A note on the Cauchy problem for $-D_0^2+2x_1D_0D_2+D_1^2+x_1^3D_2^2+\sum_{j=0}^2 b_jD_j$

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Abstract

In this note, we improve a previously proved non-solvability result of the Cauchy problem in the Gevrey class for a homogeneous second order differential operator with polynomial coefficients. We prove that the Cauchy problem for the same operator is not locally solvable for any lower order term in the Gevrey class of order greater than 5, lowering the previous Gevrey order 6.

1 Introduction

In [9] considering a second order operator in $\mathbb{R}^{1+2}$

$$P_{mod} = -D_0^2 + 2x_1D_0D_2 + D_1^2 + x_1^3D_2^2, \quad x = (x_0, x') = (x_0, x_1, x_2) \tag{1.1}$$

we have proved

Theorem 1.1. ([9, Theorem 1.1]) *The Cauchy problem for $P_{mod}+\sum_{j=0}^2 b_jD_j$ is not locally solvable at the origin in the Gevrey class of order s for any $b_0, b_1, b_2 \in \mathbb{C}$ if $s > 6$. In particular the Cauchy problem for P_{mod} is C^∞ ill-posed near the origin for any $b_0, b_1, b_2 \in \mathbb{C}$.*

Recall that the Gevrey class of order s, denoted by $\gamma^{(s)}(\mathbb{R}^n)$, is the set of all $f(x) \in C^\infty(\mathbb{R}^n)$ such that for any compact set $K \subset \mathbb{R}^n$, there exist $C > 0$, $h > 0$ such that

$$|\partial_x^\alpha f(x)| \le Ch^{|\alpha|}|\alpha|!^s, \quad x \in K, \ \alpha \in \mathbb{N}^n. \tag{1.2}$$

We say that the Cauchy problem for $P = P_{mod} + \sum_{j=0}^2 b_jD_j$ is locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (u_0, u_1) \in (\gamma^{(s)}(\mathbb{R}^2))^2$ there exists a neighborhood U_Φ, may depend on Φ, of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 \quad \text{in} \ \ U_\Phi, \\ D_0^ju(0, x') = u_j(x'), \quad x' \in U_\Phi \cap \{x_0 = 0\}, \ \ j = 0, 1 \end{cases} \tag{1.3}$$

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has a solution $u(x) \in C^2(U_\Phi)$. In this note we remark that one can improve Theorem 1.1 so that

Theorem 1.2. *The Cauchy problem for $P_{mod}+\sum_{j=0}^{2} b_j D_j$ is not locally solvable in $\gamma^{(s)}$ at the origin for any $b_0, b_1, b_2 \in \mathbb{C}$ if $s > 5$.*

Before explaining a special role played by P_{mod} in studying of the Cauchy problem for differential operators with non-effectively hyperbolic characteristics, we give a short introduction to the general context. Let P be a differential operator of order m with principal symbol $p(x,\xi)$. At a singular point ρ of $p = 0$, the Hamilton map $F_p(\rho)$ is defined as the linearization at ρ of the Hamilton vector field H_p. The first fundamental result is

Theorem 1.3. ([5], [4]) *Let ρ be a singular point of $p = 0$ and assume that $F_p(\rho)$ has no non-zero real eigenvalues. If the Cauchy problem for P is C^∞ well posed it is necessary that*

$$\mathsf{Im} P_{sub}(\rho) = 0, \quad \big|P_{sub}(\rho)\big| \le \mathrm{Tr}^+ F_p(\rho)/2 \tag{1.4}$$

where $\mathrm{Tr}^+ F_p(\rho) = \sum |\mu_j|$ and μ_j are the eigenvalues of $F_p(\rho)$, counted with multiplicity, and P_{sub} is the subprincipal symbol of P.

We call (1.4) the Ivrii-Petkov-Hörmander condition (IPH condition, for short). If the strict inequality holds in (1.4) we call it the strict Ivrii-Petkov-Hörmander condition (strict IPH condition, for short). For the sufficiency of the IPH condition, we assume that the set of singular points Σ of $p = 0$ is a C^∞ manifold and the following conditions are satisfied:

$$\begin{cases} F_p \text{ has no non-zero real eigenvalues at each point of } \Sigma, \\ \text{Near each point of } \Sigma,\ p \text{ vanishes exactly of order 2} \\ \text{and the rank of } \sum_{j=0}^{n} d\xi_j \wedge dx_j \text{ is constant.} \end{cases} \tag{1.5}$$

According to the spectral type of $F_p(\rho)$, two different possible cases may arise

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) = \{0\}, \tag{1.6}$$

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\}. \tag{1.7}$$

We say that p is of spectral type 1 (resp. type 2) near $\rho \in \Sigma$ if there is a conic neighborhood V of ρ such that (1.6) (resp. (1.7)) holds in $V \cap \Sigma$. We say that there is no transition (of spectral type) if for any $\rho \in \Sigma$ one can find a conic neighborhood V of ρ such that either (1.6) or (1.7) holds in $V \cap \Sigma$.

Theorem 1.4. ([10, Theorem 5.1]) *Assume* (1.5) *and that there is no transition and no bicharacteristic tangent to Σ. Then the Cauchy problem for P is C^∞ well posed under the strict IPH condition.*

The principal symbol $p = -\xi_0^2 + 2x_1\xi_0\xi_2 + \xi_1^2 + x_1^3\xi_2^2$ of P_{mod} is a typical example of spectral type 2 with tangent bicharacteristics (note that there is no tangent bicharacteristic if p is of spectral type 1, see [6]). The set of singular point of $p = 0$ is $\Sigma = \{x_1 = 0, \xi_0 = \xi_1 = 0\}$ which is a C^∞ manifold on which p vanishes exactly of order 2 (note that we are working near $(0, e_3)$, $e_3 = (0, 0, 1)$) and a tangent bicharacteristic is given explicitly by

$$(x_1, x_2) = (-x_0^2/4, x_0^5/80), \quad (\xi_0, \xi_1, \xi_2) = (0, x_0^3/8, 1) \tag{1.8}$$

which is parametrized by x_0. The operator P_{mod} shows how the situation becomes to be complicated when a tangent bicharacteristic exists. We now give some such results: The Cauchy problem for P_{mod} is not locally solvable in $\gamma^{(s)}$ for $s > 5$, in particular ill-posed in C^∞ while the Cauchy problem for general second order operator P of spectral type 2 satisfying $P_{sub} = 0$ on Σ (note that the subprincipal symbol of P_{mod} is identically zero) is well posed in the Gevrey class of order $1 < s \leq 5$ ([1] see also [10]). The Cauchy problem for $P_{mod} + SD_2$ with $0 \neq S \in \mathbb{C}$ is not locally solvable in $\gamma^{(s)}$ for $s > 3$ (Proposition 2.1 below) while the Cauchy problem for general second order operator P of spectral type 2 with $\operatorname{codim} \Sigma = 3$ is well posed in the Gevrey class of order $1 < s < 3$ for any lower order term even if a tangent bicharacteristic exists ([2]).

2 A family of exact solutions

First recall

Lemma 2.1. ([10, Proposition 8.1]) *The Cauchy problem for* $P_{mod} + \sum_{j=0}^{1} b_j D_j$ *is not locally solvable in* $\gamma^{(s)}$ *at the origin for any* $b_0, b_1 \in \mathbb{C}$ *if* $s > 5$.

Thus in order to prove Theorem 1.2 it suffices to prove the following result which also improves [2, Theorem 1.3].

Proposition 2.1. *The Cauchy problem for* $P_{mod} + \sum_{j=0}^{2} b_j D_j$ *is not locally solvable in* $\gamma^{(s)}$ *at the origin for any* $b_0, b_1 \in \mathbb{C}$ *and* $0 \neq b_2 \in \mathbb{C}$ *if* $s > 3$.

To prove Proposition 2.1 we repeat the proof of [2, Theorem 1.3] with obvious minor chnges. Look for a family of solutions to $(P_{mod} + \sum_{j=0}^{2} b_j D_j)U_\lambda = 0$ in the form

$$U_\lambda(x) = e^{i\xi_0 \lambda x_0} V_\lambda(x'), \quad V_\lambda(x') = e^{\pm \lambda^5 x_2 - i(b_1/2)x_1} u(\lambda^2 x_1), \;\; \xi_0 = \xi_0(\lambda) \tag{2.1}$$

that is, we look for $u(x)$ satisfying

$$u''(x) = \big(x^3 + 2\xi_0 x + b_2 \lambda - \xi_0^2 \lambda^{-2} + b_0 \xi_0 \lambda^{-3} - b_1^2 \lambda^{-4}/4\big) u(x). \tag{2.2}$$

To study solutions to (2.2) we consider

$$u''(x) = (x^3 + a_2 x + a_3) u(x), \quad x \in \mathbb{C}, \;\; a_j \in \mathbb{C}, \;\; j = 2, 3. \tag{2.3}$$

Let $\mathcal{Y}_0(x; a)$, $a = (a_2, a_3)$ be the solution given in [11, Chapter 2] to (2.3) which has asymptotic representation

$$\mathcal{Y}_0(x; a) \simeq x^{-3/4} \big[1 + \sum_{N=1}^{\infty} B_N x^{-N/2}\big] e^{-E(x,a)} \tag{2.4}$$

as x tends to infinity in any closed subsector of the open sector $|\arg x| < 3\pi/5$ where

$$E(x,a) = \frac{2}{5}x^{5/2} + a_2 x^{1/2}$$

and A_N, B_N are polynomials in (a_2, a_3). Let $\omega = e^{2\pi i/5}$ and set

$$\mathcal{Y}_k(x;a) = \mathcal{Y}_0(\omega^{-k}x; \omega^{-2k}a_2, \omega^{-3k}a_3), \quad k = 0,1,2,3,4 \tag{2.5}$$

which are also solutions to (2.3). Recall that [11, Chpater 17]

$$\mathcal{Y}_k(x;a) = C_k(a)\mathcal{Y}_{k+1}(x;a) - \omega\mathcal{Y}_{k+2}(x;a)$$

where $C_k(a)$ are entire analytic in $a = (a_2, a_3)$ and $C_k(a_2, a_3) = C_0(\omega^{-2k}a_2, \omega^{-3k}a_3)$. Choose

$$\begin{aligned} u(x) = \mathcal{Y}_0(x;a) = C_0(a)\mathcal{Y}_1(x;a) - \omega\mathcal{Y}_2(x;a), \\ a_2 = 2\xi_0, \quad a_3 = b_2\lambda - \xi_0^2\lambda^{-2} + b_0\xi_0\lambda^{-3} - b_1^2\lambda^{-4}/4 \end{aligned} \tag{2.6}$$

which solves (2.2). We require that $V_\lambda(x')$ is bounded as $\lambda \to +\infty$ when $|x'|$ remains in a bounded set. Note (2.5) and

$$\begin{aligned} (\omega^{-1}x)^{5/2} = -i|x|^{5/2}, \quad 2\omega^{-2}\xi_0(\omega^{-1}x)^{1/2} = -2i\xi_0|x|^{1/2}, \\ (\omega^{-2}x)^{5/2} = i|x|^{5/2}, \quad 2\omega\xi_0(\omega^{-2}x)^{1/2} = 2i\xi_0|x|^{1/2} \end{aligned} \tag{2.7}$$

for $x < 0$ then $|\mathcal{Y}_1(\lambda^2 x; a_2, a_3)|$ and $|\mathcal{Y}_2(\lambda^2 x; a_2, a_3)|$ behaves like $e^{2\mathsf{Re}(i\xi_0)\lambda|x|^{1/2}}$ and $e^{-2\mathsf{Re}(i\xi_0)\lambda|x|^{1/2}}$ respectively as $x \to -\infty$. Since $\omega \neq 0$, taking (2.6) into account, the requirement for boundedness implies that

$$C_0(2\xi_0, a_3) = 0, \tag{2.8}$$

$$-\mathsf{Re}(i\xi_0) = \mathsf{Im}\,\xi_0 < 0. \tag{2.9}$$

Instead of solving directly the "Stokes equation" (2.8) we go rather indirectly. Let us consider

$$H(\beta) = p^2 + x^2 + i\beta x^3$$

as an operator in $L^2(\mathbb{R})$ with the domain $D(H(\beta)) = D(p^2) \cap D(x^3)$. Here p^2 denotes the self-adjoint realization of $-d^2/dx^2$ defined in $H^2(\mathbb{R})$ and by $D(x^3)$ we mean the domain of the maximal multiplication operator by the function x^3.

Proposition 2.2. [3, Corollary 2.16, Lemma 3.1]) *Let $k \in \mathbb{N}_0$ and $\epsilon > 0$ be given. Then there is a $B > 0$ such that for $|\beta| < B$, $\mathsf{Re}\,\beta > 0$, $H(\beta)$ has exactly one eigenvalue $E_k(\beta)$ near $2k+1$. Such eigenvalues are analytic functions of β for $|\beta| < B$, $\mathsf{Re}\,\beta > 0$, and admit an analytic continuation across the imaginary axis to the whole sector $|\beta| < B$, $|\arg\beta| < \frac{5\pi}{8} - \epsilon$ and uniformly asymptotic to the following formal Taylor expansion in powers of β^2*

$$\sum_{j=0}^{\infty} a_{2j}\beta^{2j}, \quad a_0 = 2k+1 \tag{2.10}$$

near $\beta = 0$ in any closed subsector in $|\arg\beta| < \frac{5\pi}{8} - \epsilon$.

For the proof we refer to [3], [12]. Now we have

Lemma 2.2. ([2, Lemma 6.1]) *Assume that* $\mathsf{Re}\,\beta > 0$ *and* $E(\beta)$ *is an eigenvalue of the problem*

$$-u''(x) + (x^2 + i\beta x^3)u(x) = E(\beta)u(x) \tag{2.11}$$

that is (2.11) *has a solution* $0 \neq u \in D(H(\beta))$. *Then we have*

$$C_0\Big(-\frac{\omega^2}{3}\beta^{-\frac{8}{5}},\,\omega^3\big\{\frac{2}{27}\beta^{-\frac{12}{5}} + \beta^{-\frac{2}{5}}E(\beta)\big\}\Big) = 0 \tag{2.12}$$

where $\beta^{-j/5} = (\beta^{-1/5})^j$ *and the branch* $\beta^{\pm 1/5}$ *is chosen such that* $|\arg\beta^{\pm 1/5}| < \pi/10$.

Let $E(\beta)$, $|\beta| < B$, $\mathsf{Re}\,\beta > 0$ be an eigenvalue which is analytically continued to the sector $|\beta| < B$, $|\arg\beta| < 5\pi/8$ by Proposition 2.2 (though when $|\arg\beta| = \pi/2$, $H(\beta)$ admits infinitely many distinct self-adjoint extensions, see [3]). Since $C_0(a_2, a_3)$ is entire analytic in (a_2, a_3) then (2.12) holds in this sector. Thanks to Lemma 2.2, if ξ_0 satisfies

$$\begin{cases} 2\xi_0 = -\dfrac{\omega^2}{3}\beta^{-\frac{8}{5}}, \\ a_3 = \omega^3\big\{\dfrac{2}{27}\beta^{-\frac{12}{5}} + \beta^{-\frac{2}{5}}E(\beta)\big\} \end{cases} \tag{2.13}$$

then (2.8) is satisfied hence we have

$$u(\lambda^2 x) = \mathcal{Y}_0(\lambda^2 x; 2\xi_0, a_3) = -\omega\mathcal{Y}_2(\lambda^2 x; 2\xi_0, a_3).$$

Therefore we look for ξ_0 satisfying (2.13) and (2.9). Plugging $\xi_0 = -\omega^2\beta^{-8/5}/6$ into the second equation, (2.13) is reduced to

$$\begin{aligned} \frac{2}{27}\omega^3\beta^{-\frac{12}{5}} + E(\beta)\omega^3\beta^{-\frac{2}{5}} + \frac{1}{36}\omega^4\beta^{-\frac{16}{5}}\lambda^{-2} \\ + \frac{b_0}{6}\omega^2\beta^{-8/5}\lambda^{-3} = b_2\lambda - \frac{b_1^2}{4}\lambda^{-4}. \end{aligned} \tag{2.14}$$

2.1 Solving the equation (2.14)

Solve (2.14) with respect to β where $E(\beta)$ is one of $E_k(\beta)$ which is analytic in $|\arg\beta| < 5\pi/8 - \epsilon$ and admits a uniform asymptotic expansion (2.10) there by Proposition 2.2. Put $\zeta = \beta^{-2/5} = (\beta^{-1/5})^2$ so that the equation leads to

$$\omega^3\zeta^6 + \frac{27}{2}E(\zeta^{-5/2})\omega^3\zeta + \frac{3}{8}\omega^4\zeta^8\lambda^{-2} \pm \frac{9}{4}b_0\omega^2\zeta^4\lambda^{-3} = \pm\frac{27}{2}b_2\lambda - \frac{27}{8}b_1^2\lambda^{-4} \tag{2.15}$$

and we look for a solution $\zeta(\lambda)$ to (2.15) verifying

$$|\arg\zeta(\lambda)| < \pi/4 - \epsilon, \quad \mathsf{Im}\,\omega^2\zeta(\lambda)^4 > 0 \tag{2.16}$$

where the second requirement comes from (2.9). Assume that

$$0 < \arg b_2 < \pi \quad \text{or} \quad -\pi \le \arg b_2 < -\pi/2 \tag{2.17}$$

and denote $27b_2/2$ by A for notational simplicity and look for $\zeta(\lambda)$ in the form

$$\begin{cases} \zeta(\lambda) = e^{-\pi i/5} A^{1/6}\big(1+\lambda^{-5/6}z\big)\lambda^{1/6} & (0 < \arg A < \pi), \\ \zeta(\lambda) = e^{2\pi i/15} A^{1/6}\big(1+\lambda^{-5/6}z\big)\lambda^{1/6} & (-\pi \le \arg A < -\pi/2). \end{cases}$$

It is clear that $\zeta(\lambda)$ verifies (2.16) provided that z is bounded and λ is large. Note that $E(\zeta^{-5/2})$ is analytic in $|\arg\zeta| < \pi/4-\epsilon$ for large $|\zeta|$ and verifies

$$\big|E(\zeta^{-5/2}) - a_0\big| \le C|\zeta|^{-5}$$

with some $a_0 = 2k+1$, $k\in\mathbb{N}$ uniformly in $|\arg\zeta| < \pi/4-\epsilon$ when $|\zeta|\to\infty$. We insert ζ into (2.15) to get

$$\begin{aligned} A\lambda(1+\lambda^{-5/6}z)^6 + \lambda^{1/6}H(z) + d_1\lambda^{-2/3}(1+\lambda^{-5/6}z)^8 \\ +d_2\lambda^{-7/3}(1+\lambda^{-5/6}z)^4 = A\lambda + d_3\lambda^{-4}, \quad d_i\in\mathbb{C} \end{aligned} \tag{2.18}$$

where $H(z)$ is analytic in $|z| < B$ for $\lambda \ge R$ and

$$|H(z) - a_0| \le C\lambda^{-5/6}, \quad \lambda \ge R.$$

Note that B can be chosen to be arbitrarily large taking R large. Thus one can write (2.18) as

$$6Az + a_0 + \lambda^{-5/6}F(z,\lambda) = 0 \tag{2.19}$$

where $F(z,\lambda)$ is analytic in $|z| < B$ and bounded uniformly in $|\lambda| \ge R$. We may assume that $|a_0/6A| < B/2$ taking R large as noted above. By Rouché's thoerem we conclude that the equation (2.19) has a solution $z(\lambda)$ with $|z| < B$ for any $|\lambda| \ge R_1$. Returning to β ($\zeta = \beta^{-2/5}$) we conclude that (2.14) has a solution of the form

$$\begin{cases} \beta(\lambda) = iA^{-5/12}\lambda^{-5/12}\big(1+\lambda^{-5/6}z(\lambda)\big) & (0 < \arg A < \pi), \\ \beta(\lambda) = e^{-\pi i/3}A^{-5/12}\lambda^{-5/12}\big(1+\lambda^{-5/6}z(\lambda)\big) & (-\pi \le \arg A < -\pi/2) \end{cases}$$

where $|z(\lambda)| < B$ for $\lambda \ge R_1$. Plugging this $\beta(\lambda)$ into (2.13) we get

Proposition 2.3. *Assume* (2.17). *Then there exists* $\xi_0 = \xi_0(\lambda)$ *such that*

$$\begin{cases} 2\xi_0 = c\,\lambda^{2/3}(1+\lambda^{-5/6}z(\lambda)), \quad \mathsf{Im}\, c < 0, \\ C_0(2\xi_0,\, b_2\lambda - \xi_0^2\lambda^{-2} + b_0\xi_0\lambda^{-3} - b_1^2\lambda^{-4}/4) = 0 \end{cases}$$

where $|z(\lambda)| < B$ *for* $\lambda > R$.

If we write

$$2\xi_0 = \lambda^{2/3}a(\lambda), \quad b_2\lambda - \xi_0^2\lambda^{-2} + b_0\xi_0\lambda^{-3} - b_1^2\lambda^{-4}/4 = \lambda b(\lambda) \tag{2.20}$$

it is clear that $|a(\lambda)|$, $|b(\lambda)| < M$ with some $M > 0$ for $\lambda \ge R_1$.

2.2 Asymptotics of $\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda)$ for large $|x|$ and λ

Recalling (2.20) we study how $\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda)$ behaves for large $|x|$ and large λ where $|a|, |b| \le M$ is assumed. In what follows $f = o_a(1)$ means that there are positive constants $C_a > 0$ and $\delta_a > 0$ such that

$$|f| \le C_a \lambda^{-\delta_a}, \quad \lambda \to +\infty.$$

We make the asymptotic representation (2.4) slightly precise.

Proposition 2.4. *Let $\rho > 1/3$ be given. Then one can write*

$$\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda) \simeq (1 + p_\rho(x,\lambda)) e^{R_\rho(x,\lambda)} x^{-3/4} \exp\{-E_\rho(x; a, b, \lambda)\},$$
$$\mathcal{Y}_0'(x; a\lambda^{2/3}, b\lambda) \simeq (-1 + p_\rho(x,\lambda)) e^{R_\rho(x,\lambda)} x^{3/4} \exp\{-E_\rho(x; a, b, \lambda)\}$$

as x tends to infinity in any closed subsector of

$$S_\lambda = \{x; |\arg x| < 3\pi/5, |x| > \lambda^\rho\}$$

where

$$E_\rho(x; a, b, \lambda) = \frac{2}{5} x^{5/2} + a\lambda^{2/3} x^{1/2} - b\lambda x^{-1/2} + r_\rho(x, \lambda)$$

and r_ρ is a polynomial in $x^{-1/2}$ such that

$$|r_\rho(x,\lambda)| = C\lambda^{4/3 - 3\rho/2}(1 + o_\rho(1)), \quad x \in S_\lambda$$

and $p_\rho(x,\lambda)$, $R_\rho(x,\lambda)$ are holomorphic in S_λ and in any closed subsector of S_λ

$$|p_\rho(x,\lambda)| \le C_\rho \lambda^{-2(\rho - 1/3)}, \quad |R_\rho(x,\lambda)| \le C_\rho \lambda^{-1}, \quad x \in S_\lambda$$

and $R_\rho(x,\lambda) \to 0$, $p_\rho(x,\lambda) \to 0$ as $|x| \to \infty$, $x \in S_\lambda$.

Proof. We follow Sibuya [11, Chapter 2] and [2, Proposition 2.3] with needed modifications. □

Lemma 2.3. *Assume that $\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda)$ verifies*

$$\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda) = -\omega \mathcal{Y}_2(x; a\lambda^{2/3}, b\lambda) \tag{2.21}$$

and

$$\mathsf{Im}\, a = -\delta(1 + o_a(1)) \tag{2.22}$$

with some $\delta > 0$. Let $X > 0$. There exist ℓ, $c > 0$, $C > 0$ such that for any $0 \le \mu < 5/6$ we have

$$\big|(d/dx)^k \mathcal{Y}_0(\lambda^2 x,\ a\lambda^{2/3}, b\lambda)\big| \le C(1 + o_\mu(1))\lambda^\ell$$
$$\times \exp\big\{ -\delta\lambda^{5/3}|x|^{1/2}(1 + o_a(1)) + C_\mu(\lambda^\mu + \lambda^{-5/3+3\mu})\big\}$$

for $k = 0, 1$ and $|x| \ge \lambda^{-2\mu} X$.

Proof. In Proposition 2.4 choose $\rho = 2(1-\mu)$ and estimate $\mathcal{Y}_0$ in $x \le -\lambda^{-2\mu}X$ first. Recall that for $x < 0$ we have

$$\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda) = -\omega\mathcal{Y}_2(x; a\lambda^{2/3}, b\lambda) = -\omega\mathcal{Y}_0(e^{\pi i/5}|x|; \omega a\lambda^{2/3}, \omega^{-1}b\lambda).$$

Denote

$$\phi^-(x,\lambda) = E_\rho(e^{\pi i/5}\lambda^2|x|; \omega a, \omega^{-1}b, \lambda), \quad \phi^+(x,\lambda) = E_\rho(\lambda^2 x; \omega a, \omega^{-1}b, \lambda)$$

then we have

$$\phi^-(x,\lambda) = \frac{2}{5}i\lambda^5|x|^{5/2} + ia\lambda^{5/3}|x|^{1/2} + ib|x|^{-1/2} + r_\rho(e^{\pi i/5}\lambda^2|x|,\lambda). \tag{2.23}$$

Since $\left|r_\rho(e^{\pi i/5}\lambda^2|x|,\lambda)\right| \le C_\rho\lambda^{-5/3+3\mu}$ for $|x| \ge \lambda^{-2\mu}X$ and then

$$-\operatorname{Re}\phi^-(x,\lambda) \le -\delta\lambda^{5/3}|x|^{1/2}(1+o_a(1)) + C_\rho(\lambda^\mu + \lambda^{-5/3+3\mu}). \tag{2.24}$$

For $x \ge \lambda^{-2\mu}X > 0$ note that

$$\begin{aligned} -\operatorname{Re}\phi^+(x,\lambda) &\le -c\lambda^5 x^{5/2} + C\lambda^{5/3}x^{1/2} + C_\rho(\lambda^\mu + \lambda^{-5/3+3\mu}) \\ &= -c\lambda^{5-4\mu}x^{1/2}(1+o_a(1)) + C_\rho(\lambda^\mu + \lambda^{-5/3+3\mu}) \end{aligned} \tag{2.25}$$

for $\mu < 5/6$. Then the assertion follows from (2.24) and (2.25). □

Lemma 2.4. *Assume that* (2.21) *and* (2.22) *hold with some* $\delta > 0$ *and that* $\mu < 5/6$. *Let* $0 < X_1 < X_2 < 1$. *Then there exist* $C > 0$, ℓ *and* $c > 0$ *such that*

$$\left|\mathcal{Y}_0(\lambda^2 x; a\lambda^{2/3}, b\lambda)\right| \ge C\lambda^\ell e^{-c\lambda^{5/3-\mu}}, \quad \lambda^{-2\mu}X_1 \le -x \le \lambda^{-2\mu}X_2.$$

Proof. It is clear from (2.23) that there exists $C_1 > 0$ such that

$$-\lambda^{5/3-\mu}/C_1 \le -\operatorname{Re}\phi^-(x,\lambda) \le -C_1\lambda^{5/3-\mu}$$

when $\lambda^{-2\mu}X_1 \le -x \le \lambda^{-2\mu}X_2$. □

Lemma 2.5. *Under the same assumptions as in Lemma 2.3 there exist* $c > 0$, $A > 0$ *such that*

$$\left|(d/dx)^k\mathcal{Y}_0(\lambda^2 x,\ a\lambda^{2/3}, b\lambda)\right| \le C_\mu A^{k+1}(1+k^3+\lambda^{5/2})^k e^{c\lambda^{5/6}}, \ \ k \in \mathbb{N}$$

for $|x| \ge \lambda^{-2\mu}X$.

Proof. We first estimate $|(d/dx)^k\mathcal{Y}_0(\lambda^2 x; a\lambda^{2/3}, b)|$ in $x \le -\lambda^{-2\mu}X$. From Proposition 2.4 with $\rho = 2(1-\mu)$ we have

$$\mathcal{Y}_0(\lambda^2 x; a\lambda^{2/3}, b\lambda) = C(1+p_\mu(x))\lambda^{-3/2}x^{-3/4}e^{-\phi^-(x,\lambda)+R_\mu(x)}$$

where $p_\mu(x)$ and $R_\mu(x)$ are holomorphic and bounded in $|x| > \lambda^{-2\mu}X$, $|\arg x| < 3\pi/5$. Since $|x|^{-1} \le \sqrt{X}\lambda^{2\mu} \le \sqrt{X}\lambda^{5/3}$ we have

$$\begin{aligned}
&|d^k(-\phi^-(x,\lambda) + R_\mu(x))/dx^k| \\
&\le C_\mu A^k k!(\lambda^5|x|^{3/2} + \lambda^{5/3}|x|^{-1/2} + C_\mu|x|^{-3/2})|x|^{1-k} \\
&\le C_\mu A^k k!(1 + \lambda^5|x|^{3/2} + \lambda^{5/2})\lambda^{5(k-1)/3} \\
&\le C_\mu A^k k!\lambda^{5k/3}(\lambda^{10/3}|x|^{3/2} + \lambda^{5/6}), \quad |x| \ge \lambda^{-2\mu}X, \;\; k \ge 1.
\end{aligned}$$

Therefore it follows that for $x \le -\lambda^{-2\mu}X$

$$\left|d^k e^{-\phi^-(x,\lambda)+R_\mu(x)}/dx^k\right| \le CA^k\lambda^{5k/3}(\lambda^{10/3}|x|^{3/2} + \lambda^{5/6} + k)^k e^{-\mathsf{Re}\,(-\phi(x)+R_\mu(x))}.$$

Since $-\mathsf{Re}\,(-\phi^-(x,\lambda) + R_\mu(x)) \le -c\lambda^{5/3}|x|^{1/2} + C_\mu\lambda^{5/6}$ for $x \le -\lambda^{-2\mu}X$ with $c > 0$ independent of μ and

$$|x|^{3k/2}e^{-c\lambda^{5/3}|x|^{1/2}} \le C^{k+1}\lambda^{-5k}k^{3k}$$

we conclude that

$$|d^k e^{-\phi^-(x,\lambda)+R_\mu(x)}/dx^k| \le CA^k(\lambda^{5/2} + k^3)^k e^{c\lambda^{5/6}}$$

which proves the assertion for $x \le -\lambda^{-2\mu}X$. For $x \ge \lambda^{-2\mu}X$ it is enough to repeat the same arguments noting that (2.25) and $5 - 4\mu > 5/3$. □

Corollary 2.1. *Assume that $\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda)$ verifies (2.21) and (2.22). Then there exist $c > 0$, $A > 0$, $C > 0$ such that for any $\epsilon > 0$ there is λ_ϵ such that*

$$\left|(d/dx)^k\mathcal{Y}_0(\lambda^2 x,\ a\lambda^{2/3}, b\lambda)\right| \le C_\epsilon A^{k+1}(1 + k^3 + \lambda^{5/2})^k e^{c\lambda^{5/6}}, \;\; k \in \mathbb{N}$$

for $\lambda^{-5/3+\epsilon} \le |x|$, $\lambda \ge \lambda_\epsilon$.

Proof. Choose $\mu = 5/6 - \epsilon/2$ in Lemma 2.5. □

Next we estimate $\mathcal{Y}_0(\lambda^2 x; a\lambda^{2/3}, b\lambda)$ for $|x| \le \lambda^{-5/3+\epsilon}$.

Lemma 2.6. ([2, Lemma 6.5, Lemma 6.7]) *Assume that $y(x,\lambda)$ satisfies*

$$y''(x,\lambda) = (x^3 + a\lambda^{2/3}x + b\lambda)y(x,\lambda), \quad |a|,\ |b| \le M. \tag{2.26}$$

Then there are $c > 0$, $C > 0$ and $\ell_i > 0$ such that for any $T > 0$ we have

$$\begin{aligned}
\left|(d/dx)^k y(x,\lambda)\right| &\le C^{k+1}(k + \lambda^{1/3} + |x|)^{3k/2}\lambda^{\ell_1}(1+T)^{\ell_2}e^{c\lambda^{5/6}(1+\lambda^{-1/3}T)^{5/2}} \\
&\quad \times\left\{|y(T,\lambda)| + |y'(T,\lambda)|\right\}, \quad |x| \le T, \;\; k \in \mathbb{N}, \;\; \lambda \ge 1.
\end{aligned}$$

Proposition 2.5. *Assume that $\mathcal{Y}_0(x; a\lambda^{2/3}, b\lambda)$ verifies (2.21) and (2.22). Then there are ℓ, $c > 0$, $A > 0$, $C > 0$ such that for any $\epsilon > 0$ there is λ_ϵ such that*

$$\left|(d/dx)^k\mathcal{Y}_0(\lambda^2 x; a\lambda^{2/3}, b\lambda)\right| \le CA^{k+1}\lambda^\ell(k^3 + \lambda^4)^k e^{c\lambda^{5/6+\epsilon}}, \;\; k \in \mathbb{N}$$

for $\lambda \ge \lambda_\epsilon$.

Proof. Applying Lemma 2.3 with $\mu = 5/6 - \epsilon/2$. we have

$$\left|(d/dx)^k \mathcal{Y}_0(\pm\lambda^{1/3+\epsilon})\right| \le C\lambda^{\ell} e^{c_1\lambda^{5/6}}, \ \ \lambda \ge \lambda_\epsilon, \ \ k = 0, 1.$$

Since $\mathcal{Y}_0(x) = \mathcal{Y}_0(x, a\lambda^{2/3}, b\lambda)$ satisfies (2.26), choosing $T = \lambda^{1/3+\epsilon}$ in Lemma 2.6 we get

$$\left|(d/dx)^k \mathcal{Y}_0(x)\right| \le C^{k+1}\lambda^{\ell}(k + \lambda^{1/3+\epsilon})^{3k/2} e^{c\lambda^{5/6+3\epsilon}}, \ \ |x| \le \lambda^{1/3+\epsilon}, \ \ k \in \mathbb{N}$$

for $\lambda \ge \lambda_\epsilon$. This proves that

$$\left|(d/dx)^k \mathcal{Y}_0(\lambda^2 x)\right| \le C^{k+1}\lambda^{\ell}(\lambda^2 k^{3/2} + \lambda^{5/2+2\epsilon})^k e^{c\lambda^{5/6+3\epsilon}}$$

for $|x| \le \lambda^{-5/3+\epsilon}$. Since $\lambda^{k(5/2+2\epsilon)} e^{-\lambda^{5/6+3\epsilon}} \le k^{3k}$ and $\lambda^2 k^{3/2} \le C(\lambda^4 + k^3)$ combining Corollary 2.1 with the above obtained estimates we conclude the assertion. □

Recalling

$$V_\lambda(x') = e^{i\lambda^5 x_2 - i(b_1/2)x_1}\mathcal{Y}_0(\lambda^2 x_1; \lambda^{2/3}a(\lambda), \lambda b(\lambda))$$

with $\lambda^{2/3}a(\lambda) = 2\xi_0$ and $b(\lambda) = b_2 + b_0\xi_0\lambda^{-4} - \xi_0^2\lambda^{-3} - b_1^2\lambda^{-5}/4$ one has

Lemma 2.7. *There exist $c > 0$ and $A > 0$ such that for any $\epsilon > 0$ there are $C > 0$ and λ_ϵ such that*

$$\left|\partial_{x_1}^k V_\lambda(x')\right| \le CA^k(k!)^3 e^{c\lambda^{4/3}}, \quad k \in \mathbb{N}, \ \lambda \ge \lambda_\epsilon. \tag{2.27}$$

Proof. Noting $\lambda^{4k} \le C^k(k!)^3 e^{3\lambda^{4/3}}$ the assertion follows from Proposition 2.5. □

3 Proof of Theorem 1.2

First assume that $b_2 \ne 0$ satisfies (2.17). Following Section 2.1 we have a family of exact solutions $\{U_\lambda\}$ satisfying $(P_{mod} + \sum_{j=0}^2 b_j D_j)U_\lambda = 0$. We show that $\{U_\lambda\}$ does not satisfy apriori estimates derived from $\gamma^{(s)}$ local solvability of the Cauchy problem if $s > 3$. Let $h > 0$ and a compact set K be fixed and denote by $\gamma_0^{(s),h}(K)$ the set of all $f(x') \in \gamma^{(s)}(\mathbb{R}^2)$ such that $\operatorname{supp} f \subset K$ and (1.2) holds with some $C > 0$ for all $\alpha \in \mathbb{N}^2$. Note that $\gamma_0^{(s),h}(K)$ is a Banach space with the norm

$$\sup_{\alpha,x}\frac{|\partial_x^\alpha f(x')|}{h^{|\alpha|}|\alpha|!^s}.$$

Proposition 3.1 (Holmgren)**.** *Denote $D_\epsilon = \{x \in \mathbb{R}^3 \mid |x'|^2 + |x_0| < \epsilon\}$. There exists $\epsilon_0 > 0$ such that for ϵ satisfying $0 < \epsilon < \epsilon_0$ if $u(x) \in C^2(D_\epsilon)$ satisfies*

$$\begin{cases} \left(P_{mod} + \sum_{j=0}^2 b_j D_j\right)u = 0 \ \ in \ \ D_\epsilon, \\ D_0^j u(0, x') = 0 \ \ (j = 0, 1), \ \ x \in D_\epsilon \cap \{x_0 = 0\} \end{cases}$$

then $u(x) \equiv 0$ in D_ϵ.

Lemma 3.1. (e.g.[8, Proposition 4.1, Theorem 4.2], [7]) *Assume that the Cauchy problem for* $P_{mod}+\sum_{j=0}^{2} b_j D_j$ *is locally solvable in* $\gamma^{(s)}$ *at the origin. Then there exists* $\delta > 0$ *such that for any* $0 < \epsilon_1 < \delta$ *and any* $(u_j(x')) \in \gamma_0^{(s),h}(\{|x'| \le \epsilon_1\})$ *there is a unique solution* $u(x) \in C^2(D_\delta)$ *to the Cauchy problem* (1.3) *with* $U_{(u_j(x'))} = D_\delta$ *and for any compact set* $L \subset D_\delta$ *there exists* $C > 0$ *such that*

$$|u(x)|_{C^2(L)} \le C \sum_{j=0}^{1} \sup_{\alpha, x'} \frac{|\partial_{x'}^{\alpha} u_j(x')|}{h^{|\alpha|}|\alpha|!^s} \tag{3.1}$$

holds.

Since $\lambda^{5k} \le k!^s e^{s\lambda^{5/s}}$ and $U_\lambda(0,x') = V_\lambda(x')$, $D_0 U_\lambda(0,x') = \xi_0(\lambda)\lambda V_\lambda(x')$ it is clear from Lemma 2.7 that one can find $c_1 > 0$, $C > 0$ such that

$$\sum_{j=0}^{1} \sup_{\alpha, x'} \frac{|\partial_{x'}^{\alpha} D_0^j U_\lambda(0,x')|}{h^{|\alpha|}|\alpha|^{s|\alpha|}} \le C e^{c_1 \lambda^{\max\{5/s, 4/3\}}}, \quad s \ge 3. \tag{3.2}$$

On the other hand thanks to Lemma 2.4 and Proposition 2.3 there is $c_0 > 0$ such that

$$\left|U_\lambda(x_0, -\lambda^{-2\mu}, 0)\right| \ge C\lambda^{\ell} e^{c_0 \lambda^{5/3} x_0 - c\lambda^{5/3-2\mu}}, \quad x_0 > 0 \tag{3.3}$$

where μ is chosen such that $0 < \mu < 5/6$. Let $\chi(x') \in \gamma^{(s)}(\mathbb{R}^2)$ be such that $\chi(x') = 0$ for $|x'| \ge \sqrt{\epsilon_1}$ and $\chi(x') = 1$ for $|x'| \le \sqrt{\epsilon_2} < \sqrt{\epsilon_1}$. Since $\Phi_\lambda = \chi(x')(U_\lambda(0,x'), D_0 U_\lambda(0,x')) \in \gamma_0^{(s),h}(\{|x'| \le \epsilon_1\})$ there is a unique solution $u_\lambda(x) \in C^2(D_\delta)$ to the Cauchy problem (1.3) with Cauchy data $\Phi_\lambda(x')$ which satisfies (3.2). Thanks to Proposition 3.1 we see that $u_\lambda = U_\lambda$ in D_{ϵ_2}. Take a compact set $L \subset D_{\epsilon_2}$ such that the interior of L contains $(x_0, -\lambda^{-2\mu}, 0)$ with small $x_0 > 0$ and large λ. If $s > 3$ hence $\max\{5/s, 4/3\} < 5/3$ the inequalities (3.2) and (3.3) are not compatible contradicting Lemma 3.1, which proves Theorem 1.2.

When $b_2 \ne 0$ does not satisfy (2.17) we make a change of local coordinates $(x_0, x_1, x_2) \to (-x_0, x_1, -x_2)$ such that $P_{mod} + \sum_{j=0}^{2} b_j D_j$ will be

$$P_{mod} - b_0 D_0 + b_1 D_1 - b_2 D_2. \tag{3.4}$$

in the new local coordinates. Since the local solvability in $\gamma^{(s)}$ at the origin is invariant under (analytic) change of local coordinates and $-b_2$ obviously satisfies (2.17), we conclude the same assertion also in this case.

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