Hyperbolic Systems with Two Independent Variables (Lectures at Tsukuba University, 1999)

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Contents

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1 Introduction

1.1 Problems

Let us study a 2×2 system

$$Lu = \partial_t u - A(t,x)\partial_x u + B(t,x)u$$

where $t,\ x \in \mathbf{R}$ and $A(t,x),\ B(t,x)$ are 2×2 matrices which are real analytic near the origin in $\mathbf{R}^2$. Moreover we assume that $A(t,x)$ is real valued. We study the following Cauchy problem:

$$\text{(C.P.)} \qquad \left\{ \begin{array}{c} Lu = f \\ u(\tau, x) = u_0(x). \end{array} \right.$$

We start with:

Definition 1.1.1 *We say that the Cauchy problem (C.P.) is well posed near the origin if one can find a neighborhood $U \subset W$ of the origin and $\epsilon > 0$ such that for any $u_0(x) \in C^\infty(W \cap \{t = \tau\})$, $|\tau| < \epsilon$ and for any $f \in C^\infty(W)$ there is a solution $u \in C^\infty(U)$ to (C.P.).*

Remark: From the Holmgren's uniqueness theorem, the uniqueness of solutions to (C.P.) is garanteed.

Definition 1.1.2 *We say that $\partial_t - A(t,x)\partial_x$ is strongly hyperbolic near the origin if for any $B(t,x)$ the Cauchy problem (C.P.) is C^∞ well posed near the origin.*

Our main concerns are the next two questions: (A) Characterize L for which the Cauchy problem (C.P.) is C^∞ well posed. (B) Characterize strongly hyperbolic systems.

Example 1.1.1. Let us consider

$$Pv = \partial_t^2 v - a(t,x)\partial_x^2 v + b(t,x)v = f.$$

If we set $u_1 = \partial_x v$, $u_2 = \partial_t v$, $u = {}^t(u_1, u_2)$, then the equation is reduced to the following system:

$$Lu = \partial_t u - \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \partial_x u + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} u = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

If the Cauchy problem (C.P.) for P is C^∞ well posed then so is for L and vice versa. An interesting case is that $a(0,0) = 0$ and hence $\mathrm{rank} A(0,0) = 1$.

Example 1.1.2: Let us consider

$$A(t,x) = \begin{pmatrix} x^2 - t^4/2 & x^2 + xt^2 \\ -x^2 + xt^2 & -(x^2 - t^4/2) \end{pmatrix}.$$

Then we will see that for any $B(t,x)$, the Cauchy problem (C.P.) is not C^∞ well posed. On the other hand note that the eigenvalues of $A(t,x)$ are $\pm t^4/2$ which implies that L is strictly hyperbolic apart from $t=0$.
EXAMPLE 1.1.3: Let

$$A(t,x) = \begin{pmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{21}(t,x) & a_{22}(t,x) \end{pmatrix}$$

be symmetric, that is $a_{12}(t,x) = a_{21}(t,x)$. Then L is strongly hyperbolic. Note that the eigenvalues are not necessarily smooth.
EXAMPLE 1.1.4: Let us consider

$$A(t,x) = \begin{pmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{21}(t,x) & -a_{11}(t,x) \end{pmatrix}$$

where $a_{11}^2(t,x)+a_{12}a_{21}(t,x) \equiv 0$. That is the eigenvalue 0 is folded. If we factor out the common factor $K(t,x)$ among $a_{ij}(t,x)$ one can write

$$A(t,x) = \begin{pmatrix} K\sigma\rho & K\sigma^2 \\ -K\rho^2 & -K\sigma\rho \end{pmatrix}$$

where ρ and σ are relatively prime. Let us write

$$B(t,x) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then a necesary and sufficinet condition for the Cauchy problem (C.P.) to be well posed is:

$$\rho\partial_t\sigma - \sigma\partial_t\rho + b_{12}\sigma^2 - b_{21}\rho^2 + (b_{11} - b_{22})\sigma\rho \equiv 0$$

(Levi condition).
EXAMPLE 1.1.5: Let us consider

$$A(t,x) = \psi(t,x)\begin{pmatrix} 0 & 1 \\ t^2 & 0 \end{pmatrix}.$$

In this case $\partial_t - A(t,x)\partial_x$ is strongly hyperbolic for any $\psi(t,x)$.
EXAMPLE 1.1.6: Let us consider

$$A(t,x) = \psi(t,x)\begin{pmatrix} 0 & 1 \\ t^4 & 0 \end{pmatrix}.$$

Let us write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Then a necessary and sufficient condition for (C.P.) to be well posed is given by $b_{21}(0,x) = 0$. Note that the condition is independent of ψ ($\neq 0$). *In this lecture we shall provide a necessary and sufficient condition for C^∞ well posedness of (C.P.). We also give a necessary and sufficient condition in order that $\partial_t - A(t,x)\partial_x$ is strongly hyperbolic.*

Before closing this subsection we recall the Lax-Mizohata theorem:

Theorem 1.1.1 *If (C.P.) is C^∞ well posed near the origin then all eigenvalues of $A(t,x)$ are real when (t,x) varies near the origin.*

We next remark that one can assume always the trace of $A(t,x)$ is zero.

Lemma 1.1.1 *In a new system of local coordinates:*

$$s = t, \quad y = \phi(t,x), \quad \phi(0,x) = x$$

one can assume that $\mathrm{tr}A(t,x) \equiv 0$, *where* $\phi(t,x)$ *verifies*

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}A(t,x), \quad \phi(0,x) = x.$$

Proof: Easy.

In what follows we assume that $\mathrm{tr}A(t,x) \equiv 0$ and hence

$$A(t,x) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}.$$

Let us denote $h(t,x) = -\mathrm{det}A(t,x) = a_{11}^2 + a_{12}a_{21}$. Note that if all eigenvalues of $A(t,x)$ are real then

$$h(t,x) \geq 0$$

and vice versa.

1.2 Reduction to second order 2×2 quasi diagonal system

Let us take

$$T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Note that if the Cauchy problem for L is C^∞ well posed then so is for $T^{-1}LT$ and vice versa. Thus it is enough to study $T^{-1}LT$:

$$L^\sharp = T^{-1}LT = \partial_t - A^\sharp(t,x)\partial_x + B^\sharp(t,x)$$

where $A^\sharp(t,x) = T^{-1}A(t,x)T$ and $B^\sharp(t,x) = T^{-1}B(t,x)T$. More precisely

$$A^\sharp(t,x) = \begin{pmatrix} \frac{i(a_{12}-a_{21})}{2} & \frac{a_{12}+a_{21}}{2} + ia_{11} \\ \frac{a_{12}+a_{21}}{2} - ia_{11} & -\frac{i(a_{12}-a_{21}}{2} \end{pmatrix} = \begin{pmatrix} a_{11}^\sharp & a_{12}^\sharp a_{21}^\sharp \\ -a_{11}^\sharp & \end{pmatrix}.$$

It is clear that

$$\overline{a_{11}^\sharp} = -a_{11}^\sharp, \quad \overline{a_{12}^\sharp} = a_{21}^\sharp. \tag{1.2.1}$$

Lemma 1.2.1 *We have*

$$|a_{12}^\sharp| = |a_{21}^\sharp| \geq |a_{11}^\sharp|, \;\; 4|a_{12}^\sharp|^2 \geq \mathrm{tr}(A^tA) = \sum_{i,j=1}^{2} a_{ij}(t,x)^2, \;\; |a_{12}^\sharp|^2 \geq h.$$

In particular we have $a_{12}^\sharp(t,x) = 0 \Longleftrightarrow A(t,x) = O$.

Proof: Note that

$$h = (a^\sharp_{11})^2 + a^\sharp_{12}a^\sharp_{21} = |a^\sharp_{12}|^2 - |a^\sharp_{11}|^2$$

by (1.2.1). Since $h \geq 0$ it follows that $|a^\sharp_{12}|^2 \geq |a^\sharp_{11}|^2$ and $|a^\sharp_{12}|^2 \geq h$. Observing that

$$A^\sharp(A^\sharp)^* = T^{-1}A^tAT$$

we have $\mathrm{tr}(A^tA) = \mathrm{tr}(A^\sharp(A^\sharp)^*) = 2(|a^\sharp_{11}|^2 + |a^\sharp_{12}|^2) \leq 4|a^\sharp_{12}|^2$. q.e.d.

Lemma 1.2.2 *Assume that $A(t,x)$ is uniformly diagonalizable, that is for any (t,x) there is a 2×2 matrix $U(t,x)$ such that $U(t,x)^{-1}A(t,x)U(t,x)$ is diagonal, where $\|U(t,x)^{-1}\|$, $\|U(t,x)\| \leq C$, with C independent of (t,x). Then there is a $C > 0$ such that*

$$Ch(t,x) \geq \sum_{i,j=1}^{2} a_{ij}(t,x)^2.$$

Proof: By assumption there is a U such that

$$U^{-1}AU = \begin{pmatrix} \alpha(t,x) & 0 \\ 0 & -\alpha(t,x) \end{pmatrix}.$$

Hence $A = U\mathrm{diag}(\alpha, -\alpha)U^{-1}$ which shows $\|A\|^2 \leq \|U\|^2\|U^{-1}\|^2(2\alpha^2) \leq 2C^4\alpha^2$. On the other hand, since $\alpha^2 = h = -\mathrm{det}A$, we have

$$\sum_{i,j=1}^{2} a_{ij}(t,x)^2 \leq 2C^4h.$$

q.e.d.

Let us put

$$M = \partial_t + A^\sharp\partial_x + C + {}^{co}B^\sharp - A^\sharp_x$$

where ${}^{co}B^\sharp$ stands for the cofactor matrix of $B^\sharp$, $A^\sharp_x = \partial_x A^\sharp$ and C will be determined later. Actually this is the object we use in order to rduce $L^\sharp$ to second order 2×2 "quasi" diagonal system.

Note that

$$L^\sharp M = \partial_t^2 - h\partial_x^2 + (A^\sharp_t - A^\sharp C + \mathrm{tr}(AB))\partial_x$$
$$+(B^\sharp + {}^{co}B^\sharp + C - A^\sharp_x)\partial_t + L^\sharp(C + {}^{co}B^\sharp - A^\sharp_x)$$

because, for instance, we have

$$B^\sharp A^\sharp - A^{\sharp co}B^\sharp = (B^\sharp A^\sharp) + {}^{co}(B^\sharp A^\sharp) = \mathrm{tr}(A^\sharp B^\sharp) = \mathrm{tr}(AB).$$

We now want to choose C so that we have

$$A^\sharp_t - A^\sharp C + \mathrm{tr}(AB) = \mathrm{diagonal}.$$

Let us examine $A^\sharp_t - A^\sharp C + \mathrm{tr}(AB)$ which is

$$\begin{pmatrix} \partial_t a^\sharp_{11} - a^\sharp_{11}c_{11} - a^\sharp_{12}c_{21} + \mathrm{tr}(AB) & \partial_t a^\sharp_{12} - a^\sharp_{11}c_{12} - a^\sharp_{12}c_{22} \\ \partial_t a^\sharp_{21} + a^\sharp_{11}c_{21} - a^\sharp_{21}c_{11} & -\partial_t a^\sharp_{11} + a^\sharp_{11}c_{22} - a^\sharp_{21}c_{21} + \mathrm{tr}(AB) \end{pmatrix}.$$

We want to choose C so that

$$\begin{cases} \partial_t a^\sharp_{12} - a^\sharp_{11}c_{12} - a^\sharp_{12}c_{22} = 0, \\ \partial_t a^\sharp_{21} + a^\sharp_{11}c_{21} - a^\sharp_{21}c_{11} = 0 \end{cases}$$

that is

$$c_{11} = \frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}} + \frac{a^\sharp_{11}}{a^\sharp_{21}}c_{21}, \quad c_{22} = \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} - \frac{a^\sharp_{11}}{a^\sharp_{12}}c_{12}. \tag{1.2.2}$$

Lemma 1.2.3 *Assume that $C = (c_{ij})$ verifies (1.2.2). Then with*

$$\begin{cases} Y = a^\sharp_{21}\partial_t a^\sharp_{11} - a^\sharp_{11}\partial_t a^\sharp_{21} + a^\sharp_{21}\mathrm{tr}(AB), \\ Z = -a^\sharp_{12}\partial_t a^\sharp_{11} + a^\sharp_{11}\partial_t a^\sharp_{12} + a^\sharp_{12}\mathrm{tr}(AB) \end{cases}$$

we have

$$L^\sharp M = (\partial_t^2 - h\partial_x^2)I + Q\partial_x + R\partial_t + S$$

where

$$Q = \begin{pmatrix} Y/a^\sharp_{21} - hc_{21}/a^\sharp_{21} & 0 \\ 0 & Z/a^\sharp_{21} - hc_{21}/a^\sharp_{12} \end{pmatrix}$$

and $R = C - A^\sharp_x + B^\sharp + {}^{co}B^\sharp$, $S = L^\sharp({}^{co}B^\sharp - A^\sharp_x)$.

Proof: We study (2,2)-entry of $A^\sharp_t - A^\sharp C + \mathrm{tr}(AB)$:

$$-\partial_t a^\sharp_{11} + a^\sharp_{11}\left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} - \frac{a^\sharp_{11}}{a^\sharp_{12}}c_{12}\right) - a^\sharp_{21}c_{12} + \mathrm{tr}(AB)$$

$$= \frac{1}{a^\sharp_{12}}\{-a^\sharp_{12}\partial_t a^\sharp_{11} + a^\sharp_{11}\partial_t a^\sharp_{12} - ((a^\sharp_{11})^2 + a^\sharp_{12}a^\sharp_{21})c_{12} + \mathrm{tr}(AB)$$

$$= \frac{1}{a^\sharp_{12}}\{Z - hc_{12}\}.$$

We can examine the other entries similary. q.e.d.

In what follows we take $c_{12} = c_{21} = 0$ (just for simplicity, because the term $h/a^\sharp_{12}$ is harmless by Lemma 1.2.1). Recall again

$$C = \begin{pmatrix} \frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}} & 0 \\ 0 & \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \end{pmatrix}.$$

Then we see that

$$L^\sharp(C) = \begin{pmatrix} \partial_t(\frac{\partial_t a^\sharp_{21}}{as_{21}}) & 0 \\ 0 & \partial_t(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}) \end{pmatrix} + \begin{pmatrix} a^\sharp_{11}\partial_x(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}) & a^\sharp_{12}\partial_x(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}) \\ a^\sharp_{21}\partial_x(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}) & -a^\sharp_{11}\partial_x(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}) \end{pmatrix}.$$

Lemma 1.2.4 *Let us define*

$$D^\sharp = a^\sharp_{11}\partial_t a^\sharp_{12} - a^\sharp_{12}\partial_t a^\sharp_{11}.$$

Then we have $Z = D^\sharp + a^\sharp_{12}\mathrm{tr}(AB)$, $Y = \overline{D^\sharp + a^\sharp_{12}\mathrm{tr}(A\bar{B})}$.

Proof: It is clear since $\overline{a^\sharp_{11}} = -a^\sharp_{11}$, $\overline{a^\sharp_{12}} = a^\sharp_{21}$. q.e.d.

Lemma 1.2.5 *Let us put*

$$M = \partial_t + A^\sharp\partial_x + A^\sharp_x + {}^{co}B^\sharp + \tilde{C}$$

where

$$\tilde{C} = -\begin{pmatrix} \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} & 0 \\ 0 & \frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}} \end{pmatrix}.$$

Then we have

$$ML^\sharp = (\partial_t^2 - h\partial_x^2)I - h_x\partial_x + \tilde{Q}\partial_x + \tilde{R}\partial_t + \tilde{S}$$

where

$$\tilde{Q} = \begin{pmatrix} Z/a^\sharp_{12} & 0 \\ 0 & Y/a^\sharp_{21} \end{pmatrix},\ \tilde{R} = \tilde{C} + A^\sharp_x + B^\sharp + {}^{co}B^\sharp,\ \tilde{S} = M(B^\sharp).$$

Proof: Noting $A^\sharp_x A^\sharp + A^\sharp A^\sharp_x = h_x$, the proof is similar to that of Lemma 1.2.3. q.e.d.

Remark that in Lemma 1.2.3, $(\partial_t^2 - h\partial_x^2)I + Q\partial_x$ is diagonal:

$$\begin{cases} \partial_t^2 - h\partial_x^2 + (\overline{\frac{D^\sharp + a^\sharp_{12}\mathrm{tr}(A\bar{B})}{a^\sharp_{12}}})\partial_x \\ \partial_t^2 - h\partial_x^2 + (\frac{D^\sharp + a^\sharp_{12}\mathrm{tr}(AB)}{a^\sharp_{12}})\partial_x \end{cases}$$

and in Lemma 1.2.5, $(\partial_t^2 - h\partial_x^2)I + \tilde{Q}\partial_x$ is also diagonal:

$$\begin{cases} \partial_t^2 - h\partial_x^2 + (\frac{D^\sharp + a^\sharp_{12}\mathrm{tr}(AB)}{a^\sharp_{12}} - h_x)\partial_x \\ \partial_t^2 - h\partial_x^2 + (\overline{\frac{D^\sharp + a^\sharp_{12}\mathrm{tr}(A\bar{B})}{a^\sharp_{12}}} - h_x)\partial_x. \end{cases}$$

Essentially our system is reduced to a second order 2×2 "quasi" diagonal system, with singular coefficients in front of ∂_x.

In section 2, we define a finite number of pseudo-characteristic curves $t = \phi(x)$ of A. We define $f_\phi(t,x)$ for any real analytic function $f(t,x)$ defined near the origin by (see Definition 2.2.1)

$$f_\phi(t,x) = f(t + \phi(x), x).$$

We also denote by $\Gamma(f)$ the Newton polygon of f, the precise definition will be given in section 2. Then we have

Theorem 1.2.1 *In order the Cauchy problem (C.P.) for L is C^∞ well posed near the origin it is necessary and sufficient that*

$$\Gamma(t[D^\sharp + a_{12}^\sharp \mathrm{tr}(AB)]_\phi) \subset \frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi),$$
$$\Gamma(t[D^\sharp + a_{12}^\sharp \mathrm{tr}(A\bar{B})]_\phi) \subset \frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi)$$

for any pseudo-characteristic curve $t = \phi(x)$ of A (see Definition 2.2.1).

Theorem 1.2.2 *For $\partial_t - A(t,x)\partial_x$ to be strongly hyperbolic near the origin it is necessary and sufficient that*

$$\Gamma(tD_\phi^\sharp) \subset \frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi), \quad \Gamma(t[a_{ij}]_\phi) \subset \frac{1}{2}\Gamma(h_\phi)$$

for any pseudo-characteristic curves $t = \phi(x)$ of A.

2 Pseudo-characteristic curves

2.1 Zeros of non negative real analytic functions

Let $F(t,x)$ be a non negative real analytic function defined near the origin.

Lemma 2.1.1 *Let $F(t,x)$ be as above. Then there is a real valued $f(t,x)$ defined in V (a neighborhood of the origin) such that $f(t,x)$ is real analytic in $V \setminus (0,0)$ continuous in V, unique up to a non zero factor such that $f(t,x)^2 = F(t,x)$ and*

$$f(t,x) = x^n \prod_{j=1}^{l}(t - t_j(x)) \prod_{j=l+1}^{m} |t - t_j(x)|\Phi(x) \tag{2.1.1}$$

where $\Phi(0,0) \neq 0$ and $t_j(x)$ is obtained as the restriction to $\mathbf{R}$ of

$$t_j(z) = \sum_{k\geq 0} C_j k z^{k/p_j}, \quad (p_j \in \mathbf{N}).$$

Here $\mathrm{Im}\, t_j(x) \neq 0$ *for $0 < |x| < \delta$ with some $\delta > 0$ for $j \geq l+1$ (for $1 \leq j \leq l$ it may happen* $\mathrm{Im}\, t_j(x) = 0$ *in $0 < |x| < \delta$).*

Proof: Note that one can write

$$F(t,x) = x^{2n} g_1^{l_1} \cdots g_\nu^{l_\nu} h_1^{m_1} \cdots h_\mu^{m_\mu} \bar{h}_1^{m_1} \cdots \bar{h}_\mu^{m_\mu} \Phi$$

where g_i are real, that is $\bar{g}_i = g_i$ and $\bar{h}_i \neq h_i$. Here we denoted $\bar{h}(t,x) = \overline{h(\bar{t},\bar{x})}$. To see this let us factorize $F(t,x)$ as the product of irreducible factors:

$$F = x^{2n} g_1^{l_1} \cdots g_\nu^{l_\nu} k_1^{m_1} \cdots k_p^{m_p}$$

with $\bar{k}_i \neq k_i$. Since $\bar{F} = F$ we have

$$\bar{F} = x^{2n} g_1^{l_1} \cdots g_\nu^{l_\nu} \bar{k}_1^{m_1} \cdots \bar{k}_p^{m_p} = x^{2n} g_1^{l_1} \cdots g_\nu^{l_\nu} k_1^{m_1} \cdots k_p^{m_p}.$$

On the other hand from the uniqueness of the factorization $\bar{k}_j^{m_j}$ coincides with some $\bar{k}_i^{m_i}$. This proves the assertion. Taking $\delta > 0$ small enough, we may suppose that the resultant of any pair among g_i, h_j, $\bar{h}_k$ is different from zero in $0 < |x| < \delta$. We also may assume that the discriminant of every g_i, h_j, $\bar{h}_k$ is different from zero in $0 < |x| < \delta$. Factorize

$$h_i = \prod_{k=1}^{n(i)} (t - t_k(x))$$

then we have $\mathrm{Im}\, t_j(x) \neq 0$ for $x \in \mathbf{R}$, $0 < |x| < \delta$ since otherwise we would have $\bar{h}_i(t_k(\hat{x}), \hat{x}) = h_i(t_k(\hat{x}), \hat{x}) = 0$ with some $\hat{x} \in \mathbf{R}$, $0 < |\hat{x}| < \delta$ where $\mathrm{Im}\, t_k(\hat{x}) = 0$ which contradicts the assumption that the resultant of h_i and $\bar{h}_i$ is different from zero in $x \in \mathbf{R}$, $0 < |x| < \delta$. Thus one can write

$$h_i \bar{h}_i = \prod_{k=1}^{n(i)} |t - t_k(x)|^2 = \left(\prod_{k=1}^{n(i)} |t - t_k(x)| \right)^2 .$$

We turn to g_i. Let us write

$$g_i = \prod_{k=1}^{n(i)} (t - t_k(x)).$$

If There is a $x \in \mathbf{R}$, $0 < |x| < \delta$ such that $\mathrm{Im}\, t_k(x) = 0$ with some k then l_i is even (recall that the discriminant of g_i is different from zero) because $F(t, x) \geq 0$. Hence one can write

$$g_i^{l_i} = \left(\prod_{k=1}^{n(i)} (t - t_k(x))^{l_i/2} \right)^2 .$$

Finally if $\mathrm{Im}\, t_k(x) \neq 0$ for all $x \in \mathbf{R}$, $0 < |x| < \delta$ and k then, since g_i is real, $\overline{t_k(x)}$ is also a root of $g_i = 0$ so that $\overline{t_k(x)}$ coincides with some $t_i(x)$ and

$$g_i = \prod (t - t_k(x))(t - \overline{t_k(x)}) = \left(\prod |t - t_k(x)| \right)^2 .$$

This proves the assertion. q.e.d.

Remark: One can express for $x \in \mathbf{R}$, $0 < \pm x < \delta$

$$t_j(x) = \sum_{k \geq 0} C_{jk}^{\pm} (\pm x)^{k/p_j}.$$

Definition 2.1.1 *We introduce several notations:*

$$t_f^*(x) = \left(\sum |t_j(x)|^2\right)^{1/2}$$

where the sum is taken over all $t_j(x)$ *appearing in (2.1.1). We call a curve* $t = \mathrm{Re}\, t_j(x)$ *pseudo-characteristic curve of* $F(t,x) = 0$ *and set*

$$\mathcal{C}^{\pm}(F) = \{\mathrm{Re}\, t_j(x) \mid \pm x > 0\}$$

which is the set of all functions defining pseudo characteristic curves of F*.*

We may assume, after shrinking δ if necessary, that

$$\begin{aligned} \mathrm{Re}\, t_{\mu_1}(x) \le \mathrm{Re}\, t_{\mu_2}(x) \le \cdots \le \mathrm{Re}\, t_{\mu_m}(x), \quad 0 < x < \delta, \\ \mathrm{Re}\, t_{\nu_1}(x) \le \mathrm{Re}\, t_{\nu_2}(x) \le \cdots \le \mathrm{Re}\, t_{\nu_m}(x), \quad -\delta < x < 0. \end{aligned}$$

Then we put

$$\sigma_j(x) = \begin{cases} \mathrm{Re}\, t_{\mu_j}(x), & x > 0 \\ \mathrm{Re}\, t_{\nu_j}(x), & x < 0 \end{cases}$$

and define

$$\begin{aligned} s_j(x) = \frac{1}{2}\{\sigma_j(x) + \sigma_{j+1}(x)\}, \quad j = 1, 2, ..., m-1, \\ s_0(x) = 3t_f^*(x), \quad s_m(x) = 3t_f^*(x). \end{aligned}$$

We also define

$$\begin{aligned} \tilde{\omega}_j = \{(t,x) \mid |x| < \delta, s_{j-1}(x) \le t \le s_j(x)\}, \quad j = 1, ..., m \\ \tilde{\omega}(T) = \{(t,x) \mid |x| < \delta, s_m(x) \le t \le T\}. \end{aligned}$$

Note that ω_j contains a pseudo characteristic curve $t = \sigma_j(x)$.

Lemma 2.1.2 *Let* $F(t,x)$ *be as above and* $f(t,x)$ *be as in Lemma 2.1.1. Then there are* $c_i > 0$ *such that*

$$\frac{c_1}{t - t_f^*(x)} \le \frac{c_1}{t - 2t_f^*(x)} \le \frac{f_t}{f} \le \frac{c_2}{t - t_f^*(x)}$$

in $\tilde{\omega}(T)$*.*

Proof: Recall that

$$\frac{f_t}{f} = \sum_{j=1}^{l} \frac{1}{t - t_j(x)} + \sum_{j=l+1}^{m} \frac{t - \mathrm{Re}\, t_j(x)}{|t - t_j(x)|^2} + \frac{\Phi_t}{\Phi}.$$

Since

$$\sum_{j=1}^{l} \frac{1}{t - t_j(x)} = \sum_{j=1}^{l} \frac{t - \mathrm{Re}\, t_j(x) + i\mathrm{Im}\, t_j(x)}{|t - t_j(x)|^2} = \sum_{j=1}^{l} \frac{t - \mathrm{Re}\, t_j(x)}{|t - t_j(x)|^2}$$

because the left-hand side is real we get

$$\frac{f_t}{f} = \sum_{j=1}^{m} \frac{t - \mathrm{Re}\, t_j(x)}{|t - t_j(x)|^2} + \frac{\Phi_t}{\Phi}. \tag{2.1.2}$$

Hence we get

$$\frac{f_t}{f} \geq \sum_{j=1}^{m} \frac{t - \mathrm{Re}\, t_j(x)}{|t - t_j(x)|^2} - C.$$

On the other hand noting that

$$t - 2t_f^*(x) \geq \frac{t}{3} \geq \frac{1}{4}(t + |t_j(x)|) \geq \frac{1}{4}|t - t_j(x)|$$

in $\tilde{\omega}(T)$ we have

$$\frac{1}{|t - t_j(x)|} \geq \frac{1}{4(t - 2t_f^*(x))}.$$

Since $t - \mathrm{Re}\, t_j(x) \geq t - 2t_f^*(x)$ it follows that

$$\frac{f_t}{f} \geq \frac{1}{4} \sum_{j=1}^{m} \frac{1}{t - 2t_f^*(x)} - C \geq \frac{c_1}{t - 2t_f^*(x)}$$

because $0 \leq t - 2t_f^*(x) \leq T$ in $\tilde{\omega}(T)$ implies

$$-\frac{TC}{t - 2t_f^*(x)} \leq -C.$$

We turn to the right-hand inequality. Note

$$|t - t_j(x)| \geq t - |t_j(x)| \geq t - t_f^*(x)$$

and hence by (2.1.2) one has

$$\frac{f_t}{f} \leq \sum_{j=1}^{m} \frac{1}{|t - t_j(x)|} + C \leq \frac{1}{t - t_f^*(x)} + C.$$

Using $C \leq CT/(t - t_f^*(x))$ we have the desired assertion. q.e.d.

Lemma 2.1.3 *Let $F(t,x)$ be as above and $f(t,x)$ be given by Lemma 2.1.1. Then there is a $C > 0$ such that*

$$\partial_t \left(\frac{f_t}{f} \right) \leq C$$

in $\tilde{\omega}(T)$.

Proof: From (2.1.2) one has

$$\partial_t\left(\frac{f_t}{f}\right)=-\sum_{j=1}^{l}\frac{1}{(t-t_j(x))^2}-\sum_{j=l+1}^{m}\frac{(t-\mathrm{Re}\,t_j(x))^2-(\mathrm{Im}\,t_j(x))^2}{|t-t_j(x)|^4}+\partial_t\left(\frac{\Phi_t}{\Phi}\right).$$

Here we note that

$$\mathrm{Re}\,\frac{1}{(t-t_j(x))^2}=\frac{(t-\mathrm{Re}\,t_j(x))^2-(\mathrm{Im}\,t_j(x))^2}{|t-t_j(x)|^4}.$$

This shows that

$$\partial_t\left(\frac{f_t}{f}\right)=-\sum_{j=1}^{m}\frac{(t-\mathrm{Re}\,t_j(x))^2-(\mathrm{Im}\,t_j(x))^2}{|t-t_j(x)|^4}+\partial_t\left(\frac{\Phi_t}{\Phi}\right).$$

In $\tilde{\omega}(T)$ we see that

$$t-\mathrm{Re}\,t_j(x)\geq 3|t_j(x)|-\mathrm{Re}\,t_j(x)\geq 2|t_j(x)|\geq|\mathrm{Im}\,t_j(x)|$$

and hence $(t-\mathrm{Re}\,t_j(x))^2-(\mathrm{Im}\,t_j(x))^2\geq 0$. This gives

$$\partial_t\left(\frac{f_t}{f}\right)\leq\partial_t\left(\frac{\Phi_t}{\Phi}\right)\leq C$$

and hence the result. q.e.d.

Definition 2.1.2 *Let $\phi(x)\in\mathcal{C}^{\pm}(F)$. Then we define $B_\phi(t,x)$ for any real analytic $B(t,x)$ defined near the origin by*

$$B_\phi(t,x)=B(t+\phi(x),x).$$

Precisely if $\phi\in\mathcal{C}^{\pm}(F)$ then $B_\phi(t,x)$ is defined for $\pm x>0$. Then one can express $B_\phi(t,x)$ by the Puiseux series expansion:

$$B_\phi(t,x)=\sum_{i,k\geq 0}B^{\pm}_{ik}t^i(\pm x)^{k/p}$$

with some $p\in\mathbf{N}$. We define the Newton polygon $\Gamma(B_\phi)$ by

$$\Gamma(B_\phi)=\textit{convex hull of}\ \{\bigcup_{B^{\pm}_{ik}\neq 0}((i,\frac{k}{p})+\mathbf{R}^2_+))\}.$$

We say $\Gamma(B_\phi)=\emptyset$ if $B\equiv 0$.

Proposition 2.1.1 *Assume that*

$$\Gamma(tB_\phi)\subset\frac{1}{2}\Gamma(F_\phi),\quad\forall\phi\in\mathcal{C}^{\pm}(F).$$

Then there is $C>0$ such that (taking T small enough)

$$|(t-\sigma_j(x))B(t,x)|\leq C|f(t,x)|\ \textit{for}\ (t,x)\in\tilde{\omega}_j,\ j=1,...,m,$$
$$|(t-s_m(x))B(t,x)|\leq C|f(t,x)|\ \textit{for}\ (t,x)\in\tilde{\omega}(T),\ \textit{if}\ n\geq 1,$$
$$|B(t,x)|\leq C|\partial_t f(t,x)|\ \textit{for}\ (t,x)\in\tilde{\omega}(T),\ \textit{if}\ n=0.$$

Proof: We give the proof in subsection 2.3. q.e.d.

Lemma 2.1.4 *Let $n = 0$. Then there is a $C > 0$ such that*

$$\sup_{0 \le t \le t_f^*(x)} |f(t,x)| \le C|x|.$$

Proof: It is enough to show that $|F(t,x)| \le C|x|^2$ for $0 \le t \le t_f^*(x)$. By definition there is j such that

$$F(t_j(x), x) = 0, \quad t_j(x) \sim t_f^*(x).$$

If $g_i(t_j(x), x) = 0$ with $l_i \ge 2$ for some i then one gets

$$|g_i(t_j(x), x)|^{l_i} \le C|x|^2, \quad 0 \le t \le t_f^*(x).$$

To see this note that $g_i(t_j(x), x) = t_j(x)^{n(i)} + O(|x|) = 0$ and hence we have $t_j(x)^{n(i)} = O(|x|)$. This gives $g_i(t_j(x), x) = O(|x|)$ for $0 \le t \le t_f^*(x)$. If $h_i(t_j(x), x) = 0$ then it is easy to see that

$$|h_i(t,x)| \le C|x| \quad \text{for } 0 \le t \le t_f^*(x)$$

and hence $|h_i(t,x)\bar{h}_i(t,x)| \le C|x|^2$ for $0 \le t \le t_f^*(x)$. If $g_i(t_j(x), x) = 0$ with $l_i = 1$. Since $g_i(t,x) \ge 0$ then $g_i(t,x) = t^{2\bar{m}} + d_1(x)t^{2\bar{m}-1} + \cdots + d_{2\bar{m}}(x)$ where $d_{2\bar{m}}(x) = O(|x|^2)$. On the other hand every root of $g_i(t,x) = 0$ is a branch of

$$\sum_{j \ge 1} C_i (z^{1/2\bar{m}})^i.$$

Then it follows that $C_1 = 0$ and hence every root is $O(|x|^{1/\bar{m}})$. This shows that $d_j(x)(|x|^{1/\bar{m}})^{2\bar{m}-j} = O(|x|^2)$ and hence $g_i(t_j(x), x) = O(|x|^2)$. q.e.d.

Lemma 2.1.5 *Let $F(t,x)$ and $f(t,x)$ be as above. Then we have*

$$\sup_{|t| \le T, 0 < |x| < \delta} |f_x(t,x)| \le C$$

with some $C > 0$.

Proof: Recall that $f(t,x)$ is real analytic in $V \setminus (0,0)$ satisfying $f(t,x)^2 = F(t,x)$. If $g(t,x)^2 = F(t,x)$ then we have $f(t,x) = g(t,x)$ or $f(t,x) = -g(t,x)$ in $V \setminus (0,0)$. That is $f(t,x)$ is unique up to the sign. We can argue exchanging t and x to conclude that

$$f(t,x) = t^k \prod_{j=1}^{\tilde{l}} (x - s_j(t)) \prod_{j=\tilde{l}+1}^{\tilde{m}} |x - s_j(t)| \Psi(t,x)$$

where $\mathrm{Im}\, s_j(t) \ne 0$ if $j \ge \tilde{l} + 1$. Then it is clear that $f_x(t,x)$ is bounded because

$$\frac{\partial}{\partial x} |x - s_j(t)| = \frac{x - \mathrm{Re}\, s_j(t)}{|x - s_j(t)|}$$

is bounded. q.e.d.

Lemma 2.1.6 *Let $F(t,x)$ and $f(t,x)$ be as above. Let $n = 0$. Then for any $K > 0$, there is T_K such that we have*

$$\text{either } f_t(t,x) \geq Kf(t,x) > 0 \quad \text{or} \quad -f_t(t,x) \geq -Kf(t,x) > 0$$

in $\tilde{\omega}(T)$ for $0 < T \leq T_K$.

Proof: Recall that

$$f(t,x) = \prod_{j=1}^{l}(t - t_j(x)) \prod_{j=l+1}^{m} |t - t_j(x)| e(t,x).$$

Then it is easy to see

$$\bar{f}_t f + f_t \bar{f} = 2\sum_{p=1}^{m}(t - \text{Re } t_p(x)) \prod_{j \neq p} |t - t_j(x)|^2 e^2 + \prod_{j=1}^{m} |t - t_j(x)|^2 (e^2)_t.$$

On the other hand, by definition we see for $(t,x) \in \tilde{\omega}(T)$

$$t - \text{Re } t_k(x) \geq t - t_f^*(x) \geq \frac{2}{3}t \geq \frac{1}{2}|t - t_j(x)|, \quad k = 1, ..., m.$$

Then one has

$$(f^2)_t - Kf^2 \geq \sum_k (1 - CK|t - t_k(x)|)|t - t_k(x)| \prod_{j \neq k} |t - t_j(x)|^2 e^2.$$

Since $\sup_{\tilde{\omega}(T)} |t - t_k(x)| \to 0$ as $T \to 0$ we get the desired result. q.e.d.

2.2 Pseudo-characteristic curves for systems

Recall that

$$h(t,x) = x^{2n_1}(t^{2m_1} + h_1(x)t^{2m_1 - 1} + \cdots + h_{2m_1}(x))e(t,x)^2$$

where $e(0,0) \neq 0$, $h_i(0) = 0$. We apply Lemma 2.1.1 to h and we get

$$b(t,x) = x^{n_1} \prod_{i=1}^{l_1}(t - t_i(x)) \prod_{i=l_1+1}^{m_1} |t - t_i(x)| e(t,x)$$

which verifies $b^2(t,x) = h(t,x)$. We turn to study $a_{12}^\sharp(t,x)$. By Weierstrass preparation theorem one can write

$$a_{12}^\sharp(t,x) = x^{n_2}(t^{m_2} + a_1(x)t^{m_2 - 1} + \cdots + a_{m_2}(x))\Psi(t,x)$$

with $a_i(0) = 0$, $\Psi(0,0) \neq 0$. Here we note that one can express

$$a_{12}^\sharp(t,x) = x^{n_2} g_1^{\mu_1} \cdots g_p^{\mu_p} h_1^{\nu_1} \cdots h_q^{\nu_q} \Psi$$

where $\bar{g}_i = g_i$ and $\bar{h}_i \neq h_i$. By the same argument as before we conclude that if we write

$$h_i = \prod_{k=1}^{n(i)} (t - t_k(z))$$

then we have Im $t_k(x) \neq 0$ for $x \in \mathbf{R}$, $0 < |x| < \delta$ with some $\delta > 0$. This gives

$$|a_{12}^\sharp|^2 = x^{2n_2} \prod_{j=1}^{l_2} (t - t_j(x))^2 \prod_{j=l_2+1}^{m_2} |t - t_j(x)|^2 \tilde{e}(t,x)^2$$

and define $\tilde{b}(t,x)$ by

$$\tilde{b}(t,x) = x^{n_2} \prod_{j=1}^{l_2} (t - t_j(x)) \prod_{j=l_2+1}^{m_2} |t - t_j(x)| \tilde{(}t,x)$$

which is the same one given by Lemma 2.1.1 applied to $|a_{12}^\sharp|^2$.

We now study all $\{t_i(z)\}$ and $\{t_j(z)\}$ appearing in the definition of b and $\tilde{b}$. Let us take $t_1(z), ..., t_m(z)$ which are differnt ones among $\{t_i(z)\}$ and $\{t_j(z)\}$.

Definition 2.2.1 *We call the curves* $t = \text{Re}\, t_j(x)$ *pseudo characteristic curves of the reference system. Just as before one can define* $\sigma_j(x)$, $s_j(x)$, ω_j, $\omega(T)$ *etc.*

Remark: Let $F(t,x) = h|a_{12}^\sharp|^2$. Then σ_j, s_j, ω_j, $\omega(T)$ are the same ones given by Definition 2.1.1.

Remark: Note that $n_1 = 0$ implies $m_1 \geq 1$.
Proof: Let $n_1 = 0$. Note $|a_{12}^\sharp|^2 \geq h$ implies $n_2 = 0$. On the other hand $n_2 = 0$ means $m_2 \geq 1$ because $a_{12}^\sharp(0,0) = 0$. Hence $|a_{12}^\sharp|^2 \geq h$ again shows that $m_1 \geq 1$. q.e.d.

Remark: Since one can write

$$t_b^*(x) = |x|^\alpha (C_b + o(|x|)), \quad t_{\tilde{b}}^*(x) = |x|^\beta (C_{\tilde{b}} + o(|x|), \quad C_b,\ C_{\tilde{b}} > 0$$

taking $\delta > 0$ so small one may suppose that

$$\text{either} \quad 2t_{\tilde{b}}^*(x) \geq t_b^*(x) \quad \text{or} \quad 2t_b^*(x) \geq t_{\tilde{b}}^*(x)$$

in $|x| \leq \delta$.

Lemma 2.2.1 *Let* $n_1 = 0$ *and* $2t_{\tilde{b}}^*(x) \geq t_b^*(x)$ *(resp.* $2t_b^*(x) \geq t_{\tilde{b}}^*(x)$*). Then there is a* $C > 0$ *such that*

$$\frac{b_t}{b} \leq C \frac{\tilde{b}_t}{\tilde{b}} \quad (\textit{resp.}\ \frac{\tilde{b}_t}{\tilde{b}} \leq C \frac{b_t}{b}) \quad \textit{in}\ \tilde{\omega}(T).$$

Proof: Suppose $2t^*_{\tilde{b}}(x) \geq t^*_b(x)$. Clearly we have $t - 2t^*_{\tilde{b}}(x) \leq t - t^*_b(x)$ and hence by Lemma 2.1.2

$$\frac{b_t}{b} \leq \frac{C'}{t - t^*_b(x)} \leq \frac{C'}{t - 2t^*_{\tilde{b}}(x)} \leq C'' \frac{\tilde{b}_t}{\tilde{b}} \quad in \quad \tilde{\omega}(T)$$

because $\tilde{\omega}(T) \subset \tilde{\omega}_{\tilde{b}}(T) \cap \tilde{\omega}_b(T)$. The proof for the other case is similar. q.e.d.

Lemma 2.2.2 *Let $n_1 = 0$. Then there is a $C > 0$ such that*

$$\left| \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \right| \leq C \frac{\tilde{b}_t}{\tilde{b}}, \quad \left| \partial_t \left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \right) \right| \leq C \left(\frac{\tilde{b}_t}{\tilde{b}} \right)^2$$

in $\tilde{\omega}_{\tilde{b}}(T)$.

Proof: Recall that

$$a^\sharp_{12} = x^{n_2} \prod_{j=1}^{m_2} (t - t_j(x)) \Psi$$

and note that

$$\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} = \sum \frac{1}{t - t_j(x)} + \frac{\Psi_t}{\Psi}.$$

Since $|t - t_j(x)| \geq t - t^*_{\tilde{b}}(x)$ in $\tilde{\omega}_{\tilde{b}}(T)$ we have

$$\left| \frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \right| \leq \frac{c_1}{t - t^*_{\tilde{b}}(x)} + c_2 \leq \frac{c_3}{t - t^*_{\tilde{b}}(x)} \leq c_4 \left(\frac{\tilde{b}_t}{\tilde{b}} \right)$$

in $\tilde{\omega}_{\tilde{b}}(T)$, taking T small enough. Similarly we have

$$\partial_t \left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \right) = - \sum \frac{1}{(t - t_j(x))^2} + \partial_t \left(\frac{\Psi_t}{\Psi} \right)$$

it is easy to see that

$$\left| \partial_t \left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}} \right) \right| \leq \frac{c_1}{(t - t^*_{\tilde{b}}(x))^2} \leq c_2 \left(\frac{\tilde{b}_t}{\tilde{b}} \right)^2$$

in $\tilde{\omega}_{\tilde{b}}(T)$. q.e.d.

Lemma 2.2.3 *There is a $C > 0$ such that*

$$\sup_{0 \leq t \leq t^*(x)} |b(t, x)| \leq C|x|.$$

Proof: If $n_1 \geq 1$ then the assertion is trivial. Let $n_1 = 0$ and hence $m_1, m_2 \geq 1$. When $t^*(x) \sim t^*_b(x)$ then Lemma 2.1.4(or rather its proof) proves the lemma. Then we now assume that there is no j such that

$$h(t_j(x), x) = 0, \quad t_j(x) \sim t^*(x).$$

We observe the Newton polygons $\Gamma(a^\sharp_{12})$ and $\Gamma(h)$. Our assumption implies that the line with the slowest steep of $\Gamma(h)$ is steeper than that of $\Gamma(a^\sharp_{12})$. This shows that

$$h_k(x)t^*(x)^{2m_1-k} = o(t^*(x)^{2m_1}), \quad 1 \leq k \leq 2m_1.$$

On the other hand since $a^\sharp_{12}a^\sharp_{21} \geq h$, $\overline{a^\sharp_{12}} = a^\sharp_{21}$ we see that $m_1 \geq m_2$. From $a^\sharp_{12}(t(x), x) = 0$ it follows that

$$t_j(x)^{m_2} = O(|x|).$$

This shows that $t^*(x)^{2m_1} = O(|x|^2)$ and then

$$\sup_{0 \leq t \leq t^*(x)} |h(t, x)| \leq C|x|^2$$

from which we have the desired assertion. q.e.d.

Definition 2.2.2 *Let us put*

$$\rho_j(t, x) = t - \sigma_j(x), \;\; j = 1, ..., m, \;\; \rho_{m+1}(x) = t - s_m(x).$$

Lemma 2.2.4 *We have the following.*

(i) *Let $n_1 \geq 1$. Then for $j = 1, ..., m+1$ we have*

$$\sup_{0 \leq t \leq T, |x| < \epsilon} |b(t, x)\rho_{jx}(t, x)| \to 0, \quad \epsilon \to 0$$

(ii) *Let $n_1 = 0$. Then for $j = 1, ..., m+1$ we have*

$$\sup_{0 \leq t \leq t^*(x), |x| < \epsilon} |b(t, x)\rho_{jx}(t, x)| \to 0, \quad \epsilon \to 0$$

(iii) *Let $n_1 = 0$ and $2t^*_{\tilde{b}}(x) \geq t^*_b(x)$. Then for $j = 1, ..., m+1$ we have*

$$\sup_{0 \leq t \leq t^*(x), |x| < \epsilon} |\tilde{b}(t, x)\rho_{jx}(t, x)| \to 0, \quad \epsilon \to 0.$$

Proof: Remarking that $\rho_{jx}(x) = O(|x|^{\sigma-1})$ with some $\sigma > 0$ the assertions follow from Lemmas 2.2.3 and 2.1.4. q.e.d.

Lemma 2.2.5 *There is a $C > 0$ such that*

$$|\rho_j(t,x)b_t(t,x)| \le C|b(t,x)|$$

in $\tilde{\omega}_j \cap \{t \ge 0\}$, $j = 1, ..., m$ *and*

$$|\rho_{m+1}(t,x)b_t(t,x)| \le C|b(t,x)|$$

in $\tilde{\omega}(T) \cap \{t \ge 0\}$.

Proof: Recall that

$$\frac{b_t}{b} = \sum_{i=1}^{l_1} \frac{1}{t - t_i(x)} + \sum_{i=l_1+1}^{m_1} \frac{t - \text{Re } t_i(x)}{|t - t_i(x)|^2} + \frac{e_t}{e}.$$

Let $(t,x) \in \omega_j$. Then it is clear that

$$|t - \sigma_j(x)| \le |t - \text{Re } t_\mu(x)| \le |t - t_\mu(x)|$$

for all μ. This shows that

$$\left|\frac{b_t}{b}\right| \le \sum \frac{1}{|t - t_i(x)|} + c \le \frac{c'}{|t - \sigma_j(x)|} + c$$

in $\tilde{\omega}_j$. Taking T small so that

$$c \le \frac{c''}{|t - \sigma_j(x)|}$$

in $\tilde{\omega}_j$ we have the assertion. If $(t,x) \in \tilde{\omega}(T)$, then we see

$$|t - s_m(x)| \le |t - \text{Re } t_\mu(x)| \le |t - t_\mu(x)|$$

for all μ and hence the assertion follows from the same arguments as before. q.e.d.

2.3 Proof of Proposition 2.1.4

In this subsection we give a proof of Proposition 2.1.1. We may assume that $\mu_j = j$ renumbering the indices if necessary. We fix $1 \le j_0 \le m$. Assume that

$$\text{Re } t_{j_0-k-1}(x) < \text{Re } t_{j_0-k}(x) = \cdots = \text{Re } t_{j_0}(x) = \cdots = \text{Re } t_{j_0+l}(x) < \text{Re } t_{j_0+l+1}(x)$$

in $0 < x < \delta$. Put $\lambda_{j_0}(x) = \text{Re } t_{j_0}(x)$ and

$$\phi^+ = \frac{1}{2}(\text{Re } t_{j_0+l+1}(x) - \lambda(x)), \quad \phi^-(x) = \frac{1}{2}(\lambda(x) - \text{Re } t_{j_0-k-1}(x)).$$

When $\text{Re } t_{j_0}(x) = \text{Re } t_m(x)$ (resp. $\text{Re } t_{j_0} = \text{Re } t_1(x)$) we set

$$\phi^+ = \frac{1}{2}(3t^*(x) - \lambda(x)), \quad (\text{resp. } \phi^- = \frac{1}{2}(\lambda(x) + 3t^*(x))).$$

Lemma 2.3.1 *For any* $1 \leq \nu \leq m$

$$|\lambda(x) - t_\nu(x)| + \delta|\phi^\pm(x)| \sim |\lambda(x) + \delta\phi^\pm(x) - t_\nu(x)|$$

holds uniformly in $0 \leq \delta \leq 1$.

Proof: Let $\text{Re}\, t_{j_0}(x) < \text{Re}\, t_m(x)$. Note

$$|\lambda(x) - t_\nu(x) + \delta\phi^+(x)|^2 = (\lambda(x) - \text{Re}\, t_\nu(x) + \delta\phi^+(x))^2 + (\text{Im}\, t_\nu(x))^2.$$

If $\text{Re}\, t_\nu(x) \geq \text{Re}\, t_{j_0+l+1}(x)$ then

$$\begin{aligned} &|\lambda(x) - \text{Re}\, t_\nu(x) + \delta\phi^+(x)| \\ \geq |\text{Re}\, t_\nu(x) - \lambda(x)| - \delta|\phi^+(x)| \geq &\frac{1}{2}|\text{Re}\, t_\nu(x) - \lambda(x)|. \end{aligned}$$

Since $\text{Re}\, t_\nu(x) - \lambda(x) \geq 2\phi^+(x) \geq 0$ it follows that

$$|\lambda(x) - \text{Re}\, t_\nu(x) + \delta\phi^+(x)| \sim |\lambda(x) - \text{Re}\, t_\nu(x)| + \delta|\phi^+(x)|$$

which proves the assertion for ϕ^+. We next assume $\text{Re}\, t_\nu(x) \leq \text{Re}\, t_{j_0}(x)$. In this case we have

$$|\lambda(x) - \text{Re}\, t_\nu(x) + \delta\phi^+(x)| = |\lambda(x) - \text{Re}\, t_\nu(x)| + \delta\phi^+(x)$$

then one gets the assertion. The proof of the other cases are similar. q.e.d.

Write

$$B(t,x) = x^{\bar{n}}\tilde{B}(t,x)\tilde{E}(t,x)$$

where $\tilde{E}(0,0) \neq 0$. Recall that our assumption is

$$\Gamma(tx^{\bar{n}}\tilde{B}(t+\phi(x),x) \subset \Gamma(x^n \prod_{\nu=1}^{m} \Lambda_\nu(t+\phi(x),x))$$

where $\Lambda_\nu(t,x) = t - t_\nu(x)$. Let us define $\epsilon(\nu)$, $1 \leq \nu \leq m$ and ϵ by

$$|\lambda(x) - t_\nu(x)| \sim x^{\epsilon(\nu)}, \quad \phi^+(x) \sim x^\epsilon.$$

Assume that

$$\epsilon(\nu_1) \geq \cdots \geq \epsilon(\nu_l) > \epsilon \geq \epsilon(\nu_{l+1}) \geq \cdots \geq \epsilon(\nu_m).$$

From Lemma 2.3.1 it follows that

$$\prod_{j=l+1}^{m} |\Lambda_j(\lambda(x) + \delta\phi^+(x), x)| \geq c \prod_{j=l+1}^{m} x^{\epsilon(\nu_j)}$$

with some $c > 0$ uniformly in $0 \leq \delta \leq 1$. Lemma 2.3.1 again shows

$$\prod_{j=1}^{l} |\Lambda_{\nu_j}(\lambda(x) + \delta\phi^+(x), x)| \sim \prod_{j=1}^{l}(x^{\epsilon(\nu_j)} + \delta x^\epsilon) \geq c\delta^p x^{\epsilon p + \epsilon(\nu_{p+1}) + \cdots + \epsilon(\nu_l)}$$

with $c > 0$ for $p = 0, 1, ..., l$. Hence, writing

$$tx^{\bar{n}}\tilde{B}(t+\lambda(x), x) = \sum b_j(x)t^j, \quad b_{j+1}(x) = \frac{1}{j!}x^{\bar{n}}\partial_t^j\tilde{B}(\lambda(x), x)$$

the assumption implies that

$$\text{Order}\{x^{\bar{n}}\partial_t^j\tilde{B}(\lambda(x), x)\} \geq n + \sum_{i=j+2}^{m}\epsilon(\nu_i). \tag{2.3.1}$$

Lemma 2.3.2 *For* $0 \leq \delta \leq 1$ *we have*

$$|\delta\phi^{\pm}(x)x^{\bar{n}}\tilde{B}(\lambda(x) + \delta\phi^{\pm}(x), x)| \leq C|x^n\prod_{\nu=1}^{m}\Lambda_\nu(\lambda(x) + \delta\phi^{\pm}(x), x)|$$

with C independent of δ.

Proof: Let us write

$$\tilde{B}(\lambda(x) + \delta\phi^+(x), x) = \sum_{j=0}^{\bar{m}} B_j(x)\delta^j, \;\; B_j(x) = \frac{1}{j!}\phi^+(x)^j\partial_t^j\tilde{B}(\lambda(x), x).$$

From (2.3.1) it follows that

$$|x^{\bar{n}}\partial_t^j\tilde{B}(\lambda(x), x)| \leq Cx^{n+\sum_{i=j+2}^{m}\epsilon(\nu_i)}$$

and hence

$$\delta|\phi^+||\delta^j\phi^+(x)^jx^{\bar{n}}\partial_t^j\tilde{B}(\lambda(x), x)| \leq C\delta^{j+1}x^{\epsilon(\nu_{j+2})+\cdots+\epsilon(\nu_m)+(j+1)\epsilon+n}. \tag{2.3.2}$$

Let $j + 2 \leq l$ then the right-hand side of (2.3.2) is bounded by

$$C\delta^{j+1}x^{n+(j+1)\epsilon+\epsilon(\nu_{j+2})+\cdots+\epsilon(\nu_l)}\prod_{i=l+1}^{m}x^{\epsilon(\nu_i)} \leq C|x^n\prod_{\nu=1}^{m}\Lambda_\nu(\lambda(x) + \delta\phi^+(x), x)|.$$

If $j + 2 > l$ then noting

$$(j+1)\epsilon + \epsilon(\nu_{j+2}) + \cdots + \epsilon(\nu_m) \geq l\epsilon + \epsilon(\nu_{l+1}) + \cdots + \epsilon(\nu_m)$$

the right-hand side of (2.3.2) is estimated by

$$C\delta^l x^{n+l\epsilon}\left(\prod_{i=l+1}^{m}x^{\epsilon(\nu_i)}\right)\delta^{j+1-l} \leq C|x^n\prod_{\nu-1}^{m}\Lambda_\nu(\lambda(x) + \delta\phi^+(x), x)|$$

which ends the proof of the assertion for ϕ^+. The proof for ϕ^- is similar.q.e.d.

Proof of Proposition 2.1.1. Recall that

$$\tilde{\omega}_{j_0} = \{(t, x) \mid |x| < \delta, \lambda(x) - \phi^{-1}(x) \leq t \leq \lambda(x) + \phi^+(x)\}.$$

Let $(t,x) \in \tilde{\omega}_{j_0} \cap \{t \geq \lambda(x)\}$. Then there is a $0 \leq \delta \leq 1$ such that $t = \lambda(x) + \delta\phi^+(x)$. From Lemma 2.3.2 it follows

$$|(t-\lambda(x))B(t,x)| \leq C|f(t,x)|.$$

In the case $(t,x) \in \tilde{\omega}_{j_0} \cap \{t \leq \lambda(x)\}$ the proof is similar. q.e.d.

Lemma 2.3.3 *In $\omega(T)$ with small T we have*

$$|B(t,x)| \leq C\sum_{q=1}^{m} |x^n \prod_{\nu\neq q} \Lambda_\nu(t,x)|,$$

$$|(t - \mathrm{Re}\, t_m(x))B(t,x)| \leq C|x^n \prod_{\nu=1}^{m} \Lambda_\nu(t,x)|.$$

Proof: Repeating the same proof of Lemma 2.3.2 we see with $\lambda(x) = \mathrm{Re}\, t_m(x)$ that

$$|\delta\phi^+(x)x^{\bar{n}}\tilde{B}(\lambda(x)+\delta\phi^+(x),x)| \leq C|x^n \prod_{\nu=1}^{m} \Lambda_\nu(\lambda(x)+\delta\phi^+(x),x)|$$

holds for all $0 \leq \delta$. For any $(t,x) \in \omega(T)$, taking $\delta > 0$ so that $t = \lambda(x)+\delta\phi^+(x)$ the second inequality follows. Since

$$t - \mathrm{Re}\, t_m(x) \geq t - |t_m(x)| \geq \frac{2}{3}t \geq \frac{1}{3}|t - t_\nu(x)|,$$

$$|t - t_\nu(x)| \geq t - |t_\nu(x)| \geq \frac{2}{3}t \geq \frac{1}{3}(t - \mathrm{Re}\, t_m(x))$$

the first inequality follows from the second one. q.e.d.

Lemma 2.3.4 *In $\omega(T)$ with small T we have*

$$\sum_{q=1}^{m} |x^n \prod_{\nu\neq q}^{m} \Lambda_\nu(t,x)| \sim |\partial_t f(t,x)|.$$

Proof: Since it is clear that

$$|\partial_t f(t,x)| \leq C\sum_{q=1}^{m} |x^n \prod_{\nu\neq q} \Lambda_\nu(t,x)|$$

it is enough to show the converse. Note that

$$f\partial_t f = x^{2n}\sum_{\nu=1}^{m}(t - \mathrm{Re}\, t_\nu(x)) \prod_{\mu\neq\nu} |t - t_\mu(x)|^2|e|^2$$

$$+x^{2n}\prod_{\mu=1}^{m} |t - t_\mu(x)|^2 e\partial_t e.$$

On the other hand, in $\omega(T)$ we have $t - \text{Re}\, t_\nu(x) \geq c|t - t_\nu(x)|$ for $1 \leq \nu \leq m$ because

$$t - \text{Re}\, t_\nu(x) \geq \frac{2}{3}t \geq \frac{t}{3} + \frac{1}{3}|t_\nu(x)| \geq \frac{1}{3}|t - t_\nu(x)|.$$

Thus we see

$$f\partial_t f \geq cx^{2n} \sum_{\nu=1}^{m} |t - t_\nu(x)| \prod_{\mu \neq \nu} |t - t_\mu(x)|^2$$

with $c > 0$. Hence dividing $|f(t,x)|$ we get the desired assertion. q.e.d.

3 A priori estimate

3.1 Estimate in a domain bounded by pseudo-characteristic curves

Let $D \subset W$ be an open set and $\rho(t,x) \in C^\infty(D)$ where $\rho_t > 0$ in D. Put $p = \partial_t^2 - h(t,x)\partial_x^2$ and note that

$$p - h_x\partial_x = \partial_t^2 - \partial_x h \partial_x.$$

We study the energy form:

$$\begin{aligned}(pu - h_x\partial_x u)\bar{\partial_t u} + \overline{(pu - h_x\partial_x u)}\partial_t u \\ = \partial_t G_1(u) + \partial_x G_2(u) - R(u)\end{aligned}$$

where

$$\begin{aligned}G_1(u) &= |\partial_t u|^2 + h(t,x)|\partial_x u|^2, \\ G_2(u) &= -h(\partial_t u\bar{\partial_x u} + \bar{\partial_t u}\partial_x u), \\ R(u) &= h_t|\partial_x u|^2.\end{aligned}$$

Multiply $e^{-\theta t}\rho^{\pm N}$ to the energy form and integrate over D:

$$\begin{aligned}2\int_D e^{-\theta t}\rho^{\pm N}|pu - h_x\partial_x u||\partial_t u|dxdt \\ \geq \int_D [\partial_t(e^{-\theta t}\rho^{\pm N}G_1(u)) + \partial_x(e^{-\theta t}\rho^{\pm N}G_2(u))]dxdt \\ \mp N\int_D e^{-\theta t}\rho^{\pm N-1}(\rho_t G_1(u) + \rho_x G_2(u))dxdt \\ +\theta\int_D e^{-\theta t}\rho^{\pm N}G_1(u)dxdt - \int_D e^{-\theta t}\rho^{\pm N}R(u)dxdt\end{aligned}$$

where $\theta > 0$ and N is even. Note that

$$\begin{aligned}N\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_x G_2(u)|dxdt \leq \frac{N}{4}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t||\partial_t u|^2 dxdt \\ +4N\int_D |e^{-\theta t}\rho^{\pm N-1}h^2\rho_x^2\rho_t^{-1}||\partial_x u|^2 dxdt\end{aligned}$$

by the Cauchy-Schwarz inequality. Similarly we have

$$
\begin{aligned}
&2\int_D e^{-\theta t}\rho^{\pm N}|pu-h_x\partial_x u||\partial_t u|dxdt\\
&\le \frac{4}{N}\int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}||pu-h_x\partial_x u|^2 dxdt\\
&\quad +\frac{N}{4}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t||\partial_t u|^2 dxdt.
\end{aligned}
$$

We choose $\pm$ so that $\mp\rho^{\pm N-1}\rho_t>0$ in D, that is if $\rho>0$ in D we take ρ^{-N} and if $\rho<0$ in D then we take ρ^N. Using these inequalities we get

$$
\begin{aligned}
&\frac{4}{N}\int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}||pu-h_x\partial_x u|^2 dxdt\\
&\ge \int_D [\partial_t(e^{-\theta t}\rho^{\pm N}G_1(u))+\partial_x(e^{-\theta t}\rho^{\pm N}G_2(u))]dxdt\\
&\quad +\frac{N}{4}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t|(2|\partial_t u|^2+h(t,x)|\partial_x u|^2)dxdt\\
&\quad +\int_D \sigma(t,x)|e^{-\theta t}\rho^{\pm N-1}||\partial_x u|^2 dxdt+\theta\int_D e^{-\theta t}\rho^{\pm N}G_1(u)dxdt
\end{aligned}
$$

where

(3.1.1) $$\sigma(t,x)=\frac{3N}{4}h\rho_t-4Nh^2\rho_x^2\rho_t^{-1}-C|\rho h_t|.$$

We turn to $\partial_t u\cdot\bar{u}+\bar{\partial_t u}\cdot u=\partial_t|u|^2$. Multiply $e^{-\theta t}\rho^{\pm N-2}\rho_t^2$ we get $\mathrm{C}_1 N^{-1}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t||\partial_t u|^2 dxdt \ge \int_D \partial_t(e^{-\theta t}\rho^{\pm N-2}\rho_t^2|u|^2)dxdt$
$+\frac{N}{4}\int_D e^{-\theta t}\rho^{\pm N-3}\rho_t^3|u|^2 dxdt+\theta\int_D e^{-\theta t}\rho^{\pm N-2}\rho_t^2|u|^2 dxdt$.Let us put

(3.1.1) $$E(u)=|\partial_t u|^2+h(t,x)|\partial_x u|^2+cN^2\rho^{-2}\rho_t^2|u|^2 \quad (c=(4C_1)^{-1})$$

and

(3.1.2) $$\Gamma(u)=-(e^{-\theta t}\rho^{\pm N}E(u))dx+(e^{-\theta t}\rho^{\pm N}G_2(u))dt$$

and summarize:

Proposition 3.1.1 *Assume $\rho\in C^\infty(D)$, $\rho\ne 0$, $\rho_t>0$ in D and N is even. Choose $\pm$ so that $\mp\rho^{\pm N-1}\rho_t>0$ in D. Then we have*

$$
\begin{aligned}
&\frac{4}{N}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t^{-1}||pu-h_x\partial_x u|^2 dxdt\\
&\ge \int_{\partial D}\Gamma(u)+\int_D \sigma(t,x)|e^{-\theta t}\rho^{\pm N-1}||\partial_x u|^2 dxdt\\
&\quad +\frac{N}{4}\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t|[E(u)-C'N\rho^{-1}\rho_{tt}|u|^2]dxdt\\
&\quad +\theta\int_D e^{-\theta t}\rho^{\pm N}E(u)dxdt
\end{aligned}
$$

where
$$\sigma(t,x)=\frac{3N}{4}h\rho_t-4Nh^2\rho_x^2\rho_t^{-1}-C|\rho h_t|$$

Definition 3.1.1 *We define $\rho_{A,D}(t,x)$ by*

(1) $\rho_{A,D}(t,x)=\rho_j=t-\sigma_j(x)$ *if* $D=\omega_j\cap\{t\geq 0\}$, $j=1,2,...,m$

(2) $\rho_{A,D}(t,x)=\rho_{m+1}=t-s_m(x)$ *if* $D=\omega(T)\cap\{t\geq 0\}$ *and* $n_1\geq 1$

(3) $\rho_{A,D}(t,x)=b(t,x)$ *if* $D=\omega(T)\cap\{t\geq 0\}$, $n_1=0$ *and* $2t_b^*(x)\geq t_{\tilde b}^*(x)$

(4) $\rho_{A,D}(t,x)=\tilde b(t,x)$ *if* $D=\omega(T)\cap\{t\geq 0\}$, $n_1=0$ *and* $2t_{\tilde b}^*(x)\geq t_b^*(x)$

where we have set

$$\omega_j=\{(t,x)\mid |x|<\delta(T-t),s_{j-1}(x)\leq t\leq s_j(x)\},\quad j=1,...,m$$
$$\omega(T)=\{(t,x)\mid |x|<\delta(T-t),s_m(x)\leq t\}.$$

Remark: We may suppose that $b>0$, $\tilde b>0$ in $\omega(T)$.

Lemma 3.1.1 *Let $D=\omega_j\cap\{t\geq 0\}$ or $D=\omega(T)\cap\{t\geq 0\}$ and $\rho=\rho_{A,D}$. Then there are $c>0$, $C>0$ such that, taking T small, we have*

$$\sigma(t,x)\geq cNb(t,x)^2\rho_t,\quad C\rho^{-2}\rho_t^2\geq\rho^{-1}\rho_{tt}\quad in\quad D$$

for $N\geq N_0$.

Proof: We first study the case $n_1\geq 1$. In this case, by definition, $\rho_{A,D}=t-\sigma_j$ or $t-s_m$. Note
$$\rho h_t=2\rho bb_t.$$
From Lemma 2.2.5 we have $|\rho b_t|\leq Cb$ in D and hence $|\rho h_t|\leq Cb^2$ in D. On the other hand, from Lemma 2.2.4 we see that

$$\sup_{0\leq t\leq t^*(x),|x|<\epsilon}|b(t,x)\rho_x|\to 0\quad as\quad \epsilon\to 0\quad if\quad D=\omega_j\cap\{t\geq 0\},\ \ j\leq m$$
$$\sup_{0\leq t\leq T,|x|<\epsilon}|b(t,x)\rho_x|\to 0\quad as\quad \epsilon\to 0\quad if\quad D=\omega(T)\cap\{t\geq 0\}.$$

Noticing $\rho_t=1$ it is clear that, taking T small,

$$\sigma(t,x)\geq CNb^2$$

with some $C>0$. Since $\rho_{tt}=0$ the second inequality is trivial.

We turn to the case $n_1=0$. Let $2t_b^*(x)\geq t_{\tilde b}^*(x)$. Recall that

$$b_t\sigma(t,x)=b^2\left[\frac{3N}{4}b_t^2-4Nb^2b_x^2-Cb_t^2\right]$$

because $\rho = b$. By Lemma 2.1.7 we have $b_t \geq Kb > 0$ for any given K if taking T small in $\tilde{\omega}_b(T)$. Since b_x is bounded we get (Lemma 2.1.6)

$$b_t \sigma(t,x) \geq CNb^2 b_t^2 \quad in \quad \tilde{\omega}_b(T).$$

Since $\omega(T) \subset \tilde{\omega}_b(T)$ it is clear that $\sigma(t,x) \geq CNb^2 b_t$ in D. We turn to the second inequality. By Lemma 2.1.1 we see

$$\partial_t \left(\frac{b_t}{b}\right) \leq C \quad in \quad \tilde{\omega}_b(T).$$

This shows that $b_{tt}b^{-1} \leq C + b_t^2 b^{-2}$ in $\tilde{\omega}_b(T)$. From Lemma 2.1.7 again we have

$$b^{-2}b_t^2 \geq b^{-1}b_t \geq C \quad in \quad \tilde{\omega}_b(T)$$

taking T small and hence we get

$$b_{tt}b^{-1} \leq 2b_t^2 b^{-2} \quad in \quad \tilde{\omega}_b(T).$$

Since $\omega(T) \subset \tilde{\omega}_b(T)$, we have the second inequality. Finally we study the case $n_1 = 0$ and $2t_{\tilde{b}}^*(x) \geq t_b^*(x)$. Recall

$$\tilde{b}_t \sigma(t,x) = b^2 \left[\frac{3N}{4}\tilde{b}_t^2 - 4Nb^2\tilde{b}_x^2 - C\tilde{b}\tilde{b}_t b^{-1} b_t\right]$$

because $\rho = \tilde{b}$. Since $\tilde{b}_x$ is bounded (Lemma 2.1.6) and $\tilde{b} \geq b$ it follows from Lemma 2.1.7 that

$$b^2\tilde{b}_x^2 \leq C\tilde{b}^2 \leq K^{-1}\tilde{b}_t^2 \quad in \quad \tilde{\omega}_{\tilde{b}}(T)$$

for any K taking T small. This shows that the second term can be cancelled against the first term. On the other hand, since $2t_{\tilde{b}}^* \geq t_b^*$, from Lemma 2.2.1 we see that

$$b_t b^{-1} \leq C\tilde{b}_t \tilde{b}^{-1} \quad in \quad \tilde{\omega}(T)$$

and hence $b_t b^{-1}\tilde{b}\tilde{b}_t \leq C\tilde{b}_t^2$ in $\tilde{\omega}(T)$. This shows that

$$\sigma(t,x) \geq cNb^2\tilde{b}_t^2 = cNb^2\rho_t \quad in \quad \omega(T).$$

By Lemma 2.1.1 we see

$$\partial_t \left(\frac{\tilde{b}_t}{\tilde{b}}\right) \leq C \quad in \quad \tilde{\omega}_{\tilde{b}}(T)$$

and hence $\tilde{b}_{tt}\tilde{b}^{-1} \leq C + \tilde{b}_t^2\tilde{b}^{-2}$ in $\tilde{\omega}_{\tilde{b}}(T)$. From Lemma 2.1.7 we get

$$\tilde{b}^{-2}\tilde{b}_t^2 \geq \tilde{b}^{-1}\tilde{b}_t \geq c \quad in \quad \tilde{\omega}_{\tilde{b}}(T)$$

with T small. Then one has

$$\tilde{b}_{tt}\tilde{b}^{-1} \leq 2\tilde{b}_t^2\tilde{b}^{-2} \quad in \quad \tilde{\omega}_{\tilde{b}}(T).$$

Noting $\rho = \tilde{b}$ and $\omega(T) \subset \tilde{\omega}_{\tilde{b}}(T)$, this gives the desired assertion. q.e.d.

We summarize:let us denote

$$\omega_j^u = \{(t,x) \in \omega_j \mid t \geq \sigma_j(x)\}, \quad \omega_j^d = \{(t,x) \in \omega \mid t \leq \sigma_j(x)\}.$$

Proposition 3.1.2 *We take ρ^{-N} with $\rho = \rho_{A,D}$ if $D = \omega(T)$, ρ^N with $\rho = \rho_{A,D}$ if $D = \omega_j^d \cap \{t \geq 0\}$ and ρ^{-N} with $\rho = \rho_{A,D}$ if $D = \omega_j^u \cap \{t \geq 0\}$. Then there is $C_1 > 0$ such that*

$$\frac{4}{N}\int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}||pu - h_x\partial_x u|^2 dxdt \geq \int_{\partial D}\Gamma(u)$$
$$+c_1 N\int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t|E(u)dxdt + \theta\int_D e^{-\theta t}\rho^{\pm N}E(u)dxdt.$$

Lemma 3.1.2 *Assume that*

$$\Gamma(tY_\phi) \subset \frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi),\ \Gamma(tZ_\phi) \subset \frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi),\ \ \forall \phi \in \mathcal{C}^\pm(A).$$

Let $\rho = \rho_{A,D}$ and $D = \omega_j \cap \{t \geq 0\}$ or $D = \omega(T) \cap \{t \geq 0\}$. Then taking T small we have

$$\left|\rho(t,x)\frac{Y(t,x)}{a_{21}^\sharp}\right|,\ \ \left|\rho(t,x)\frac{Z(t,x)}{a_{12}^\sharp}\right| \leq Cb(t,x)\rho_t(t,x)$$

in D.

Proof: We prove the asertion for Z because the proof for Y is a repetition. From Proposition 2.1.4 with $F = h|a_{12}^\sharp|^2$, $B = Z$ we have if $D = \omega_j \cap \{t \geq 0\}$

$$|\rho(t,x)Z(t,x) \leq C|b(t,x)\tilde{b}(t,x)| \quad in \quad D.$$

On the other hand, since $|\tilde{b}(t,x)| = |a_{12}^\sharp(t,x)|$, $\rho_t = 1$ we get the desired assertion. Let $D = \omega(T) \cap \{t \geq 0\}$ and $n_1 \geq 1$. Then the proof is same. Let $D = \omega(T) \cap \{t \geq 0\}$ and $n_1 = 0$. Proposition 2.1.4 gives

$$|Z(t,x)| \leq C|\partial_t(b\tilde{b})|.$$

This shows that

$$\left|\frac{Z(t,x)}{a_{12}^\sharp(t,x)}\right| \leq C\left|\frac{\partial_t(b\tilde{b})}{\tilde{b}}\right| = C\left(b_t + \frac{b\tilde{b}_t}{\tilde{b}}\right).$$

When $2t_b^*(x) \geq t_{\tilde{b}}^*(x)$ from Lemma 2.2.1 it follows that

$$\frac{\tilde{b}_t}{\tilde{b}} \leq c\frac{b_t}{b} \quad in \quad \omega(T)$$

and hence we have

$$|\frac{Z}{a_{12}^{\sharp}}| \le c'(b_t + b_t) \le 2c'b_t.$$

Remarking that $\rho = b$ we get

$$|\rho \frac{Z}{a_{12}^{\sharp}}| \le c'' b\rho_t \quad in \quad \omega(T).$$

We turn to the case $2t^*_{\tilde{b}}(x) \ge t^*_b(x)$. By lemma 2.2.1 we have

$$\frac{b_t}{b} \le c\frac{\tilde{b}_t}{\tilde{b}} \quad in \quad \omega(T).$$

Hence we get

$$|\frac{Z}{a_{12}^{\sharp}}| \le C'(\frac{\tilde{b}+b}{\tilde{b}} + \frac{b\tilde{b}_t}{\tilde{b}}).$$

Since $\rho = \tilde{b}$ we see that

$$|\rho \frac{Z}{a_{12}^{\sharp}}| \le C'' b\rho_t \quad in \quad \omega(T).$$

q.e.d.

Lemma 3.1.3 *Let $D = \omega_j \cap \{t \ge 0\}$ or $D = \omega(T) \cap \{t \ge 0\}$ and $\rho = \rho_{A,D}$. Then we have*

$$\left|\frac{\partial_t a_{12}^{\sharp}}{a_{12}^{\sharp}}\right| \le C\frac{\rho_t}{\rho}, \quad \left|\partial_t(\frac{\partial_t a_{12}^{\sharp}}{a_{12}^{\sharp}})\right| \le C(\frac{\rho_t}{\rho})^2 \quad in \quad D.$$

Proof: Let $D = \omega_j \cap \{t \ge 0\}$. Since

$$\frac{\partial_t a_{12}^{\sharp}}{a_{12}^{\sharp}} = \sum \frac{1}{t - t_j(x)} + \frac{\Psi_t}{\Psi}$$

and for $(t,x) \in \omega_j$ we have

$$|t - \sigma_j(x)| \le |t - \mathrm{Re}\, t_\mu(x)| \le |t - t_\mu(x)|$$

for all μ. It is clear that

$$|\rho(t,x)\frac{\partial_t a_{12}^{\sharp}}{a_{12}^{\sharp}}| \le C \quad in \quad \omega_j$$

taking T small. This proves the assertion because $\rho_t = 1$. Similar arguments prove the second inequality when $D = \omega_j \cap \{t \ge 0\}$ or $D = \omega(T)$, $n_1 \ge 1$. Let

$D=\omega(T)$ and $n_1=0$. Assume that $2t^*_{\tilde b}(x)\geq t^*_b(x)$. Then from Lemma 2.2.2 it follows that

$$|\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}|\leq C\frac{\tilde b_t}{\tilde b},\ \ |\partial_t(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}})|\leq C(\frac{\tilde b_t}{\tilde b})^2\quad in\quad \tilde\omega_{\tilde b}(T)$$

and this proves the assertion because $\tilde b=\rho$. When $2t^*_b(x)\geq t^*_{\tilde b}(x)$ then using

$$\frac{\tilde b_t}{\tilde b}\leq C\frac{b_t}{b}\quad in\quad \tilde\omega(T)$$

(Lemma 2.2.1) we get the desired assertion. q.e.d.

We pass to $ML^\sharp u=f$. Assume that u verifies $ML^\sharp u=f$. Recall that

$$ML^\sharp=\begin{pmatrix} p+(Z/a^\sharp_{12}-h_x)\partial_x & 0\\ 0 & p+(Y/a^\sharp_{21}-h_x)\partial_x\end{pmatrix}+\tilde R\partial_t+\tilde S$$
$$=(p-h_x\partial_x)I+\tilde Q\partial_x+\tilde R\partial_t+\tilde S$$

where

$$\tilde Q=\begin{pmatrix} Z/a^\sharp_{12} & 0\\ 0 & Y/a^\sharp_{21}\end{pmatrix},\ \ \tilde R=\tilde C+A^\sharp_x+B^\sharp+{}^{co}B^\sharp,\ \ \tilde S=M(B^\sharp)$$
$$\tilde C=-\mathrm{diag}\left(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}},\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}\right).$$

We assume that the hypothesis in Lemma 3.1.2 holds.

Lemma 3.1.4 *Let $D=\omega_j\cap\{t\geq 0\}$ or $D=\omega(T)$ and $\rho=\rho_{A,D}$. Then we have*

$$\rho^2\rho_t^{-1}|\tilde Q|^2\leq C(\tilde Q)\rho_t b(t,x)^2\quad in\quad D$$

with some $C(\tilde Q)$.

Proof: It is clear from Lemma 3.1.2. q.e.d.

Lemma 3.1.5 *Let $D=\omega_j\cap\{t\geq 0\}$ or $D=\omega(T)$ and $\rho=\rho_{A,D}$. Then we have*

$$\rho^2\rho_t^{-1}|\tilde R|^2\leq C(\tilde R)\rho_t,\ \ \rho^2\rho_t^{-1}|\tilde S|^2\leq C(\tilde S)\rho_t$$

with some $C(\tilde R)>0$, $C(\tilde S)>0$.

Proof: It is clear from Lemma 3.1.3. q.e.d.

Note that

$$\rho^{\pm N+1}\rho_t^{-1}|pu-h_x\partial_x u|^2\leq 2\rho^{\pm N+1}\rho_t^{-1}|ML^\sharp u|^2$$
$$+\rho^{\pm N+1}\rho_t^{-1}\{C|\tilde Q|^2|\partial_x u|^2+C|\tilde R|^2|\partial_t u|^2+C|\tilde S|^2|u|^2\}$$
$$\leq 2\rho^{\pm N+1}\rho_t^{-1}|ML^\sharp u|^2+\rho^{\pm N-1}\rho_t\{C(\tilde Q)b^2|\partial_x u|^2+C(\tilde R)|\partial_t u|^2+C(\tilde S)|u|^2\}$$

by Lemmas 3.1.4 and 3.1.5. Taking $N^2 \geq c_1 C(\tilde{Q}), c_1 C(\tilde{C})$ and $\theta \geq c_2 C(\tilde{S})$, it follows from Proposition 3.1.2 that

$$\frac{8}{N}\int_D e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}|ML^\sharp u|^2 dxdt \geq \int_{\partial D}\Gamma(u)$$
$$+\frac{c_1}{2}\int_D e^{-\theta t}\rho^{\pm N-1}\rho_t E(u)dxdt + \frac{\theta}{2}\int_D e^{-\theta t}\rho^{\pm N}E(u)dxdt$$

where $D = \omega_j^u \cap \{t \geq 0\}$ or $D = \omega_j^d \cap \{t \geq 0\}$ or $D = \omega(T)$.

3.2 Estimates of higher order derivatives

We start with

Lemma 3.2.1 *Let $D = \omega_j \cap \{t \geq 0\}$ or $D = \omega(T)$ and $\rho = \rho_{A,D}$. Then we make $\rho\rho_t^{-1}$ as small as we please in D taking T small.*

Proof: Clear. q.e.d.

Lemma 3.2.2 *Let $D = \omega_j^u \cap \{t \geq 0\}$ or $D = \omega_j^d \cap \{t \geq 0\}$ or $D = \omega(T)$ and $\rho = \rho_{A,D}$. Then we have*

$$c_3 N^{-1}\int_D e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}|(M + nA_x^\sharp)(L^\sharp - nA_x^\sharp)u|^2 dxdt$$
$$\geq \int_{\partial D}\Gamma(u) + c_2\int_D e^{-\theta t}\rho^{\pm N-1}\rho_t E(u)dxdt + c_2\theta\int_D e^{-\theta t}\rho^{\pm N}E(u)dxdt$$

for any $N \geq N_0(\tilde{Q}, \tilde{R}) + n$, $\theta \geq \theta_0(\tilde{S}, n)$, $n \in \mathbf{N}$.

Proof: Note that

$$(M + nA_x^\sharp)(L^\sharp - nA_x^\sharp) = p - h_x\partial_x + \hat{Q}\partial_x + \tilde{R}\partial_t + \hat{S}$$

where $\hat{Q} = \tilde{Q} - nh_x I$, $\hat{S} = \tilde{S} + nA_x^\sharp B^\sharp - nM(A_x^\sharp) - n^2(A_x^\sharp)^2$ since $A^\sharp A_x^\sharp + A_x^\sharp A^\sharp = h_x$. Let $\epsilon > 0$ be given. Taking T small one may suppose that

$$\rho^2\rho_t^{-1}|nh_x|^2 \leq \epsilon n^2\rho_t b^2$$

since $h_x = 2bb_x$ and b_x is bounded by Lemma 3.2.1.

It is clear that $C(\hat{Q}) \leq 2(C(\tilde{Q}) + \epsilon n^2)$ and $C(\hat{S}) \leq 2(C(\tilde{S}) + cn^4)$ with some $c > 0$. Then taking $\epsilon > 0$, $N_0(\tilde{Q})$, $\theta_0(\tilde{S}, n)$ suitably so that

$$N \geq N_0(\tilde{Q}) + n,\ \theta \geq \theta_0(\tilde{S}, n) \Longrightarrow N^2 \geq c_1 C(\hat{Q}),\ c_1 C(\tilde{R}),\ \ \theta \geq c_2 C(\hat{S})$$

(note that $c_1 C(\hat{Q}) \leq 2c_1 C(\tilde{Q}) + 2c_1\epsilon n^2 \leq (\sqrt{2c_1 C(\tilde{Q})} + n)^2$ if $2c_1\epsilon \leq 1$). Then we get the assertion applying the previous inequality. q.e.d.

Proposition 3.2.1 *One can find $N_0 > 0$ such that for any $n \in \mathbf{N}$ there is $\theta_1(n)$ such that with $D = \omega_j^u \cap \{t \geq 0\}$, $D = \omega_j^d \cap \{t \geq 0\}$, $D = \omega(T)$ we have*

$$\sum_{k+l\leq n} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_t^k \partial_x^l u|^2 dxdt + \sum_{l\leq n} \int_{\partial D} \Gamma(\partial_x^l u)$$
$$\leq C \sum_{k+l\leq n+1} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_t^k \partial_x^l L^\sharp u|^2 dxdt$$
$$+ C \sum_{k+l\leq n} \int_D |e^{-\theta t}\rho^{\pm N-1}||\partial_t^k \partial_x^l L^\sharp u|^2 dxdt$$

for any $N \geq N_0 + n$, $\theta \geq \theta_1(n)$ where $\rho = \rho_{A,D}$.

Proof: Take $N_0 = N(\tilde{Q}, \tilde{R})$. Then from Lemma 3.2.2 it follows

$$c_2\theta \int_D |e^{-\theta t}\rho^{\pm N}|E(\partial_x^q u)dxdt + \int_{\partial D} \Gamma(\partial_x^q u)$$

is estimated by

$$c_3 N^{-1} \int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}|(M + qA_x^\sharp)(L^\sharp - qA_x^\sharp)\partial_x^q u|^2 dxdt.$$

Since $|\tilde{C}| \leq c(\rho_t/\rho)$ in D and

$$(L^\sharp - qA_x^\sharp)\partial_x^q u = \partial_x^q L^\sharp u - \sum_{j=0}^{q-1} B_j \partial_x^j u$$

this is bounded by constant (depend on q) times

$$\int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}|(|\partial_t \partial_x^q L^\sharp u|^2 + |\partial_x^{q+1} L^\sharp u|^2)dxdt$$
$$+ \int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t||\partial_x^q L^\sharp u|^2 dxdt$$
$$+ \sum_{i+j\leq q, i\leq 1} \int_D |e^{-\theta t}\rho^{\pm N+1}\rho_t^{-1}||\partial_t^i \partial_x^j u|^2 dxdt$$
$$+ \sum_{j\leq q-1} \int_D |e^{-\theta t}\rho^{\pm N-1}\rho_t||\partial_x^j u|^2 dxdt.$$

The third and fourth terms are estimated by

$$C \sum_{j=0}^{q} \int_D |e^{-\theta t}\rho^{\pm N}|E(\partial_x^j u)dxdt.$$

Hence, taking θ large and summing up over $q = 0, 1, ..., n$ we get

$$\frac{c_2}{2}\theta \sum_{j=0}^{n} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_x^j u|^2 dxdt + \sum_{j=0}^{n} \int_{\partial D} \Gamma(\partial_x^j u)$$
$$\leq C \sum_{j=0}^{n} \int_D |e^{-\theta t}\rho^{\pm N}|(|\partial_t \partial_x^j L^\sharp u|^2 + |\partial_x^{j+1} L^\sharp u|^2) dxdt$$
$$+ C \sum_{j=0}^{n} \int_D |e^{-\theta t}\rho^{\pm N-1}||\partial_x^j L^\sharp u|^2 dxdt$$

where we have used $E(u) \geq c|u|^2$ with some $c > 0$. Note that

$$\partial_t^k \partial_x^l u = \partial_t^{k-1} \partial_x^l L^\sharp u + \sum_{i \leq k-1, j \leq l+1} c_{ij} \partial_t^i \partial_x^j u.$$

We consider

$$\sum_{k+l \leq n, k \geq 1} \lambda^k \mu^l |\partial_t^k \partial_x^l u|^2$$

with $\lambda > 0$, $\mu > 0$ small and $\sum_l \mu^l < +\infty$. Since

$$\sum_{k+l \leq n, k \geq 1} \lambda^k \mu^l \sum_{i \leq k-1, j \leq l+1} |\partial_t^i \partial_x^j u|^2$$
$$\leq C \sum_{j=0}^{n} |\partial_x^j u|^2 + C\lambda\mu^{-1} \sum_{i+j \leq n, i \geq 1} \lambda^i \mu^j |\partial_t^i \partial_x^j u|^2$$

taking $\lambda\mu^{-1}$ small enough so that the second term in the right-hand side cancells against to the left-hand side we get

$$\sum_{k+l \leq n, k \geq 1} \lambda^k \mu^l |\partial_t^k \partial_x^l u|^2 \leq C \sum_{k+l \leq n, k \geq 1} \lambda^k \mu^l |\partial_t^{k-1} \partial_x^l L^\sharp u|^2$$
$$+ C \sum_{j=0}^{n} |\partial_x^j u|^2.$$

Now multiplying $|e^{-\theta t}\rho^{\pm N}|$ and integrating over D we have

$$\sum_{k+l \leq n, k \geq 1} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_t^k \partial_x^l u|^2 dxdt$$
$$\leq C \sum_{k+l \leq n, k \geq 1} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_t^{k-1} \partial_x^l L^\sharp u|^2 dxdt$$
$$+ C \sum_{j=0}^{n} \int_D |e^{-\theta t}\rho^{\pm N}||\partial_x^j u|^2 dxdt.$$

Since we have already estimated

$$\theta\sum_{j=0}^{n}\int_D |e^{-\theta t}\rho^{\pm N}||\partial_x^j u|^2 dxdt$$

plugging this estimate into above inequality, we get the desired estimate. q.e.d.

3.3 A priori estimate

Recall that $A^\sharp(0,0)=0$ because $a^\sharp_{12}(0,0)=0$.

Proposition 3.3.1 *Let $r(t,x)=t-\theta(x)$, $L^\sharp u=f$. Assume that*

(1) $|A^\sharp(t,x)|\le C|x|$ *in* $0\le t\le t^*(x)$,

(2) $|\theta^{(\alpha)}(x)|\le C_\alpha |x|^{\delta-\alpha}$ *with some* $\delta>0$ *for* $\alpha=0,1,...,Q$,

(3) $\partial_t^\alpha u(0,x)=0$, $\partial_t^\alpha f(0,x)=0$ *for* $\alpha=0,1,...,Q$.

Then for any $q\in\mathbf{N}$ with $2q+1\le Q$ there is a $w_q(t,x)$ verifying the followings:

$$L^\sharp(u-w_q)=r^qF,\quad u-w_q=r^{q+1}V$$

where

$$|\partial_t^k\partial_x^l(u-w_q)|^2\le C|x|^{-2l}|r|^{2(q+1-k-l)}t^*(x)^{2(Q-q-k-l-1)+1}\times\sum_{\beta=0}^{l}\int_0^{t^*(x)}|\partial_t^{Q+1}\partial_x^\beta u|^2dt$$

for $0\le t\le t^*(x)$, $k+l+q+1\le Q$, $k+l\le q+1$,

$$\begin{aligned}|\partial_t^k\partial_x^l(L^\sharp u-L^\sharp w_q)|^2 &\le C|x|^{-2(l+1)}|r|^{2(q-k-l)}\\ &\times t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l+1}\int_0^{|\theta(x)|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt\\ &+C|x|^{-2l}t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l}\int_0^{|\theta(x)|}|\partial_t^{Q+1}\partial_x^\beta f|^2dxdt\\ &+C|t|^{2(Q-k)}\int_0^t|\partial_t^{Q+1}\partial_x^l f|^2dxdt\end{aligned}$$

for $q+l+1\le Q$, $k+l\le q$ *and*

$$|\partial_t^k\partial_x^l w_q|^2\le C|x|^{-2l}t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l}\int_0^{|\theta(x)|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt$$

for $q+l\le Q$.

We first show the following lemma.

Lemma 3.3.1 *Let $\psi(t,x) \in C^\infty$. Then one can write*

$$\psi(t,x) = \sum_{j=0}^{q} \psi_j(x) r^j + r^{q+1}\psi_q(t,x)$$

where $\psi_j(x)$, $\psi_q(t,x)$ verifies

$$|\partial_x^l \psi_j(x)| \le C_{jl}|x|^{-l}, \quad l = 0,1,....,$$
$$|\partial_t^k \partial_x^l \psi_q(t,x)| \le C_{qkl}|x|^{-l}, \quad l = 0,1,....$$

Moreover if $\partial_t^\alpha \psi(0,x) = 0$, $\alpha = 0,1,...,Q$ then we have

$$|\partial_x^l \psi_j|^2 \le C \sum_{\beta=0}^{l} |x|^{-2l}|\theta(x)|^{2(Q-j-l)+1} \int_0^{|\theta(x)|} |\partial_t^{Q+1}\partial_x^\beta \psi(\tau,x)|^2 d\tau$$

for $j + l \le Q$ and

$$|\partial_t^k \partial_x^l \psi_q|^2 \le C \sum_{\beta=0}^{l} |x|^{-2l} t^*(x)^{2(Q-l-q-k-1)+1} \int_0^{t^*(x)} |\partial_t^{Q+1}\partial_x^\beta \psi|^2 d\tau$$

for $0 \le t \le t^*(x)$, $k + l + q + 1 \le Q$.

Proof: Since

$$\psi_j(x) = \frac{1}{j!}\partial_t^j \psi(\theta(x),x), \quad \psi_q(t,x) = \frac{1}{q!}\int_0^1 (\partial_t^{q+1}\psi)(\theta(x) + \tau(t-\theta(x)),x)d\tau$$

the first two inequalities are clear. Assume that $\partial_t^\alpha \psi(0,x) = 0$, $\alpha = 0,1,...,Q$, then

$$\partial_t^\alpha(\partial_t^j \psi)(0,x) = 0, \quad \alpha = 0,1,...,Q-j$$

and hence we see

$$\partial_t^j \psi(t,x) = \frac{t^{Q-j+1}}{(Q-j)!}\int_0^1 (1-s)^{Q-j}\partial_t^{Q+1}\psi(st,x)ds.$$

This shows that

$$\psi_j(x) = \frac{\theta(x)^{Q-j+1}}{(Q-j)!j!}\int_0^1 (1-s)^{Q-j}\partial_t^{Q+1}\psi(s\theta(x),x)ds.$$

Noting that

$$\left|\int_0^1 \partial_t^{Q+1}\partial_x^\beta \psi(s\theta(x),x)ds\right|^2 \le C\int_0^1 |\partial_t^{Q+1}\partial_x^\beta \psi(s\theta(x),x)|^2 ds$$
$$= C|\theta(x)|^{-1}\int_0^{|\theta(x)|} |\partial_t^{Q+1}\partial_x^\beta \psi(\tau,x)|^2 d\tau$$

we get the third inequality. Remarking that

$$\partial_t^k \psi_q(t,x) = \frac{1}{q!}\int_0^1 \tau^k (\partial_t^{q+k+1}\psi)(\theta(x)+\tau(t-\theta(x)),x)d\tau,$$

$$\partial_t^{q+k+1}\psi(t,x) = c\int_0^t (t-u)^{Q-k-q-1}\partial_t^{Q+1}\psi(u,x)du$$

we see

$$\partial_t^k \psi_q(t,x) = c'\int_0^1 \tau^k[\tau t+(1-\tau)\theta(x)-u]^{Q-q-k-1}d\tau \times \int_0^{\theta(x)+\tau(t-\theta(x))} \partial_t^{Q+1}\psi(u,x)du.$$

By the same arguments we get

$$\int_0^1 (\partial_t^{q+k+1}\partial_x^l\psi)(\theta(x)+\tau(t-\theta(x)),x)$$

$$= c'\int_0^1 \tau^k[\tau t+(1-\tau)\theta(x)-u]^{Q-q-k-1}d\tau \int_0^{\theta(x)+\tau(t-\theta(x))} \partial_t^{Q+1}\partial_x^l\psi(u,x)du.$$

Since $|\tau t+(1-\tau)\theta(x)| \le \tau t^*(x)+(1-\tau)t^*(x) = t^*(x)$ for $0\le t\le t^*(x)$ then

$$|\partial_t^k\partial_x^l\psi_q| \le \sum_{l_1+l_2=l}\int_0^1 d\tau \int_0^{t^*(x)} t^*(x)^{Q-q-k-l_1-1}|x|^{-2l_1}|\partial_t^{Q+1}\partial_x^{l_2}\psi(u,x)|du$$

and hence

$$|\partial_t^k\partial_x^l\psi_q|^2 \le t^*(x)[t^*(x)^{Q-q-k-l-1}]^2\int_0^{t^*(x)} |\partial_t^{Q+1}\partial_x^l\psi(u,x)|^2du$$

$$= t^*(x)^{2(Q-q-k-l-1)+1}\int_0^{t^*(x)} |\partial_t^{Q+1}\partial_x^l\psi(u,x)|^2du$$

which is the desired inequality. q.e.d.

Proof of Proposition 3.3.1 From Lemma 3.3.1 one can write

$$u(t,x) = \sum_{j=0}^q u_j(x)r^j + r^{q+1}V(t,x),\ f(t,x) = \sum_{j=0}^{q-1} f_j(x)r^j + r^qF_{q-1}(t,x).$$

Let us put

$$w_q(t,x) = \sum_{j=0}^q u_j(x)r(t,x)^j.$$

From Lemma 3.3.1 it follows that

$$|\partial_t^k\partial_x^lV|^2 \le C\sum_{\beta=0}^l |x|^{-2l}t^*(x)^{2(Q-l-q-k-1)+1}\int_0^{t^*(x)} |\partial_t^{Q+1}\partial_x^\beta u|^2d\tau$$

for $0 \le t \le t^*(x)$, $k+l+q+1 \le Q$. Hence we get

$$|\partial_t^k \partial_x^l (r^{q+1}V)|^2 \le C|x|^{-2l}|r|^{2(q+1-k-l)} t^*(x)^{2(Q-q-k-l-1)+1} \times \sum_{\beta=0}^{l} \int_0^{t^*(x)} |\partial_t^{Q+1}\partial_x^\beta u|^2 dt$$

for $0 \le t \le t^*(x)$, $k+l+q+1 \le Q$, $k+l \le q+1$. It is clear that one can write $L^\sharp(u - w_q) = r^q F$. We show the third estimate. From Lemma 3.3.1 we see

$$|u_j^{(l)}|^2 \le C|x|^{-2l}|\theta(x)|^{2(Q-j-l)+1} \sum_{\beta=0}^{l} \int_0^{|\theta|} |\partial_t^{Q+1}\partial_x^\beta u|^2 dt$$

for $j+l \le Q$ where $u_j^{(l)} = \partial_x^l u_j$. Since

$$|\partial_t^k \partial_x^l w_q| \le C \sum_{0 \le j \le q, l_1+l_2=l} |u_j^{(l_1)}||x|^{-l_2}$$

then noting $|\theta(x)| \le t^*(x)$ we have

$$|\partial_t^k \partial_x^l w_q|^2 \le C|x|^{-2l}|\theta(x)|^{2(Q-q-l)+1} \sum_{\beta=0}^{l} \int_0^{|\theta|} |\partial_t^{Q+1}\partial_x^\beta u|^2 dt$$
$$\le C|x|^{-2l} t^*(x)^{2(Q-l-q)+1} \sum_{\beta=0}^{l} \int_0^{|\theta|} |\partial_t^{Q+1}\partial_x^\beta u|^2 d\tau.$$

This is the third assertion. Finally we prove the second estimate. From $L^\sharp u = f$ and $L^\sharp(u - w_q) = r^q F$ we see $L^\sharp w_q = f - r^q F$. Hence we have

$$L^\sharp w_q = \sum_{j=0}^{q-1} f_j(x) r^j \mod O(r^q).$$

We now study $L^\sharp w_q$.

$$L^\sharp w_q = \sum_{j=0}^{q-1} f_j r^j + \sum_{\mu \ge q} \left(\sum_{i+j=\mu, i,j \le q} -A_i^\sharp u_j' + B_i^\sharp u_j + \sum_{i+j=\mu+1, i,j \le q} j A_i^\sharp u_j \theta' \right) r^\mu$$
$$+ r^{q+1} A_q \left(\sum_{j=0}^{q} -u_j' r^j + j u_j r^{j-1} \theta' \right) + r^{q+1} B_q (\sum_{j=0}^{q} u_j r^j).$$

Note that

$$\left| \partial_t^k \partial_x^l \sum_{i+j=\mu, i,j \le q} -A_i^\sharp u_j' r^\mu + B_i^\sharp u_j r^\mu \right|^2$$

is bounded by

$$\sum_{l_1+l_2=l,k_1\le k} r^{2(\mu-k_1-l_1)}|x|^{-2(l+1)}|\theta(x)|^{2(Q-j-l_2-1)+1}\sum_{\beta=0}^{l_2+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt$$

and hence by

$$r^{2(\mu-k-l)}|x|^{-2(l+1)}t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt. \tag{3.3.1}$$

Similary the term

$$\left|\partial_t^k\partial_x^l\sum_{i+j=\mu+1,i,j\le q}A_i^\sharp u_j\theta' r^\mu\right|^2$$

is estimated by

$$r^{2(\mu-k-l)}|x|^{-2(l+1)}t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt$$

and

$$|\partial_t^k\partial_x^{\sum_{j=0}^q} r^{q+1}A_q(u_j' r^j - ju_j\theta' r^{j-1})|^2$$

is bounded by

$$\sum_{l_1+l_2=l,k_1\le k} r^{2(q+1-k_1-l_1)}|x|^{-2(l+1)}|\theta|^{2(Q-q-l_2-1)+1}\sum_{\beta=0}^{l_2+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt$$

and again by

$$r^{2(q+1-k-l)}|x|^{-2(l+1)}t^*(x)^{2(Q-q-l-1)+1}\sum_{\beta=0}^{l+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt. \tag{3.3.2}$$

One can estimate the term

$$|\partial_t^k\partial_x^l\sum_{j=0}^q r^{q+1}B_q u_j r^j|^2$$

by the same argument. We summarize:

$$|\partial_t^k\partial_x^l(L^\sharp w_q - \sum_{j=0}^{q-1} f_j r^j)|^2 \le Cr^{2(q-k-l)}|x|^{-2(l+1)}t^*(x)^{2(Q-q-l-1)+1} \times\sum_{\beta=0}^{l+1}\int_0^{|\theta|}|\partial_t^{Q+1}\partial_x^\beta u|^2dt.$$

Since

$$L^\sharp u - L^\sharp w_q = -(L^\sharp w_q - \sum_{j=0}^{q-1} f_j r^j) + r^q F_{q-1}$$

it remains to estimate $|\partial_t^k \partial_x^l r^q F_{q-1}|$. From Lemma 3.3.1 it follows that

$$|\partial_t^k \partial_x^l r^q F_{q-1}|^2 \le C|\partial_t^k \partial_x^l f|^2 + C|\partial_t^k \partial_x^l \sum_{j=0}^{q-1} f_j r^j|^2$$

$$\le C|\partial_t^k \partial_x^l f|^2 + C|x|^{-2l} t^*(x)^{2(Q-q-l)+1} \sum_{\beta=0}^{l} \int_0^{|\theta|} |\partial_t^{Q+1} \partial_x^\beta f|^2 dt.$$

Noting

$$|\partial_t^k \partial_x^l f|^2 \le C|t|^{2(Q-k)} \int_0^t |\partial_t^{Q+1} \partial_x^l f|^2 dt$$

we conclude the proof. q.e.d.

We prepare some notations:

$$\Omega_\nu = \{(t,x) \mid |x| \le \bar\delta(T-t), 0 \le t \le s_\nu(x)\},\ \nu = 0, ..., m,$$
$$\tilde\Omega_{\nu+1} = \{(t,x) \mid |x| \le \bar\delta(T-t), 0 \le t \le \sigma_{\nu+1}(x)\}, \nu = 0, ..., m-1.$$

Similarly

$$\omega_\nu^- = \{(t,x) \mid |x| \le \bar\delta(T-t), s_{\nu-1}(x) \le t \le \sigma_\nu(x)\},\ \nu = 1, ..., m,$$
$$\omega_\nu^+ = \{(t,x) \mid |x| \le \bar\delta(T-t), \sigma_\nu(x) \le t \le s_\nu(x)\},\ \nu = 1, ..., m.$$

Now we introduce the inductive hypothesis: Inductive hypothesis: For any $n \in \mathbf{N}$ there are $Q_\nu = Q_\nu(n) \ge n$ and $q_\nu = q_\nu(n) \ge n$ such that

$$L^\sharp u = f,\ \ \partial_t^\alpha u(0,x) = 0,\ \ \partial_t^\alpha f(0,x) = 0,\ \ \alpha = 0, 1, ..., Q_\nu$$
$$\Longrightarrow \sum_{k+l\le n} \int_{\Omega_\nu} |\partial_t^k \partial_x^l u|^2 dxdt \le C \sum_{k+l\le q_\nu(n)} \int_{\Omega_m} |\partial_t^k \partial_x^l f|^2 dxdt. (H_\nu)$$

Let $\kappa > 0$ be so that $t^*(x) = O(|x|^\kappa)$. In Proposition 3.3.1 we take $\theta = s_\nu$ and construct w_q and study the eqaution

$$L^\sharp(u - w_q) = f - L^\sharp w_q$$

in $\Omega = \omega_{\nu+1}^- \cap \{t \ge 0\} = \tilde\omega_{\nu+1}^-$.In Proposition 3.2.3, taking $N = 2(N_0 + n)$, $\theta = \theta_1(n + N/2)$ we get

$$\sum_{l\le n+N/2} \int_{\partial\tilde\omega_{\nu+1}^-} \Gamma(\partial_x^l(u - w_q)) + \sum_{k+l\le n+N/2} \int_{\tilde\omega_{\nu+1}^-} e^{-\theta t} |\rho_{\nu+1}^N| |\partial_t^k \partial_x^l (u - w_q)|^2 dxdt$$
$$\le C \sum_{k+l\le n+1+N/2} \int_{\tilde\omega_{\nu+1}^-} e^{-\theta t} |\rho_{\nu+1}^N| |\partial_t^k \partial_x^l L^\sharp (u - w_q)|^2 dxdt$$
$$+ C \sum_{k+l\le n+N/2} \int_{\tilde\omega_{\nu+1}^-} e^{-\theta t} |\rho_{\nu+1}^{N-1}| |\partial_t^k \partial_x^l L^\sharp (u - w_q)|^2 dxdt.$$

Taking q, Q so that

$$(3.3.3) \qquad 2(q-n) \geq N+1, \quad 2\kappa(Q-q-n-1-N/2) \geq 2n$$

Proposition 3.3.1 shows

$$\partial_t^{\alpha}\partial_t^k\partial_x^l(u-w_q)(s_\nu(x),x)=0, \;\; k+l\leq n, \;\alpha\leq N/2+1$$

because we have $q+1-(k+\alpha)-l \geq q+1-(N/2+n+1) = q-(N/2+n) > 0$ and $2\kappa(Q-q-k-\alpha-l-1)-2l \geq 2\kappa(Q-q-n-N/2-2)-2l \geq 2n-2l \geq 0$.

Lemma 3.3.2 *Assume that* $(\partial_t^{\alpha}u)(s_\nu(x),x)=0$*,* $\alpha=0,1,...,p+N/2+1$*. Then there is* $C(N)>0$ *such that*

$$\int_{\omega_{\nu+1}^-} e^{-\theta t}|\rho_{\nu+1}|^N|\partial_t^{p+N/2}u|^2 dxdt \geq C(N)\int_{\omega_{\nu+1}^-} e^{-\theta t}|\partial_t^p u|^2 dxdt.$$

Proof: Note that

$$\partial_t|\partial_t^p u|^2 = \partial_t^{p+1}u\cdot\partial_t^{\bar{p}}u + \partial_t^{\overline{p+1}}u\cdot\partial_t^p u.$$

Multiply $-\rho^{2p+1}$ to the equation get

$$\begin{aligned}&-\partial_t(\rho^{2p+1}|\partial_t^p u|^2)+(2p+1)\rho^{2p}|\partial_t^p u|^2\\ &= -\rho^{2p+1}(\partial_t^{p+1}u\cdot\partial_t^{\bar{p}}u+\partial_t^{\overline{p+1}}u\cdot\partial_t^p u).\end{aligned}$$

Integrating over $\omega_{\nu+1}^-$ we get

$$\begin{aligned}-\int_{\omega_{\nu+1}^-}\partial_t(\rho^{2p+1}|\partial_t^p u|^2)dxdt + (2p+1)\int_{\omega_{\nu+1}^-}\rho^{2p}|\partial_t^p u|^2 dxdt\\ \leq 2\int_{\omega_{\nu+1}^-}\rho^{2p+2}|\partial_t^{p+1}u|^2 dxdt + \frac{1}{2}\int_{\omega_{\nu+1}^-}\rho^{2p}|\partial_t^p u|^2 dxdt\end{aligned}$$

so that

$$\begin{aligned}(2p+\frac{1}{2})\int_{\omega_{\nu+1}^-}\rho^{2p}|\partial_t^p u|^2 dxdt + \int_{\partial\omega_{\nu+1}^-}(\rho^{2p+1}|\partial_t^p u|^2)dx\\ \leq 2\int_{\omega_{\nu+1}^-}\rho^{2p+2}|\partial_t^{p+1}u|^2 dxdt.\end{aligned}$$

Since $\partial_t^p u=0$ on $t=s_\nu(x)$ we get

$$\int_{\partial\omega_{\nu+1}^-}(\rho^{2p+1}|\partial_t^p u|^2)dx \geq 0.$$

Hence we have

$$(2p+\frac{1}{2})\int_{\omega_{\nu+1}^-}\rho^{2p}|\partial_t^p u|^2 dxdt \leq 2\int_{\omega_{\nu+1}^-}\rho^{2p+2}|\partial_t^{p+1}u|^2 dxdt.$$

Inductively we get the assertion. q.e.d.

Since $|x| = \bar{\delta}(T-t)$ is space-like, Lemma 3.3.2 gives

$$\sum_{k+l\leq n}\int_{\tilde{\omega}^-_{\nu+1}} e^{-\theta t}|\partial_t^k\partial_x^l(u-w_q)|^2dxdt$$

$$\leq C\sum_{k+l\leq n+1+N/2}\int_{\tilde{\omega}^-_{\nu+1}} e^{-\theta t}|\rho^N_{\nu+1}||\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt$$

$$+C\sum_{k+l\leq n+N/2}\int_{\tilde{\omega}^-_{\nu+1}} e^{-\theta t}|\rho^{N-1}_{\nu+1}||\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt$$

and hence assuming that q, Q verify

$$2\kappa(Q-q-l-1)\geq 2(n+2+\frac{N}{2}) \tag{3.3.4}$$

we have from Prposition 3.3.1 that

$$\sum_{k+l\leq n+1+N/2}\int_{\tilde{\omega}^-_{\nu+1}} e^{-\theta t}|\rho^N_{\nu+1}||\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt$$

$$\leq C\sum_{k+l\leq Q+n+N/2+3}\int_{\Omega_\nu} e^{-\theta t}|\partial_t^k\partial_x^l u|^2dxdt$$

$$+C\sum_{k+l\leq Q+3+n+N/2}\int_{\Omega_m} e^{-\theta t}|\partial_t^k\partial_x^l f|^2dxdt.$$

We choose q, Q so that (recall $N = 2(N_0+n)$)

$$q\geq N_0+2n+1,\quad \kappa Q\geq(\kappa+1)(N_0+2n+q+2) \tag{3.3.5}$$

then it is easy to check that these q, Q verify (3.3.3) and (3.3.4). We summarize: if $\partial_t^\alpha u(0,x) = 0$, $\partial_t^\alpha f(0,x) = 0$ for $\alpha = 0, 1, ..., \tilde{Q}_\nu(n)$ then we have

$$\sum_{k+l\leq n}\int_{\tilde{\Omega}_{\nu+1}}|\partial_t^k\partial_x^l u|^2dxdt\leq C\sum_{k+l\leq\tilde{q}_\nu(n)}\int_{\Omega_m}|\partial_t^k\partial_x^l f|^2dxdt$$

where $\tilde{q}_\nu(n) = q_\nu(Q+2n+N_0+3)$, $\tilde{Q}_\nu(n) = Q_\nu(Q+2n+N_0+3)$. We go to the next step. Let $\theta = \sigma_{\nu+1}$ we consider $L^\sharp(u-w_q) = f - L^\sharp w_q$ in the region $\tilde{\omega}^+_{\nu+1} = \omega^+_{\nu+1}\cap\{t\geq 0\}$. From Proposition 3.3.1 it follows

$$\sum_{l\leq n}\int_{\partial\tilde{\omega}^+_{\nu+1}}\Gamma(\partial_x^l(u-w_q))+\sum_{k+l\leq n}\int_{\tilde{\omega}^+_{\nu+1}} e^{-\theta t}|\rho^{-N}_{\nu+1}||\partial_t^k\partial_x^l(u-w_q)|^2dxdt$$

$$\leq C\sum_{k+l\leq n+1}\int_{\tilde{\omega}^+_{\nu+1}} e^{-\theta t}|\rho^{-N}_{\nu+1}||\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt$$

$$+C\sum_{k+l\leq n}\int_{\tilde{\omega}^+_{\nu+1}} e^{-\theta t}|\rho^{-N-1}_{\nu+1}||\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt.$$

From Proposition 3.3.1 we have

$$(\rho_{\nu+1}^{-N}\partial_x^l(u-w_q))(\sigma_{\nu+1}(x),x)=0,\quad l\le n$$

if

$$(q-n)\ge N+1,\quad 2\kappa(Q-q-n-1)\ge 2(n+1).$$

Since $\partial\tilde{\omega}_{\nu+1}^+$ is space-like, thanks to Proposition 3.3.1, the above inequality yields

$$\sum_{k+l\le n}\int_{\tilde{\omega}_{\nu+1}^+}|\partial_t^k\partial_x^l(u-w_q)|^2dxdt$$
$$\le C\sum_{k+l\le Q+n+2}\left(\int_{\tilde{\Omega}_{\nu+1}}|\partial_t^k\partial_x^l u|^2dxdt+\int_{\Omega_m}|\partial_t^k\partial_x^l f|^2dxdt\right).$$

Then by induction hypothesis one has:

$$\sum_{k+l\le n}\int_{\Omega_{\nu+1}}|\partial_t^k\partial_x^l u|^2dxdt\le C\sum_{k+l\le q_{\nu+1}(n)}\int_{\Omega_m}|\partial_t^k\partial_x^l f|^2dxdt$$

for any u, f with

$$\partial_t^\alpha u(0,x)=0,\ \ \partial_t^\alpha f(0,x)=0,\quad \alpha=0,1,...,Q_{\nu+1}(n)$$

where $q_{\nu+1}(n)=\tilde{q}_\nu(Q+n+2)$, $Q_{\nu+1}(n)=\tilde{Q}_\nu(Q+n+2)$. This proves $(H_{\nu+1})$. Finally we derive an energy inequality in $\omega(T)$. We remark that

$$C\rho\ge(|x|^{c_1}+|t|^{c_2})\quad\text{in}\quad\omega(T)$$

with some $c_i>0$ when $n_1=0$ because we have

$$C\rho\ge\prod|t-t_j(x)|\ge\prod(t-|t_j(x)|)\ge\prod\frac{2}{3}t$$
$$\ge\prod\frac{1}{3}(t+t^*(x))\ge\prod\frac{1}{3}(|t|+|x|^\kappa).$$

When $n_1\ge 1$ we see

$$\rho=\rho_{m+1}=t-s_m(x)\ge\frac{2}{3}t+\frac{t}{3}-s_m\ge\frac{1}{2}(t+t^*(x))\ge\frac{1}{2}(|t|+|x|^\kappa).$$

Take $\theta=s_m(x)$ and q, Q are large in Proposition 3.3.1, then one gets

$$\sum_{k+l\le n+1}\int_{\omega(T)}e^{-\theta t}\rho^{-N}|\partial_t^k\partial_x^l L^\sharp(u-w_q)|^2dxdt$$
$$\le C\sum_{k+l\le Q+n+3}\left(\int_{\Omega_m}|\partial_t^k\partial_x^l u|^2dxdt+\int_S|\partial_t^k\partial_x^l f|^2dxdt\right)$$

where $S = \{(t,x) \mid |x| \le \bar{\delta}(T-t), 0 \le t \le T\}$. Hence we have

$$\sum_{k+l\le n} \int_{\omega(T)} |\partial_t^k \partial_x^l u|^2 dxdt$$
$$\le C \sum_{k+l\le Q+n+3} \left(\int_{\Omega_m} |\partial_t^k \partial_x^l u|^2 dxdt + \int_S |\partial_t^k \partial_x^l f|^2 dxdt \right).$$

Thus we have proved

Proposition 3.3.2 *Let W be an open neighborhood of the origin and assume that $(C^{\pm})$ are verified. Then there are $\bar{\delta}$, T such that for any $n \in \mathbf{N}$ one can find $q(n)$, $Q(n)$ so that we have*

$$\sum_{k+l\le n} \int_S |\partial_t^k \partial_x^l u|^2 dxdt \le C_n \sum_{k+l\le q(n)} \int_S |\partial_t^k \partial_x^l L^\sharp u|^2 dxdt$$

for any $u \in C^\infty(W)$ with $\partial_t^\alpha u(0,x) = 0$, $\alpha = 0, 1, ..., Q(n)$.

Now we prove

Theorem 3.3.1 *Assume that $(C^{\pm})$ are verified. Then the Cauchy problem (C.P.) is C^∞ well posed.*

Proof: We first rewrite the energy inequality in Proppposition 3.3.2. Let $u_0(x)$ be given and assume that $L^\sharp u = f$. We define $u_j(x)$, $j \ge 1$ so that

$$U_N(t,x) = \sum_{j=0}^{N} \frac{1}{j!} u_j(x) t^j$$

verifies with $L^\sharp(u - U_N) = F_N$ that

$$u - U_N = O(t^N), \quad F_N = O(t^{N-1}).$$

From Proposition 3.3.2 we get

$$\sum_{k+l\le n} \int_S |\partial_t^k \partial_x^l (u - U_N)|^2 dxdt \le C \sum_{k+l\le q(n)} \int_S |\partial_t^k \partial_x^l F_N|^2 dxdt$$

if $N \ge Q(n) + 1$. This shows that

$$\sum_{k+l\le n} |\partial_t^k \partial_x^l u|^2 dxdt \le C \sum_{k+l\le n} \int_S |\partial_t^k \partial_x^l U_N|^2 dxdt$$
$$+ C \sum_{k+l\le q(n)} \int_S |\partial_t^k \partial_x^l F_N|^2 dxdt.$$

Note that

$$\sum_{k+l\leq n}\int_S |\partial_t^k\partial_x^l U_N|^2 dxdt \leq C\sum_{l\leq n+N}\int_{S\cap\{t=0\}} |\partial_x^l u_0|^2 dx$$
$$+C\sum_{k+l\leq n+N}\int_S |\partial_t^k\partial_x^l f|^2 dxdt.$$

Thus we get

$$\sum_{k+l\leq n}\int_S |\partial_t^k\partial_x^l u|^2 dxdt \leq C\sum_{l\leq \tilde{q}(n)}\int_{S\cap\{t=0\}} |\partial_x^l u_0|^2 dx$$
$$+C\sum_{k+l\leq \tilde{q}(n)}\int_S |\partial_t^k\partial_x^l f|^2 dxdt. \tag{3.3.6}$$

Let us choose polynomials $\{p_k(x)\}$ and $\{q_k(t,x)\}$ so that

$$\sup_{l\leq\tilde{q}(n),x\in S\cap\{t=0\}} |\partial_x^l(u_0(x)-p_k(x))| \to 0 \quad k\to\infty,$$
$$\sup_{l\leq\tilde{q}(n),(t,x)\in S} |\partial_x^l(f(t,x)-q_k(t,x))| \to 0 \quad k\to\infty.$$

By the Cauchy-Kowalevsky theorem the Cauchy problem

$$\begin{cases} L^\sharp u_k = f \\ u_k(0,x) = p_k \end{cases}$$

has a solution u_k in a fixed domain W (independent of k). Let us take S so that $S\subset W$. Then

$$\sum_{k+l\leq n}\int_S |\partial_t^k\partial_x^l u_i|^2 dxdt \leq C\sum_{l\leq \tilde{q}(n)}\int_{S\cap\{t=0\}} |\partial_x^l p_i|^2 dx$$
$$+C\sum_{k+l\leq \tilde{q}(n)}\int_S |\partial_t^k\partial_x^l q_i|^2 dxdt.$$

Thus $\{u_i\}$ is a Cauchy sequence and hence there exists $u\in C^n(S)$ such that

$$\sum_{k+l\leq n}\int_S |\partial_t^k\partial_x^l (u_i-u)|^2 dxdt \to 0, \quad i\to\infty.$$

This is a desired solution to our Cauchy problem. q.e.d.

4 Necessary condition

4.1 Dilation

Definition 4.1.1 *Let $\gamma \in_{+}$. We say $\phi(x) \in \mathcal{G}^{\pm}(\gamma)$ if $\phi(x)$ is defined in $0 < \pm x < \gamma(\phi)$ with some $\gamma(\phi) > 0$ and expressed by convergent Puiseux series*

$$\phi(x) = \sum_{j=0} C_j (\pm x)^{j/p}, \quad C_j \in \mathbf{R}, \quad 0 < \pm x < \gamma(\phi)$$

with some $p \in \mathbf{N}$. We also define $\sigma(\phi)$ for $\phi \in \mathcal{G}^{\pm}(\gamma)$ by

$$C^{-1}(\pm x)^{\sigma(\phi)} \le |\phi(x)| \le C(\pm x)^{\sigma(\phi)}$$

with a $C > 0$.

Definition 4.1.2 *Let $f(t, x)$ be real analytic near the origin and $f(0, 0) = 0$. Let $p, q \in_{+}$ and $\phi \in \mathcal{G}^{\pm}(\gamma)$. We define $\mu(f_\phi; p, q)$ by*

$$f_\phi(s^p t, s^q x) = s^\mu (f^0(t, x) + o(s)), \quad s \to 0$$

where $f^0(t, x)$ does not vanish identically. Let $f(t, x)$, $g(t, x)$ be real analytic near the origin. Then we define

$$\mu([\frac{f}{g}]_\phi; p, q) = \mu(f_\phi; p, q) - \mu(g_\phi; p, q).$$

Remark: $\mu(f_\phi; p, q)$ is uniquely determined by $\Gamma(f_\phi)$: Write

$$f_\phi(s^p t, s^q x) = \sum C_{ij} (s^p t)^i (s^q x)^{j/\alpha} = \sum C_{ij} s^{pi + qj/\alpha} t^i x^{j/\alpha}$$

then we see

$$\mu = \min_{C_{ij} \ne 0} \{pi + \frac{qj}{\alpha}\}.$$

This means that the line $pt + qx/\alpha = \mu$ is tangent to $\Gamma(f_\phi)$. It is obvious that $\mu([fg]_\phi; p, q) = \mu(f_\phi; p, q) + \mu(g_\phi; p, q)$. We introduce the following condition. Let γ be so that $t^*(x) \sim |x|^\gamma$. For any p, $q \in_{+}$ and $\phi \in \mathcal{G}^{\pm}(\gamma)$ with

$$p \ge \sigma(\phi) q, \quad \mu(h_\phi; p, q) > 2q(1 - \sigma(\phi))$$

$(C^{\pm})$ we have

$$2p + 2\mu([\frac{Y}{a_{21}^\sharp}]_\phi; p, q) \ge \mu(h_\phi; p, q), \quad 2p + 2\mu([\frac{Z}{a_{12}^\sharp}]_\phi; p, q) \ge \mu(h_\phi; p, q).$$

Here $\sigma(\phi)q$ should be read as q if $\phi \equiv 0$.

Lemma 4.1.1 *Let $f(t,x)$ be real analytic near the origin and $f(0,0)=0$. Then*

$$\mu([\frac{\partial_t f}{f}]_\phi;p,q)\geq -p,\quad \mu([\frac{\partial_t^2 f}{f}]_\phi;p,q)\geq -2p.$$

Moreover

$$\mu([\partial_t(\frac{\partial_t f}{f})]_\phi;p,q)\geq -2p.$$

Proof: Let $f_\phi(s^p t,s^q x)=s^{\mu(f_\phi;p,q)}(f^0(t,x)+o(1))$. On the other hand, writing

$$f_\phi(t,x)=x^{\tilde n}(t^{\tilde m}+f_1(x)t^{\tilde m-1}+\cdots+f_{\tilde m}(x))$$

we have

$$\begin{aligned}t\partial_t f_\phi(t,x)=x^{\tilde n}(\tilde m t^{\tilde m}+(\tilde m-1)f_1(x)t^{\tilde m-1}+\cdots+f_{\tilde m-1}(x)t)\Phi_\phi(t,x)\\ +x^{\tilde n}(t^{\tilde m}+\cdots+f_{\tilde m}(x))t(\partial_t\Phi)_\phi(t,x).\end{aligned}$$

It is clear that $\Gamma(t\partial_t f_\phi)\subset\Gamma(f_\phi)$ by definition. This gives

$$(t\partial_t f_\phi)(s^p t,s^q x)=s^{\mu^*}(c^0(t,x)+o(1)),\ \ \mu^*\geq\mu(f_\phi;p,q).$$

Since $\partial_t f_\phi=(\partial_t f)_\phi$ we see

$$s^p t(\partial_t f)_\phi(s^p t,s^q x)=s^{\mu^*}(c^0(t,x)+o(1))$$

and hence $(\partial_t f)_\phi(s^p t,s^q x)=s^{\mu^*-p}(c^0(t,x)/t+o(1))$. This proves that

$$\mu([\partial_t f]_\phi;p,q)=\mu^*-p$$

and hence $\mu([\partial_t f]_\phi;p,q)-\mu(f_\phi;p,q)=\mu^*-\mu-p\geq -p$.

The second inequality is proved similary because $\Gamma(t^2\partial_t^2 f_\phi)\subset\Gamma(f_\phi)$. We turn to the final inequality. Note that

$$\partial_t(\frac{\partial_t f}{f})=\frac{\partial_t^2 f}{f}-(\frac{\partial_t f}{f})^2,\quad \mu([(\frac{\partial_t f}{f})^2]_\phi;p,q)=2\mu([\frac{\partial_t f}{f}]_\phi;p,q).$$

Then we conclude that

$$\mu([\partial_t(\frac{\partial_t f}{f})]_\phi;p,q)\geq\min\{\mu([\frac{\partial_t^2 f}{f}]_\phi;p,q),\mu([(\frac{\partial_t f}{f})^2]_\phi;p,q)\}\geq -2p$$

which is the desired asserion. q.e.d.Recall that $L^\sharp M=p+Q\partial_x+R\partial_t+S$ and

$$Q=\mathrm{diag}(\frac{Y}{a_{21}^\sharp},\frac{Z}{a_{21}^\sharp}),\ \ R=C-A_x^\sharp+B^\sharp+{}^{co}B^\sharp,$$

$$S=L^\sharp(C)+L^\sharp({}^{co}B^\sharp-A_x^\sharp),\ \ C=\mathrm{diag}(\frac{\partial_t a_{21}^\sharp}{a_{21}^\sharp},\frac{\partial_t a_{12}^\sharp}{a_{12}^\sharp}).$$

Lemma 4.1.2 *Let $S = (s_{ij})$, $R = (r_{ij})$. Then we have*

$$\mu([s_{ij}]_\phi; p, q) \geq -2p, \quad \mu([r_{ij}]_\phi; p, q) \geq -p.$$

Proof: It suffies to study $L^\sharp(C) = \partial_t C - A^\sharp \partial_x C$. Since

$$\partial_t C = \mathrm{diag}(\partial_t(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}), \partial_t(\frac{\partial_t a^\sharp_{12}}{a^\sharp_{12}}))$$

the assertion $\mu([\partial_t C]_\phi; p, q) \geq -2p$ follows from Lemma 4.1.1. Note

$$A^\sharp \partial_x C = \begin{pmatrix} a^\sharp_{11}\partial_x(\partial_t a^\sharp_{21}/a^\sharp_{21}) & a^\sharp_{12}\partial_x(\partial_t a^\sharp_{12}/a^\sharp_{12}) \\ a^\sharp_{21}\partial_x(\partial_t a^\sharp_{21}/a^\sharp_{21}) & -a^\sharp_{11}\partial_x(\partial_t a^\sharp_{12}/a^\sharp_{12}) \end{pmatrix}.$$

We study the (1,1)-th entry:

$$a^\sharp_{11}\partial_x(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}) = \frac{a^\sharp_{11}}{a^\sharp_{21}}\partial_x\partial_t a^\sharp_{21} - \partial_x a^\sharp_{21}\frac{a^\sharp_{11}}{a^\sharp_{21}}\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}}.$$

Since $|a^\sharp_{11}/a^\sharp_{21}| \leq 1$ we see that

$$\mu([a^\sharp_{11}\partial_x(\frac{\partial_t a^\sharp_{21}}{a^\sharp_{21}})]_\phi; p, q) \geq -2p.$$

Similarly we get $\mu([A^\sharp \partial_x C]_\phi; p, q) \geq -p$. We turn to R. From Lemma 4.1.1 it follows immediately that

$$\mu([R]_\phi; p, q) \geq -p.$$

q.e.d.

Lemma 4.1.3 *We have*

$$\mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) \geq 0, \quad \mu([\frac{Z}{a^\sharp_{12}}]_\phi; p, q) \geq 0.$$

Proof: We consider $Y/a^\sharp_{21}$. The argument for $Z/a^\sharp_{12}$ is same. Note that

$$\frac{Y}{a^\sharp_{21}} = \partial_t a^\sharp_{11} - (\frac{a^\sharp_{11}}{a^\sharp_{21}})\partial_t a^\sharp_{21} + \mathrm{tr}(AB).$$

Then the assertion follows because $\mu([a^\sharp_{11}/a^\sharp_{21}]_\phi; p, q) \geq 0$. q.e.d.

In what follows we assume that (C^+) does not hold, that is: There are p, $q \in_+$, $\phi \in \mathcal{G}^+(\gamma)$ with $p \geq \sigma(\phi)q$, $\mu(h_\phi; p, q) > 2q(1 - \sigma(\phi))$ such that

$$(4.1.1) \quad 2p + 2\mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) < \mu(h_\phi; p, q), \quad \mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) \leq \mu([\frac{Z}{a^\sharp_{12}}]_\phi; p, q).$$

Proposition 4.1.1 *Assume that* (C^+) *does not hold. Then there are* p, $q \in_+$, $\phi \in \mathcal{G}^+(\gamma)$ *with* $p \geq \sigma(\phi)q$, $1 > q(1-\sigma(\phi))$, $\mu(h_\phi; p, q) \geq 2$ *such that*

$$\mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) + p < 1, \ \ 2q(1-\sigma(\phi)) - 1 - p - \mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) < 0$$

where $q\sigma(\phi)$ *should read as* p *if* $\phi \equiv 0$.

Proof: Let $\phi \neq 0$. Then we replace p, q in (4.1.1) by

$$\frac{2p}{\mu(h_\phi; p, q)}, \quad \frac{2q}{\mu(h_\phi; p, q)}.$$

Then remarking that $\mu(h_\phi; \kappa p, \kappa q) = \kappa\mu(h_\phi; p, q)$ we may suppose that in (4.1.1)

$$p \geq \sigma(\phi)q, \quad 1 > q(1-\sigma(\phi)), \quad \mu(h_\phi; p, q) = 2,$$
$$p + \mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) < 1.$$

In the case $\phi \equiv 0$ we make the same replacement.

Let us put

$$f(p) = 1 - p - \mu([\frac{Y}{a^\sharp_{12}}]_\phi; p, q),$$
$$g(p) = 2q(1-\sigma(\phi)) - 1 - p - \mu([\frac{Y}{a^\sharp_{12}}]_\phi; p, q).$$

Suppose that $g(p) \geq 0$. Otherwise nothing to be proved. We note that $p < 1$ because

$$p + \mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) < 1, \ \ \mu([\frac{Y}{a^\sharp_{21}}]_\phi; p, q) \geq 0.$$

Remark that

$$f(p) - g(p) = 2(1 - q(1-\sigma(\phi))) > 0.$$

On the other hand we see $f(1) \leq 0$ and $g(1) < 0$ since $\mu([Y/a^\sharp_{12}]_\phi; p, q) \geq 0$. Write

$$\mu([\frac{Y}{a^\sharp_{12}}]_\phi; p, q) = \mu(Y_\phi; p, q) - \mu([a^\sharp_{12}]_\phi; p, q)$$

then we see that $\mu([Y/a^\sharp_{12}]_\phi; p, q)$ is continuous with respect to p. Then there exists $p \leq p^* < 1$ such that

$$g(p^*) = 0, \quad g(p) < 0, \quad p^* < p < 1.$$

Since $f(p^*) > g(p^*) = 0$ one can take $\hat{p}$ so close to p^* ($p^* < \hat{p}$) so that $f(\hat{p}) > 0$ and $g(\hat{p}) < 0$. This $\hat{p}$ is a desired one. q.e.d.

Remark: Since $p \geq \sigma q$, $1 > q(1-\sigma)$ this shows that $1 + p > q$.

Lemma 4.1.4 *Assume that* $p \geq \sigma(p)q$, $\mu + p < 1$, $2q(1 - \sigma(\phi)) - 1 - p - \mu < 0$. *Set* $\delta = (1 + p - q)^{-1}$ *and* $2\sigma_1 = 1 - \delta\mu + \delta q - 2\delta p$. *Then we have*

$$\sigma_1 - \delta q\sigma(\phi) - 1 + \delta p < 0.$$

In particular $\sigma_1 < 1 - \delta(p - \sigma(\phi)q) \leq 1$.

Proof: We plug $1 = \delta(1 + p - q)$ into $1 - \delta\mu + \delta q - 2\delta p$ then we get

$$2\sigma_1 = \delta(1 + p - q) - \delta\mu + \delta q - 2\delta p = \delta - \delta\mu - \delta p = \delta(1 - \mu - p).$$

We compute $\delta(2q(1 - \sigma) - 1 - p - \mu) < 0$ which is

$$\begin{aligned} 2q\delta - 2q\delta\sigma - \delta - \delta p - \delta\mu &= \delta(1 - p - \mu) - 2q\delta\sigma - 2\delta(1 - q) \\ &= 2\sigma_1 - 2q\delta\sigma - 2 + 2\delta p \end{aligned}$$

because $\delta(1 - q) = 1 - \delta p = 2(\sigma_1 - \delta q\sigma - 1 + \delta p)$. This proves the assertion. q.e.d.

4.2 Construction of an asymptotic solution

From Proposition 4.1.1 we may suppose that $p, q \in_+$ and $\mu = \mu([Y/a^{\sharp}_{21}]_\phi; p, q)$ verifies

$$\begin{aligned} &\mu \geq 2, \quad p \geq \sigma q, \ 1 > q(1 - \sigma), \\ &\mu + p < 1, \quad 2q(1 - \sigma) - 1 - p - \mu < 0 \end{aligned} \tag{4.2.1}$$

where if $\phi \equiv 0$ then $q\sigma$ should be read as p. Let $\phi \in \mathcal{G}^+(\gamma)$. Take local coordinates $x = (x_1, x_2)$ so that

$$x_1 = t - \phi(x), \quad x_2 = x.$$

Let P be a differential operator defined near the origin which is expressed as $P(t, x, \partial_t, \partial_x)$ in the local coordinates (t, x). Let P_ϕ be the representation of P in the coordinates (x_1, x_2). Let

$$(L^{\sharp}M)_\phi = \sum_{i,j=1}^{2} h^{(ij)}(x)_{ij} + \sum_{i=1}^{2} B^{(i)}(x)_i + F(x)$$

where $_i = /x_i$ and $h^{(ij)}$ has the form

$$\begin{aligned} h^{(11)}(x) = 1 - h_\phi(x)\phi'(x_2)^2, \ h^{(12)}(x) = 2h_\phi(x)\phi'(x_2), \ h^{(22)}(x) = -h_\phi(x), \\ B^{(2)}(x) = Q_\phi(x), \ B^{(1)}(x) = h_\phi(x)\phi''(x_2) - \phi'(x_2)Q_\phi(x) + R_\phi(x), \\ F(x) = S_\phi(x). \end{aligned}$$

Recall that $L^{\sharp}M = \partial_t^2 - h\partial_x^2 + Q\partial_x + R\partial_t + S$ and

$$h_\phi(t, x) = x^{2n_1}(t^{2m_1} + h_1(x)t^{2m_1 - 1} + \cdots + h_{2m_1}(x))e(t, x)^2 = \tilde{h}e(t, x)^2.$$

One can write

$$\tilde{h}(x) = \sum_{j=0}^{2m_1} \tilde{h}_j(x_2)x_1^j, \quad \tilde{h}_j(x_2) = \bar{h}_j x_2^{\sigma_j}(1 + O(x_2^{1/\theta}))$$

where $\theta = \theta(\phi)$. This shows that

$$\begin{aligned} h_\phi(x) &= \sum_{j=0}^{2m_1} e_\phi^2(x)\bar{h}_j x_1^j x_2^{\sigma_j}(1 + O(x_2^{1/\theta})) \\ &= \sum_{(\alpha,\beta)\in M(\phi)} h_{\alpha\beta}(x)x_1^\alpha x_2^\beta(1 + O(x_2^{1/\theta})). \end{aligned}$$

It is clear that

$$\lim_{x_1\to 0, x_2\downarrow 0} h_{\alpha\beta}(x) = h^*_{\alpha\beta} \neq 0 \quad \text{for} \quad (\alpha,\beta) \in M(\phi)$$

and the Newton polygon $\Gamma(\phi)$ is given by $\{(\alpha,\beta) \mid (\alpha,\beta) \in M(\phi)\}$. Note that

$$\mu(h_\phi; p, q) \geq 2 \Longrightarrow \alpha p + \beta q \geq 2, \quad \forall(\alpha,\beta) \in M(\phi).$$

Then we get

$$\begin{aligned} h^{(22)}(x) &= -\sum_{(\alpha,\beta)\in M(\phi)} h_{\alpha\beta}(x)x_1^\alpha x_2^\beta(1 + O(x_2^{1/\theta})), \\ h^{(12)}(x) &= 2\sum_{(\alpha,\beta)\in M(\phi)} ch_{\alpha\beta}(x)x_1^\alpha x_2^{\beta+(\sigma-1)}(1 + O(x_2^{1/\theta})), \\ h^{(11)}(x) &= 1 - \sum_{(\alpha,\beta)\in M(\phi)} c^2 h_{\alpha\beta}x_1^\alpha x_2^{\beta+2(\sigma-1)}(1 + O(x_2^{1/\theta})). \end{aligned}$$

We make a dilation: $x_1 = \lambda^{-\delta p}y_1$, $x_2 = \lambda^{-\delta q}y_2$. Let P_λ be the reprezentation of P in the coordinates $y = (y_1, y_2)$

$$\begin{aligned} \lambda^{-2\delta p}(L^\sharp M)_{\phi,\lambda} = h_\lambda^{(11)}(y)_1^2 + h_\lambda^{(12)}(y)\lambda_1^{\delta q-\delta p}{}_2 \\ +h_\lambda^{(22)}(y)\lambda_2^{2\delta q-2\delta p2} + B_\lambda^{(1)}(y)\lambda_1^{-\delta p} \\ +B_\lambda^{(2)}(y)\lambda_2^{\delta q-2\delta p} + F_\lambda(y)\lambda^{-2\delta p} \end{aligned}$$

where $f_\lambda(y) = f(\lambda^{-\delta p}y_1, \lambda^{-\delta q}y_2)$. Let us take τ as the least common denominator of δ, p, q, σ, σ_1, $1/\theta$.

Lemma 4.2.1 *We have*

$$\begin{aligned} \lambda^{2\sigma_1}h_\lambda^{(11)}(y) &= \lambda^{2\sigma_1}(1 + O(\lambda^{-1/\tau})), \\ \lambda^{\delta q-\delta p+\sigma_1+1}h_\lambda^{(12)}(y) &= O(\lambda^{-1/\tau}), \\ \lambda^{2\delta q-2\delta p+2}h_\lambda^{(22)}(y) &= O(1). \end{aligned}$$

Proof: Note that $-\delta\alpha p - \delta\beta q - 2\delta q(\sigma-1) = -\delta(\alpha p + \beta q - 2q(1-\sigma))$. From $\mu(h_\phi; p, q) \geq 2$ we see $\alpha p + \beta q \geq 2$ if $(\alpha, \beta) \in M(\phi)$ and hence it follows that $\alpha p + \beta q - 2q(1-\sigma) > 2q(1-\sigma)$. That is

$$\lambda^{2\sigma_1} h_\lambda^{(11)}(y) = \lambda^{2\sigma_1}(1 + O(\lambda^{-1/\tau})).$$

We next study $\lambda^{\delta q - \delta p + \sigma_1 + 1} h_\lambda^{(12)}(y)$. Recall that

$$\begin{aligned} &-\alpha\delta p - \beta\delta q - \delta q(\sigma - 1) + \delta q - \delta p + \sigma_1 + 1 \\ &= \delta(-\alpha p - \beta q) - \delta q\sigma + 2\delta q - \delta p + \sigma_1 + 1 \\ &= \delta(-\alpha p - \beta q) - \delta q\sigma - 2 + 2\delta(1+p) - \delta p + \sigma_1 + 1 \\ &= \delta(2 - \alpha p - \beta q) + (\sigma_1 - \delta q\sigma - 1 + \delta p) < 0 \end{aligned}$$

by Lemma 4.1.4 and the fact $\alpha p + \beta q \geq 2$ for $(\alpha, \beta) \in M(\phi)$. This proves that

$$\lambda^{\delta q - \delta p + \sigma_1 + 1} h_\lambda^{(12)}(y) = O(\lambda^{-1/\tau}).$$

Finally we study $\lambda^{2\delta q - 2\delta p + 2} h_\lambda^{(22)}(y)$. Then we see

$$\begin{aligned} &2\delta q - 2\delta p + 2 - \alpha\delta p - \beta\delta q \\ &= 2\delta q - 2\delta p + 2\delta(1 + p - q) - \alpha\delta p - \beta\delta q = \delta(2 - \alpha p - \beta q) \leq 0 \end{aligned}$$

because $(\alpha, \beta) \in M(\phi)$ and hence the assertion. q.e.d.

Lemma 4.2.2 *We have*

$$\begin{aligned} \lambda^{\delta q - 2\delta p + 1} B_\lambda^{(2)}(y) = \lambda^{\delta q - 2\delta p + 1} Q_{\phi,\lambda}(y) = \lambda^{2\sigma_1}[Q_\phi^0(y) + O(\lambda^{-1/\tau})] \\ \text{diadonal of } \lambda^{-\delta p + \sigma_1} B_\lambda^{(1)} = O(\lambda^{2\sigma_1 - 1/\tau}), \\ \text{off diadonal of } \lambda^{-\delta p + \sigma_1} B_\lambda^{(1)} = O(\lambda^{-\delta p + \sigma_1}), \quad \lambda^{-2\delta p} F_\lambda = O(1) \end{aligned}$$

where $Q_{\phi,\lambda}(y) = \lambda^{\mu([Y/a_{21}^\sharp]_\phi; -\delta p, -\delta q)}[Q_\phi^0(y) + O(\lambda^{-1/\tau})]$.

Proof: By definition $\delta q - 2\delta p + 1 = 2\sigma_1 + \delta\mu([Y/a_{21}^\sharp]_\phi; p, q)$. Noting that the fact $\mu([Y/a_{21}^\sharp]_\phi; -\delta p, -\delta q) = -\delta\mu([Y/a_{21}^\sharp]_\phi; p, q)$ we get the first assertion. We next study $\lambda^{-\delta p + \sigma_1} B_\lambda^{(1)}(y)$. Recall

$$B^{(1)}(x) = h_\phi(x)\phi''(x_2) - \phi'(x_2) Q_\phi(x) + R_\phi(x).$$

Note that $\lambda^{-\delta p + \sigma_1}(h_\phi \phi'')_\lambda$ yields the power $-\delta p + \sigma_1 - \delta\alpha p - \delta\beta q - \delta q(\sigma - 2)$. We plug $2\delta q = 2\delta(1+p) - 2$ and hence this gives the power

$$-\delta(\alpha p + \beta q - 2) + (\sigma_1 - \delta q\sigma - 1 + \delta p) - 1 < -1$$

by Lemma 4.1.4. This shows $\lambda^{-\delta p+\sigma_1}(h_\phi \phi'')_\lambda = O(\lambda^{-1})$. We turn to the term $\lambda^{-\delta p+\sigma_1}(\phi' Q_\phi)_\lambda$:

$$\begin{aligned} -\delta p + \sigma_1 - \delta q(\sigma - 1) + \mu([Y/a_{21}^\sharp]_\phi; -\delta p, -\delta q) \\ = -\delta\mu + \delta(1+p) - 1 - \delta q\sigma - \delta p + \sigma_1 \\ = \delta(1 - p - \mu) - 1 - \delta q\sigma + \delta p + \sigma_1 \\ = 2\sigma_1 + (\sigma_1 - \delta q\sigma - 1 + \delta p) < 2\sigma_1 \end{aligned}$$

by Lemma 2.1.5. This gives $\lambda^{-\delta p+\sigma_1}(\phi' Q_\phi)_\lambda = O(\lambda^{2\sigma_1 - 1/\tau})$. Recall $R = C + G$ with smooth G. From Lemma 4.1.1 it follows that $C_{\phi,\lambda} = O(\lambda^{\delta p})$ and hence $\lambda^{-\delta p+\sigma_1}R_{\phi,\lambda}(y) = O(\lambda^{\sigma_1})$. Finally we consider $\lambda^{-2\delta p}F_\lambda$. Since $S = L^\sharp(C)+$ smooth term and $F = S_\phi$ it is enough to consider $L^\sharp(C)$. From Lemma 4.1.2 it follows that

$$S_{\phi,\lambda} = O(\lambda^{2\delta p})$$

and hence the desired result. q.e.d.

Let us define ν by $\nu = \sigma_1 \tau$.

Proposition 4.2.1 *Assume that there are $\phi \in \mathcal{G}^+(\gamma)$, p, $q \in_+$ with $p \geq \sigma(\phi)q$, $\mu(h_\phi; p, q) > 2q(1 - \sigma(\phi))$ ($q\sigma(\phi) = p$ if $\phi \equiv 0$) such that we have either*

$$2p + 2\mu([\frac{Y}{a_{21}^\sharp}]_\phi; p, q) < \mu(h_\phi; p, q)$$

or

$$2p + 2\mu([\frac{Z}{a_{12}^\sharp}]_\phi; p, q) < \mu(h_\phi; p, q).$$

Then there is $\hat{y} = (\hat{y}_1, \hat{y}_2)$, $\hat{y}_2 > 0$ such that for any neighborhood $U(\hat{y})$ of $\hat{y}$ and any $N \in \mathbf{N}$ there is $\bar{y} \in U(\hat{y})$, a neighborhood W of $\bar{y}$ and $l^j(y)$, $1 \leq j \leq \nu$ and $u_n(y)$, $0 \leq n \leq N$ defined in W such that

$$E(y,\lambda)^{-1}\lambda^{-2\delta p}L^\sharp_{\phi,\lambda}U_\lambda = O(\lambda^{2\sigma_1 - (\nu+N+1)/\tau})$$

where

$$E(y,\lambda) = \exp\{i(\mu y_2 \lambda + \sum_{j=1}^{\nu} l^j(y)\lambda^{\sigma_j})\},$$

$$U_\lambda = E(y,\lambda)\lambda^\kappa \sum_{n=0}^{N} \lambda^{-n/\tau}u_n(y), \quad \sigma_j = \frac{\nu+1-j}{\tau}, \quad \kappa = \kappa(p,q).$$

and $\operatorname{Im} l^1(y) \geq (y_2 - \bar{y}_2)^2 + \delta_0(\bar{y}_1 - y_1)$ in $W \cap \{y_1 \leq \bar{y}_1\}$ with some $\delta_0 > 0$ and $u_0(\bar{y}) \neq 0$.

Proof: Recall that

$$\lambda^{\delta q-2\delta p+1}B_\lambda^{(2)}(y)=\lambda^{\delta q-2\delta p+1}Q_{\phi,\lambda}(y)$$
$$=\lambda^{\delta q-2\delta p+1}\begin{pmatrix} [Y/a^\sharp_{21}]_{\phi,\lambda} & 0 \\ 0 & [Z/a^\sharp_{12}]_{\phi,\lambda}\end{pmatrix}$$
$$=\lambda^{2\sigma_1}\begin{pmatrix} \sum_{j=0} C^1_j(y)\lambda^{-j/\tau} & 0 \\ 0 & \sum_{j=0} C^2_j(y)\lambda^{-j/\tau}\end{pmatrix}$$

where $C^i_j(y)$ are defined in a neighborhood of $\hat{y}$ and we may suppose $C^1_0(\hat{y})\neq 0$. We look for U_λ in the form

$$U_\lambda=M_{\phi,\lambda}u_\lambda,\quad u_\lambda=E(y,\lambda)\sum_{n=0}^N \lambda^{-n/\tau}v_n(y).$$

We study

$$E(y,\lambda)^{-1}(L^\sharp M)_{\phi,\lambda}E(y,\lambda)\sum_{n=0}^N v_n(y)\lambda^{-n/\tau}.$$

This turns out to be

$$\lambda^{2\sigma_1+2\delta p}\{\sum_{j=1}^{\nu}\mathcal{L}_j(l^1,...,l^j)\lambda^{-(j-1)/\tau}\sum_{n=1} v_n\lambda^{-n/\tau}$$
$$(4.2.2)\quad +\sum_{n=0}^N(2\sqrt{-1}l^1_{y_1}\frac{}{y_1}v_n+R_n(l^1,...,l^\nu,v_0,...,v_{n-1}))\lambda^{-(n+\nu)/\tau}\}$$
$$+O(\lambda^{2\sigma_1+2\delta p-(\nu+N+1)/\tau})$$

where

$$\mathcal{L}_j=\begin{pmatrix}\mathcal{L}^1_j & 0\\ 0 & \mathcal{L}^2_j\end{pmatrix},\quad \mathcal{L}^i_1(l^1)=-(l^1_{y_1})^2+\sqrt{-1}C^i_0(y),\quad v_n=\begin{pmatrix}v^I_n\\ v^{II}_n\end{pmatrix}$$
$$\mathcal{L}^i_j(l^1,...,l^j)=-2l^1_{y_1}l^j_{y_1}+K^i_j(l^1,...,l^{j-1}),\ j\geq 2$$

and K^i_j, R_n are non linear differential operators with real analytic coefficients. More precisely

$$\mathcal{L}^i_j=\Phi_j(C^i_0,...,C^i_{j-2},l^1,...,l^j)+\sqrt{-1}C^i_{j-1}(y),\quad 1\leq j\leq\nu$$

where Φ_j is independent of i. To see this it is enough to note that non diagonal part of the coefficients does not enter to the determination.

Let $U(\hat{y})$ be given. We devide the cases into two:

(1) $C^1_j(y)=C^2_j(y)$ in U for $0\leq j\leq\nu-1$,

(2) there exists $k\leq\nu-1$ and $\bar{y}\in U$ such that

$$C^1_j(y)=C^2_j(y)\quad\text{in}\quad U,\quad 0\leq j\leq k-1,\quad C^1_k(\bar{y})\neq C^2_k(\bar{y}).$$

In case (2) we choose $W_1 = W_1(\bar{y}) \subset U$ so taht

$$|C_k^1(y) - C_k^2(y)| \geq c > 0 \quad \text{in} \quad W_1.$$

We first define $l^j(y)$. Take $\mu \in \mathbf{R}$ and $W_2 \subset W_1$ so taht

$$-\zeta^2 + \sqrt{-1}\mu C_0^1(y) = 0$$

has a root $F(y)$ with Im $F(y) < -\delta_0 < 0$ in W_2. Note that $|F(y)| \sim \sqrt{|\mu|}$. We next solve the Cauchy problem

$$l^1_{y_1} = F(y), \quad l^1|_{y_1=\bar{y}_1} = \sqrt{-1}(y_2 - \bar{y}_2)^2.$$

This gives that $|l^1_{y_1}| \sim \sqrt{|\mu|}$. We define $l^j(y)$ succesively by solving

$$\begin{cases} \mathcal{L}_j^1(l^1, ..., l^j) = -2l^1_{y_1} l^j_{y_1} + K_j^1(l^1, ..., l^{j-1}) = 0 \\ l^j|_{y_1=\bar{y}_1} = 0 \end{cases}$$

for $2 \leq j \leq \nu$. In the case (1) we have clearly that

$$\mathcal{L}_j^2(l^1, ..., l^j) = 0 \quad \text{in} \quad W_2 \quad \text{for} \quad j = 1, ..., \nu$$

and in the case (2) we have

$$\mathcal{L}_j^2(l^1, ..., l^j) = 0 \ \text{ in } \ W_2 \ \text{ for } \ j = 1, ..., k, \ \ |\mathcal{L}_{k+1}^2(l^1, ..., l^{k+1})| \geq c' > 0 \ \text{ in } \ W_2.$$

We observe the second component of (4.2.3) which is equal to, up to the factor $\lambda^{2\sigma_1+2\delta p}$

$$\sum_{n=0}^{N+\nu-k-1} \{\mathcal{L}_{k+1}^2(l^1, ..., l^{k+1})v_n^{II} + \tilde{R}_n^2(l^1, ..., l^\nu, v_0, ..., v_{n-1})\}\lambda^{-(n+k)/\tau}$$
$$+O(\lambda^{-(\nu+N-k)/\tau}).$$

We remark that

$$\tilde{R}_n^2(l^1, ..., l^\nu, v_0, ..., v_{n-1})|_{v_0^{II}=\cdots=v_{n-1}^{II}} = 0 \ \text{ for } \ n \leq \nu - k - 1.$$

Hence this second component is reduced to

$$\mathcal{L}_{k+1}^2(l^1, ..., l^{k+1})v_n^{II} + \tilde{R}_n^2(l^1, ..., l^\nu, v_0, ..., v_{n-1}) = 0.$$

On the other hand the first component is

$$\sum_{n=0}^{N}(2\sqrt{-1}l^1_{y_1}\frac{\partial}{\partial y_1}v_n^I + R_n^1(l^0, ..., l^\nu, v_1, ..., v_{n-1}))\lambda^{-(n+\nu))/\tau}$$
$$+O(\lambda^{-(\nu+N+1)/\tau})$$

and hence we are led to the equation

$$2\sqrt{-1}l^1_{y_1}\frac{\partial}{\partial y_1}v^I_n + R^1_n(l^1,...,l^\nu,v_0,...,v_{n-1}) = 0.$$

We summarize:

$$\begin{cases}\mathcal{L}^2_{k+1}(l^1,...,l^{k+1})v^{II}_n + \tilde{R}^2_n(l^1,...,l^\nu,v_0,...,v_{n-1}) = 0\\ 2\sqrt{-1}l^1_{y_1}\frac{\partial}{\partial y_1}v^I_n + R^1_n(l^1,...,l^\nu,v_0,...,v_{n-1}) = 0.\end{cases}$$

We solve this system with initial conditions

$$v^I_0|_{y_1=\bar{y}_1} \neq 0, \;\; v^I_n|_{y_1=\bar{y}_1} = 0, \;\; n = 1,2,...,N, \;\; v^{II}_0 = 0.$$

Since v^{II}_0 verifies the first equation, then one can solve the system successively.

We turn to the case (1). Up to the factor $\lambda^{2\sigma_1+2p}$ we see

$$\sum_{n=0}^{N}\{2\sqrt{-1}l^1_{y_1}\frac{\partial}{\partial y_1}v_n + R_n(l^1,...,l^\nu,v_0,...,v_{n-1})\}\lambda^{-(n+\nu)/\tau}$$
$$+O(\lambda^{-(N+\nu+1)/\tau}).$$

Hence we are led to

$$\begin{cases}2\sqrt{-1}l^1_{y_1}\frac{\partial}{\partial y_1}v_n + R_n(l^1,...,l^\nu,v_0,...,v_{n-1}) = 0\\ v_0|_{y_1=\bar{y}_1} = \begin{pmatrix}\neq 0\\ \\ 0\end{pmatrix}, \;\; v_n|_{y_1=\bar{y}_1} = 0, \;\; n = 1,2,...,N.\end{cases}$$

Lemma 4.2.3 *Let v_n be as above. Let us write*

$$U_\lambda = M_{\phi,\lambda}E(y,\lambda)\sum_{n=0} v_n\lambda^{-n/\tau} = E(y,\lambda)\lambda^{\tilde{\kappa}}\sum_{n=0} u_n\lambda^{-n/\tau}.$$

Then U_λ is non trivial, that is there is a $\tilde{\kappa} \in_+$ independent of N such that $u_0(\bar{y}) \neq 0$.

Proof: Recall $M = \partial_t + A^\sharp\partial_x$ where

$$\begin{pmatrix}\lambda^{-\alpha}(a(y)+O(\lambda^{-1/\tau}) & \lambda^{-\beta}(b(y)+O(\lambda^{-1/\tau})\\ \lambda^{-\beta}(b(y)+O(\lambda^{-1/\tau}) & -\lambda^{-\alpha}(a(y)+O(\lambda^{-1/\tau})\end{pmatrix}$$

where $\beta \leq \alpha$ and $b(\bar{y}) \neq 0$. Recall also

$$\lambda^{-2\delta p}M_{\phi,\lambda} = \lambda^{-\delta p}(I - \phi'(\lambda^{-\delta q}y_2)A^\sharp_{\phi,\lambda})_1 + \lambda^{\delta q-2\delta p}A^\sharp_{\phi,\lambda^2} + \lambda^{-2\delta p}\tilde{C}_{\phi,\lambda}.$$

We observe

$$\begin{aligned}-\delta q(\sigma-1) - \delta p + \sigma_1 &= -\delta q\sigma + \delta q - \delta p + \sigma_1 = -\delta q\sigma + \delta - 1 + \sigma_1\\ &= \delta(1-p-\mu) + \delta p + \delta\mu - \delta q\sigma - 1 + \sigma_1\\ &< \delta(1-p-\mu) + \delta\mu = 1 + \delta q - 2\delta q\end{aligned}$$

by Lemma dafive. This proves that

$$\lambda^{-\delta p+\sigma_1}\phi'(\lambda^{-\delta q}y_2) = o(\lambda^{1+\delta q-2\delta p}).$$

Since $\tilde{C}_{\phi,\lambda} = O(\lambda^{\delta p})$ we get $\lambda^{-2\delta p}\tilde{C}_{\phi,\lambda} = O(\lambda^{-\delta p})$ and hence

$$\lambda^{-2\delta p}\tilde{C}_{\phi,\lambda} = \lambda^{\delta p}\left[\begin{pmatrix} c(y) & 0 \\ 0 & \bar{c(y)} \end{pmatrix} + O(\lambda^{-1/\tau})\right] = \lambda^{\delta p}(c^0(y) + O(\lambda^{-1/\tau}).$$

We note that $\delta q - 2\delta p + 1 = 2\sigma_1 + \delta\mu$ by Lemma 4.1.4. Let us set

$$\kappa = \max\{2\sigma_1 + \delta\mu - \beta, -\delta p\}.$$

Then we conclude that

$$\lambda^{-2\delta p}E(y,\lambda)^{-1}M_{\phi,\lambda}E(y,\lambda) = \lambda^{\kappa}\{\mu\begin{pmatrix} \lambda^{\beta-\alpha}a(y) & b(y) \\ \bar{b}(y) & -\lambda^{\beta-\alpha}a(y) \end{pmatrix} + O(\lambda^{-1/\tau})\}$$

when $\kappa = 2\sigma_1 + \delta\mu - \beta > -\delta p$. Since

$$v_0(\bar{y}) = \begin{pmatrix} v_0^I(\bar{y}) \\ 0 \end{pmatrix} \quad \text{(case (2))} \quad \text{or} \quad v_0(\bar{y}) = \begin{pmatrix} v_0^I(\bar{y}) \\ v_0^{II}(\bar{y}) \end{pmatrix}$$

choosing $v_0^{II}(y)$ suitably we get the assertion. If $\kappa = 2\sigma_1 + \delta\mu - \beta = -\delta p$ then we see

$$\begin{aligned} &\lambda^{-2\delta p}E(y,\lambda)^{-1}M_{\phi,\lambda}E(y,\lambda) \\ &= \lambda^{\kappa}\{\mu\begin{pmatrix} \lambda^{\beta-\alpha}a(y) & b(y) \\ \bar{b}(y) & -\lambda^{\beta-\alpha}a(y) \end{pmatrix} \\ &+ \begin{pmatrix} l_{y_1}^1 + c(y) & 0 \\ 0 & l_{y_1}^1 + \bar{c}(y) \end{pmatrix} + O(\lambda^{-1/\tau})\}. \end{aligned}$$

Then choosing

$$v_0(\bar{y}) = \begin{pmatrix} v_0^I(y) \\ 0 \end{pmatrix}$$

the assertion follows clearly. Finally if $\kappa = -\delta p > 2\sigma_1 + \delta\mu - \beta$ then

$$\begin{aligned} &\lambda^{-2\delta p}E(y,\lambda)^{-1}M_{\phi,\lambda}E(y,\lambda) \\ &= \lambda^{\kappa}\{\begin{pmatrix} l_{y_1}^1 + c(y) & 0 \\ 0 & l_{y_1}^1 + \bar{c(y)} \end{pmatrix} + O(\lambda^{-1/\tau})\}. \end{aligned}$$

Since $l_{y_1}^1 = F(y) = \{\sqrt{-1}\mu C_0^1(y)\}^{1/2}$, it is clear that one can choose μ so that $l_{y_1}^1 + c(y) \neq 0$ and hence the result. q.e.d.

4.3 Proof of necessity

Theorem 4.3.1 *Assume that the Cauchy problem (C.P.) is C^∞ well posed near the origin. Then $(C^{\pm})$ are verified.*

Let us fix $\gamma > 0$. Denote

$$D(r, M) = \{(t, x) \mid 0 < x < r, 0 < t < Mx^\gamma\},$$
$$\Delta(\hat{t}, \hat{x}; c) = \{(t, x) \mid (t - \hat{t}) + c^{-1}|x - \hat{x}| \leq 0, 0 \leq t \leq \hat{t}\}.$$

Assume that $\gamma \in_+$ verifies

$$|h(t, x)| \leq C(M)^2 r^2 \quad \text{in} \quad (t, x) \in D(r, M).$$

Let us put

$$\mu = \begin{cases} C(M)^{-1}(2M)^{-1} & \text{if} \quad \gamma \geq 1 \\ C(M)^{-1}(2M)^{-1}\hat{x}^{1-\gamma} & \text{if} \quad \gamma < 1. \end{cases}$$

Then we have

Lemma 4.3.1 *There is a $T = T(M, \gamma)$ such that*

$$(\hat{t}, \hat{x}) \in D(\mu M),\ 0 < \hat{x} < T \Longrightarrow \Delta(\hat{t}, \hat{x}; C(M)\mu) \subset D(\mu, M).$$

Proof: Let $\gamma \geq 1$ and choose T so that

$$0 < \hat{x} < T \Longrightarrow \gamma \hat{x}^{\gamma - 1} < 2.$$

With this choice of T we have

$$\gamma M \hat{x}^{\gamma-1} < [C(M)\mu]^{-1} = \begin{cases} 2M & \text{if} \quad \gamma \geq 1 \\ 2M\hat{x}^{\gamma-1} & \text{if} \quad \gamma < 1 \end{cases}$$

if $0 < \hat{x} < T$ and hence the assertion is clear. q.e.d.

Remark: Since $|h(t, x)| \leq C(M)^2 r^2$ for $(t, x) \in D(r, M)$ we see that the dependence domain of $(\hat{t}, \hat{x})$ is $\Delta(\hat{t}, \hat{x}; C(M)\mu)$ if $(\hat{t}, \hat{x}) \in D(\mu, M)$, $0 < \hat{x} < T$. Thus we have

$$Lu = 0 \ \text{in} \ \Delta(\hat{t}, \hat{x}; C(M)\mu),\ \ u(t, x) = 0 \ \text{in} \ t \leq 0 \Longrightarrow u(\hat{t}, \hat{x}) = 0.$$

Now let $\phi(x)$ be a C^∞ function in $(0, r(\phi))$ and let

$$T_\phi : U \cap \{x > 0\} \ni (t, x) \mapsto (x_1, x_2) = (t - \phi(x), x) \in W \cap \{x_2 > 0\}$$

be a diffeomorphism. Let L_ϕ be a representation of L in the coordinates (x_1, x_2). Put

$$E = E(M, \gamma, \phi) = \{(x_1, x_2) \mid 0 < x_2 < \delta_0, 0 < x_1 < Mx_2^\gamma - \phi(x_2)\}$$

then we have the following lemma with a suitable δ_0.

Proposition 4.3.1 *Assume that the Cauchy problem (C.P.) is C^∞ well posed near the origin. Then for any T there are $M(> M_0)$, a neighborhood $\tilde{W}$ of the origin, $C > 0$ and $l \in \mathbf{N}$ such that*

$$\sup_{0 \le x_1 \le T} |u| \le C \sup_{0 \le x_1 \le T, |\alpha| \le l} |D^\alpha L^\sharp_\phi u|$$

for any $u \in C_0^\infty(\tilde{W} \cap E)$.

We now admit this proposition. Let

$$y_1 = \lambda^{\delta p} x_1, \;\; y_2 = \lambda^{\delta q} x_2, \;\; \delta, p, q \in_+$$

be a dilation such that $p \ge \gamma q$. Let $L_{\phi,\lambda}$ be the representation of L_ϕ in the coordinates (y_1, y_2):

$$L_{\phi,\lambda}(y, D) = L_\phi(\lambda^{-\delta p} y_1, \lambda^{-\delta q} y_2, \lambda^{\delta p} D_{y_1}, \lambda^{\delta q} D_{y_2}).$$

Then we have

Proposition 4.3.2 *Let $B > 0$ be given and let $p \ge \gamma q$, $\phi \in \mathcal{C}^+(A)$ and $1+p > q$. Assume that the Cauchy problem for L is C^∞ well posed near the origin. Then there are $C > 0$, $l \in \mathbf{N}$, $\lambda_0 = \lambda_0(B, \sigma, \phi)$ such that*

$$\sup_{0 \le y_1 \le \bar{y}_1} |u| \le C \lambda^{\delta k l} \sup_{0 \le y_1 \le \bar{y}_1, |\beta| \le l} |D_y^\beta (L^\sharp_{\phi,\lambda} u)|$$

for any $u \in C_0^\infty(\{0 < y_1, y_2 < B\})$, $k = \max(p, q)$, $\delta = (1 + p - q)^{-1}$, $\lambda \ge \lambda_0$.

Proof: Let $u \in C_0^\infty(\{0 < y_i < B\})$ and $u_\lambda(y) = u(\lambda^{\delta p} y_1, \lambda^{\delta q} y_2)$. Then there are λ_0 and M_0 so that $u_\lambda \in C_0^\infty(\tilde{W} \cap E)$ if $\lambda \ge \lambda_0$, $M \ge M_0$ and $u \in C_0^\infty(\{y_i < B\})$. Applying Proposition 4.3.1 we get

$$\sup_{0 \le y_1 \le T} |u_\lambda| \le C \sup_{0 \le y_1 \le T, |\alpha| \le l} |D^\alpha (L_\phi u_\lambda)|.$$

Taking $T = \lambda^{-\delta p} \bar{y}_1$ we get the desired inequality. q.e.d.

Proof of necessity: From Proposition 4.3.1 we can construct an asymptotic solution U_λ. Take $\chi(y) \in C_0^\infty(W)$ so that $\chi(y) = 1$ on a neighborhood of $\bar{y}$. Set $u_\lambda = \chi(y) U_\lambda(y)$ then we have

$$\sup_{0 \le y_1 \le \bar{y}_1, |\alpha| \le l} |D^\alpha (L^\sharp_{\phi,\lambda} u_\lambda)| \le C_l \lambda^{2\sigma_1 + 2\delta p + l - (\nu + N + 1)/\tau}.$$

On the other hand since $u_\lambda(\bar{y}) \ge c \lambda^\kappa$ with some $c > 0$, taking N large we get a contradiction. q.e.d.

5 Equivalence of conditions

5.1 Equivalence of conditions

The aim of this section is to prove

Proposition 5.1.1 *The condition* $(C^{\pm})$ *is equivalent to*

$$\Gamma(tZ_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp_{12}|^2]_\phi), \quad \Gamma(tY_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp_{12}|^2]_\phi), \quad \forall \phi \in \mathcal{G}^{\pm}(\gamma).$$

Remark: Actually we prove that the condition $(C^+; Y)$, the condition obtained from (C^+) dropping the requirements on Z, is equivalent to

$$\Gamma(tY_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp_{12}|^2]_\phi), \quad \forall \phi \in \mathcal{G}^{+}(\gamma).$$

Proof: Let p, $q \in_+$, $\phi \in \mathcal{G}^{\pm}(\gamma)$, $p \geq \sigma(\phi)q$, $\mu(h_\phi; p, q) > 2q(1-\sigma(\phi))$. Note that

$$\Gamma(tY_\phi) \subset \frac{1}{2}\Gamma([h|a^\sharp_{12}|^2]_\phi)$$

implies that

$$\begin{aligned} p + \mu([Y]_\phi; p, q) &\geq \frac{1}{2}\mu([h|a^\sharp_{12}|^2]_\phi; p, q) \\ &= \frac{1}{2}\mu(h_\phi; p, q) + \frac{1}{2}\mu([|a^\sharp_{12}|^2]_\phi; p, q) \\ &= \frac{1}{2}\mu(h_\phi; p, q) + \mu(a^\sharp_{12,\phi}; p, q). \end{aligned}$$

By definition, this shows that

$$p + \mu\left(\left[\frac{Y}{a^\sharp_{12}}\right]_\phi; p, q\right) \geq \frac{1}{2}\mu(h_\phi; p, q).$$

We show that $(C^{\pm})$ implies $\Gamma(tZ_\phi) \subset \Gamma([h|a^\sharp_{12}|^2]_\phi)$. Note that

$$\begin{aligned} (tZ_\phi)(s^p x_1, s^q x_2) &= (a^\sharp_{12})_\phi(s^p x_1, s^q x_2)\{t(\frac{Z}{a^\sharp_{12}})_\phi\}(s^p x_1, s^q x_2) \\ &= s^\nu(c^0(x) + o(1)) \end{aligned}$$

with $\nu = \mu([a^\sharp_{12}]_\phi; p, q) + \mu([Z/a^\sharp_{12}]_\phi; p, q) + p$. Let

$$[h|a^\sharp_{12}|^2]_\phi(s^p x_1, s^q x_2) = s^\kappa(d^0(x) + o(1))$$

with $\kappa = 2\mu([a^\sharp_{12}]_\phi; p, q) + \mu(h_\phi; p, q)$. Thus

$$2\nu - \kappa = 2p + 2\mu([\frac{Z}{a^\sharp_{12}}]_\phi; p, q) - \mu(h_\phi; p, q).$$

Hence (C^+) implies that $2\nu \geq \kappa$, that is

$$2\mu(tZ_\phi; p, q) \geq \mu([h|a_{12}^\sharp|^2]_\phi; p, q) \tag{5.1.1}$$

for any p, $q \in_+$ and for any $\phi \in \mathcal{G}^\pm(\gamma)$ which is verifying $p \geq \sigma(\phi)q$ and $\mu(h_\phi; p, q) > 2q(1-\sigma(\phi))$ (if $\phi = 0$ then $q\sigma(\phi)$ should read as p). Take $\phi \in \mathcal{G}^+(\gamma)$. Denote by

$$\{(j, \beta_j(\phi))\}_{j=0}^r, \quad \{(j, \gamma_j(\phi))\}_{j=0}^{\tilde{r}}$$

the points which consists in the boundary of $\frac{1}{2}\Gamma([h|a_{12}^\sharp|^2]_\phi)$ and $\Gamma(Z_\phi)$ respectively where $\beta_r(\phi) = n$, $\gamma_{\tilde{r}}(\phi) = \tilde{n}$, $n = n_1 + n_2$ and $r = m_1 + m_2$. Set

$$\epsilon_j(\phi) = \beta_{j-1}(\phi) - \beta_j(\phi), \quad 1 \leq j \leq r$$
$$\delta_j(\phi) = \gamma_{j-1}(\phi) - \gamma_j(\phi), \quad 1 \leq j \leq \tilde{r}.$$

Note that the boundary points of $\Gamma(tZ_\phi)$ consists of $\{(j+1, \gamma_j(\phi))\}_{j=0}^{\tilde{r}}$. Then it is enough to show that

$$\gamma_j(\phi) \geq \beta_{j+1}(\phi), \quad \forall j \geq 0.$$

Let

$$\epsilon_1(\phi) \geq \cdots \geq \epsilon_\ell(\phi) \geq \sigma(\phi) > \epsilon_{\ell+1}(\phi) \geq \cdots \geq \epsilon_r(\phi).$$

Let $\alpha p_j + \beta q_j = 1$ be the line which is tangent to $\frac{1}{2}\Gamma((h|a_{12}^\sharp|^2)_\phi)$ along the segment joining $(j-1, \beta_{j-1}(\phi))$ and $(j, \beta_j(\phi))$. That is

$$\frac{p_j}{q_j} = \epsilon_j(\phi).$$

Hence we have

$$\frac{p_j}{q_j} = \epsilon_j(\phi) \geq \sigma(\phi) \quad \text{for} \quad 1 \leq j \leq \ell$$

that is $p_j \geq \epsilon_j(\phi) q_j$ for $1 \leq j \leq \ell$.

Lemma 5.1.1 *We have*

$$\frac{1}{2}\Gamma((h|a_{12}^\sharp|^2)_\phi) \subset \text{convex hull of}\{((r,n) + \mathbf{R}_+^2) \cup ((0, n+1) + \mathbf{R}_+^2)\}.$$

Proof: Let us write

$$h|a_{12}^\sharp|^2 = x^{2(n_1+n_2)} \prod^{2(m_1+m_2)} (t - t_\nu(x))\hat{e}(t,x).$$

It is clear that

$$\Gamma((h|a_{12}^\sharp|^2)_\phi) = \Gamma(x^{2n} \prod^{2m} (t + \phi(x) - t_\nu(x))).$$

Recall that there is ν_0 such that $t_{\nu_0}(x) \sim t^*(x)$ and this implies that

$$C|t_{\nu_0}(x)| \geq |\phi(x) - t_\nu(x)| \quad \text{for any} \quad 1 \leq \nu \leq 2m.$$

Hence we have

$$\Gamma(x^{2n}\prod_{\nu=1}^{2m}(t+\phi(x)-t_\nu(x)))\subset\Gamma(x^{2n}\prod_{\nu=1}^{2m}(t-t_{\nu_0}(x))).$$

On the other hand, from the proof of Lemma 2.1.5 we see that

$$t_{\nu_0}(x)^{2m}=O(|x|^2)$$

(note that $r=m_1+m_2=m$). Since

$$\Gamma(x^{2n}\prod_{\nu=1}^{2m}(t-t_{\nu_0}(x)))\subset \text{ convex hull of}\{((r,n)+\mathbf{R}_+^2)\cup((0,n+1))+\mathbf{R}_+^2)\}$$

this proves the assertion. q.e.d.

Lemma 5.1.1 shows that

$$\frac{1}{q_j}\geq n+1$$

and hence $q_j\leq 1$. Since $\sigma(\phi)>0$ we get $1>q_j(1-\sigma(\phi))$. Then the condition (C^+) is verified for $p=p_j$, $q=q_j$. Thus we get from (5.1.1) that

$$\Gamma(Z_\phi)\text{ lies right side of the line }(\alpha+1)p_j+\beta q_j=1,\quad 1\leq j\leq \ell.$$

This proves that

$$\gamma_j(\phi)\geq\beta_{j+1}(\phi),\quad 0\leq j\leq\ell-1. \tag{5.1.2}$$

We now show that $\tilde n\geq n$. If $n=0$ nothing to be proved. If $n\geq 1$ then with $\phi=0$, $q=s/n$, $p=(1-s)/r$ one can apply (5.1.1) because

$$1+p=\frac{1-s+r}{r}>\frac{s}{n}.$$

Thus one gets

$$\tilde n\geq\frac{n}{s}.$$

Letting $s\uparrow 1$ we conclude that $\tilde n\geq n$. Then we have

$$\gamma_j(\phi)\geq\tilde n\geq n=\beta_{j+1}(\phi)\quad\text{for}\quad r-1\leq j.$$

Then it remains to prove

$$\gamma_j(\phi)\geq\beta_{j+1}(\phi)\quad\text{for}\quad \ell\leq j\leq r-2.$$

Assume now that there were j with $\ell\leq j\leq r-2$ such that

$$\gamma_j(\phi)<\beta_{j+1}(\phi).$$

Let us define $j^* = \max\{j \mid \gamma_j(\phi) < \beta_{j+1}(\phi)\}$. By definition we have

$$\gamma_{j^*+1}(\phi) \geq \beta_{j^*+2}(\phi) \quad \text{and} \quad \gamma_{j^*}(\phi) < \beta_{j^*+1}(\phi).$$

This implies that

$$\delta_{j*+1}(\phi) = \gamma_{j^*}(\phi) - \gamma_{j^*+1}(\phi) < \beta_{j^*+1}(\phi) - \beta_{j^*+2}(\phi) = \epsilon_{j^*+2}(\phi) < \sigma(\phi).$$

Take $\psi \in \mathcal{G}^+(\gamma)$ so that

$$\sigma(\psi) = \epsilon_{j^*+2}(\phi).$$

Since $\sigma(\psi - \phi) = \sigma(\phi)$, $\delta_{j^*+1}(\phi) < \sigma(\psi)$, we can apply the following lemma to get

$$\delta_{j+1}(\psi) = \delta_{j+1}(\phi) \quad \text{for} \quad j \geq j^*.$$

Lemma 5.1.2 *Let $f(t,x) = x^n \prod^m (t - t_\nu(x))$ and $\{(j, \beta_j(\phi))\}$ be on the boundary of $\Gamma(f_\phi)$. Assume that $\sigma(\psi - \phi) = \sigma(\psi)$. Let $\epsilon_j(\phi) = \beta_{j-1}(\phi) - \beta_j(\phi)$.*

(1) *Assume $\sigma(\psi) > \epsilon_{k+1}(\phi)$ then we have*

$$\epsilon_j(\psi) = \epsilon_j(\phi) \quad \textit{for} \quad j \geq k+1.$$

(2) *Assume $\sigma(\psi) \geq \epsilon_{k+1}(\phi)$ then we have*

$$\epsilon_j(\psi) \geq \epsilon_j(\phi) \quad \textit{for} \quad j \geq k+1.$$

Proof: (1) Take $\ell \leq k$ so that $\epsilon_\ell(\phi) > \epsilon_{\ell+1}(\phi) = \cdots = \epsilon_{k+1}(\phi)$. Then it is clear by definition of $\epsilon_j(\phi)$ that $(\ell, \beta_\ell(\phi))$ is a vertex of $\Gamma(f_\phi)$. Recall that

$$f_\phi(t,x) = x^n \prod^m (t + \phi(x) - t_\nu(x)) = \sum_{j=1}^m C_j^\phi(x) t^j.$$

By definition we get

$$C_j^\phi(x) = O(|x|^{\beta_m(\phi) + \sum_{i=m}^{j+1} \epsilon_i(\phi)}).$$

When $j = \ell$, since $(\ell, \beta_\ell(\phi))$ is a vertex of $\Gamma(f_\phi)$ we see

$$|C_\ell^\phi(x)| = |x|^{\beta_m(\phi) + \sum_{i=m}^{\ell+1} \epsilon_i(\phi)}(c + o(1)) \tag{5.1.3}$$

with $c \neq 0$. We observe that

$$\frac{1}{\ell!}\left(\frac{}{t}\right)^\ell f_\phi(t,x)|_{t=\psi-\phi} = \frac{1}{\ell!}\left(\frac{}{t}\right)^\ell f_\psi(t,x)|_{t=0} = C_\ell^\psi(x).$$

Then we see that

$$C_\ell^\psi(x) = \sum_{j \geq \ell} \frac{j!}{(j-\ell)!} C_j^\phi(x)(\psi - \phi)^{j-\ell}.$$

On the other hand, we have

$$C_j^{\psi}(x) = O(|x|^{\beta_m(\phi)+\sum_{i=m}^{j+1}\epsilon_i(\phi)}) \quad \text{for} \quad j \geq \ell+1$$

in general, and hence this shows that

$$\Gamma(f_\phi) \cap \{x \leq \beta_\ell(\phi)\} = \Gamma(f_\psi) \cap \{x \leq \beta_\ell(\phi)\} \tag{5.1.4}$$

and hence the assertion.

(2) Take $\ell \geq k+1$ so that $\epsilon_{k+1}(\phi) = \cdots = \epsilon_\ell(\phi) > \epsilon_{\ell+1}(\phi)$ if exists. Since $\sigma(\psi) \geq \epsilon_{k+1}(\phi) > \epsilon_{\ell+1}(\phi)$ the same assertion proving (1) shows . We turn to $\epsilon_j(\phi)$, $\epsilon_j(\psi)$ for $j < \ell+1$. Since

$$C_j^{\psi}(x) = O(|x|^{\beta_m(\phi)+\sum_{i=m}^{j+1}\epsilon_i(\phi)}) \quad \text{for} \quad j \geq \ell$$

and $\Gamma(f_\phi)$ is convex, this proves that

$$\epsilon_j(\psi) \geq \epsilon_j(\phi) \quad \text{for} \quad j = k+1, ..., \ell+1$$

and the assertion. Thus we have

$$\gamma_{j^*}(\psi) = \sum_{j=j^*+1}^{r} \delta_j(\psi) + \tilde{n} = \sum_{j=j^*+1}^{r} \delta_j(\phi) + \tilde{n} = \gamma_{j^*}(\phi).$$

Since $\sigma(\psi) \geq \epsilon_{j^*+2}(\phi)$, applying Lemma 5.1.2 again, we get

$$\epsilon_j(\psi) \geq \epsilon_j(\phi) \quad \text{for} \quad j \geq j^*+2.$$

Thus we conclude that

$$\epsilon_i(\psi) \geq \epsilon_{j^*+2}(\psi) \geq \epsilon_{j^*+2}(\phi) = \sigma(\psi) \quad \text{for} \quad 0 \leq i \leq j^*+2.$$

We now apply the same argument to prove (5.1.2) (note that we do not use $\sigma(\phi) > \epsilon_{\ell+1}(\phi)$ to prove (5.1.2)). We conclude that

$$\gamma_j(\psi) \geq \beta_{j+1}(\psi) \quad \text{for} \quad 0 \leq j \leq j^*+1.$$

On the other hand one has

$$\gamma_{j^*}(\phi) = \gamma_{j^*}(\psi) \geq \beta_{j^*+1}(\psi) \geq \beta_{j^*+1}(\phi)$$

where the last inequality follows from

$$\beta_{j^*+1}(\psi) = \sum_{j=j^*+2}^{r} \epsilon_j(\psi) + n \geq \sum_{j=j^*+2}^{r} \epsilon_j(\phi) + n = \beta_{j^*+1}(\phi).$$

Thus we have a contradiction. q.e.d.

When $\sigma(\phi) > \epsilon_1(\phi)$ we repeat the same arguments. We show $\gamma_j(\phi) \geq \beta_{j+1}(\phi)$ for $0 \leq j \leq r-2$. Suppose that there were $0 \leq j \leq r-2$ such that $\gamma_j(\phi) < \beta_{j+1}(\phi)$. Set

$$j^* = \max\{j \mid \gamma_j(\phi) < \beta_{j+2}(\phi)\}.$$

Then by definition we have $\sigma(\phi) > \epsilon_{j^*+2}(\phi) > \delta_{j^*+2}(\phi)$. Take $\psi \in \mathcal{G}^+(\gamma)$ such that $\sigma(\psi) = \epsilon_{j^*+2}(\phi)$ and hence

$$\sigma(\psi - \phi) = \sigma(\psi), \quad \sigma(\psi) > \delta_{j^*+1}(\phi).$$

Then by Lemma 5.1.1 we see that $\delta_{j+1}(\phi) = \delta_{j+1}(\psi)$ for $j \geq j^*$. Hence one has

$$\gamma_{j^*}(\psi) = \sum_{j=j^*+1}^{r} \delta_j(\psi) + \tilde{n} = \sum_{j=j^*+1}^{r} \delta_j(\phi) + \tilde{n} = \gamma_{j^*}(\phi).$$

Note that $\sigma(\psi) = \epsilon_{j^*+2}(\phi) \geq \cdots$, we apply Lemma 5.1.1 to get

$$\epsilon_j(\psi) \geq \epsilon_j(\phi) \quad \text{for} \quad j^* + 2 \leq j \leq r$$

and then

$$\beta_{j^*+1}(\psi) = \sum_{j=j^*+2}^{r} \epsilon_j(\psi) + n \geq \sum_{j=j^*+2}^{r} \epsilon_j(\phi) + n = \beta_{j^*+1}(\phi).$$

Since $\epsilon_i(\psi) \geq \epsilon_{j^*+2}(\psi) \geq \epsilon_{j^*+2}(\phi) = \sigma(\psi)$ for $0 \leq i \leq j^* + 2$, the same arguments as before give that

$$\gamma_j(\psi) \geq \beta_{j+1}(\psi) \quad \text{for} \quad 0 \leq j \leq j^* + 1.$$

This clearly gives a contradiction because

$$\gamma_{j^*}(\phi) = \gamma_{j^*}(\psi) \geq \beta_{j^*+1}(\psi) \geq \beta_{j^*+1}(\phi).$$

When $\epsilon_r(\phi) \geq \sigma(\phi)$ taking the line given by

$$tp_j + xq_j = 1 \quad \text{with} \quad \frac{p_j}{q_j} = \sigma_j(\phi) \geq \sigma(\phi) \quad (0 \leq j \leq r-2)$$

one can conclude that

$$\gamma_j(\phi) \geq \beta_{j+1}(\phi) \quad \text{for} \quad 0 \leq j \leq r-2$$

and hence the result. q.e.d.

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