v_2) \le 0$ by (3.8). Let $V = \mathrm{span}_{\mathbb{R}}\{v_1, v_2\}$ so that $Q(V) \le 0$. This proves that v_1 and v_2 are colinear by Lemma 3.3.2. Then one can write $v = \alpha f$ with some $\alpha \in \mathbb{C}$, $f \in S$. Since $Fv = i\mu v$ this gives a contradiction. This proves $Q(v, \bar{v}) > 0$. We now suppose that there are v, $w \in V_{i\mu}$ such that

$$Fv = i\mu v, \quad Fw = i\mu w + v.$$

Then we have $Q(v, \bar{v}) = \sigma(v, F\bar{v}) = -i\mu\sigma(v, \bar{v})$. On the other hand from

$$\sigma(v, F\bar{w}) = -i\mu\sigma(v, \bar{w}) + \sigma(v, \bar{v}) = -\sigma(Fv, \bar{w}) = -i\mu\sigma(v, \bar{w})$$

it follows that $\sigma(v, \bar{v}) = 0$. This implies $Q(v, \bar{v}) = 0$ which is a contradiction. □

By Lemma 3.3.3, $Q(v, \bar{v})$, $v \in V_{i\mu}$ induce a inner product in $V_{i\mu}$. Choose a basis $\{e_1, \ldots, e_s\}$ for $V_{i\mu}$ which is orthogonal with respect to this inner product:

$$V_{i\mu} = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_s\}, \quad Q(e_i, \bar{e_j}) = 0,\ i \neq j.$$

Let us put

$$V_i = \mathrm{span}_{\mathbb{R}}\{\mathrm{Re}\, e_i, \mathrm{Im}\, e_i\}.$$

Note that $\dim V_i = 2$. Since $Q(\pm\bar{e}_i, \pm e_j) = 0$ $(i \neq j)$ we see that V_i are Q orthogonal each other: $Q(V_i, V_j) = 0$, $i \neq j$. This proves that

$$\mathrm{Re}(V_{i\mu} + V_{-i\mu}) = \sum_{j=1}^{s} \oplus V_j.$$

We now compute

$$Q(x\,\mathrm{Re}\,e_i + \xi\,\mathrm{Im}\,e_i) = \sigma(x\,\mathrm{Re}\,e_i + \xi\,\mathrm{Im}\,e_i, F(x\,\mathrm{Re}\,e_i + \xi\,\mathrm{Im}\,e_i))$$
$$= \mu\sigma(x\,\mathrm{Re}\,e_i + \xi, -x\,\mathrm{Im}\,e_i + \xi\,\mathrm{Re}\,e_i) = -\mu\sigma(\mathrm{Re}\,e_i, \mathrm{Im}\,e_i)(x^2 + \xi^2)$$

where $-\mu\sigma(\mathrm{Re}\,e_i, \mathrm{Im}\,e_i) > 0$ by Lemma 3.3.3. We now normalize e_i so that $\sigma(\mathrm{Re}\,e_i, \mathrm{Im}\,e_i) = -1$ and thus we obtain

$$Q(x\,\mathrm{Re}\,e_i + \xi\,\mathrm{Im}\,e_i) = \mu(x^2 + \xi^2). \tag{3.9}$$

We are left with $\mathrm{Re}\,V_0 = V$. It is sufficient to study Q on V so that we may assume that $V = S$ and F is nilpotent on S (recall that V is symplectic).

Lemma 3.3.4. *Let F be nilpotent on S. Then there are symplectic subspaces V_i, $\dim V_i = 2$ such that*

$$S = (\sum \oplus V_i) \oplus W$$

where the sum being σ, Q orthogonal and

$$Q(xf_i + \xi e_i) = \pm x^2, \quad V_i = \mathrm{span}_{\mathbb{R}}\{f_i, e_i\} \tag{3.10}$$

and moreover F, restricted on W, verifies the followings:

(a) $\sigma(v, w) = 0, \quad \forall v, w \in \mathrm{Ker}F$

(b) $F^2 v = 0 \Longrightarrow Q(v) = 0.$

Proof. Assume that there is a $v \in S$ such that $F^2 v = 0$, $Q(v) = \sigma(v, Fv) \neq 0$. Set $V = \mathrm{span}\{v, Fv\}$ then V is a symplectic subspace. Decomposing $S = V \oplus V^\sigma$ we see that the sum is Q orthogonal for

$$Q(v, w) = \sigma(v, Fw) = -\sigma(Fv, w) = 0, \quad v \in V,\ w \in V^\sigma$$

because V is F invariant. Note that

$$Q(xv + \xi Fv) = \sigma(xv + \xi Fv, xFv) = x^2 Q(v).$$

We next assume that there are v, $w \in \mathrm{Ker}F$ such that $\sigma(v, w) \neq 0$. Let us denote $V = \mathrm{span}\{v, w\}$. Then the same arguments as above show that $S = V \oplus V^\sigma$ where the sum is Q orthogonal and V is symplectic. Note that

$$Q(xv + \xi w) = \sigma(xv + \xi w, F(xv + \xi w)) = 0.$$

This ends the proof. □

Lemma 3.3.5. *Let F be nilpotent on S and verifies* (a) *and* (b) *in Lemma 3.3.4. Then* $\dim S = 4$ *and one can choose a symplectic basis* $\{-F^3v, -Fv, v, F^2v\}$ *so that*

$$Q(-x_1F^3v - x_2Fv + \xi_1 v + \xi_2 F^2 v) = x_2^2 - 2\xi_1\xi_2. \tag{3.11}$$

Proof. Let us study the map

$$F : \mathrm{Ker}F^2 \to \mathrm{Ker}F.$$

Note that $\mathrm{Im}\,F = (\mathrm{Ker}F)^\sigma \supset \mathrm{Ker}F$ by (a). Then for any $w \in \mathrm{Ker}F$ there is v such that $Fv = w$ and hence $F^2v = 0$, that is $v \in \mathrm{Ker}F^2$. This shows that F is surjective on $\mathrm{Ker}F$. Thus we get

$$\mathrm{Ker}F \simeq \mathrm{Ker}F^2/\mathrm{Ker}F.$$

On the other hand from (b) we have $Q(v) = 0$ for any $v \in \mathrm{Ker}F^2$. Then Lemma 3.3.2 shows that $\dim(\mathrm{Ker}F^2/\mathrm{Ker}F) \leq 1$. Thus we get

$$\dim \mathrm{Ker}F = 1.$$

Take v so that

$$S = \mathrm{span}\{v, Fv, \ldots, F^{N-1}v\}, \;\; F^jv \neq 0, \; 1 \leq j \leq N-1, \;\; F^Nv = 0.$$

It is clear that N is even and $N > 2$. We show that $N = 4$. To see this we first note that

$$Q(F^jv, F^kv) = \sigma(F^jv, F^{k+1}v) = (-1)^j\sigma(v, F^{j+k+1}v) = 0 \tag{3.12}$$

if $j + k + 1 \geq N$. Let $V = \mathrm{span}\{F^{N-3}v, F^{N-2}v\}$ then $V \cap \mathrm{RadQ} \;= \{0\}$ by definition. If $(N-3) + (N-3) + 1 \geq N$ then $Q(V) = 0$ by (3.12) which contradicts to $\dim V = 2$ by Lemma 3.3.2. Thus $(N-3) + (N-3) + 1 \leq N - 1$ and hence $N = 4$. Let us set

$$W = \mathrm{span}\{Fv, F^2v\}$$

which verifies $W \cap \mathrm{Rad}\; Q = \{0\}$. We have $Q(Fv, F^2v) = -\sigma(v, F^3v) > 0$ by Lemma 3.3.2. We normalize v so that $\sigma(v, F^3v) = -1$. Put $w = v + tF^2v$ with $t \in \mathbb{R}$. Then one has

$$\begin{aligned}\sigma(w, Fw) &= \sigma(v + tF^2w, Fv + tF^3v)\\ &= \sigma(v, Fv) + 2t\sigma(v, F^3v) = \sigma(v, Fv) - 2t.\end{aligned}$$

Then we take t so that $\sigma(w, Fw) = 0$. Then it is easy to check that $-F^3w$, $-Fw$, w, F^2w become a symplectic basis and verify the assertion. □

Remark 3.3.4. If we choose a symplectic basis $\{-F^3v + Fv, -F^3v, -F^2v, v + F^2v\}$ then we have

$$\begin{aligned}Q(x_1(-F^3v + Fv) - x_2F^3v - \xi_1F^2v + \xi_2(v + F^2v))\\ = -2\xi_2^2 + 2\xi_2\xi_1 + x_1^2.\end{aligned}$$

Theorem 3.3.1. *Let S be a symplectic vector space of dimension $2n$ and Q be a non negative definite quadratic form on S. Then one can choose symplectic coordinates (x, ξ) so that*

$$Q(x,\xi) = \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{k+1}^{k+l} x_j^2$$

where $\mu_j > 0$. Let Q be a hyperbolic quadratic form. Then one can choose symplectic coordinates (x, ξ) so that

$$Q(x,\xi) = \sum_{1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{k+1}^{k+l} x_j^2 + q(x,\xi)$$

where $\mu_j > 0$ and $q(x, \xi)$ is

(I) *$k + l < n - 1$ and $q(x, \xi) = x_n^2 - 2\xi_{n-1}\xi_n$*

(II) *$k + l < n$ and $q(x, \xi) = -x_n^2$ or $q(x, \xi) = 2\lambda x_n \xi_n$, $\lambda > 0$*

Proof. By Lemma 3.3.2 only one of three cases of (I) and (II) may occur. □

Corollary 3.3.1. *Let Q be hyperbolic quadratic form. Then one can choose symplectic coordinates so that*

(1) $Q = -\xi_0^2 + \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{l} \xi_j^2$

(2) $Q = (-\xi_0^2 + 2\xi_0\xi_1 + x_1^2)/\sqrt{2} + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{l} \xi_j^2$

(3) $Q = \lambda(x_0^2 - \xi_0^2)/\sqrt{2} + \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{l} \xi_j^2$ *where (1) and (2) are not effectively hyperbolic while (3) is effectively hyperbolic and*

$$2\sum \mu_j = \mathrm{Tr}^+ F.$$

In (1) and (3) we have $\mathrm{Ker}F^2 \cap \mathrm{Im}F^2 = \{0\}$ while one has $\mathrm{Ker}F^2 \cap \mathrm{Im}F^2 \neq \{0\}$ in the case (2).

Corollary 3.3.2. *Assume that Q is hyperbolic. Then the following conditions are equivalent.*

(i) *F has non zero real eigenvalues*

(ii) *there is $v \in V_0^\sigma$ ($v \in S$) such that $Q(v) < 0$*

(iii) *there is $v \in (\mathrm{Ker}F)^\sigma$ such that $Q(v) < 0$*

(iv) *for any $v \in \mathrm{Ker}F$ there is $w \in S$ such that $\sigma(v, w) = 0$, $Q(w) < 0$.*

Proof. The implication (i)⟹(ii) is clear from the proof of Theorem 3.3.1. The implications (ii)⟹(iii)⟹(iv) are trivial. We now prove (iv)⟹(i). By Theorem 3.3.1, Q has one of the forms in (I) and (II) in suitable symplectic coordinates. Suppose now (I) occurs. Working in $\{x_{n-1}, x_n, \xi_{n-1}, \xi_n\}$ space $\mathrm{Ker}F$ is given by $\{x_n = \xi_{n-1} = \xi_n = 0\}$. If $\sigma(v, w) = 0$, $\forall v \in \mathrm{Ker}F$ means that

the ξ_{n-1} coordinate of w is zero. Hence we get $Q(w) \geq 0$ and this shows that if (iv) holds then (I) never occurs.
Suppose that the former case of (II) occurs. Working in $\{x_n, \xi_n\}$ space we have $\mathrm{Ker}F = \{x_n = 0\}$. If $\sigma(v, w) = 0$, $\forall v \in \mathrm{Ker}F$ then we see that the x_n coordinate of w is zero and hence $Q(w) = 0$. This shows that the former case of (II) also never happens if (iv) holds.
Thus we proved that (iv) implies that only the latter case of (II) happens. This proves the assertion. □

Proof of Lemma 3.1.3: Apply Corollary 3.3.2 to $Q = -F_p(z)$ which is hyperbolic quadratic form. Then (iii) of Corollary 3.3.2 shows that (i) and (ii) are equivalent. It is sufficient to prove the equivalence of (ii) and (iii). Recall that $\mathrm{Ker}F_p(z) = \Lambda_z$ in this case. Assume $\Gamma_z \cap \Lambda_z^\sigma = \emptyset$. Then by the Hahn-Banach theorem there is $0 \neq Y \in T_z(T^*\Omega)$ such that

$$\sigma(Y, X) \leq 0, \ \ \forall X \in \Gamma_z, \quad \sigma(Y, x) \geq 0, \ \ \forall X \in \Lambda_z^\sigma.$$

This implies that $Y \in C_z \cap \Lambda_z$ which would give a contradiction to (iii). Thus (iii)$\Longrightarrow$(ii). Suppose $0 \neq Y \in \Gamma_z \cap \Lambda_z^\sigma$. Then it is clear that $\langle Y\rangle^\sigma \supset \Lambda_z$ and $\langle Y\rangle^\sigma \cap C_z = \{0\}$ because Γ_z is open where $\langle Y\rangle = \mathbb{R}\cdot Y$. This implies obviously $C_z \cap \Lambda_z = \{0\}$ and hence (ii)$\Longrightarrow$(iii). □

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.
[3] L.Hörmander: The Cauchy problem for differential equations with double characteristics. J.Analyse Math. **32** (1977) 118-196.

3.4 Proof of Theorem 3.1.2

In the subsequent sections, we prove Theorem 3.1.2 for second order operators with 3 independent variables. These restrictions simplify a little bit the proof without losing the essential features of the proof (while we could not assume $n = 2$ unless losing some essential structures of the proof). Let us start with a second order differential operator defined near the origin of $\mathbb{R}^3$:

$$P(x, D) = P_2(x, D) + P_1(x, D) + P_0(x).$$

In the subsequent sections, we take $-P$ rather than P. Taking a new system of local coordinates we may assume that

$$P_2(x, \xi) = -\xi_0^2 + \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j.$$

EXERCISE: Find such a system of local coordinates. Let us set

$$Q_2(x,\xi') = \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j.$$

Then Theorem 2.1.3 shows that $Q_2(x,\xi') \geq 0$, x near the origin, $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2$ is necessary in order that the Cauchy problem for P is well posed. Hence we assume this in what follows. Let $z = (0,\bar{\xi}) \in T_0^*\Omega \setminus \{0\}$ be a double characteristic. Then by a linear change of coordinates $x' = (x_1,x_2)$ we may assume that

$$(0,\bar{\xi}) = (0,e_2).$$

Since z is a double characteristic we have $Q_2(0,e_2) = 0$. Let us write

$$Q_2(x,\xi') = a_{11}(x)\xi_1^2 + a_{12}(x)\xi_1\xi_2 + a_{22}(x)\xi_2^2.$$

From Corollary 1.6.3 it follows that

$$\partial_x^\beta \partial_{\xi'}^{\alpha'} Q_2(0,e_2) = 0, \quad |\alpha' + \beta| < 2$$

and hence $a_{12}(x) = O(|x|)$ and $a_{22}(x) = O(|x|^2)$. With

$$a_{12}(x) = a_{12}^{(1)}(x) + O(|x|^2), \;\; a_{22}(x) = a_{22}^{(2)}(x) + O(|x|^3)$$

one can write

$$\begin{aligned} Q_2(x,\xi') = a_{11}(0)\xi_1^2 + a_{12}^{(1)}(x)\xi_1\xi_2 + a_{22}^{(2)}(x)\xi^2 \\ +O(|x|\xi_1^2 + |x|^2|\xi_1\xi_2| + |x|^3\xi_2^2). \end{aligned}$$

We denote by $a_{12}^{[1]}(x)$, $a_{22}^{[2]}(x)$ which are obtained from $a_{12}^{(1)}(x)$, $a_{22}^{(2)}(x)$ removing the terms containing x_2. To simplify notation we set

$$f = |x|\xi_1^2 + |x|^2|\xi_1\xi_2| + |x|^3\xi_2^2 + |x_2||\xi_1\xi_2| + |x||x_2|\xi_2^2.$$

Then one can write

$$Q_2(x,\xi') = a_{11}(0)\xi_1^2 + a_{12}^{[1]}(x)\xi_1\xi_2 + a_{22}^{[2]}(x) + O(f).$$

We summarize:

Lemma 3.4.1. *Let $(0,e_2) \in T_0^*\Omega \setminus \{0\}$ be a double characteristic of the principal symbol $P_2(x,\xi) = -\xi_0^2 + Q_2(x,\xi')$. Then $Q_2(x,\xi')$ can be written as*

$$\begin{aligned} Q_2(x,\xi') = Q(x,\xi') + O(f), \\ Q(x,\xi') = a\xi_1^2 + a^{[1]}(x)\xi_1\xi_2 + a^{[2]}(x)\xi_2^2 \end{aligned}$$

where $a^{[1]}(x) = a^{[1]}(x_0,x_1)$ is linear in (x_0,x_1) and $a^{[2]}(x) = a^{[2]}(x_0,x_1)$ is a quadratic form in (x_0,x_1).

We want to find a system of local coordinates on which $-\xi_0^2 + Q(x, \xi')$ looks as simple as possible. To do so we make some preliminary observations. Let (x, ξ) be canonical coordinates on $T^*\mathbb{R}^2$.

Lemma 3.4.2. *Any symplectic isomorphism on $T^*\mathbb{R}^2$ which preserves the plane $x = 0$ is obtained by composing the following symplectic isomorphisms:*

(i) $(x, \xi) \mapsto (x, \xi + Ax)$, *$A$ is a 2×2 symmetric matrix,*

(ii) *symplectic isomorphism caused by a linear change of the coordinates x.*

Remark 3.4.1. Let $T : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$ be a symplectic isomorphism and write

$$(x, \xi) \mapsto (T_{11}x + T_{12}\xi, T_{21}x + T_{22}\xi)$$

where T_{ij} are 2×2 matrices. Then T preserves the plane $x = 0$ if and only if $T_{12} = 0$. In other words, symbols of differential operators are transformed to symbols of differential operators under this T.

Proof. Let (x, ξ) be canonical coordinates so that σ is given by

$$\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle.$$

Let $T : (x, \xi) \mapsto (T_{11}x + T_{12}\xi, T_{21}x + T_{22}\xi)$ be a symplectic isomorphism. By assumption $T_{12} = 0$. Let $J : (x, \xi) \mapsto (\xi, -x)$ be a linear map on $T^*\mathbb{R}^2$. Then it is easy to see that

$$T \text{ is symplectic } \iff J = {}^tTJT.$$

Then from

$${}^tTJT(x, \xi) = ({}^tT_{11}T_{21}x + {}^tT_{11}T_{22}\xi - {}^tT_{21}T_{11}x, -{}^tT_{22}T_{11}x) = (\xi, -x)$$

it follows that

$${}^tT_{11}T_{21} - {}^tT_{21}T_{11} = 0, \quad {}^tT_{11}T_{22} = E. \tag{3.13}$$

Now we decompose T as

$$(x, \xi) \to (T_{11}x, {}^tT_{11}^{-1}\xi) \to (T_{11}x, {}^tT_{11}^{-1}\xi + AT_{11}x)$$

where A should satisfy $AT_{11} = T_{21}$. Let us examine that A is symmetric:

$${}^tA = {}^t(T_{21}T_{11}^{-1}) = ({}^tT_{11}^{-1}\,{}^tT_{21}) = T_{21}T_{11}^{-1} = A$$

by (3.13). □

Here we note that symplectic isomorphism

$$(\tilde{x}, \tilde{\xi}) \mapsto (\tilde{x}, \tilde{\xi} + A\tilde{x}), \quad \tilde{x} = (x_0, x_1),\ \tilde{\xi} = (\xi_0, \xi_1)$$

with a 2×2 symmetric A can be obtained by a quadratic change of the coordinates $x = (\tilde{x}, x_2)$. Indeed let

$$y_j = x_j\ (j = 0, 1),\ \ y_2 = x_2 + \frac{1}{2}\langle A\tilde{x}, \tilde{x}\rangle. \tag{3.14}$$

In fact this change of the coordinates x gives

$$(x,\xi)\mapsto(\tilde{x},x_2+\frac{1}{2}\langle A\tilde{x},\tilde{x}\rangle,\tilde{\xi}-\xi_2A\tilde{x},\xi_2).$$

In the coordinates $(\tilde{x},\tilde{\xi})$, this is reduced to

$$(\tilde{x},\tilde{\xi})\mapsto(\tilde{x},\tilde{\xi}-\xi_2A\tilde{x}).$$

Taking Lemma 3.4.2 into account, in what follows we use changes of coordinates x as (3.14) and the next linear changes:

$$y_0=x_0,\;\; y_1=kx_0+lx_1+mx_2,\;\; y_2=x_2 \quad (0\neq l) \tag{3.15}$$

giving the change of canonical coordinates (x,ξ) as:

$$(x,\xi)\mapsto(x_0,l^{-1}(x_1-kx_0-mx_2),x_2,\xi_0+k\xi_1,l\xi_1,m\xi_1+\xi_2).$$

The reason why we restrict ourselves to these linear changes is, one thing our techniques of proof require that the planes $x_0=cnst.$ be preserved and another thing is the point $(0,e_2)$, already normalized, should be preserved.

Theorem 3.4.1. *Assume that P_2 is not effectively hyperbolic at $(0,e_2)$. Then by repeated use of changes* (3.14) *and* (3.15), $-\xi_0^2+Q_2(x,\xi')$ *is transformed to one of the followings:*

(I) $-\xi_0^2+2\xi_0L_0(x_1\xi_2,\xi_1)+Q'(x_1\xi_2,\xi_1)+O(f)$ *where either*

$$\begin{cases} Q'(x_1,\xi_2)=\mu(x_1^2+\xi_1^2),\ \mu>0,\ \ L_0 \text{ is linear in } (x_1,\xi_1) \\ Q'(x_1,\xi_1)=x_1^2,\quad (\text{or } \xi_1^2),\ \ L_0=0 \\ Q'(x_1,\xi_1)=0,\ \ L_0=0 \end{cases}$$

(II) $-\xi_0^2+2c\xi_0\xi_1+(x_1\xi_2)^2+O(f),\ \ c\neq 0$

(III) $-\xi_0^2+2cx_1\xi_2\xi_0+\xi_1^2+k\xi_1\xi_0+O(f),\ \ c\neq 0.$

Remark 3.4.2. Except for the first case in (I), we have $\mathrm{Tr}^+F_p(z)=0$. In the first case of (I), $\mathrm{Tr}^+F_p(z)=2\mu$.

EXERCISE: Check this remark.

Proof. We first remark that

Lemma 3.4.3. *By* (3.14) *and* (3.15), $O(f)$ *is transformed to* $O(f)$.

Proof. Easy. ☐

From Lemma 3.4.3 it is enough to study

$$Q(x,\xi')=a\xi_1^2+a^{[1]}(x)\xi_1\xi_2+a^{[2]}(x)\xi_2^2.$$

Lemma 3.4.4. *Let $l(x_1\xi_2, \xi_1)$ be a linear function in $(x_1\xi_2, \xi_1)$. Then there is a $k \in \mathbb{R}$ so that*

$$l(x_1\xi_2, \xi_1 - kx_1\xi_2) = a\xi_1 \quad \text{or} \quad ax_1\xi_2$$

with some a. That is, by a quadratic change of coordinates x, $l(x_1\xi_2, \xi_1)$ becomes either $a\xi_1$ or $ax_1\xi_2$.

Proof. Easy. □

If $a^{[2]}(x)$ does not contain the term x_0^2 then $a^{[2]}(x) = \alpha x_1^2$ with $\alpha \geq 0$ and hence

$$Q(x, \xi') = a(x_1\xi_2)^2 \quad \text{or} \quad a(\xi_1 - \beta x_1\xi_2)^2 + b(x_1\xi_2)^2.$$

If the second case occurs, by Lemma 3.4.4 one can choose coordinates x so that

$$Q(x, \xi_1 + \beta x_1\xi_2, \xi_2) = a\xi_1^2 + b(x_1\xi_2)^2.$$

Making a change of coordinates $x_1 \to kx$, $\xi_1 \to k^{-1}\xi_1$ with $k^4 = a/b$ we get (I). Assume that $a^{[2]}(x)$ contains x_0^2. Then one can write

$$Q(x, \xi') = a(x_0\xi_2 - l(x_1\xi_2, \xi_1))^2 + q(x_1\xi_2, \xi_1)$$

where $q(x_1\xi_2, \xi_1) \geq 0$. Let us set

$$Q'(x_1\xi_2, \xi_1) = al(x_1\xi_2, \xi_1)^2 + q(x_1\xi_2, \xi_1) \tag{3.16}$$

which is a quadratic form in $(x_1\xi_2, \xi_1)$ so that

$$Q(x, \xi') = a(x_0\xi_2)^2 - 2ax_0\xi_2 l(x_1\xi_2, \xi_1) + Q'(x_1\xi_2, \xi_1).$$

Lemma 3.4.5. *Assume that p is not effectively hyperbolic at $(0, e_2)$. Then* $\dim \mathrm{Rad}Q' = 0$.

Proof. If $\dim \mathrm{Rad}Q' = 2$ then we have $l = 0$ and $q = 0$ so that $p = -\xi_0^2 + a(x_0\xi_2)^2$ which is effectively hyperbolic at $(0, e_2)$. Thus this can not occur. If $\dim \mathrm{Rad}Q' = 1$ then there is a linear function ϕ such that

$$Q'(x_1\xi_2, \xi') = \phi(x_1\xi_2, \xi_1)^2.$$

From Lemma 3.4.4 one can take k so that

$$Q'(x_1\xi_2, \xi_1 - kx_1\xi_2) = A\xi_1^2 \quad (\text{or } A(x_1\xi_2)^2) \quad A > 0.$$

This implies that

$$l(x_1\xi_2, \xi_1 - kx_1\xi_2) = \alpha\xi_1 \quad (\ \alpha(x_1\xi_2)^2)$$

because $q \geq 0$. Hence after a quadratic change of coordinates p has the form

$$-\xi_0^2 + a(x_0\xi_2)^2 - 2a\alpha x_0\xi_2\xi_1 + A\xi_1^2,$$
$$-\xi_0^2 + a(x_0\xi_2)^2 - 2a\alpha x_0\xi_2 x_1\xi_2 + A(x_1\xi_2)^2.$$

If p takes the first form in (3.17), making a change $\xi_1 \to \rho^{-1}\xi_1$, $x_1 \to \rho x_1$ and letting $\rho \to \infty$, p turns to be

$$-\xi_0^2 + a(x_0\xi_2)^2.$$

Since the eigenvalues of the Hamilton map are invariant under symplectic transformation, we conclude that this p is effectively hyperbolic at $(0, e_2)$. As for the second form, make a change $\xi_1 \to \rho\xi_1$, $x_1 \to \rho^{-1}x_1$ we get the same result. Thus we have proved the lemma. □

Lemma 3.4.6. *Assume that p is not effectively hyperbolic at $(0, e_2)$. Then we have* $\dim \mathrm{Rad}Q' = 0$ *and* $\dim \mathrm{Rad}q = 1$.

Proof. It remains to prove $\dim \mathrm{Rad}q = 1$. Suppose that q is positive definite. Then from

$$(x_0\xi_2)^2 \le 2(x_0\xi_2 - l)^2 + 2l^2 \le M\{(x_0\xi_2 - l)^2 + q\}$$

we see that

$$a(x_0\xi_2 - l(x_1\xi_2, \xi_1))^2 + q(x_1\xi_2, \xi_1) \ge c(x_0\xi_2)^2$$

with some $c > 0$ and this implies that p is effectively hyperbolic at $(0, e_2)$. If $\dim \mathrm{Rad}q = 2$ then $q = 0$ and hence

$$p = -\xi_0^2 + a(x_0\xi_2 - l(x_1\xi_2, \xi_1))^2.$$

This is again effectively hyperbolic. This proves the assertion. □

Exercise: In the above proof, check the assertion that p is effectively hyperbolic.

By Lemma 3.4.4 with some k one has

$$q(x_1\xi_2, \xi_1 - kx_1\xi_2) = \begin{cases} m(x_1\xi_2)^2 \\ m\xi_1^2 \end{cases}$$

with $m > 0$. Let us set

$$l(x_1\xi_2, \xi_1 - kx_1\xi_2) = sx_1\xi_2 + \xi_1$$

and hence we have

$$\begin{aligned} p &= -\xi_0^2 + a(x_0\xi_2 - sx_1\xi_2 - t\xi_1)^2 + m(x_1\xi_2)^2 \\ p &= -\xi_0^2 + a(x_0\xi_2 - sx_1\xi_2 - t\xi_1)^2 + m\xi_1^2. \end{aligned} \tag{3.17}$$

Since $\mathrm{Rad}Q' = \{0\}$ it follows that $t \ne 0$ in the first case and $s \ne 0$ in the second case. We first study the second case in (3.17). Make a linear change of the coordinates x: $y_0 = x_0$, $y_1 = x_0 - sx_1$, $y_2 = x_2$, the second p turns out

$$-(\xi_0 + \xi_1)^2 + a(x_1\xi_2 + ts\xi_1)^2 + ms^2\xi_1^2.$$

Again make a linear change $\xi_1 \to s^{-1}\xi_1$, $x_1 \to sx_1$ one gets

$$-\xi_0^2 - 2s^{-1}\xi_1\xi_0 + \{a(sx_1\xi_2 + t\xi_1)^2 + (m - s^{-2})\xi_1^2\}.$$

Here we prove that

$$R(x_1\xi_2, \xi_1) = a(sx_1\xi_2 + t\xi_1)^2 + (m - s^{-2})\xi_1^2$$

is non negative definite. If $m \geq s^{-2}$ the assertion is clear. If $m - s^{-2} < 0$ make a change of coordinates $\xi_0 \to \rho^{-1}\xi_0$, $x_0 \to \rho x_0$ and let $\rho \to \infty$. Then p goes to

$$a(sx_1\xi_2 + t\xi_1)^2 - (s^{-2} - m)\xi_1^2.$$

This is effectively hyperbolic and a contradiction. Thus we have proved that R is non negative definite.
Let $m = s^{-2}$. Then we have

$$p = -\xi_0^2 - 2s^{-1}\xi_1\xi_0 + a(sx_1\xi_2 + t\xi_1)^2.$$

It $t \neq 0$, by the quadratic change of coordinates $\xi_1 \to \xi_1 - (s/t)x_1\xi_2$ we have

$$-\xi_0^2 + 2t^{-1}x_1\xi_2\xi_0 + at^2\xi_1^2 - 2s^{-1}\xi_1\xi_0.$$

Again by the change of coordinates $\xi_1 \to (\sqrt{a}t)^{-1}\xi_1$, $x_1 \to \sqrt{a}tx_1$ we have (III). If $t = 0$ then

$$p = -\xi_0^2 - 2s^{-1}\xi_1\xi_0 + as^2(x_1\xi_2)^2$$

this is (II). Let $m > s^{-2}$ so that R is positive definite:

$$R(x_1\xi_2, \xi_1) = \alpha(\xi_1 + \beta x_1\xi_2)^2 + \gamma(x_1\xi_2)^2, \quad \alpha, \gamma > 0.$$

Make the change $x_1 \to x_1$, $\xi_1 \to \xi_1 - \beta x_1\xi_2$ then p turns to

$$-\xi_0^2 - 2s^{-1}(\xi_1 - \beta x_1\xi_2)\xi_0 + \alpha\xi_1^2 + \gamma(x_1\xi_2)^2.$$

Again making the change $\xi_1 \to \lambda\xi_1$, $x_1 \to \lambda^{-1}x_1$ with $\lambda^4 = \gamma/\alpha$ we have (I).
We turn to the first case in (3.17). Make the quadratic change of the coordinates x:

$$\xi_1 \to \xi_1 + t^{-1}x_0\xi_2, \;\; \xi_o \to \xi_0 + t^{-1}x_1\xi_2, \;\; x_j \to x_j \; (j = 0, 1)$$

then p goes to

$$-(\xi_0 + t^{-1}x_1\xi_2)^2 + a(sx_1\xi_2 + t\xi_1)^2 + m(x_1\xi_2)^2.$$

Again make the quadratic change of coordinates $\xi_1 \to \xi_1 - (s/t)x_1\xi_2$, $x_j \to x_j$ $j = 0, 1$ then p becomes

$$-\xi_0^2 - 2t^{-1}x_1\xi_2\xi_0 + at^2\xi_1^2 + (m - t^{-2})(x_1\xi_2)^2.$$

It is clear that $m - t^{-2} \geq 0$ because p is not effectively hyperbolic. If $m - t^{-2} = 0$ then (III) if $m - t^{-2} > 0$ we get (I). These prove the assertion. □

3.5 Proof of Theorem 3.1.2 (continued)

Recall that P has the form

$$P = -D_0^2 + Q_2(x, D') + \sum_{j=1}^{2} b_j(x) D_j + c(x)$$

where $Q_2(x, \xi')$ has the form in Theorem 3.4.1. Since the subprincipal symbol is well defined on multiple characteristics we have

$$P_{\text{sub}}(z) = b_2(z). \tag{3.18}$$

Exercise: Show (3.18). We make the following change of coordinates

depending on a large parameter λ

$$x_0 = y_0 \lambda^{-s/2+\kappa},\ x_1 = y_1 \lambda^{-s/2+\sigma},\ x_2 = y_2 \lambda^{-s} \tag{3.19}$$

where κ, σ are chosen so that

(I) $Q' = (x_1\xi_2)^2$ (resp. $Q' = \xi_1^2$) we take $\kappa = 0$, $\sigma = -\nu$ (resp. $\kappa = 0$, $\sigma = \nu$) Otherwise we take $\kappa = 0$, $\sigma = \nu$,

(II) $\kappa = 1$, $\sigma = -1$,

(III) $\kappa = 1$, $\sigma = 0$,

where $\nu \geq 1$, $s \gg \nu$. With $b = b_2(0)$ we study

$$P_\lambda(y, D) = \lambda^{-s} P(\lambda^{-s/2+\kappa} y_0, \lambda^{-s/2+\sigma} y_1, \lambda^{-s} y_2, \lambda^{s/2-\kappa} D_0, \lambda^{s/2-\sigma} D_1, \lambda^s D_2).$$

Lemma 3.5.1. *Let $|\kappa|$, $|\sigma| \leq \nu$. We have*

$$P_\lambda(y, D) = Q_\infty(\tilde{y} D_2, D; \lambda) + O(\lambda^{-s/2+3\nu})$$

where $O(\lambda^{-s/2+3\nu})$ is a differential operator of order 2 with coefficients which are $O(\lambda^{-s/2+3\nu})$ and $Q_\infty(\tilde{y} D_2, D; \lambda)$, $\tilde{y} = (y_0, y_1)$, $D = (D_0, D_1, D_2)$ has the form

(I)

$$\begin{cases} Q_\infty(\tilde{x}, \xi; \lambda) = -\xi_0^2 + 2\xi_0 L_0(x_1, \xi_1) + \alpha(x_1^2 + \xi_1^2) + b\xi_2 \\ Q_\infty(\tilde{x}, \xi; \lambda) = -\xi_0^2 + b\xi_2 + \lambda^{-2\nu} x_1^2 \text{ or } (\lambda^{-2\nu} \xi_1^2) \end{cases}$$

(II) $Q_\infty(\tilde{x}, \xi; \lambda) = -\lambda^{-2}\xi_0^2 + 2c\xi_1\xi_0 + \lambda^{-2} x_1^2 + b\xi_2$

(III) $Q_\infty(\tilde{x}, \xi; \lambda) = -\lambda^{-2}\xi_0^2 + 2c\lambda^{-1} x_1 \xi_0 + \xi_1^2 + k\lambda^{-1}\xi_1\xi_0 + b\xi_2$.

Proof. Taking $|\kappa|$, $|\sigma| \leq \nu$ into account, it is easy to see that by changes of type (3.14) and (3.15)

$$\lambda^{-s} O(f) \to O(\lambda^{-s/2+3\nu}).$$

On the other hand it is also clear that

$$\sum_{j=0}^{1} b_j(x)D_j + c(x) \to O(\lambda^{-s/2}).$$

Thus one concludes that

$$P_\lambda(y,D) = -\lambda^{-2\kappa}D_0^2 + Q_2(\lambda^{-s/2+\kappa}y_0, \lambda^{-s/2+\sigma}y_1, \lambda^{-s}y_2, \lambda^{s/2-\sigma}D_1, \lambda^s D_2) \\ +bD_2 + O(\lambda^{-s/2+3\nu}).$$

Now the result follows easily because of the choice of κ, σ. □

We study the first case of (I). Let us set

$$E_\lambda = \exp\left[i\lambda^2(y_2 + \frac{i}{2}y_1^2) + i\lambda\phi\right]$$

where ϕ will be determined later. Let us write

$$Q'(y_1D_2, D_1) = \alpha(y_1^2D_2^2 + D_1^2).$$

Remarking that

$$Q'(y_1, \partial_{y_1}(iy_1^2/2)) = 0$$

by computation one gets

$$\begin{aligned} E_\lambda^{-1}Q'E_\lambda &= 2\alpha y_1\lambda^3(y_1\phi_{y_2} + i\phi_{y_1}) \\ &+\alpha\lambda^2(2y_1^2D_2 + 2iy_1D_1 + y_1^2\phi_{y_2}^2 + \phi_{y_1}^2 + 1) \\ &+\alpha\lambda(2y_1^2\phi_{y_2}D_2 + 2\phi_{y_1}D_1 + y_1^2(D_2\phi_{y_2}) + (D_1\phi_{y_1})) \\ &+Q'(y_1D_2, D_1). \end{aligned} \tag{3.20}$$

Similar computations give

$$\begin{aligned} E_\lambda^{-1}D_0L_0(y_1D_2, D_1)E_\lambda &= \lambda^3\phi_{y_0}L_0(y_1, iy_1) \\ &+\lambda^2\{L_0(y_1, iy_1)D_0 + \phi_{y_0}L_0(y_1\phi_{y_2}, \phi_{y_1})\} \\ &+\lambda\{L_0(y_1\phi_{y_2}, \phi_{y_1})D_0 + L_0(y_1(D_0\phi_{y_2}), D_0\phi_{y_1}) + \phi_{y_0}L_0(y_1D_2, D_1)\} \\ &+D_0L_0(y_1D_2, D_1). \end{aligned} \tag{3.21}$$

Noting

$$E_\lambda D_0^2 E_\lambda = \lambda^2\phi_{y_0}^2 + \lambda\{2\phi_{y_o}D_0 + (D_0\phi_{y_0})\} + D_0^2$$

we summarize:

$$\begin{aligned} E_\lambda^{-1}Q_\infty(\tilde{y}D_2, D)E_\lambda &= 2\lambda^2\{\alpha y_1(y_1\phi_{y_2} + i\phi_{y_1}) + \phi_{y_0}L_0(y_1, iy_1)\} \\ &+\lambda^2\{2\alpha y_1(y_1D_2 + iD_1) + 2L_0(y_1, iy_1)D_0 + \alpha y_1^2\phi_{y_2}^2 + \alpha\phi_{y_1}^2 + \alpha \\ &+2\phi_{y_0}L_0(y_1\phi_{y_2}, \phi_{y_1}) - \phi_{y_0}^2 + b\} \\ &+\lambda\{2\alpha(y_1^2\phi_{y_2}D_2 + \phi_{y_1}D_1) + 2L_0(y_1\phi_{y_2}, \phi_{y_1})D_0 - 2\phi_{y_0}D_0 - D_0\phi_{y_0}\} \\ &+Q_\infty(\tilde{y}D_2, D) \end{aligned}$$

where $Q_\infty(\tilde{x},\xi) = -\xi_0^2 + 2\xi_0 L_0(x_1,\xi_1) + Q'(x_1,\xi_1) + b\xi_2$.

Exercise: Check the above computations.

Let us define

$$\Lambda = 2\alpha y_1^2 D_2 + 2i\alpha y_1 D_1 + 2L_0(y_1, iy_1)D_0.$$

Hence one has

$$i\Lambda\phi = 2\alpha y_1^2\phi_{y_2} + 2i\alpha y_1\phi_{y_1} + 2L_0(y_1, iy_1)\phi_{y_0}.$$

We also define

$$\begin{aligned}\bar{c}_0 &= Q'(y_1\phi_{y_2},\phi_{y_1}) - \phi_{y_0}^2 + 2\phi_{y_0}L_0(y_1\phi_{y_2},\phi_{y_1}) + \alpha + b,\\ C_0 &= \alpha y_1^2(D_2\phi_{y_2}) + \alpha(D_1\phi_{y_1}) + 2L_0(y_1(D_0\phi_{y_2}), D_0\phi_{y_1}) - (D_0\phi_{y_0}),\\ C &= (L_0(y_1\phi_{y_2},\phi_{y_1}) - \phi_{y_0}, \alpha\phi_{y_1} + A\phi_{y_0}, \alpha y_1^2\phi_{y_2} + By_1\phi_{y_0})\end{aligned}$$

where $L_0 = AD_1 + By_1D_2$. Introducing

$$R = 2\langle C, D\rangle + C_0$$

one can write as

Lemma 3.5.2. *We have*

$$E_\lambda^{-1}Q_\infty(\tilde{y}D_2, D)E_\lambda = i\lambda^3\Lambda\phi + \lambda^2(\Lambda + \bar{c}_0) + \lambda R + Q_\infty(\tilde{y}D_2, D).$$

Assume that $\Lambda\phi = 0$. Dividing y_1 this yields

$$2\alpha y_1\phi_{y_2} + 2i\alpha\phi_{y_1} + 2L_0(1,i)\phi_{y_0} = 0. \tag{3.22}$$

Since Λ vanishes on $y_1 = 0$ we impose that $\bar{c}_0$ vanishes on $y_1 = 0$ (otherwise one can not solve the equation like $(\Lambda + \bar{c}_0)v = 0$). This requirement turns

$$\alpha\phi_{y_1}^2 - \phi_{y_0}^2 + 2A\phi_{y_0}\phi_{y_1} + \alpha + b = 0 \quad \text{on} \quad y_1 = 0. \tag{3.23}$$

From (3.22) it follows that $i\alpha\phi_{y_1} + L_0(1,i)\phi_{y_0} = 0$ on $y_1 = 0$. Then one can solve ϕ_{y_1} as $\phi_{y_1} = i\alpha^{-1}L_0(1,i)\phi_{y_0}$. Plugging this into (3.23) to get

$$\{-\alpha^{-1}L_0(1,i)^2 - 1 + 2Ai\alpha^{-1}L_0(1,i)\}\phi_{y_0}^2 + \alpha + b = 0.$$

Since $L_0(1,i) = iA + B$, the coefficient of $\phi_{y_0}^2$ is equal to $-1 - \alpha^{-1}(A^2 + B^2)$ and hence we get

Lemma 3.5.3. *Assume that $\Lambda\phi = 0$. Then in order that $\bar{c}_0 = 0$ on $y_1 = 0$ it is necessary and sufficient*

$$(\alpha + b) - \{1 + \alpha^{-1}(A^2 + B^2)\}\phi_{y_0}^2 = 0 \quad \text{on} \quad y_1 = 0.$$

We note that one can write

$$E_\lambda^{-1} O(\lambda^k) E_\lambda = \sum_{j=0} \lambda^{k+4-j} r_j(y, D)$$

where $r_j(y, D)$ are differential operators of second order. Then taking s large so that

$$E_\lambda^{-1} O(\lambda^{-s/2+3\nu}) E_\lambda = \sum_{j=0} \lambda^{-j} a_j(y, D)$$

and hence we have

$$E_\lambda P_\lambda(y, D) E_\lambda = E_\lambda Q_\infty(\tilde{y} D_2, D) E_\lambda + \sum_{j=0} \lambda^{-j} P_j(y, D).$$

Now the (second) eikonal equation that we have to solve is:

$$i\Lambda\phi = 2y_1\{\alpha y_1 \phi_{y_2} + i\alpha\phi_{y_1} + L_0(1, i)\phi_{y_0}\} = 0.$$

Since $\alpha \neq 0$ we solve this equation giving a initial data ψ on $y_1 = 0$:

$$\begin{cases} i\alpha\phi_{y_1} + L_0(1, i)\phi_{y_0} + \alpha y_1 \phi_{y_2} = 0 \\ \phi = \psi \quad \text{on} \quad y_1 = 0. \end{cases}$$

Taking Lemma 3.5.3 into account we impose ψ to verify

$$\psi_{y_0}^2 = \frac{\alpha + b}{1 + \alpha^{-1}(A^2 + B^2)}.$$

Assume now

$$\frac{\alpha + b}{1 + \alpha^{-1}(A^2 + B^2)} \in \mathbb{C} \setminus [0, \infty). \tag{3.24}$$

Then it is easy to see that there is $\gamma \in \mathbb{C}$ with $\operatorname{Im} \gamma < 0$ such that

$$\gamma^2 = \frac{\alpha + b}{1 + \alpha^{-1}(A^2 + B^2)} \in \mathbb{C} \setminus [0, \infty).$$

Then we take $\psi = \gamma y_0 + i y_2^2$. Then for the solution ϕ with initial data ψ we have $\bar{c}_0 = 0$ on $y_1 = 0$.

Lemma 3.5.4. *Assume* (3.24)*. Then there is a solution ϕ to $\Lambda\phi = 0$ such that $\bar{c}_0 = 0$ on $y_1 = 0$ and*

$$\operatorname{Im}[\lambda^2(y_2 + \frac{i}{2} y_1^2) + \lambda\phi] \geq -C^2 + \lambda(\operatorname{Im} \gamma) y_0 + \lambda y_2^2 + \frac{1}{4}\lambda^2 y_1^2$$

with some C where $\operatorname{Im} \gamma < 0$.

Proof. It remains to prove the last inequality. Since

$$\phi = \gamma y_0 + i y_2^2 + \phi_{y_1}(y_0, 0, y_2) y_1 + O(|y_1|^2)$$

it is clear that

$$\operatorname{Im}\phi \geq (\operatorname{Im}\gamma)y_0 + y_2^2 - C|y_1|.$$

Hence we have

$$\begin{aligned}\operatorname{Im}[\lambda^2(y_2 + \frac{i}{2}y_1^2) + \lambda\phi] &\geq \frac{1}{2}\lambda^2 y_1^2 + \lambda(\operatorname{Im}\gamma)y_0 + \lambda y_2^2 - C\lambda|y_1| \\ &= \frac{1}{4}(\lambda|y_1| - 2C)^2 - C^2 + \lambda(\operatorname{Im}\gamma)y_0 + \lambda y_2^2 + \frac{\lambda^2}{4}y_1^2 \\ &\geq -C^2 + \lambda(\operatorname{Im}\gamma)y_0 + \lambda y_2^2 + \frac{\lambda^2}{4}y_1^2\end{aligned}$$

which proves the assertion. □

We now recall that we are looking for a null asymptotic solution to $P_\lambda(y, D)$ of the form

$$E_\lambda \sum_{j=0} v_j(y)\lambda^{-j}.$$

Then the transport equations are

$$(\Lambda + \bar{c}_0)v_{l+1} + Rv_l + \sum_{j=1}^{l} P_j v_{l-j} = 0, \quad l = -1, 0, 1, \ldots \tag{3.25}$$

where $P_j(y, D)$ are differential operators of order two. We first note that the coefficient of D_0 in R is different from zero, that is

$$L_0(y_1\phi_{y_2}, \phi_{y_1}) - \phi_{y_0} \neq 0. \tag{3.26}$$

To see this we plug $y_1\phi_{y_2} = -i\phi_{y_1} - \alpha^{-1}L_0(1, i)\phi_{y_0}$, which results from (3.22), into (3.26) to get

$$\begin{aligned}L(-i\phi_{y_1} - \alpha^{-1}L_o(1, i)\phi_{y_0}, \phi_{y_1}) - \phi_{y_0} &= -\alpha^{-1}BL_0(1, i)\phi_{y_0} - \phi_{y_0} \\ = -(1 + \alpha^{-1}BL_0(1, i))\phi_{y_0} &= -(1 + \alpha^{-1}B^2 + i\alpha^{-1}AB)\phi_{y_0}.\end{aligned}$$

This is different from zero as claimed because $\phi_{y_0} \neq 0$.
To see the structure of the equations (3.25) we make the linear change of coordinates

$$x_0 = y_0 + ky_1, \ \ x_1 = y_1, \quad k = -L(1, i)/i\alpha.$$

Switching to the new coordinates x we have

$$\frac{1}{2}\Lambda = \alpha x_1^2 D_2 + i\alpha x_1 D_1$$

that is, Λ contains no D_0. In the coordinates x, R can be written as

$$R = a(x)D_o + b(x)D_1 + c(x)D_2.$$

Here we note that

$$a(0) \neq 0. \tag{3.27}$$

EXERCISE: Check (3.27). Since we are interested in asymptotic solutions, in

what follows, we assume that all coefficients occurring in Λ, R, P_j are formal power series in x. In particular

$$\bar{c}_0 = \sum_{j-1} c_j(x_b)x_1^j, \quad x_b = (x_0, x_2)$$

and $c_j(x_b)$ are formal power series in x_b.

Lemma 3.5.5. *Let f be a formal power series in x. Then the equation*

$$(\Lambda + \bar{c}_0)w = f$$

is solvable (in formal power series in x) if and only if $f(x) = 0$ on $x_1 = 0$.

Proof. Necessity is obvious. Let

$$w = \sum_{j=0} w_j(x_b)x_1^j.$$

Then it is clear that

$$(\Lambda + \bar{c}_0)w = \sum_{j=1} (i^{-1}\alpha j w_j(x_b) + \alpha\partial_{x_2} w_{j-2}(x_b) + c_j(x_b))x_1^j.$$

Thus if $f = \sum_{j=1} f_j(x_b)x_1^j$, it is enough to choose $w_j(x_b)$ such that

$$i^{-1}\alpha j w_j(x_b) + \alpha\partial_{x_2} w_{j-2}(x_b) + c_j(x_b) = f_j(x_b), \ j = 1, 2, \ldots \tag{3.28}$$

with $w_{-1}(x_b) = 0$. □

Remark 3.5.1. In (3.28), $w_0(x_b)$ is free.

Lemma 3.5.6. *Let v be a formal power series in x. Then for any $N \in \mathbb{N}$ there exists a formal power series w in x such that*

$$\begin{cases} (\Lambda + \bar{c}_0)w = O(|x|^N) \\ Rw + v = O(|x_b|^N) \quad \text{on} \quad x_1 = 0. \end{cases}$$

Proof. Take a w which verifies $(\Lambda + \bar{c}_0)w \sim 0$. Recall that $w_0(x_b)$ is still free. We show that one can choose $w_0(x_b)$ so that the second equation will be satisfied. Writing $R = \tilde{a}(x_b)D_0 + \tilde{b}(x_b)D_1 + \tilde{c}(x_b)D_2 + x_1\tilde{R}$ it is clear that

$$Rw|_{x_1=0} = (\tilde{a}(x_b)D_0 + \tilde{c}(x_b)D_2)w_0(x_b) + i^{-1}\tilde{b}(x_b)w_1(x_b).$$

On the other hand, $w_1(x_b)$ verifies

$$i\alpha w_1(x_b) + c_1(x_b)w_0(x_b) = 0$$

and hence $w_1(x_b) = i\alpha^{-1}c_1(x_b)w_0(x_b)$. Thus we get

$$\begin{aligned} Rw|_{x_1=0} &= \{\tilde{a}(x_b)D_0 + \tilde{c}(x_b)D_2 + \alpha^{-1}\tilde{b}(x_b)c_1(x_b)\}w(0, x_b) \\ &= K(x_b, D_b)w(0, x_b). \end{aligned}$$

We take $w^*(x_b)$ as a formal power series solution to

$$K(x_b, D_b)w^*(x_b) = -v|_{x_1=0} + O(|x_b|^N).$$

If we take $w_0(x_b) = w^*(x_b)$, this is a desired solution. □

Proposition 3.5.1. *Let $N \in \mathbb{N}$ be fixed. Then we can find $v_j(x)$, $j = 0, 1, \dots$ with $v_0(0) \neq 0$ which are formal power series in x such that*

$$(\Lambda + \bar{c}_0)v_{l+1} + Rv_l + \sum_{j=1}^{l} P_j v_{l-j} = O(|x|^N). \tag{3.29}$$

Proof. Assume that we have solved $v_0, \dots, v_l$ with $v_0(0) \neq 0$. From Lemma 3.5.6 one can find w such that

$$\begin{cases} (\Lambda + \bar{c}_0)w = O(|x|^N) \\ Rw + Rv_l + \sum_{j=1}^{l} P_j v_{l-j} = O(|x_b|^N) \quad \text{on} \quad x_1 = 0. \end{cases}$$

We now change v_l to $v_l + w$ and denote it by the same v_l again. Then it is clear that v_l verifies (3.29) with l replaced by $l - 1$ and moreover

$$Rv_l + \sum_{j-1}^{l} P_j v_{l-1} = O(|x_b|^N) \quad \text{on} \quad x_1 = 0.$$

Hence by Lemma 3.5.5 there exists v_{l+1} satisfying (3.29). Then the proof carried out by induction. □

Completion of the proof of Theorem 3.1.2 in the first case of (I): As we have proved, assuming (3.24), one can construct an asymptotic null solution to $P_\lambda(y, D)$ so that for any $N \in \mathbb{N}$ there is $p(N) \in \mathbb{N}$ such that

$$P_\lambda(y, D)E_\lambda \sum_{j=0}^{p(N)} v_j(y)\lambda^{-j} = E_\lambda O(|x| + \lambda^{-1})^N. \tag{3.30}$$

By Lemma 3.5.4, we have

$$|E_\lambda| \leq \exp\{-\frac{1}{4}\lambda^2 y_1^2 - \lambda y_2^2 - \lambda(\operatorname{Im}\gamma)y_0 + C^2\}$$

and it is clear that

$$|y|^N |E_\lambda| \leq C_N \lambda^{-N} \quad \text{in} \quad y_0 \leq 0.$$

Then the right-hand side of (3.30) is $O(\lambda^{-N})$. Thanks to Lemma 3.5.4, repeating the same arguments as in the proof of Theorem 2.1.4 or Theorem 2.1.6, we conclude that this asymptotic solution gives a contradiction to the inequality in Corollary 2.3.2. Thus we proved that the condition

$$\frac{\alpha+b}{1+\alpha^{-1}(A^2+B^2)}\in\mathbb{R}_+ \tag{3.31}$$

is necessary for the well-posedness. This means that $b\in\mathbb{R}$ and $b\geq -\alpha$ is necessary. Changing $\xi_2\to-\xi_2$ and repeating the same arguments we get $-b\geq\alpha$ is also necessary. Thus recalling that $\alpha=\mathrm{Tr}^+F_p(z)/2$ and $b=P_{\rm sub}(z)$ we have finished the proof in the first case of (I). □

In the remaining cases of (I) we have

$$Q_\infty(\tilde{x},\xi;\lambda)=-\xi_0^2+b\xi_2+\left\{\begin{array}{l}\lambda^{-2\nu}x_1^2\\ \lambda^{-2\nu}\xi_1^2.\end{array}\right.$$

We take as E_λ

$$E_\lambda=\exp\{i\lambda^2y_2+i\lambda\phi\}$$

where ϕ has to satisfy

$$\phi_{y_0}^2=b.$$

Then the construction of asymptotic solution which contradicts the inequality in Corollary 2.3.2 is just a repetition of those we did in the proof of Theorem 2.1.4. Thus one can prove that

$$b\geq 0$$

is necessary for the well-posedness. Then changing $\xi_2\to-\xi_2$, we conclude that $b=0$ is necessary.

3.6 Proof of Theorem 3.1.2 (continued)

In this section we study the cases (II) and (III) in Theorem 3.5.1. Let

$$E_\lambda=\exp\{i\lambda^2y_2+i\lambda\phi\}$$

and compute $E_\lambda^{-1}Q_\infty(\tilde{y}D_2,D;\lambda)E_\lambda$. Then we have

Proposition 3.6.1. *In (II) we have*

$$\begin{aligned}\lambda^{-1}E_\lambda^{-1}Q_\infty E_\lambda&=\lambda\{2c\phi_{y_0}\phi_{y_1}+y_1^2+b\}\\ &+\{2c\phi_{y_0}D_1+2c\phi_{y_1}D_0+2y_1^2\phi_{y_2}+b\phi_{y_2}+2c(D_0\phi_{y_1})\}\\ &+\lambda^{-1}h^{(1)}(y,D)+\lambda^{-2}h^{(2)}(y,D)+\lambda^{-3}h^{(3)}(y,D).\end{aligned}$$

In (III), we have

$$\begin{aligned}\lambda^{-1}E_\lambda^{-1}Q_\infty E_\lambda&=\lambda\{2cy_1\phi_{y_0}+\phi_{y_1}^2+b\}\\ &+\{2cy_1D_0+2c\phi_{y_1}D_1+2cy_1\phi_{y_2}\phi_{y_0}+k\phi_{y_2}+\phi_{y_1}\phi_{y_0}+(D_1\phi_{y_1})\}\\ &+\lambda^{-1}h^{(1)}(y,D)+\lambda^{-2}h^{(2)}(y,D)+\lambda^{-3}h^{(3)}(y,D)\end{aligned}$$

where $h^{(j)}(y,D)$ are differential operators of order at most two.

EXERCISE: Check the proposition. We first study the case (II). Assume

$$b \in \mathbb{C} \setminus [0, \infty). \tag{3.32}$$

The phase function ϕ should verify

$$2c\phi_{y_0}\phi_{y_1} + y_1^2 + b = 0. \tag{3.33}$$

We divide the cases into two: (i) $\operatorname{Im} b \neq 0$. We solve the equation (3.33) under the condition

$$\phi = \gamma y_1 + iy_1^2 + iy_2^2 \quad \text{on} \quad y_0 = 0$$

where $\gamma \in \mathbb{R}$ has to be chosen so that $\operatorname{Im} \phi_{y_0}(0) < 0$. This is possible because

$$\phi_{y_0} = -\frac{b}{2c\gamma} \quad \text{on} \quad y_0 = 0,\ y_1 = 0$$

and $\operatorname{Im} b \neq 0$. Then we have

$$\phi_{y_0}(y) = -\frac{b}{2c\gamma} + O(|y_a|), \quad y_a = (y_1, y_2)$$

and hence

$$\phi(y) = \gamma y_1 + iy_1^2 + iy_2^2 + \left(-\frac{b}{2c\gamma} + O(|y_a|)\right) y_0 + O(y_0^2).$$

Therefore we get

$$\operatorname{Im} \phi \geq y_1^2 + y_2^2 + (-\operatorname{Im} \frac{b}{2c\gamma} + C|y_a|)y_0 \quad \text{for} \quad y_0 \leq 0.$$

Writing the right-hand side as

$$\begin{aligned} &|y_a|^2 - (\operatorname{Im} \frac{b}{2c\gamma})y_0 + \frac{1}{2}(\epsilon^{-1}y_0 + \epsilon C|y_a|)^2 - \frac{\epsilon^{-2}}{2}y_0^2 - \frac{\epsilon^2 C^2}{2}|y_a|^2 \\ &= (1 - \frac{\epsilon^2 C^2}{2})|y_a|^2 - (\operatorname{Im} \frac{b}{2c\gamma} + \frac{\epsilon^{-2}}{2}y_0)y_0 + \frac{1}{2}(\epsilon^{-1}y_0 + \epsilon C|y_a|)^2 \end{aligned}$$

it is clear that $-\operatorname{Im} \phi$ attains its strict maximum at $y = 0$ in $\{y \mid |y|^2 < \delta, y_0 \leq 0\}$ where $\delta > 0$ is enough small.

(ii) $b = -\gamma^2$, $\gamma < 0$. We solve the equation (3.33) under the condition

$$\phi = (\frac{i\gamma}{c})y_0 + iy_2^2 \quad \text{on} \quad y_1 = \gamma.$$

Since $\phi_{y_0} = i\gamma/c$ on $y_1 = \gamma$ it follows that $\phi_{y_1} = 0$ on $y_1 = \gamma$ and hence

$$2y_1 + 2c\phi_{y_0}\phi_{y_1y_1} = 0 \quad \text{on} \quad y_1 = \gamma.$$

This proves that $\phi_{y_1y_1} = i$ on $y_1 = \gamma$. Thanks to the above observations one has

$$\phi = (\frac{i\gamma}{c})y_0 + iy_2^2 + \frac{i}{2}(y_1 - \gamma)^2 + O(|y_1 - \gamma|^3).$$

Thus we have

$$-\operatorname{Im}\phi \geq \frac{\gamma}{c}y_0 + y_2^2 + \frac{1}{2}(y_1 - \gamma)^2 - C|y_1 - \gamma|^3$$

and it is clear that, in a set $\{y \mid y_0 \leq 0, y_0^2 + |y_1 - \gamma|^2 + y_2^2 \leq \delta\}$, $\delta > 0$ is small, $-\operatorname{Im}\phi$ attains its strict maximum at $(0, \gamma, 0)$. Note that we solve the transport equations in the direction y_1 near $(0, \gamma, 0)$ which is possible because $\phi_{y_0} \neq 0$. We turn to the case (III). Assume (3.32). The equation that we have to solve is

$$2cy_1\phi_{y_0} + \phi_{y_1}^2 + b = 0. \tag{3.34}$$

We devide the cases into two again. (i) $\operatorname{Im} b \neq 0$. Take y_1^* so that

$$\operatorname{Im}\frac{b}{cy_1^*} > 0$$

and work near the point $y^* = (0, y_1^*, 0)$. We solve the equation (3.34) imposing the condition

$$\phi = (y_1 - y_1^*) + i(y_1 - y_1^*)^2 + iy_2^2 \quad \text{on} \quad y_0 = 0.$$

Noticing $\phi = (y_1 - y_1^*) + i(y_1 - y_1^*)^2 + iy_2^2 + \phi_{y_0}(0, y_a)y_0 + O(y_0^2)$ we get

$$\operatorname{Im}\phi \geq (y_1 - y_1^*)^2 + y_2^2 + (\operatorname{Im}\phi_{y_0}(0, y_1^*, 0) + C|y_a - y_a^*|)y_0$$

with some $C > 0$ in $y_0 \leq 0$ because $\operatorname{Im}\phi_{y_0}(y^*) < 0$. Writing with $\alpha = \operatorname{Im}\phi_{y_0}(y^*)$ we have

$$\begin{aligned}(y_1 - y_1^*)^2 + y_2^2 + \alpha y_0 + \frac{1}{2}(\epsilon^{-1}y_0 + \epsilon C|y_a - y_a^*|)^2 - \frac{\epsilon^{-2}}{2}y_0^2 - \frac{\epsilon^2C^2}{2}|y_a - y_a^*|^2 \\ = (1 - \frac{\epsilon^2C^2}{2})|y_a - y_a^*|^2 + (\alpha - \frac{\epsilon^{-2}}{2}y_0)y_0 + \frac{1}{2}(\epsilon^{-1}y_0 + \epsilon C|y_a - y_a^*|)^2.\end{aligned}$$

Thus $-\operatorname{Im}\phi$ attains its strict maximum at y^* in a set $\{y \mid |y - y^*| < \delta, y_0 \leq 0\}$ if $\delta > 0$ is small enough. Note that the principal part of the transport equation is $2\phi_{y_1}D_1 + 2cy_1D_0$ and $y_1 \neq 0$ in an open set where we are working. (ii) $b = -\gamma^2$. We solve the equation (3.34) under the condition

$$\phi = -iy_0 + iy_2^2 \quad \text{on} \quad y_1 = 0.$$

That is, one solves the equation $\phi_{y_1} = \sqrt{\gamma^2 - 2cy_1\phi_{y_0}}$. It is clear that

$$\phi_{y_1} = (\gamma + i\frac{cy_1}{\gamma}) + O(y_1^2).$$

Note that the sign of γ is still free. One can write

$$\phi = -iy_0 + iy_2^2 + (\gamma + i\frac{cy_1}{\gamma})y_1 + O(y_1^3)$$

and hence $\operatorname{Im}\phi \geq -y_0 + y_2^2 + (c/\gamma)y_1^2 - C|y_1|^3$. Choose the sign of γ so that $c/\gamma > 0$. Then $-\operatorname{Im}\phi$ attains its strict maximum at $y = 0$ in a set $\{y \mid |y| < \delta, y_0 \leq 0\}$ where $\delta > 0$ is small. Noting that ϕ_{y_1} is different from zero in an open set we are working, we solve the transport equations in the direction y_1.

In both cases (II) and (III), assuming (3.32) one can construct an asymptotic solution which gives a contradiction. Hence the condition $b \geq 0$ is necessary for the well-posedness. As before, changing $\xi_2 \to -\xi_2$, we conclude that $b \leq 0$ is also necessary. Thus we have proved the assertion. □

References

[1] L.Hörmander: The Analysis of Linear Partial Differential Operators, III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.

[2] V.Ja.Ivrii, V.M.Petkov: Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed. Uspehi Mat. Nauk. **29** (1974) 3-70.

[3] L.Hörmander: The Cauchy problem for differential equations with doube characteristics. J.Analyse Math. **32** (1977) 118-196.

Chapter 4

Appendix

4.1 A lemma on changes of local coordinates

To prove Theorem 3.1.5 in full generality, we need to extend Lemma 3.4.5. Let G_k be the group of linear transformations on

$$T^*\mathbb{R}^k = \{(x,\xi) \mid x = (x_1,\dots,x_k), \xi = (\xi_1,\dots,\xi_k) \in \mathbb{R}^k\}$$

which is generated by

$$(x,\xi) \mapsto (Ax, {}^tA^{-1}\xi) \tag{4.1}$$

with a non singular real $k \times k$ matrix A and

$$(x,\xi) \mapsto (x, \xi + Ax) \tag{4.2}$$

with a real symmetric matrix A. Note that (4.1) consists of a linear change of coordinates $y = Ax$ and the corresponding contragredient change of coordinates $\eta = {}^tA^{-1}\xi$ in the dual space. It is also noted that (4.2) consists of a quadratic change of coordinates in x space with one more coordinate and the corresponding change of coordinates in the dual space.

Lemma 4.1.1. *Let $l_i(x,\xi)$, $i = 1,\dots,2k-1$ be linear functions in (x,ξ) which are linearly independent. Then there exists $T \in G_k$ such that the line*

$$\{(x,\xi) \mid l_i(T(x,\xi)) = 0, 1 \le i \le 2k-1\}$$

is either x_k axis or ξ_k axis.

Proof. We proceed by induction on k. When $k = 1$ let $l_1 = \alpha x_1 + \beta\xi_1$. If $\beta \neq 0$ then by $(x_1,\xi_1) \to (x_1, \xi_1 - (\alpha/\beta)x_1)$, l_1 turns to be $\beta\xi_1$. If $\beta = 0$ then $l_1 = \alpha x_1$ and nothing to be proved. Assume that the assertion is true in $T^*\mathbb{R}^{k-1}$. Let

$$l_1(x,\xi) = a_1x_1 + \cdots + a_kx_k + b_1\xi_1 + \cdots + b_k\xi_k.$$

If $(b_1,\dots,b_k)\neq(0,\dots,0)$, by $(x,\xi)\to(x,\xi-(a/b)x)$, where

$$\frac{a}{b}x=(\frac{a_ix_i}{b_i}),\quad b_i\neq 0$$

and renaming the coordinates if necessary, one can assume

$$l_1=a_1\xi_1+\cdots+a_l\xi_l+a_{l+1}x_{l+1}+\cdots+a_kx_k.$$

Let A be the real symmetric $k\times k$ matrix such that the first row is

$$(0,\dots,0,-a_1^{-1}a_{l+1},\dots,-a_1^{-1}a_k)$$

and the remaining rows are zero. Then by $(x,\xi)\to(x,\xi+Ax)$, l_1 is reduced to $l_1=\xi_1$. If $(b_1,\dots,b_k)=(0,\dots,0)$, then by the linear change $a_1x_1+\cdots+a_kx_k\to x_1$, l_1 becomes x_1. (i) When $l_1=\xi_1$. Subtracting

constant times l_1 from l_j $(j\geq 2)$ we may assume that l_j $(j\geq 2)$ contains no ξ_1. If l_j $(j\geq 2)$ contains no x_1, then it is clear that

$$\{l_j=0,\ 2\leq j\leq 2k-1\}=\{x_2=\cdots=x_k=0,\xi_2=\cdots=\xi_k=0\}$$

because l_j $(2\leq j\leq 2k-1)$ are linearly independent. Then exchanging x_k and x_1 we get the desired assertion. So we may assume that now

$$l_2=x_1+q(x_2,\dots,x_k,\xi_2,\dots,\xi_k).$$

By subtracting constant times l_2 from l_j $(j\geq 3)$ we may assume that l_j $(j\geq 3)$ contains neither x_1 nor ξ_1. By the induction hypothesis, we may assume that $E=\{l_3=\cdots=l_{2k-1}=0\}$ is either ξ_k axis or x_k axis. We first assume that E is the ξ_k axis. Let us write

$$l_2=x_1+a\xi_k+\tilde{q}(x_2,\dots,x_k,\xi_2,\dots,\xi_{k-1}).$$

If $a=0$ nothing to be proved. Let $a\neq 0$. By

$$\xi_1\to\xi_1-\frac{1}{a}x_k,\quad \xi_k\to\xi_k-\frac{1}{a}x_1$$

we have

$$l_1=\xi_1-\frac{1}{a}x_k,\ \ l_2=a\xi_k+\tilde{q}(x_2,\dots,x_k,\xi_2,\dots,\xi_{k-1}).$$

Since this change preserves x and $\xi_2,\dots,\xi_{k-1}$ coordinates, one conclude that the set $\{l_j=0,1\leq j\leq 2k-1\}$ is x_1 axis. To get the assertion it is enough to exchange x_1 and x_k.
Let E be the x_k axis. Write

$$l_2=x_2+ax_k+\tilde{q}(x_2,\dots,x_{k-1},\xi_2,\dots,\xi_k).$$

If $a=0$ nothing to be proved. Let $a\neq 0$. Make the change of coordinates

$$y_1=x_1+ax_k,\ y_j=x_j,\ (2\leq j\leq k),\ \xi_j=\eta_j,\ (1\leq j\leq k-1),\ \xi_k=\eta_k+a\eta_1.$$

Then we get $l_1 = \xi_1$, $l_2 = x_1 + \tilde{q}(x_2, \ldots, x_{k-1}, \xi_2, \ldots, \xi_{k-1}, \xi_k + a\xi_1)$. This proves

$$\{l_j = 0, 1 \leq j \leq 2k-1\}$$
$$= \{\xi_1 = 0, l_2 = 0, x_2 = \cdots = x_{k-1} = 0, \xi_2 = \cdots = \xi_{k-1} = \xi_k + a\xi_1\}.$$

The left-hand side is x_k axis. When $l_1 = x_1$. Subtracting constant times l_1 from l_j $(j \geq 2)$ we may assume that l_j $(j \geq 2)$ contains no x_1. If l_j $(j \geq 2)$ contains no ξ_1 then $\{l_j = 0, j \geq 2\}$ is the (x_1, ξ_1) space and exchanging ξ_1 and ξ_k we get the result. Thus one can assume

$$l_2 = \xi_1 + q(x_2, \ldots, x_k, \xi_2, \ldots, \xi_k).$$

Subtracting constant times l_2 from l_j $(j \geq 3)$ we may suppose that l_j $(j \geq 3)$ contains neither x_1 nor ξ_1. By the induction hypothesis one can assume that E is either ξ_k axis of x_k axis. The rest of the proof is similar as those before. □

EXERCISE: Complete the proof of Lemma 4.1.1.

4.2 A lemma on positive definite quadratic forms

In Section 3, the phase function $\phi = iy_1^2/2$ has to be chosen so that

$$Q(y_1, \phi_{y_1}) = 0, \quad Q = y_1^2 + D_1^2.$$

In the general case that $Q(x, \xi)$ is a positive definite quadratic form in $T^*\mathbb{R}^d$ we ask ourselves whether one can find a real nonsingular A and a real symmetric B such that with

$$\tilde{Q}(x, \xi) = Q(A^{-1}x, {}^tA\xi + BA^{-1}x)$$

we have $\tilde{Q}(x, ix) = 0$. That is $\phi(x) = \exp\{i/2 \sum_{j=1}^n x_j^2\}$ verifies $Q(x, \phi_x(x)) = 0$.
Let $Q(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)$ be a positive definite quadratic form on $T^*\mathbb{R}^d$. F_Q is defined by

$$Q(u, v) = \sigma(u, F_Q v) \quad \forall u, v \in T^*\mathbb{R}^d.$$

As we have seen in Section 3, all eigenvalues of F_Q are non-zero pure imaginary. Let us set

$$V^+ = \sum_{\mu > 0} \oplus V_{i\mu}$$

where the sum is taken over all eigenvalues with positive imaginary part. From Lemma 3.3.3 it follows that

$$Q(V^+, V^+) = 0. \tag{4.3}$$

Lemma 4.2.1. *The maps* $V^+ \ni v \mapsto \operatorname{Re} v \in T^*\mathbb{R}^d$, $V^+ \ni v \mapsto \operatorname{Im} v \in T^*\mathbb{R}^d$ *are both bijective.*

Proof. As we have seen in Lemma 3.3.5, for $v \in V^+$ we have $Q(v, \bar{v}) > 0$. Writing $v = \sum_\mu v_\mu$, $v_\mu \in V_{i\mu}$, one has

$$Q(v, \bar{v}) = \sum_\mu 2\mu\sigma(\operatorname{Im} v_\mu, \operatorname{Re} v_\mu) > 0$$

(see Lemma 3.3.5). If $\operatorname{Re} v = 0$ then $Q(v, \bar{v}) = 0$ and hence $v = 0$. Since the dimension of V^+ over $\mathbb{R}$ is $2d$ this is a bijection. □

From Lemma 4.2.1 for any $v \in T^*\mathbb{R}^d$ there is a $w \in V^+$ such that $\operatorname{Re} w = v$ so that one can define the bijective map J as

$$J : T^*\mathbb{R}^d \ni v \mapsto \operatorname{Im} w \in T^*\mathbb{R}^d, \quad \operatorname{Re} w = v, \ w \in V^+.$$

Here we remark that

$$(x, \xi) + i(y, \eta) \in V^+ \Longleftrightarrow J \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} y \\ \eta \end{pmatrix}. \tag{4.4}$$

Note that $\sigma(Ju, v) = -\sigma(u, Jv)$ and $J^2 = -I$. EXERCISE: Prove $\sigma(Ju, v) = -\sigma(u, Jv)$.

Lemma 4.2.2. *Let* $A(v, w) = \sigma(Jw, v)$, v, $w \in T^*\mathbb{R}^d$. *Then* A *is positive definite.*

Proof. Let $v \in T^*\mathbb{R}^d$ then there is $u \in V^+$ such that $\operatorname{Re} u = v$. Then

$$\sigma(Jv, v) = \sigma(J \operatorname{Re} u, \operatorname{Re} u) = \sigma(\operatorname{Im} u, \operatorname{Re} u) = \sum_\mu \sigma(\operatorname{Im} u_\mu, \operatorname{Re} u_\mu) > 0.$$

□

Write

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}.$$

Lemma 4.2.3. J_{21} *(resp.* J_{12}*) is positive (resp. negative) definite on* $\mathbb{R}^d$.

Proof. Since

$$\sigma(J(x, \xi), (y, \eta)) = \langle J_{21}x, y\rangle + \langle J_{22}\xi, y\rangle - \langle J_{11}x, \eta\rangle - \langle J_{12}\xi, \eta\rangle$$

and J is skew symmetric with respect to σ we conclude that J_{12} and J_{21} are symmetric and ${}^tJ_{11} = -J_{22}$. From Lemma 4.2.1 it follows that J_{21} is positive definite and J_{12} is negative definite. □

Lemma 4.2.4. *Assume that* $Q(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)$ *is positive definite on* $T^*\mathbb{R}^d$*. Then there are a real non singular* A *and a real symmetric* B *such that*

$$\tilde{Q}(x,\xi) = Q(A^{-1}x, {}^tA\xi + BA^{-1}x)$$

verifies $\tilde{Q}(x, ix) = 0$ *for all* $x \in \mathbb{R}^d$*.*

Proof. Recall that

$$\tilde{Q}(x, ix) = Q((A^{-1}x, BA^{-1}x) + i(0, {}^tAx)).$$

From (4.3) it is enough to show that

$$(A^{-1}x, BA^{-1}x) + i(0, {}^tAx) \in V^+.$$

By (4.4) it is equivalent to

$$J \begin{pmatrix} A^{-1}x \\ BA^{-1}x \end{pmatrix} = \begin{pmatrix} 0 \\ {}^tAx \end{pmatrix}.$$

This is again equivalent to

$$\begin{pmatrix} A^{-1}x \\ BA^{-1} \end{pmatrix} = -J \begin{pmatrix} 0 \\ {}^tAx \end{pmatrix} \tag{4.5}$$

because $J^2 = -I$. Then (4.5) turns to $A^{-1} = -J_{12}{}^tA$, $BA^{-1} = -J_{22}{}^tA$. That is

$$I = -AJ_{12}{}^tA, \quad B = -J_{22}{}^tAA. \tag{4.6}$$

We first show that there is A such that $A^{-1} = -J_{12}{}^tA$. Since $\langle -J_{12}u, v\rangle$, u, $v \in \mathbb{R}^d$ becomes an inner product on $\mathbb{R}^d$, we choose an orthonormal basis $e_1, \ldots, e_d$ with respect to this inner product. Define $A = (e_1, \ldots, e_d)$ then it is clear that ${}^tA$ is a desired one. We next show that $B = -J_{22}{}^tAA$ is symmetric. Since $AJ_{12} = -{}^tA^{-1}$ then we get

$$B = J_{22}J_{12}^{-1}.$$

Note that $J_{11}J_{12} + J_{12}J_{22} = 0$ because $J^2 = -I$. Hence $J_{12}{}^tJ_{11} + {}^tJ_{22}J_{12} = 0$. Then

$${}^tB = {}^tJ_{12}^{-1}{}^tJ_{22} = -{}^tJ_{12}^{-1}J_{11} = {}^tJ_{12}^{-1}(J_{12}J_{22}J_{12}^{-1}) = J_{22}J_{12}^{-1} = B.$$

so that B is symmetric. □

Exercise: In $\mathbb{R}^4$ we consider

$$Q = \xi_1^2 + 2(x_1 - \xi_2)^2 + \frac{1}{2}x_2^2 + 4\xi_2^2.$$

Compute A, B explicitly.

4.3 References

[1] V.Ja.Ivrii: Linear hyperbolic equations. in Partial Differential Equations IV, Encyclopaedia Math. Sci. **33**, Springer, 1988 pp.149-235.
[2] L.Gårding: Hyperbolic differential operators. Perspectives in Mathematics, Anniversary of Oberwolfach, Birkhäuser, 1984 pp.215-247.
[3] L.Gårding: Hyperbolic equations in the twentieth century. Semin. Congr. **3**. Soc. Math. France, Paris, 1998 pp.37-68.
[4] R.Melrose: The Cauchy problem and propagation of singularities. Seminar on Nonlinear Partial Differential Equations, Springer, 1984 pp.185-201.