Effectively Hyperbolic Cauchy Problem (Lectures at De Giorgi Center)

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Contents

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1 Cauchy problem and multiple characteristics

1.1 Multiple characteristics

Let P be a differential operator of order m on $\Omega \subset \mathbf{R}^n$:

$$P(x,D) = D_1^m + \sum_{|\alpha|\le m, \alpha_1<m} a_\alpha(x) D^\alpha = \sum_{j=0}^{m} P_j(x,D)$$

where $P_j(x,D)$ denotes the homogeneous part of degree j and $x = (x_1, x') = (x_1, x_2, ..., x_n)$, $D = (D_1, D') = (D_1, D_2, ..., D_n)$,

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

Let us denote $p(x,D) = P_m(x,D)$ and hence

$$p(x,\xi) = \xi_1^m + \sum_{|\alpha|=m, \alpha_1<m} a_\alpha(x)\xi^\alpha$$

is the principal symbol of $P(x,D)$. Assume that $p(x,\xi)$ is hyperbolic with respect to $\theta = (1,0,...,0)$, that is

$$p(x, \xi - i\theta) \neq 0, \quad \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^n. \tag{1.1.1}$$

Definition 1.1.1: We say that the Cauchy problem for P is well posed near 0 if there exist $\epsilon > 0$ and a neighborhood ω of the origin such that: for every τ,

$|\tau| \leq \epsilon$ and every $f(x) \in C_0^\infty(\omega)$ vanishing in $x_1 < \tau$ there is a unique solution $u(x) \in C^\infty(\omega)$ to $Pu = f$ in ω vanishing in $x_1 < \tau$.

Recall that $z^0 \in T^*\Omega \setminus \{0\}$ is called a characteristic if

$$p(z^0) = 0.$$

We say that z^0 is a characteristic of order r if

$$d^j p(z^0) = 0, \quad j < r, \quad d^r p(z^0) \neq 0.$$

If every characteristic is simple, that is $dp(z^0) \neq 0$ then p is strictly hyperbolic. As far as the wellposedness is concerned, the strictly hyperbolic Cauchy problem is well studied. We are concerned with multiple characteristics. We first recall a general necessary condition for the Cauchy problem to be well posed:

Theorem 1.1.1 ([7]) *Let $z^0 = (0, \xi^0) \in T_0(T^*\Omega) \setminus \{0\}$ be a characteristic of order r of p. In order that the Cauchy problem for P is C^∞ well posed near 0 it is necessary that $P_{m-j}(x, \xi)$ vanishes of ordeer $m - 2j$ at z^0.*

Corollary 1.1.1 *If $p(x, D)$ is strongly hyperbolic near 0, that is the Cauchy problem for P is C^∞ well posed for any lower order term $P_j(x, D)$, $j < m$ then every multiple characteristic is at most double.*

Recall that $\sigma = d\xi \wedge dx$ is the canonical 2-form on $T^*\Omega$. Let $f \in C^\infty(\Omega)$ then the Hamilton vector field H_f is defined by

$$df(\cdot) = \sigma(\cdot, H_f)$$

and hence

$$H_f = \frac{\partial f}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial \xi}$$

in the canonical coordinates (x, ξ). By the definition a bicharacteristic of p is a null integral curve of the Hamilton system:

$$\begin{cases} \dot{x} = \frac{\partial p}{\partial \xi}(x, \xi) \\ \dot{\xi} = -\frac{\partial p}{\partial x}(x, \xi) \end{cases} \tag{1.1.2}$$

Let z^0 be a multiple characteristic, say a characteristic of order r. The localization of p at z^0 is defined by

$$p(z^0 + \mu z) = \mu^r (p_{z^0}(z) + o(1)), \quad \mu \to 0. \tag{1.1.3}$$

The localization $p_{z^0}(z)$ is a homogeneous polynomial of degree r on $T_{z^0}(T^*\Omega)$ which is hyperbolic with respect to $(0, \theta)$ (see §§8.1). Thus we can define the hyperbolic cone $\Gamma_{z^0} = \Gamma(p_{z^0}, (0, \theta))$ of p_{z^0} as the connected component of the set

$$\{z \in T_{z^0}(T^*\Omega) \mid p_{z^0}(z) \neq 0\}$$

containing $(0,\theta)$ (see §§8.1). Let $C_{z^0} = C(p_{z^0}, (0,\theta))$ be the dual cone of Γ_{z^0} with respect to the symplectic 2-form $\sigma = d\xi \wedge dx$:

$$C_{z^0} = \{X \in T_{z^0}(T^*\Omega) \mid \sigma(X,Y) \le 0, \ \ \forall Y \in \Gamma_{z^0}\}.$$

C_{z^0} is the "minimal" cone containing all bicharacteristics with limit point z^0. More precisely

Lemma 1.1.1 *Let $z^0 \in T^*\Omega \setminus \{0\}$ be a multiple characteristic of p. Assume that there are simple characteristics z_j and positive numbers λ_j such that*

$$z_j \to z^0, \quad \lambda_j H_p(z_j) \to X(\neq 0), \quad j \to \infty.$$

Then $X \in C_{z^0}$.

The proof is based on the following "semi-continuity" of the hyperbolic cones Γ_z: (Theorem 8.2.1)

Lemma 1.1.2 *Let z^0 be a multiple characteristic of p. Let $K \subset \Gamma_{z^0}$ be a compact set. Then there is a $\delta > 0$ such that we have*

$$K \subset \Gamma_{z^0} \quad \textit{for} \quad |z - z^0| < \delta.$$

Proof of Lemma 1.1.1: Let K be any compact set in Γ_{z^0}. Then for sufficiently large j we have $K \subset \Gamma_{z_j}$ thnaks to Lemma 1.1.2. Since

$$dp(z_j; Y) = \sigma(Y, H_p(z_j)) > 0$$

for any $Y \in K$ it follows that $\sigma(Y,X) \ge 0$. Hence $X \in C_{z^0}$.

Definition 1.1.2: Let $t(x,\xi)$ be homogeneous of degree 0 in ξ and smooth near z^0. We say that $t(x,\xi)$ is a time function for p with respect to Γ_{z^0} at z^0 if $t(z^0) = 0$ and

$$-H_t(z^0) \in \Gamma_{z^0}. \tag{1.1.4}$$

Note that $t(x,\xi)$ is a time function at z^0 for p if and only if

$$C_{z^0} \cap T_{z^0}\{t(z) = 0\} = \{0\}.$$

We define the linearity of p_{z^0} by

$$\Lambda_{z^0} = \Lambda(p_{z^0}) = \{X \in T_{z^0}(T^*\Omega) \mid p_{z^0}(tX + Y) = p_{z^0}(Y), \ \forall t \in \mathbf{R}, \forall Y\}.$$

For a linear subspace $W \subset T_{z^0}(T^*\Omega)$ we denote $W^\sigma = \{X \in T_{z^0}(T^*\Omega) \mid \sigma(X,Y) = 0, \forall Y \in W\}$. Then we have

Lemma 1.1.3 ([13]) *The following four conditions are equivalent:*

(i) $C_{z^0} \cap \Lambda_{z^0} = \{0\}$

(ii) *there is a subspace* H *of codimension* 1 *such that* $H \cap C_{z^0} = \{0\}$, $\Lambda_{z^0} + \langle (0,\theta)\rangle \subset H$

(iii) $\Gamma_{z^0} \cap \Lambda^\sigma_{z^0} \cap \langle (0,\theta)\rangle^\sigma \neq \emptyset$

(iv) $\Gamma_{z^0} \cap \Lambda^\sigma_{z^0} \neq \emptyset$.

Proof: We show (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$ (i). (i)$\Longrightarrow$(ii). We first assume $\Theta = (0,\theta) \in \Lambda + \Lambda^\sigma$ so that $\Theta = X_1 + X_2$, $X_1 \in \Lambda$, $X_2 \in \Lambda^\sigma$. Since $\Gamma + \Lambda \subset \Gamma$ and $\Gamma \cap \Lambda = \emptyset$ it follows that $0 \neq X_2 \in \Gamma$ and $X_2 \neq 0$. It is clear that $\sigma(X_2, \Theta) = 0$ and hence $\Theta \in \langle X_2\rangle^\sigma$. Noting that $X_2 \in \Lambda^\sigma$, $X_2 \in \Gamma$ we get $\Lambda \subset \langle X_2\rangle^\sigma$ and $\langle X_2\rangle^\sigma \cap C = \{0\}$ for Γ is open. Then Then $\langle X_2\rangle^\sigma$ is a desired subspace. Consider the case $\Theta \notin \Lambda + \Lambda^\sigma$ and hence $(\Lambda + \Lambda^\sigma) \cap \langle \Theta\rangle = \{0\}$. We show that $\Gamma \cap \Lambda^\sigma \neq \emptyset$. Indeed if $\Gamma \cap \Lambda^\sigma = \emptyset$ then there is $0 \neq Y \in T_{z^0}(T^*\Omega)$ such that $\sigma(Y, X) \leq 0$ for any $X \in \Gamma$ and $\sigma(Y, X) \geq 0$ for any $X \in \Lambda^\sigma$. These imply that $Y \in C$ and $Y \in \Lambda$ which is a contradiction. Thus we can take $0 \neq Z \in \Gamma \cap \Lambda^\sigma$. Note that

$$\Lambda \subset \langle Z\rangle^\sigma, \quad \langle Z\rangle^\sigma \cap C = \{0\}. \tag{1.1.5}$$

Setting $T = \langle Z\rangle^\sigma \cap (\Lambda + \Lambda^\sigma)$ we have

$$\Lambda \subset T, \quad T \cap C = \{0\}. \tag{1.1.6}$$

We examine that $\dim T = \dim(\Lambda + \Lambda^\sigma) - 1$. Indeed from $\Gamma + \Lambda \subset \Gamma$ it follows that

$$C \subset \Lambda^\sigma. \tag{1.1.7}$$

From (1.1.5) and (1.1.7) it follows that $\Lambda + \Lambda^\sigma \not\subset \langle Z\rangle^\sigma$ and this shows that the desired assertion. Take a subspace $V \subset T_{z^0}(T^*\Omega)$ so that $T_{z^0}(T^*\Omega) = (\Lambda + \Lambda^\sigma) \oplus V$ and write $\Theta = Y_1 + Y_2$, $Y_1 \in \Lambda + \Lambda^\sigma$, $0 \neq Y_2 \in V$. Again we take a subspace $W \subset T_{z^0}(T^*\Omega)$ so that

$$V = \langle Y_2\rangle \oplus W.$$

Note that $\dim T = \dim(\Lambda + \Lambda^\sigma) - 1$ implies that $H = T + \langle\Theta\rangle + W$ is a subspace of codimension 1. Then this is a desired one. In fact we have $H \cap C = \{0\}$ by (1.1.6) and (1.1.7). On the other hand it is obvious that $\Lambda + \langle\Theta\rangle \subset H$. (ii)$\Longrightarrow$(iii): Take $0 \neq Y \in T_{z^0}(T^*\Omega)$ so that $\langle Y\rangle = H^\sigma$. Then it is clear that $\langle Y\rangle \subset \Lambda^\sigma \cap \langle\Theta\rangle^\sigma$. We show that Y or $-Y$ belongs to Γ. If not we would have $\langle Y\rangle \cap \Gamma = \emptyset$. Then there is $0 \neq Z \in T_{z^0}(T^*\Omega)$ such that $\sigma(Z, X) \leq 0$ for any $X \in \Gamma$ and $\sigma(Z, X) \geq 0$ for any $X \in \langle Y\rangle$. This shows that $Z \in C$ and $Z \in \langle Y\rangle^\sigma = H$ and contradicts (ii). (iii)$\Longrightarrow$(iv): trivial. (iv)$\Longrightarrow$(i): Suppose $0 \neq Y \in \Gamma \cap \Lambda^\sigma$. Then it is clear that $\Lambda \subset \langle Y\rangle^\sigma$ and $\langle Y\rangle^\sigma \cap C = \{0\}$ because Γ is open. This implies ovbiously $C \cap \Lambda = \{0\}$.

Let $z^0 = (x^0, \xi^0)$ be a characteristic of order r of p. Assume that $P(x, \xi)$ verifies the Ivrii-Petkov necessary condition (Theorem 1.1.1). We define P_{z^0} as the non trivial lowest order term of the expansion (cf. Definition 8.1.3)

$$\mu^{2m} P(x^0 + \mu x, \mu^{-2}(\xi^0 + \mu\xi)) = \mu^r\big[P_{z^0}(x, \xi) + o(1)\big], \ \ \mu \to 0$$

which is invariantly defined as a polynomial on $T_{z^0}(T^*\Omega)$.

Theorem 1.1.2 ([14]) *Assume that the Cauchy problem for P is C^∞ well posed near 0 and let $z^0 \in T^*\Omega \setminus \{0\}$ be a multiple characteristic of p. Then the localization $P_{z^0}(z)$ is a hyperbolic polynomial with respect to $(0,\theta)$ (see Definition 8.1.1): there is $T > 0$ such that with $\Theta = (0,\theta)$*

$$z \in \mathbf{R}^{2n},\ \tau \in \mathbf{C},\ \ P_{z^0}(z + \tau\Theta) = 0 \Longrightarrow |\mathrm{Im}\tau| < T.$$

1.2 Effective hyperbolicity

Let z^0 be a double characteristic of p and let Q be the Hessian of $p/2$ at z^0. Since z^0 is a singular (stationary) point of the Hamilton system (1.1.2), to study the behavior of bicharacteristics it is natural to consider the linearization of the system at z^0. We consider

$$dp(z^0 + \mu X; Y) = \sigma(Y, H_p(z^0 + \mu X))$$

which gives

$$Q(X,Y) + O(\mu) = \sigma(Y, F_p(z^0)X) + O(\mu), \quad \mu \to 0.$$

We say that $F_p(z^0)$ is the Hamilton map of p at z^0. In local coordinates (x,ξ), $F_p(z^0)$ is given by

$$F_p(z^0) = \begin{pmatrix} \frac{\partial^2 p}{\partial\xi\partial x}(z^0) & \frac{\partial^2 p}{\partial\xi\partial\xi}(z^0) \\ -\frac{\partial^2 p}{\partial x\partial x}(z^0) & -\frac{\partial^2 p}{\partial x\partial\xi}(z^0) \end{pmatrix}. \tag{1.2.1}$$

Lemma 1.2.1 *All eigenvalues of $F_p(z^0)$ are pure imaginary possibly except for a pair of non zero real eigenvalues $\pm\lambda$ ($\lambda \in \mathbf{R} \setminus \{0\}$).*

DEFINITION 1.2.1: We say that p is effectively hyperbolic at z^0 if $F_p(z^0)$ has a non zero real eigenvalue.

Theorem 1.2.1 ([7]) *Let z^0 be a double characteristic. If the Cauchy problem is C^∞ well posed for any lower order term then p is effectively hyperbolic at z^0.*

Lemma 1.2.2 *The following conditions are equivalent:*

(i) $F_p(z^0)$ *has a real non-zero eigenvalue,*

(ii) $(\mathrm{Ker}F_p(z^0))^\sigma \cap \Gamma_{z^0}(p_{z^0}, \Theta) \neq \emptyset$.

Since $\mathrm{KerHess}p(z^0) = \Lambda(p_{z^0})$ for a double characteristic z^0 we see that

Lemma 1.2.3 *The following conditions are equivalent:*

(i) *p is effectively hyperbolic at z^0,*

(ii) $C(p_{z^0}, \Theta) \cap \mathrm{KerHess}p(z^0) = \{0\}$,

(iii) $\Gamma(p_{z^0}, \Theta) \cap (\mathrm{KerHess}p(z^0))^\sigma \cap \langle\Theta\rangle^\sigma \neq \emptyset$.

We analyze implications of (iii) of Lemma 1.2.3 for second order operator P:

$$P(x, D) = D_1^2 + 2A_1(x, D')D_1 + A_2(x, D'). \tag{1.2.2}$$

Writing the principal symbol $p(x, \xi)$ in the following form

$$p(x, \xi) = (\xi_1 - a(x, \xi'))^2 - q(x, \xi') \tag{1.2.3}$$

we see that the hypotheses on hyperbolicity reduces to that $a(x, \xi')$ is real valued and $q(x, \xi') \geq 0$. Moreover $z^0 = (x^0, \xi^0)$ is a double characteristic if and only if $q(x^0, \xi^{0\prime}) = 0$ and $\xi_1^0 = a(x^0, \xi^{0\prime})$.

Lemma 1.2.4 ([13]) *Assume that p is effectively hyperbolic at z^0. Then there is a time function $t(x, \xi')$ at z^0 with rspect to Γ_{z^0} satisfying*

$$q(x, \xi') \geq ct(x, \xi')^2|\xi'|^2 \tag{1.2.4}$$

near z^0 with a positive constant c.

Proof: Let us write $\rho = (x^0, \xi^0)$, $\rho' = (x^0, \xi^{0\prime})$. Recall that $p(x, \xi)$ has the form (1.2.3). Since $a(x, \xi')$ is non negative near ρ' the Morse lemma shows that there are functions $b_j(x, \xi')$ $(1 \leq j \leq \nu)$, homogeneous of degree 1 in ξ', C^∞ in a conic neighborhood of $z^{0\prime}$ such that

$$q(x, \xi') \geq \sum_{j=1}^{\nu} b_j(x, \xi')^2 \quad \text{near } \rho' \tag{1.2.5}$$

$$q_{\rho'}(x, \xi') = \sum_{j=1}^{\nu} db_j(x, \xi')^2$$

where $q_{\rho'}(x, \xi')$ is the quadratic part of the Taylor expansion of q at ρ'. Hence we have

$$(\mathrm{KerHess}p(\rho))^\sigma = \langle H_{\xi_1 - a}(\rho), H_{b_j}(\rho') \mid 1 \leq j \leq \nu\rangle.$$

Taking $0 \neq X \in (\mathrm{KerHess}p(\rho))^\sigma \cap \Gamma(p_\rho, \Theta) \cap \langle\Theta\rangle^\sigma$ we have with real constants α_j that

$$X = \sum_{j=1}^{\nu} \alpha_j H_{b_j}(\rho') + \alpha_0 H_{\xi_1 - a}(\rho).$$

Since $\Theta = H_{x_1}$ and $X \in \langle\Theta\rangle^\sigma$ it follows tht $\alpha_0 = 0$. We set

$$t(x, \xi') = -\sum_{j=1}^{\nu} \alpha_j \frac{b_j(x, \xi')}{|\xi'|}$$

so that $-H_t(\rho') = X \in \Gamma_{z^0}$. (1.2.4) follows from (1.2.5).

Lemma 1.2.5 *Assume that the conclusion of Lemma 1.2.4 holds. Then p is effectively hyperbolic at z^0.*

Proof: Since the doubly characteristic set of p is contained in the set $q^{-1}(0)$, it follows that $\mathrm{KerHess}p(z^0) \subset T_{z^0}\{t(x,\xi') = 0\}$. On the other hand, since $t(x,\xi')$ is a time function at z^0 with respect to Γ_{z^0} we have

$$C_{z^0} \cap \mathrm{KerHess}p(z^0) = \{0\}$$

and hence p is effectively hyperbolic at z^0 by Lemma 1.2.3.

We return to the operator $P(x,D)$ of order m. Let us define the symbols $h_j(x,\xi)$, $j = 0,1,...,m$ by

$$|p(x,\xi - is\theta)|^2 = \sum_{j=1}^{m} s^{2(m-j)} h_j(x,\xi). \tag{1.2.6}$$

Factorizing $p(x,\xi) = \prod_{j=1}^{m} q_j(x,\xi)$, $q_j(x,\xi) = \xi_1 - \lambda_j(x,\xi')$ where $\lambda_j(x,\xi')$ are homogeneous of degree 1 in ξ', real valued (not necessarily smooth) we have

$$h_j(x,\xi) = \sum_{1\le \ell_1<\ell_2<\cdots \ell_j\le m} |q_{\ell_1}(x,\xi)|^2 \cdots |q_{\ell_j}(x,\xi)|^2 \tag{1.2.7}$$

and $h_0(x,\xi) = 1$, $h_m(x,\xi) = |p(x,\xi)|^2$. We now have

Lemma 1.2.6 *Assume that z^0 is a double characteristic and p is effectively hyperbolic at z^0. Then there is a time function $t(x,\xi')$ at z^0 with respect to Γ_{z^0} satisfying*

$$h_{m-1}(x,\xi) \ge c h_{m-2}(x,\xi) t(x,\xi')^2 |\xi'|^2 \tag{1.2.8}$$

near z^0 with a positive constant c.

Proof: We first note that $p(x,\xi)$ is factorized as

$$p(x,\xi) = p_{m-2}(x,\xi) p_2(x,\xi)$$

in a conic neighborhood of $z^0 = (x^0,\xi^{0\prime})$ where $p_{m-2}(x,\xi)$, $p_2(x,\xi)$ are hyperbolic with respect to θ, that is (1.1.1) is verified in a conic neighborhood of $(x^0,\xi^{0\prime})$ and of homogeneous of degree $m-2$ and 2 respectively. Moreover $p_{m-2}(z^0) \neq 0$ and $p_2(z^0) = 0$, $dp_2(z^0) = 0$. It is easy to see that p_2 is effectively hyperbolic at z^0 and by Lemma 1.2.4 there is a time function $t(x,\xi')$ such that

$$q_2(x,\xi') \ge c t(x,\xi')^2 |\xi'|^2$$

where $p_2 = (\xi_1 - a_1)^2 - q_2$. Since $|p_2(x,\xi - is\theta)|^2 = s^4 + 2s^2[q_2(x,\xi') + (\xi_1 - a_1)^2] + |p_2(x,\xi)|^2$ it is clear from (1.2.7) that

$$\begin{aligned} h_{m-1}(x,\xi) &\ge |p_{m-2}(x,\xi)|^2 [q_2(x,\xi') + (\xi' - a_1)^2] \\ &\ge c t(x,\xi')^2 h_{m-2}(x,\xi) |\xi'|^2 \end{aligned}$$

near z^0 because $p_{m-2}(z^0) \neq 0$.

1.3 Generalized flows

The next definition of generalized characteristics is found in [11].

Definition 1.3.1: A Lipschitz continuous curve $z(t) : [a, b] \to T^*\mathbf{R}^n$ is called a generalized bicharacteristic if

$$\dot{z}(t) \in C_{z(t)} \setminus \{0\}, \quad a.e.\ t.$$

Recall that we denote by $\Gamma(p(x, \cdot), \theta)$ the hyperbolic cone of $p(x, \cdot)$ with respect to

$$\theta = (1, 0, ..., 0).$$

For a subset $\Gamma \subset \mathbf{R}^n$ we denote

$$\Gamma^* = \{x \in \mathbf{R}^n \mid \langle x, y\rangle \geq 0, \forall y \in \Gamma\}.$$

Definition 1.3.2: A Lipschitz continuous curve $x(t) : [a, b] \ni t \mapsto x(t) \in \mathbf{R}^n$ is called a generalized characteristic curve if

$$\dot{x}(t) \in \Gamma(p(x(t), \cdot), \theta)^* \setminus \{0\}, \text{ a.e. } t.$$

For any $\xi \in \mathbf{R}^n$ we see that (Lemma 8.1.4)

$$\{0\} \times \Gamma(p(x, \cdot), \theta) \subset \{0\} \times \Gamma(p_\xi(x, \cdot), \theta) \subset \Gamma_{(x,\xi)}.$$

This shows that

$$C_{(x,\xi)} \subset \Gamma(p(x, \cdot), \theta)^* \times \mathbf{R}^n, \ \forall \xi \in \mathbf{R}^n.$$

It is clear that $\Gamma_{(x,0)} = \mathbf{R}^n \times \Gamma(p(x, \cdot), \theta)$ and hence

$$C_{(x,0)} = \Gamma(p(x, \cdot), \theta)^* \times \{0\}.$$

We note that

$$\Gamma_{(x,\lambda\xi)} = \lambda\Gamma_{(x,\xi)} = \{(y, \lambda\eta) \mid (y, \eta) \in \Gamma_{(x,\xi)}\}$$

for $\lambda > 0$. It is also clear that

$$C_{(x,\lambda\xi)} = \lambda C_{(x,\xi)}. \tag{1.3.1}$$

Lemma 1.3.1 *Let $[0, T] \ni t \mapsto z(t)$ be a generalized bicharacteristic with $z(0) = (x^0, 0)$. Then we have $z(t) = (x(t), 0)$ for all $t \in [0, T]$.*

Proof: Let us write $z(t) = (x(t), \xi(t))$ and set $t^* = \sup\{t \mid \xi(t) = 0\}$. Let K be a compact neighborhood of $x(t^*)$. From the semi-continuity of the hyperbolic cone there is a compact set $A \subset T^*\mathbf{R}^n$ such that

$$x \in K, \ |\xi| = 1 \Longrightarrow A \subset \Gamma_{(x,\xi)}, \ \ (0, \theta) \in A^\circ.$$

Hence we have for $x \in K$, $|\xi| = 1$

$$C_{(x,\xi)} \subset A^{\sigma} \subset \{(y,\eta) \mid |y'|, |\eta| \le C y_1\}$$

with some $C > 0$. If t is close to t^* then $x(t) \in K$ and $\dot{z}(t) \in C_{z(t)}$ and then it follows that

$$(\dot{x}(t), \dot{\xi}(t)/|\xi(t)|) \in C_{(x(t),\xi(t)/|\xi(t)|)} \subset A^{\sigma}$$

and hence

$$|\dot{\xi}(t)|/|\xi(t)| \le C x_1(t)$$

which gives $|\xi(t)| \le |\xi(t^*)| e^{x_1(t) - x_1(t^*)}$. This proves that $\xi(t) = 0$ if $|t - t^*|$ is enough small, This shows that $t^* = T$.

Let us write

$$p(x,\xi) = \prod_{j=1}^{m} (\xi_1 - \lambda_j(x,\xi'))$$

where $\lambda_1(x,\xi') \le \cdots \le \lambda_m(x,\xi')$. Then

Proposition 1.3.1 ([1]) *Assume that $p(x,\xi)$ is hyperbolic with respect to θ. Then $\lambda_j(x,\xi')$ are Lipschitz continuous.*

A simpler proof is found in [16].

1.4 Generalized characteristic curves and the Hamilton-Jacobi equation

In the last two subsections we follow J-L.Joly, G.Métivier and J.Rauch [8] and write $x_1 = t$, $x = (x_2, ..., x_n) \in \mathbf{R}^{n-1}$ which is more conveinient. To simplify notations we denote

$$\Gamma(t,x) = \Gamma(p(t,x;\cdot), \theta), \quad \Gamma^*(t,x) = \Gamma(p(t,x;\cdot),\theta)^*.$$

Recall that (see §§8.1)

$$\Gamma(t,x) = \{(\tau,\xi) \mid \tau > \lambda_{\max}(t,x,\xi)\} \tag{1.4.1}$$

where $\lambda_{\max}(t,x,\xi) = \max_j \lambda_j(t,x,\xi)$ and

$$\Gamma^*(t,x) = \{(s,y) \mid s \ge 0, s\lambda_{\max}(t,x,\xi) + \langle y, \xi \rangle \ge 0, \forall \xi\}.$$

Let us introduce

$$\Gamma_1^*(t,x) = \Gamma^*(t,x) \cap \{t = 1\}$$

which is a convex compact set in $\mathbf{R}^{n-1}$. Note that if $[a,b] \ni s \mapsto (t(s), x(s))$ is a generalized characteristic curve (abreviated as g.c.curve) then $\dot{t}(s) > 0$ and hence we can take t as a parameter. Now $[a,b] \ni t \mapsto (t, x(t))$ is a g.c.curve if and only if

$$\dot{x}(t) \in \Gamma_1^*(t, x(t)), \quad \text{a.e. } t \in [a,b]. \tag{1.4.2}$$

We call $x(t)$ verifying (1.4.2) a g.c.curve again.

Lemma 1.4.1 *We have*

$$\lambda_{\max}(t,x,-\xi) = \max_{y\in\Gamma_1^*(t,x)} \langle y,\xi\rangle.$$

Proof: We first note that

$$\begin{aligned}\Gamma(t,x) &= \{(\tau,\xi) \mid \langle(\Gamma^*(t,x)\setminus\{0\}),(\tau,\xi)\rangle > 0\}\\ &= \{(\tau,\xi) \mid \tau + \langle\Gamma_1^*(t,x),\xi\rangle > 0\}.\end{aligned}$$

This shows that

$$(\tau,\xi)\in\Gamma(t,x) \Longleftrightarrow \tau + \min_{y\in\Gamma_1^*(t,x)}\langle y,\xi\rangle > 0 \tag{1.4.3}$$

which proves the assertion.

Let us denote

$$\mathcal{C}(T,X) = \{\text{g.c.curve } \ x(t) : [0,T]\to\mathbf{R}^{n-1} \ \text{ with } \ x(T)=X\}.$$

Let $\psi(x)$ be uniformly Lipschitz continuous on $\mathbf{R}^{n-1}$ with $\|\psi\|_{L^\infty} < \infty$ and define

$$\Psi(T,X) = \inf\{\psi(x(0)) \mid x(\cdot)\in\mathcal{C}(T,X)\}. \tag{1.4.4}$$

Lemma 1.4.2 *In (1.4.4), the infimum is attained.*

Proof: Let $x^n(\cdot)\in\mathcal{C}(T,X)$ be such that $\psi(x^n(0))\to\Psi(T,X)$. From the Ascoli-Arzela theorem there is a subsequence which converges to $x(\cdot)$ in $C([0,T])$. It is clear that $x(t)$ is Lipschitian. We show that $\dot{x}(t)\in\Gamma_1^*(t,x(t))$ a.e. $t\in[0,T]$. Let $\hat{t}\in[0,T]$ be fixed. We may assume that $x^n(t)\to x(t)$ in $C([0,T])$ and hence

$$\frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} \dot{x}^n(s)ds \to \frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} \dot{x}(s)ds.$$

Consider the Riemann sum of the above integral:

$$\frac{1}{\epsilon}\sum \dot{x}^n(s_i)(s_{i+1}-s_i).$$

Let $|x^n(t)-x(t)| \le \epsilon_1$ with small $\epsilon_1 > 0$ where $\epsilon_1\to 0$ as $n\to\infty$. From Proposition 1.3.1 there is $v_i\in\Gamma_1^*(\hat{t},x(\hat{t}))$ such that

$$|\dot{x}^n(s_i)-v_i| \le C\{|s_i-\hat{t}| + |x^n(s_i)-x(\hat{t})|\} \le C'|s_i-\hat{t}| + \epsilon_1.$$

Since $\Gamma_1^*(\hat{t},x(\hat{t}))$ is compact convex we have $\sum v_i(s_{i+1}-s_i)/\epsilon\in\Gamma_1^*(\hat{t},x(\hat{t}))$ and hence we have

$$\Big|\frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} \dot{x}^n(s)ds - v\Big| \le C\epsilon + \epsilon_1 \tag{1.4.5}$$

with $v \in \Gamma_1^*(\hat{t}, x(\hat{t}))$. Letting $n \to \infty$ one gets

$$\Big| \frac{1}{\epsilon} \int_{\hat{t}-\epsilon}^{\hat{t}} \dot{x}(s) ds - v \Big| \le C\epsilon.$$

Since ϵ is arbitrary this proves that $\dot{x}(\hat{t}) \in \Gamma_1^*(\hat{t}, x(\hat{t}))$.

Lemma 1.4.3 *We have*

(i) *Let $T > 0$ and $x(t)$ be a g.c.curve with $\Psi(T, x(T)) = \psi(x(0))$. Then we have*

$$\Psi(t, x(t)) = \psi(x(0)), \quad \forall t \in [0, T].$$

(ii) *For any $t \in [0, T]$ we have*

$$\Psi(T, X) = \min_{x(\cdot) \in \mathcal{C}(T,X)} \Psi(t, x(t)).$$

Proof: (i) Easy by the definition. (ii) Let $x(\cdot) \in \mathcal{C}(T, X)$ and $\hat{t} \in [0, T]$. Take $y(\cdot) \in \mathcal{C}(\hat{t}, x(\hat{t}))$ such that $\Psi(\hat{t}, x(\hat{t})) = \psi(y(0))$. Then $\Psi(\hat{t}, x(\hat{t})) \ge \Psi(T, X)$ because $z(t) \in \mathcal{C}(T, X)$ where $z(t)$ is given by

$$z(t) = y(t), \quad 0 \le t \le \hat{t}, \quad z(t) = x(t), \quad \hat{t} \le t \le T.$$

On the other hand take $x(t) \in \mathcal{C}(T, X)$ such that $\Psi(T, X) = \psi(x(0))$. Then $\Psi(T, X) \ge \Psi(\hat{t}, x(\hat{t}))$ because $x(t) : [0, \hat{t}] \ni t \mapsto x(t)$ is in $\mathcal{C}(\hat{t}, x(\hat{t}))$. This proves the assertion.

Lemma 1.4.4 *Let $\bar{x}(\cdot) \in \mathcal{C}(T, X_1)$. Then there are C and $\delta > 0$ such that for any X_2 with $|X_1 - X_2| < \delta$ there exists $x(\cdot) \in \mathcal{C}(T, X_2)$ such that*

$$|x(0) - \bar{x}(0)| \le C|X_1 - X_2|.$$

Proof: From the proof of Lemma 1.4.2 we see that there exist $\bar{v}_k \in \Gamma_1^*(Tk/n, \bar{x}(Tk/n))$ such that

$$\int_{\frac{T}{n}(k-1)}^{\frac{T}{n}k} \frac{d}{ds}\bar{x}(s) ds = \bar{x}(\frac{T}{n}k) - \bar{x}(\frac{T}{n}(k-1)) = \frac{T}{n}\bar{v}_k + R_k, \ \ |R_k| \le C\big(\frac{T}{n}\big)^2.$$

Set $x^n(t) = X_2 - v_n(T - t)$ for $T(n-1)/n \le t \le T$ where $v_n \in \Gamma_1^*(T, X_2)$ with $|v_n - \bar{v}_n| \le C|X_1 - X_2|$. Define $x^n(t)$ for $T(j-1)/n \le t \le Tj/n$ by

$$x^n(t) = x^n(\frac{T}{n}j) - v_j(\frac{T}{n}j - t)$$

where $v_j \in \Gamma_1^*(Tj/n, x^n(Tj/n))$ which verifies

$$|v_j - \bar{v}_j| \leq C|x^n(\frac{T}{n}j) - \bar{x}(\frac{T}{n}j)|.$$

Then it follows that

$$\begin{aligned}|x^n(0) - \bar{x}(0)| &\leq (1 + C\frac{T}{n})^n |X_1 - X_2| + C(\frac{T}{n})^2 \sum_{k=0}^{n-1} (1 + C\frac{T}{n})^k \\ &\leq (1 + C\frac{T}{n})^n |X_1 - X_2| + \frac{T}{n}\big[(1 + C\frac{T}{n})^n - 1\big].\end{aligned}$$

By the Ascoli-Arzela theorem again, a subsequence of $\{x^n(\cdot)\}$ converges a Lipschitz continuous curve $x(t)$ which verifies $x(T) = X_2$ and

$$|x(0) - \bar{x}(0)| \leq e^{CT}|X_1 - X_2|.$$

We examine that $x(\cdot) \in \mathcal{C}(T, X_2)$. We repeat the proof of Lemma 1.4.2. We may assume that $x^n(t) \to x(t)$ in $C([0,T])$ and hence

$$\frac{1}{\epsilon}\int_{t-\epsilon}^t \dot{x}^n(s)ds \to \frac{1}{\epsilon}\int_{t-\epsilon}^t x(s)ds.$$

Consider the Riemann sum of the integral of $\dot{x}^n(s)$:

$$\frac{1}{\epsilon}\sum \dot{x}^n(s_i)(s_{i+1} - s_i).$$

By the definition there is t_i with $|s_i - t_i| \leq T/n$ and $\dot{x}^n(s_i) = \dot{x}^n(t_i) \in \Gamma_1^*(t_i, \dot{x}^n(t_i))$. Thus there exists $v_i \in \Gamma_1^*(t, x(t))$ such that

$$\begin{aligned}|\dot{x}^n(s_i) - v_i| = |\dot{x}^n(t_i) - v_i| &\leq C\big\{|t - t_i| + |x^n(t_i) - x(t)|\big\} \\ \leq C\big\{|t - t_i| + |x^n(t_i) - x^n(t)|\big\} + C\epsilon_1 &\leq C'|t - t_i| + C\epsilon_1 \\ &\leq C'|t - s_i| + C''\frac{T}{n} + C\epsilon_1\end{aligned}$$

where $\epsilon_1 \to 0$ as $n \to \infty$. The rest of the proof is just a repetition of the proof of Lemma 1.4.2.

Lemma 1.4.5 *$\Psi(t, x)$ is Lipschitz continuous.*

Proof: Let $\Lambda = \|\nabla_x \psi\|_{L^\infty}$. Take $\bar{x}(\cdot) \in \mathcal{C}(T, X_1)$ such that $\Psi(T, X_1) = \psi(\bar{x}(0))$. By Lemma 1.4.4 there exists a $x(\cdot) \in \mathcal{C}(T, X_2)$ such that

$$|x(0) - \bar{x}(0)| \leq C|X_1 - X_2|.$$

By the definition one has $\Psi(T, X_2) \leq \psi(x(0))$. Then it follows that

$$\Psi(T, X_2) \leq \psi(x(0)) \leq \psi(\bar{x}(0)) + C\Lambda|X_1 - X_2| = \Psi(T, X_1) + C\Lambda|X_1 - X_2|.$$

Reversing the role of X_1, X_2 one has

$$\Psi(T, X_1) \leq \Psi(T, X_2) + C\Lambda|X_1 - X_2|$$

and hence $|\Psi(T, X_1) - \Psi(T, X_2)| \leq C\Lambda|X_1 - X_2|$.

We next consider $\Psi(T_1, X) - \Psi(T_2, X)$ where we may assume $T_1 > T_2$. Take $x(\cdot) \in \mathcal{C}(T_1, X)$ such that $\Psi(T_1, X) = \psi(x(0))$. Then from Lemma 1.4.3 it follows that $\Psi(T_1, X) = \Psi(T_2, x(T_2))$ and hence

$$\begin{aligned} |\Psi(T_1, X) - \Psi(T_2, X)| &= |\Psi(T_2, x(T_2)) - \Psi(T_2, X)| \\ &\leq C\Lambda|x(T_2) - x(T_1)| \leq C\Lambda C'|T_2 - T_1|. \end{aligned}$$

Proposition 1.4.1 *$\Psi(t, x)$ satisfies the Hamilton-Jacobi initial value problem:*

$$\begin{cases} \partial_t \Psi + \lambda_{\max}(t, x, -\nabla_x \Psi) = 0, \quad a.e.\ (t, x) \\ \Psi(0, x) = \psi(x). \end{cases} \tag{1.4.6}$$

Proof: Rademacher's theorem implies that it suffices to show (1.4.6) at points (T, X) where Ψ is differentiable. Let $x(\cdot) \in \mathcal{C}(T, X)$ then

$$x(T - \epsilon) = X - \int_{T-\epsilon}^{T} \dot{x}(s)ds$$

and hence

$$\Psi(T - \epsilon, x(T - \epsilon)) = \Psi(T, X) - \epsilon\Big(\Psi_t(T, X) + \langle \frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds, \Psi_x(T, X)\rangle\Big) + o(\epsilon).$$

Here we have used the fact that

$$\frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds$$

is bounded as $\epsilon \to 0$ which is shown in Lemma 1.4.2. We now apply Lemma 1.4.3 (ii) with $t = T - \epsilon$ to get

$$\begin{aligned} \min_{x(\cdot)\in\mathcal{C}(T,X)} \Big[\Psi(T, X) - \epsilon\Big(\Psi_t(T, X) + \langle \frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds, \Psi_x(T, X)\rangle\Big) \\ + o(\epsilon)\Big] = \Psi(T, X) \end{aligned}$$

and hence

$$\max_{x(\cdot)\in\mathcal{C}(T,X)} \Big[\Psi_t(T, X) + \langle \frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds, \Psi_x(T, X)\rangle + \frac{o(\epsilon)}{\epsilon}\Big] = 0.$$

Since for any $v \in \Gamma_1^*(T, X)$ we can choose $x(\cdot) \in \mathcal{C}(T, X)$ with

$$\frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds \to v, \quad \epsilon \to 0.$$

To examine this, in the construction of $x^n(\cdot)$ in Lemma 1.4.4 we choose $v_j \in \Gamma_1^*(Tj/n, x^n(Tj/n))$ so that

$$|v_j - v| = \mathrm{dist}(v, \Gamma_1^*(\frac{T}{n}j, x^n(\frac{T}{n}j))), \quad v_n = v, \;\; x^n(T) = X.$$

Since $|x^n(t) - X| \le C\epsilon$ for $T - \epsilon \le t \le T$ we see that

$$|v_j - v| \le C\big(\frac{Tj}{n} + |X - x^n(\frac{Tj}{n})|\big) \le C'\epsilon$$

for $T - \epsilon \le Tj/n \le T$. This shows that

$$|\dot{x}^n(t) - v| \le C\epsilon \quad \text{for } \; T - \epsilon \le t \le T$$

and hence

$$\Big|\frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}^n(s)ds - v\Big| \le C\epsilon.$$

Let $x(\cdot)$ be a limit of a convergent subsequence of $\{x^n(\cdot)\}$ then we have

$$\Big|\frac{1}{\epsilon}\int_{T-\epsilon}^{T} \dot{x}(s)ds - v\Big| \le C\epsilon.$$

On the other hand the same arguments proving Lemma 1.4.2 shows $x(\cdot) \in \mathcal{C}(T, X)$ and hence the assertion. Thus we get

$$\Psi_t(T, X) + \max_{v \in \Gamma_1^*(T,X)} \langle v, \Psi_x(T, X)\rangle = 0$$

which proves the assertion thanks to Lemma 1.4.1.

2 Existence and uniqueness

In this section we essentially follow [9].

2.1 Local hyperbolic apriori estimates

Let $\zeta(x) \in C^\infty(\mathbf{R}^n)$ be such that $\nabla\zeta(x) \in \mathcal{B}(\mathbf{R}^n)$ and

$$\zeta(x) = \langle\tilde{\theta}, x - x^0\rangle + k|x - x^0|^2$$

near x^0 where $\tilde{\theta} \in \Gamma(p(x^0, \cdot), \theta)$ and $k > 0$. We denote by $P_\gamma(x, D)$

$$P_\gamma(x, D) = e^{-\gamma\zeta(x)} P(x, D) e^{\gamma\zeta(x)}.$$

Our local hyperbolic apriori estimate is stated as:

Lemma 2.1.1 *For any $\delta > 0$ there exists $\phi \in C_0^\infty(\mathbf{R}^n)$ which is equal to 1 near x^0, $\chi(s) \in C_0^\infty(\mathbf{R})$, $\chi(s) = 1$ in $|s| \le 1$, $\chi(s) = 0$ in $|s| \ge 2$ such that: for any $s \ge 0$ there are $\ell'(s)$, $\ell''(s)$ such that we have for every $\ell \in \mathbf{R}$*

$$\|\langle D\rangle_\gamma^\ell \phi u\| \le C\Big\{\|\langle D\rangle_\gamma^{\ell+\ell'(s)}\chi((x_1 - x_1^0)/3\delta)P_\gamma u\| \tag{2.1.1}$$
$$+\|\langle D\rangle_\gamma^{\ell-s}u\| + \|\langle D\rangle_\gamma^{\ell+\ell''(s)}\chi((x_1 - x_1^0)/3\delta)\chi(|D|/2\gamma)u\|\Big\}$$

for $u \in C_0^\infty(\{x_1 > x_1^0 - \delta\})$, $\gamma \ge \gamma_0$.

Here we stress that the "hyperbolicity" is reflected by $\nabla\zeta(x^0) \in \Gamma(p(x^0,\cdot),\theta)$ and the function space $C_0^\infty(\{x_1 > x_1^0 - \delta\})$.

Let us fix a bounded domain $\Omega \subset \mathbf{R}^n$ and assume that $\nabla\zeta(x) \in \Gamma(p(x,\cdot),\theta)$ for $x \in \overline{\Omega}$. Since $\overline{\Omega}$ is compact there are $c > 0$ and γ_0 such that

$$|P_\gamma(x,\xi)| \ge c\gamma^m$$

with some $c > 0$ for any $x \in \overline{\Omega}$, $|\xi| \le C\gamma$ if $\gamma \ge \gamma_0$. Then for any N there are $Q_N \in S(\langle\xi\rangle_\gamma^{-m}, g)$, $R_N \in S(\langle\xi\rangle_\gamma^{-N}, g)$ with $g = |dx|^2 + \langle\xi\rangle_\gamma^{-2}|d\xi|^2$ such that

$$Q_N(x,D)P_\gamma(x,D) = \chi(|D|/2\gamma)\Phi(x) - R_N(x,D)$$

where $\Phi(x) \in C_0^\infty(\mathbf{R}^n)$, $\Phi(x) = 1$ on $\overline{\Omega}$ and supported in a small neighborhood of $\overline{\Omega}$. This enables us to remove the last term in the right-hand side of (2.1.1). Since $\{x_1 = t\} \cap \overline{\Omega}$ is compact we have

Lemma 2.1.2 *Assume that (2.1.1) holds everywhere $x \in \mathbf{R}^n$. Then for any $t \in \mathbf{R}$ and any $\delta > 0$ there is a $\delta' \ge 0$ such that for any $s \ge 0$ there is $\ell'(s) \in \mathbf{R}$ verifying: for every $\ell \in \mathbf{R}$ we have*

$$\|\langle D\rangle_\gamma^\ell \chi((x_1 - t)/\delta')u\| \le C\{\|\langle D\rangle_\gamma^{\ell+\ell'(s)}\chi((x_1 - x_1^0)/\delta)P_\gamma(x,D)u\| + \|\langle D\rangle_\gamma^{\ell-s}u\|\}$$

where $C = C(\ell,s)$ for any $u \in C_0^\infty(\Omega \cap \{x_1 > t - \delta'\})$, $\gamma \ge \gamma_0$.

2.2 Uniqueness results

We show that

Proposition 2.2.1 *Assume that (2.1.1) holds for every $x \in \mathbf{R}^n$. Then for any $x^0 \in \mathbf{R}^n$ there is a neighborhood U of x^0 such that*

$$u \in \mathcal{D}',\ \ \mathrm{supp}Pu \cap \overline{V} = \emptyset,\ \ \mathrm{supp}u \cap \{x \in \partial V \mid \zeta(x) \le \zeta(x^0)\} = \emptyset$$

with some $V \subset\subset U$ implies $x^0 \notin \mathrm{supp}u$.

From Lemma 2.1.2 it is easy to see

Lemma 2.2.1 *There are a neighborhood U of x^0 and $\ell' \in \mathbf{R}$ such that for any $\ell \in \mathbf{R}$ there exist $\tilde{m}$, γ_0, C such that we have*

$$\|\langle D\rangle^{\ell} u\| \leq C\gamma^{\tilde{m}}\|\langle D\rangle^{\ell+\ell'} P_\gamma u\|$$

for $u \in C_0^\infty(U)$, $\gamma \geq \gamma_0$.

Proof of Proposition 2.2.1: From the assumption one can find $\phi(x) \in C_0^\infty(U)$ and small $\delta > 0$ such that $\phi = 1$ near x^0 and

$$\mathrm{supp}\phi \cap \mathrm{supp}Pu = \emptyset, \;\; \mathrm{supp}u \cap \{x \in \mathbf{R}^n \mid x \in \mathrm{supp}\nabla\phi, \zeta(x) \leq \zeta(x^0) + \delta\} = \emptyset.$$

Consider $v_\gamma = e^{-\gamma\zeta(x)}\phi(x)u(x)$ for which we have

$$\|\langle D\rangle^{\ell} v_\gamma\| \leq C\gamma^{\tilde{m}}\|\langle D\rangle^{\ell+\ell'} P_\gamma v_\gamma\|.$$

Since

$$P_\gamma v_\gamma = e^{-\gamma\zeta(x)}P\phi(x)u(x) = e^{-\gamma\zeta(x)}[P,\phi]u + e^{-\gamma\zeta(x)}\phi(x)Pu$$

this proves that

$$\|\langle D\rangle^{\ell}\chi(x)\phi(x)u\| \leq C\gamma^{\tilde{m}}e^{-\gamma\delta'}\|\langle D\rangle^{s}\phi_1 u\|$$

with some $\delta' > 0$, $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi(x^0) \neq 0$, $\phi_1 \in C_0^\infty(\mathbf{R}^n)$. From this we obtain the assertion letting $\gamma \to \infty$.

In the rest of the subsection we use again the notation in subsection 1.4 so that $t = x_1$, $x = (x_2, ..., x_n)$. Let us take a bounded open set Ω_0 in $\mathbf{R}^{n-1}$ and denote by Ω the set of all (t,x), $t \geq 0$ such that for every g.c.curve $x(t) : [0,t] \ni t \mapsto \mathbf{R}^{n-1}$ with $x(t) = x$ one has $x(0) \in \Omega_0$. Then we have

Theorem 2.2.1 *Assume that (2.1.1) is verified everywhere. Assume that $Pu = 0$ in Ω where $u \in \mathcal{D}'$ is vanishing in $x_1 \leq 0$. Then $u = 0$ in Ω.*

Proof: We again follow [8]. Let us define $\psi(x)$ by

$$\psi(x) = e^{-|x|}\mathrm{dist}(x, \complement\Omega_0)$$

which is uniformly Lipschitian and define $\Psi(t,x)$ by (1.4.4). It is clear that

$$\Omega = \{(t,x) \mid t \geq 0, \Psi(t,x) > 0\}.$$

Then to prove Theorem 2.2.1 it is enough to show that

$$t \geq 0, \;\; \Psi(t,x) > 0 \Longrightarrow u \text{ is zero near } (t,x).$$

Set $\Psi^\delta = \Psi - \delta t$ then it is clear

$$\lim_{t\geq 0, t+|x|\to\infty} \Psi^\delta(t,x) \leq 0$$

since Ψ is bounded by the definition and $\Psi = 0$ for large $|x|$. Recall that from Proposition 1.4.1 we have

$$\Psi_t^\delta + \lambda_{\max}(t, x, -\nabla_x \Psi^\delta) = -\delta, \quad \Psi^\delta(0, x) = \psi(x). \tag{2.2.1}$$

We now regularize Ψ^δ by

$$\Psi^{\epsilon,\delta}(t, x) = \int \epsilon^{-n} \rho(\frac{t-s}{\epsilon}, \frac{x-y}{\epsilon}) \Psi^\delta(s, y) ds dy = J_\epsilon(\Psi^\delta)$$

where $\rho \in C_0^\infty(\mathbf{R}^n)$ with $\int \rho(t, x) dt dx = 1$. We rewrite (2.2.1) as

$$\Psi_t^\delta + \langle v(t, x), \nabla_x \Psi^\delta \rangle \le -\delta$$

for all bounded $v(t, x)$ with $v(t, x) \in \Gamma_1^*(t, x)$. Let us assume that $v(t, x)$ is Lipschitian then it is easy to see that

$$\|\langle [J_\epsilon, v], \nabla_x \Psi^\delta \rangle\|_{L^\infty} \le C\epsilon.$$

Note that C comes from the Lipschitz constant of v. Since $\|\Psi^{\epsilon,\delta} - \Psi^\delta\|_{L^\infty} \le C\epsilon$ one can take $\epsilon > 0$ small so that we have

$$\begin{aligned} &\Psi_t^{\epsilon,\delta} + \langle v(t, x), \nabla_x \Psi^{\epsilon,\delta} \rangle < -\delta/2, \\ &\sup_{x \in \mathrm{C}\Omega_0} \Psi^{\epsilon,\delta}(0, x) < \Psi^{\epsilon,\delta}(T, X)/2. \end{aligned} \tag{2.2.2}$$

Since for any $\bar{v} \in \Gamma_1^*(\bar{t}, \bar{x})$ one can take a Lipschitz continuous $v(t, x)$ with $v(\bar{t}, \bar{x}) = \bar{v}$ (note that the Lipschitz constant can be taken to be independent of $(\bar{t}, \bar{x})$ and $\bar{v}$) it follows from (2.2.2) that

$$\Psi_t^{\epsilon,\delta} + \lambda_{\max}(t, x, -\nabla_x \Psi^{\epsilon,\delta}) \le -\delta/2. \tag{2.2.3}$$

Let us set

$$\Phi = \Psi^{\epsilon,\delta} - \Psi^{\epsilon,\delta}(T, X)/2.$$

Then we conclude that

$$\begin{aligned} &\Phi_t + \lambda_{\max}(t, x, -\nabla_x \Phi) \le -\delta/2, \\ &\Phi(T, X) > 0, \quad \Phi(0, x) < 0, \quad x \in \mathrm{C}\Omega_0. \end{aligned} \tag{2.2.4}$$

It is easy to see that $\{t \ge 0, \Phi(t, x) > 0\} \subset \Omega$ if $\delta > 0$ and $\epsilon > 0$ are chosen to be small. Note that there is $R > 0$ such that for any $c > 0$ the surface $\{\Phi(t, x) = c, t \ge 0\}$ is contained in $\{t \ge 0, t^2 + |x|^2 \le R^2\}$. Let $c_{\max} = \max_{t \ge 0, x \in \mathbf{R}^{n-1}} \Phi(t, x)$. Since $d\Phi(t, x) \ne 0$ by (2.2.4) the maximum can not be attained at an interior point and hence

$$\max_{x \in \mathbf{R}^{n-1}} \Phi(0, x) = c_{\max}$$

and $\Phi(t, x) < c_{\max}$ for $t > 0$. Let

$$\Lambda = \{c \in [\frac{1}{2}\Phi(T, X), c_{\max}] \mid u = 0 \text{ on } \Phi \ge c, t \ge 0\}.$$

It is clear that $c_{\max} \in \Lambda$ by the assumption. Let us show that Λ is open. Take $(\hat{t}, \hat{x}) \in \{\Phi = c, t \geq 0\}$. Denote

$$\zeta(t,x) = \Phi(t,x) - c - k|(t,x) - (\hat{t},\hat{x})|^2, \quad k > 0.$$

Then $d\zeta(\hat{t},\hat{x}) \in \Gamma(p(\hat{t},\hat{x},\cdot),\theta)$ and with a small neighborhood V of $(\hat{t},\hat{x})$ one has

$$\partial V \cap \{\zeta(t,x) \leq 0\} \cap \mathrm{supp} u = \emptyset.$$

From Proposition 2.2.1 it follows that $u = 0$ near $(\hat{t},\hat{x})$. This proves that Λ is open. Since Λ is closed we conclude that $\Lambda = [\Phi(T,X)/2, c_{\max}]$ which proves the assertion.

2.3 Existence theorem

Recall

$$P(x,D) = D_1^m + \sum_{|\alpha| \leq m, \alpha_1 < m} a_\alpha(x) D^\alpha = \sum_{j=0}^{m} P_j(x,D)$$

where $a_\alpha(x) \in \mathcal{B}^\infty(\mathbf{R}^n)$ and

$$p(x, \xi - i\theta) \neq 0, \quad \forall x \in \mathbf{R}^n, \forall \xi \in \mathbf{R}^n.$$

With $\Omega^+ = \Omega \cap \{x_1 > 0\}$ let us set $\Omega_\delta^+ = \{x \in \mathbf{R}^n \mid \mathrm{dist}(x, \overline{\Omega^+}) < \delta\}$ and take $R > 0$ so that $\Omega_{2\delta}^+ \subset \{|x| < R\}$. Let $\phi_\delta(x) \in C^\infty(\mathbf{R}^n)$ to be $0 \leq \phi_\delta(x) \leq 1$, $\phi_\delta(x) = 1$ on Ω_δ^+, $\phi_\delta(x) = 0$ in the complement of $\Omega_{2\delta}^+$ and moreover

$$0 \leq \phi_\delta(x) < 1, \quad x \notin \overline{\Omega_\delta^+}. \tag{2.3.1}$$

We take $\alpha_\delta(x) \in C^\infty(\mathbf{R}^n;\mathbf{R}^n)$ such that $\alpha_\delta(x) = x$, $|x| < R$ and $\alpha_\delta(x) = Rx/|x|$ for $|x| > R + \delta$ and define $\tilde{P}$ as

$$\tilde{P} = (1 - (1 - \phi_\delta(x))^2 |\xi'|^2 \frac{\partial^2}{\partial \xi_1^2})^{[m/2]+1} p(\alpha_\delta(x),\xi) + \phi_{\delta/2}(x) \sum_{j=0}^{m-1} P_j(x,\xi).$$

Then

Lemma 2.3.1 *We have*

(i) *for every multiple characteristic z of $\tilde{p}$ we have $\tilde{p}_z = p_z$,*

(ii) *the coefficients of $\tilde{P}$ are in $\mathcal{B}^\infty(\mathbf{R}^n)$ and $\tilde{p}$ is strictly hyperbolic with respect to θ in the complement of $\Omega_{2\delta}^+$.*

Proof: Recall that every characteristic root of

$$(1 + s\frac{\partial}{\partial \xi_1}) p(\alpha_\delta(x),\xi)$$

is at most $m-1$-th folded if $s \neq 0$. This shows that the multiplicity of every characteristic root of

$$(1 - s\frac{\partial}{\partial \xi_1})(1 + s\frac{\partial}{\partial \xi_1})p(\alpha_\delta(x), \xi) = (1 - s^2\frac{\partial^2}{\partial \xi_1^2})p(\alpha_\delta(x), \xi)$$

is at most $m-2$ folded for every $x \in \mathbf{R}^n$ and every $\xi' \in \mathbf{R}^{n-1}$ if $s \neq 0$. Take $s = (1-\phi_\delta(x))|\xi'|$ then we conclude that for x in the complement of $\Omega^+_{2\delta}$ we have $s = |\xi'|$ and hence $\tilde{p}$ is strictly hyperbolic. If $x \notin \overline{\Omega^+_\delta}$ and hence $(1-\phi_\delta(x)) \neq 0$, then $z = (x, \xi)$ is never multiple characteristic of $\tilde{p}$. This shows that $\tilde{p}_z = p_z$ for any multiple characteristic z of $\tilde{p}$ because $d^j\phi_\delta(x) = 0$ for any $j \neq 0$ if $x \in \overline{\Omega^+_\delta}$.

Our aim in this lecture is to prove

Theorem 2.3.1 *Assume that there is a neighborhood U of $\overline{\Omega}_+$ such that $p(x,\xi)$ is effectively hyperbolic at every double characteristic (x,ξ) with $x \in U$. Then, taking $\delta > 0$ small, $\tilde{P}_\gamma$, ${}^t\tilde{P}_\gamma$ verify (2.1.1) at every $x \in \mathbf{R}^n$.*

Admitting this result we show

Theorem 2.3.2 *Assume that $\tilde{P}_\gamma$ and ${}^t\tilde{P}_\gamma$ satisfy (2.1.1) at every $x \in \mathbf{R}^n$. Then for every $f \in \mathcal{D}'$ with* supp$f \subset \{x_1 \geq 0\}$*, the Cauchy problem*

$$\begin{cases} P(x,D)u(x) = f \quad \textit{in} \quad \Omega \\ \mathrm{supp} u \subset \{x_1 \geq 0\} \end{cases}$$

has a solution $u \in \mathcal{D}'$.

Proof: To prove the assertion it suffices to show the existence of u vanishing in $x_1 < -c$ with some $c > 0$ such that $\tilde{P}u = f$. In fact applying Proposition 2.2.1 and the method of sweeping out (see the proof of Theorem 2.2.1) we conclude that $u = 0$ in $x_1 \leq 0$. Since $P = \tilde{P}$ in $\Omega^+_{\delta/2}$ we obtain the desired assertion.

We drop the tilde and write P for $\tilde{P}$. We may assume that $p(x,\xi)$ is strictly hyperbolic outside Ω^+_δ (replacing δ by $\delta/2$). We choose a special $\zeta(x)$ as $\zeta(x) = x_1$ and hence

$${}^tP_\gamma(x,D) = {}^tP(x, D + i\gamma\theta).$$

From Lemma 2.1.2 for ${}^tP$ it follows that for any $t \in \mathbf{R}$ and any $\delta > 0$ there is $\delta' > 0$ such that for any $s \geq 0$ there is $\ell'(s) \in \mathbf{R}$ verifying: for every $\ell \in \mathbf{R}$ we have

$$\|\langle D\rangle^\ell_\gamma \chi((x_1 - t)/\delta')u\| \leq C\{\|\langle D\rangle^{\ell+\ell'(s)}_\gamma \chi((x_1-t)/\delta)\,{}^tP_\gamma(x,D)u\| + \|\langle D\rangle^{\ell-s}_\gamma u\|\}$$

where $C = C(\ell, s)$ for any $u \in C_0^\infty(\tilde{\Omega} \cap \{x_1 < t + \delta\})$, $\gamma \geq \gamma_0$ and $\Omega \subset\subset \tilde{\Omega}$. Choose T so that $\tilde{\Omega} \subset \{x_1 < T\}$. Then one can find a finite number of points

$t_1 > t_2 > \cdots > t_L \geq -2\delta$ and intervals $I_j = [t_j - \delta_j/2, t_j + \delta_j/2]$ such that

$$[-2\delta, T] \subset \bigcup_{j=1}^{L} I_j,$$

and for every $\ell \in \mathbf{R}$ there is $\gamma_0 = \gamma_0(\ell, s)$ such that for any $\theta(t) \in C_0^\infty([t_j - \delta_j, t + \delta_j])$ one has

$$\|\langle D\rangle_\gamma^\ell \theta(x_1)u\| \leq C\{\|\langle D\rangle_\gamma^{\ell+\ell'(s)}\chi_1(x_1)^t P_\gamma u\| + \|\langle D\rangle_\gamma^{\ell-s}u\|$$

with $C = C(\ell, s, \theta)$ for $u \in C_0^\infty(\tilde{\Omega} \cap \{x_1 < t_j + \delta_j\})$ where $\chi_1(x_1) = 0$ in $x_1 \leq -4\delta$. Choose $\Psi_j \in C_0^\infty([t_j - \delta_j, t_j + 2\delta_j/3])$ so that

$$\sum_{j=1}^{L} \Psi_j(t) = 1 \quad \text{on} \quad -2\delta \leq t \leq T.$$

Let $\phi_j(t) = 1$ in $t \leq t_j + 2\delta_j/3$ then one has

$$\|\langle D\rangle_\gamma^\ell \Psi_j u\| \leq C\{\|\langle D\rangle_\gamma^{\ell+\ell'(s)}\chi_1(x_1)^t P_\gamma \phi_j u\| + \|\langle D\rangle_\gamma^{\ell-s}\phi_j u\|\}.$$

Since we can take ϕ_j so that $\mathrm{supp}\phi_j' \subset \{\Psi_{j-1} = 1\}$ and hence

$${}^t P_\gamma \phi_j u = \phi_j {}^t P_\gamma u + [{}^t P_\gamma, \phi_j] \sum_{k=1}^{j-1} \Psi_k u.$$

This proves that

$$\|\langle D\rangle_\gamma^\ell \Psi_j u\| \leq C\{\|\langle D\rangle^{\ell+\ell'}\chi_1(x_1)^t P_\gamma u\|$$
$$+\|\langle D\rangle_\gamma^{\ell-s}u\| + \|\langle D\rangle_\gamma^{\ell+\ell'+m-1}\sum_{k=1}^{j-1}\Psi_k u\|\}.$$

By induction we see that for any $s \geq 0$ there is $\ell' = \ell'(s)$ such that for any $\ell \in \mathbf{R}$ one has

$$\|\langle D\rangle_\gamma^\ell \sum_{j=1}^{L}\Psi_j u\| \leq C\{\|\langle D\rangle_\gamma^{\ell+\ell'}\chi_1(x_1)^t P_\gamma u\| + \|\langle D\rangle_\gamma^{\ell-s}u\|\} \tag{2.3.2}$$

where $C = C(\ell, s)$ for $u \in C_0^\infty(\tilde{\Omega})$, $\gamma \geq \gamma_0(\ell, s)$. Hence choosing $\Phi(x_1)$ so that $\Phi(x_1) = 0$ in $x_1 \leq -2\delta$ and $\Phi(x_1) = 1$ in $x_1 \geq -\delta$ we have

$$\|\langle D\rangle_\gamma^\ell \Phi(x_1)u\| \leq C\{\|\langle D\rangle_\gamma^{\ell+\ell'}\chi_1(x_1)^t P_\gamma u\| + \|\langle D\rangle_\gamma^{\ell-s}u\|\}. \tag{2.3.3}$$

We now introduce several spaces associated with $X = \{x \in \mathbf{R}^n \mid x_1 > -4\delta\}$:

$$H_{(m,s)}(\mathbf{R}^n) = \{u \in \mathcal{S}' \mid \langle D\rangle_\gamma^m \langle D'\rangle_\gamma^s u \in L^2(\mathbf{R}^n)\},$$
$$\overline{H}_{(m,s)}(X) = \{u \in \mathcal{D}'(X) \mid u = U|_X \text{ for some } U \in H_{(m,s)}(\mathbf{R}^n)\},$$
$$\dot{H}_{(m,s)}(\overline{X}) = \{u \in H_{(m,s)}(\mathbf{R}^n) \mid \mathrm{supp} u \subset \overline{X}\}$$

where $\overline{H}_{(m,s)}(X)$ is equipped with the norm

$$\|u\|_{\overline{H}_{(m,s)}(X)} = \inf\{\|\langle D\rangle_\gamma^m \langle D'\rangle_\gamma^s u\| \mid U|_X = u, U \in H_{(m,s)}(\mathbf{R}^n)\}.$$

Then $\overline{C}_0^\infty(X)$ and $C_0^\infty(X)$ are dense in $\overline{H}_{(m,s)}(X)$ and $\dot{H}_{(m,s)}(\overline{X})$ respectively and the spaces $\overline{H}_{(m,s)}(X)$, $\dot{H}_{(m,s)}(\overline{X})$ are dual with respect to an extension of sesquilinear form $\int u\bar{v}dx$ for $u \in \overline{C}_0^\infty(X)$ and $v \in C_0^\infty(X)$. To prove the existence of u verifying $Pu = f$ and vanishing in $x_1 \le -4\delta$ we derive *hyperbolic* apriori estimate for ${}^tP_\gamma$, that is apriori estimate in the space $\overline{H}_{(m,s)}(X)$.

Note that (2.3.3) implies that

$$\|\Phi u\|_{\overline{H}_{(\ell,0)}(X)} \le C\big\{\|{}^tP_\gamma u\|_{\overline{H}_{(\ell+\ell',0)}(X)} + \gamma^{-1}\|u\|_{\overline{H}_{(\ell,0)}(X)}\big\}. \tag{2.3.4}$$

On the other hand, since ${}^tP$ is strictly hyperbolic in $x_1 \le -\delta$, then for every $s \in \mathbf{R}$ there are γ_0 and $C > 0$ such that

$$\|(1-\Phi)u\|_{\overline{H}_{(m,s)}(X)} \le C\|{}^tP_\gamma(1-\Phi)u\|_{\overline{H}_{(0,s)}(X)} \tag{2.3.5}$$

for $u \in C_0^\infty(\tilde{\Omega})$, $\gamma \ge \gamma_0$. Taking $s = \ell - m + 1$ in (2.3.5) we get

$$\|(1-\Phi)u\|_{\overline{H}_{(\ell,0)}(X)} \le C\{\|{}^tP_\gamma u\|_{\overline{H}_{(0,\ell-m+1)}(X)} + \|[{}^tP_\gamma, 1-\Phi]u\|_{\overline{H}_{(0,\ell-m+1)}(X)}\}$$

for $\ell \le m-1$. On the other hand, from Theorem B.2.9 in [5] it follows that

$$\|[{}^tP_\gamma, 1-\Phi]u\|_{\overline{H}_{(0,\ell-m+1)}(X)} \le C\|u\|_{\overline{H}_{(m-1,\ell-m+1)}(x_1>-2\delta)} \le C'\big\{\|{}^tP_\gamma u\|_{\overline{H}_{(-1,\ell-m+1)}(x_1>-2\delta)} + \|u\|_{\overline{H}_{(\ell,0)}(x_1>-2\delta)}\big\}$$

if $\ell \le m-1$. Since it is clear that

$$\|u\|_{\overline{H}_{(\ell,0)}(x_1>-2\delta)} \le \|\langle D\rangle_\gamma^\ell \sum_{j=1}^L \Psi_j(x_1)u\|$$

we get

$$\|(1-\Phi)u\|_{\overline{H}_{(\ell,0)}(X)} \le C\Big\{\|{}^tP_\gamma u\|_{\overline{H}_{(\ell-m+1,0)}(X)} + \|{}^tP_\gamma u\|_{\overline{H}_{(0,\ell-m+1)}(X)} + \|\langle D\rangle_\gamma^\ell \sum_{j=1}^L \Psi_j u\|\Big\} \tag{2.3.6}$$

for $\ell \le m-1$, $u \in C_0^\infty(\tilde{\Omega})$, $\gamma \ge \gamma_0$. Applying the same arguments to $D_1^k[{}^tP_\gamma u]$, we conclude that (2.3.6) holds for $\ell \ge m-1$. Since the last term on the right-hand side of (2.3.6) is estimated using (2.3.2), combining (2.3.4) and (2.3.6), we can take $\ell' \in \mathbf{R}$ such that for every $\ell \in \mathbf{R}$ the estimate (hyperbolic estimate)

$$\|u\|_{\overline{H}_{(\ell,0)}(X)} \le \big\{\|{}^tP_\gamma u\|_{\overline{H}_{(\ell+\ell',0)}(X)} + \|{}^tP_\gamma u\|_{\overline{H}_{(0,\ell+\ell')}(X)}\big\}$$

holds. Let $f \in \dot{H}_{(-\ell,0)}(\overline{X})$ where $\ell \in \mathbf{R}$ is fixed. Then we have

$$|\langle f, v\rangle| \le \|\langle D\rangle_\gamma^{-\ell} f\|\|v\|_{\overline{H}_{(\ell,0)}(X)}$$
$$\le C\|\langle D\rangle_\gamma^{-\ell} f\|\big\{\|{}^t P_\gamma v\|_{\overline{H}_{(\ell+\ell',0)}(X)} + \|{}^t P_\gamma v\|_{\overline{H}_{(0,\ell+\ell')}(X)}\big\}$$

for $v \in C_0^\infty(\tilde{\Omega})$ and $\gamma \ge \gamma_0$. By the Hahn-Banach theorem there is a $u \in \dot{H}_{(-\ell-\ell',0)}(\overline{X})$ (resp. $u \in \dot{H}_{(0,-\ell-\ell')}(\overline{X})$) such that

$$\langle u, {}^t P_\gamma v\rangle = \langle f, v\rangle, \quad \forall v \in C_0^\infty(\tilde{\Omega})$$

if $\ell + \ell' \ge 0$ (resp. $\ell + \ell' < 0$) and hence $P(e^{\gamma x_1} u) = e^{\gamma x_1} f$ in $\tilde{\Omega}$. This proves the assertion (starting with $e^{-\gamma x_1} f$ instead of f).

2.4 Microlocal hyperbolic apriori estimates

We first give a heuristic argument. We are working near x^0 but P_γ may have different character for different fiber direction ξ. Let us take

$$\chi(x,\xi) = \langle\xi\rangle^{-a\rho(x,\xi)}, \quad a > 0$$

with $\rho(x,\xi) = x_1 - \tilde{\rho}(x,\xi')$ where $\tilde{\rho}(x,\xi')$ attains a local strict maximum at $(x^0,\hat{\xi}')$. Since P is hyperbolic we may naturally expect that the estimate

$$\|\langle D\rangle^{\ell_1}\chi u\| \le C\Big\{\|\langle D\rangle^{\ell_2}\chi P_\gamma u\| + \|\langle D\rangle^{\ell_3}(1-\psi)u\|\Big\} \tag{2.4.1}$$

holds for any $a > 0$ for every u vanishing in $x_1 < -c$ with small $c > 0$ since in a small neighborhood of $(x^0,\hat{\xi})$ every propagation cone $z - C_z$ with vertex z on the surface $\rho(x,\xi) = \rho(x^0,\hat{\xi})$ is contained in the region $\rho < 0$ and we can expect that u in $\rho < 0$ is determined by Pu in $\rho < 0$ because $u = 0$ in $x_1 < x_1^0 - c$.

We now assume that (2.4.1) holds and derive local hyperbolic apriori estimate. Let $u = 0$ in $x_1 \le x_1^0 - \delta$ with small $\delta > 0$ and let ϕ be such that $\phi = 1$ near $(x^0,\hat{\xi})$ and $(1-\psi)\phi = 0$. Note that

$$\langle D\rangle^{\ell_2}\chi P_\gamma \phi u = \langle D\rangle^{\ell_2}\chi[P_\gamma,\phi]u + \chi\langle D\rangle^{\ell_2}\phi P_\gamma u.$$

Take $\delta > 0$ small so that we have

$$\rho(x,\xi) \ge 2\delta \quad \text{if} \quad (x,\xi) \in \mathrm{supp}\nabla_{(x,\xi)}\phi, \;\; x_1 \ge x_1^0 - \delta.$$

This shows that

$$\|\langle D\rangle^{\ell_1}\chi\phi u\| \le C\Big\{\|\langle D\rangle^{\ell_2+ac}P_\gamma u\| + \|\langle D\rangle^{\ell_3-2a\delta}u\|\Big\}$$

for any a. Let us take $\phi_1(x)$, $\phi_2(\xi)$, cut off functions around x^0 and $\hat{\xi}$, so that

$$\rho(x,\xi) \le \delta$$

on supp$\phi_1(x)\phi_2(\xi)$. Then one has

$$\|\langle D\rangle^{\ell_1-a\delta}\phi_1(x)\phi_2(D)u\| \le C\Big\{\|\langle D\rangle^{\ell_1}\chi\phi u\| + \|\langle D\rangle^{-N}u\|\Big\}$$

and hence we get

$$\|\langle D\rangle^{\ell_1-a\delta}\phi_1(x)\phi_2(D)u\| \le C\Big\{\|\langle D\rangle^{\ell_2+ac}P_\gamma u\| + \|\langle D\rangle^{\ell_3-2a\delta}u\|\Big\}.$$

Starting with $\langle D\rangle^t u$ instead of u we get

$$\|\langle D\rangle^{\ell_1-a\delta+t}\phi_1(x)\phi_2(D)u\| \le C\Big\{\|\langle D\rangle^{\ell_2+ac+t}P_\gamma u\| + \|\langle D\rangle^{\ell_3-2a\delta+t}u\|\Big\}.$$

Taking t and a so that $\ell_1 - a\delta + t = \ell$, $\ell_3 - 2a\delta + t = \ell - s$ we get the deisred local hyperbolic apriori estimate.

We now make precise our "hyperbolic microlocal apriori estimate": Assume that there exist $\rho(z)$, $\rho(z^0) = 0$, supported in a conic neighborhood of z^0 with $|\xi| \ge h$, homogeneous of degree 0 in ξ, which is equal to $x_1 - \tilde\rho(x',\xi)$ in another conic neighborhood of z^0 where $\tilde\rho(x',\xi)$ attains a local strict maximum x_1^0 at $((x^0)',\xi^0)$ and $\phi(z) \in C^\infty(T^*\mathbf{R}^n \setminus 0)$, positively homogeneous of degree 0 in ξ which is 1 in a conic neighborhood of z^0 and $\ell_i \in \mathbf{R}$ such that for any $a \ge 0$ we have "*microlocal hyperbolic apriori estimate*"

$$\|\langle D\rangle_h^{\ell_1}v\| \le C\{\|\langle D\rangle_h^{\ell_2}\chi^- P_\gamma \chi^+ v\| + \|\langle D\rangle_h^{\ell_3}(1-\phi_h)v\|\} \tag{2.4.2}$$

for $v \in C_0^\infty(\mathbf{R}^n)$, $h = \gamma \ge \gamma_0$ where $\chi^\pm = \mathrm{Op}(\langle\xi\rangle^{\pm a\rho})$ and $\phi_h(\xi) = [1 - \chi(|\xi|/h)]\phi(x,\xi)$, $\chi(s) \in C^\infty(\mathbf{R})$ be such that $\chi(s) = 1$ for $|s| \le 1$ and $\chi(s) = 0$ in $|s| \ge 2$. It is clear that in (2.4.2) one can replace ϕ by $\tilde\phi$ with supp$\tilde\phi \subset \{\phi = 1\}$.

According to the above heuristic arguments we derive local hyperbolic apriori estimate assuming microlocal hyperbolic apriori estimate (2.4.2):

Proposition 2.4.1 *Assume that microlocal hyperbolic apriori estimate (2.4.2) holds for every (x,ξ) with $x \in \mathbf{R}^n$, $|\xi| = 1$. Then local hyperbolic apriori estimate (2.1.1) holds at every $x \in \mathbf{R}^n$.*

Proof: Assume that hyperbolic microlocal apriori estimate holds for every (x^0,ξ^0) with $|\xi^0| = 1$. We may assume $\rho(z) = x_1 - \tilde\rho(x',\xi)$ in a conic neighborhood U_1 of z^0 and supp$\phi \subset U_1$ and $\phi = 1$ on $U_2 \subset\subset U_1$. Let $U_3 \subset\subset U_2$ and choose $\delta > 0$ so that we have

$$x_1 \ge x_1^0 - \delta,\ \ (x,\xi) \in U_1 \setminus U_3 \Longrightarrow \rho(x,\xi) \ge 2\delta.$$

Take $\psi \in C^\infty(T^*\mathbf{R}^n \setminus 0)$, of degree 0 in ξ, $\psi = 1$ in a conic neighborhood of $U_3 \cap \{|x_1 - x_1^0| \le 2\delta\}$ and supp$\psi \subset U_2 \cap \{|x_1 - x_1^0| < 3\delta\}$. Let us set $\psi_h = [1 - \chi(|\xi|/h)]\psi$. Note that $\chi^+\chi^- = 1 + r$ where $r \in S(\langle\xi\rangle_h^{-1+d}, g)$ with $g = \langle\xi\rangle_h^{2d}|dx|^2 + \langle\xi\rangle_h^{-2+2d}|d\xi|^2$ for any small $d > 0$. Hence for any N there are $e(z)$ and R such that

$$\chi^+\chi^-(1 + e(x,D)) = 1 + R(x,D)$$

where $e \in S(\langle\xi\rangle_h^{-1+2d}, g)$ and $R \in S(\langle\xi\rangle_h^{-N(1-2d)}, g)$. For any $u \in C_0^\infty(\mathbf{R}^n)$ with $u = 0$ in $x_1 \le x_1^0 - \delta$ we set $v = \chi^-(1+e)\psi_h \langle D\rangle_h^t u$. Then it is clear that

$$\chi^- P_\gamma \chi^+ v = \chi^- \psi_h \langle D\rangle_h^t P_\gamma u + \chi^-[P_\gamma, \psi_h\langle D\rangle_h^t]u + \chi^- P_\gamma R\psi_h\langle D\rangle_h^t u.$$

Here we note that

$$|\chi^-(z)| \le C\langle\xi\rangle_h^{-2a\delta}, \quad x_1 \ge x_1^0 - \delta,\ z \in \text{supp}\nabla\psi,\ |\xi| \ge h$$

and

$$|\chi^-(z)| \le C\langle\xi\rangle_h^{ac_1}, \quad x_1 \ge x_1^0 - \delta,\ z \in \text{supp}\psi,\ |\xi| \ge h/2$$

where

$$c_1 = -\inf_{z \in U_1, x_1 \ge x_1^0 - \delta} \rho(z).$$

Then noting that $(1-\phi)\psi = 0$ we get from (2.4.2) that

$$\begin{aligned} \|\langle D\rangle_h^{\ell_1} v\| \le C\{&\|\langle D\rangle_h^{\ell_2 + ac_1 + t}\chi((x_1 - x_1^0)/3\delta)P_\gamma u\| \\ &+\|\langle D\rangle_h^{\ell_2 - 2a\delta + m - 1 + t}u\| + \|\langle D\rangle_h^{-N}u\| \\ &+\|\langle D\rangle_h^{\ell_3 + ac_1 + t}\chi((x_1 - x_1^0)/3\delta)\chi(|D|/2h)u\|\}. \end{aligned} \tag{2.4.3}$$

Take $\phi(x) \in C_0^\infty(\mathbf{R}^n)$, $\tilde\psi(\xi) \in C^\infty(\mathbf{R}^n \setminus 0)$, homogeneous of degree 0 so that $\text{supp}\phi(x)\tilde\psi(\xi) \subset U_2$, $\phi(x)\tilde\psi(\xi) = 1$ in a conic neighborhood of z^0 and $\rho(z) < \delta$ if $z \in \text{supp}\phi(x)\tilde\psi(\xi)$, which is possible because $\rho(z^0) = 0$. Note that there are $\tilde e \in S(\langle\xi\rangle_h^{-1+2d}, g)$, $\tilde R \in S(\langle\xi\rangle_h^{-N(1-2d)}, g)$ such that $(1+\tilde e)\chi^+ v = (1+\tilde R)\psi_h\langle D\rangle_h^t u$. This proves that

$$\|\langle D\rangle_h^{\ell_1 - a\delta + t}\phi(x)\tilde\psi(D)u\| \le C\{\|\langle D\rangle_h^{\ell_1} v\| + \|\langle D\rangle_h^{-N}u\|\}. \tag{2.4.4}$$

We now take t and $a \ge 0$ so that $t = \ell - \ell_1 + a\delta$ and $\ell_2 - \ell_1 + m - 1 - a\delta = -s$ then (2.4.3) and (2.4.4) give

$$\begin{aligned} \|\langle D\rangle_h^{\ell}\phi(x)\tilde\psi(D)u\| \le C\{&\|\langle D\rangle_h^{\ell + \ell'(s)}\chi((x_1 - x_1^0)/3\delta)P_\gamma u\| \\ +\|\langle D\rangle_h^{\ell - s}u\| + &\|\langle D\rangle_h^{\ell + \ell''(s)}\chi((x_1 - x_1^0)/3\delta)\chi(|D|/2h)u\|\} \end{aligned}$$

where $\ell'(s) = \ell_2 + ac_1 - \ell_1 + a\delta$, $\ell''(s) = \ell_3 + ac_1 - \ell_1 + a\delta$.

2.5 How to derive microlocal hyperbolic estimates, a heuristic argument

We shall derive microlocal hyperbolic apriori estimate near a double characteristic z^0 where $p(z)$ is effectively hyperbolic. In virtue of Lemma 1.2.6 there exists a time function $t(x,\xi)$ at z^0 such that

$$h_{m-1}(x,\xi) \ge c h_{m-2}(x,\xi) t(x,\xi)^2 |\xi'|^2$$

in a conic neighborhood of z^0. A main feature of the (microlocal) Cauchy problem for P is that the solutions may lose the regularity when they cross the hypersurface $\{t(x,\xi)=0\}$. A natural idea to study such a Cauchy problem is to transform the original equation $Pu=f$ to $W^{-1}PWv=W^{-1}f$ with $u=Wv$ where W is a suitable operator with an inverse W^{-1}. We choose a W which behaves like $W\approx\langle D\rangle^M$ in $t(x,\xi)<0$ and $W\approx 1$ in $t(x,\xi)>0$. Then if u loses the regularity by $\langle D\rangle^M$ when it crosses the hypersurface $t(x,\xi)=0$ then $v=W^{-1}u$ loses no regularity and hence we could expect that v could be verified by an easier equation, that is the equation $P_Wv=W^{-1}f$, $P_W=W^{-1}PW$ may be easier to solve.

A typical W is given by

$$W(x,\xi)=T(x,\xi)^M,\ T(x,\xi)=\big(t(x,\xi)^2+\lambda\langle\xi\rangle^{-1}\big)^{1/2}+t(x,\xi)$$

where $\lambda\geq 1$ is a large parameter. We see that $T(x,\xi)\approx\lambda^{1/2}\langle\xi\rangle^{-1}$ in $t(x,\xi)<0$ and $T(x,\xi)\approx 1$ in $t(x,\xi)>0$. Moreover we have for $|\alpha|=1$

$$\partial_x^\alpha T=\partial_x^\alpha t\phi^{-1}T,\ \ \partial_\xi^\alpha T=\partial_\xi^\alpha t\phi^{-1}T+\lambda\partial_\xi\langle\xi\rangle^{-1}\phi^{-1}/2$$
$$\phi(x,\xi)=(t(x,\xi)^2+\lambda\langle\xi\rangle^{-1})^{1/2}$$

and hence

$$\nabla_{(x,\xi)}T/T\sim\phi^{-1}\nabla_{(x,\xi)}t.$$

It is easy to see that

$$|\partial_x^\beta\partial_\xi^\alpha T(x,\xi)|\leq C_{\alpha\beta}T(x,\xi)\phi(x,\xi)^{-|\beta|}\psi(x,\xi)^{-|\alpha|} \tag{2.5.1}$$

where $\psi(x,\xi)=\phi(x,\xi)\langle\xi\rangle$. That is, introducing the metric

$$g_{(x,\xi)}(y,\eta)=\phi(x,\xi)^{-2}|y|^2+\psi(x,\xi)^{-2}|\eta|^2$$

(2.5.1) is stated as

$$|T(x,\xi)|_k^g/T(x,\xi)\leq C_{\alpha\beta}$$

or one can write $T(x,\xi)\in S(T(x,\xi),g)$. This metric g is equivalent to the one which defines the class $S_{1,0}$ in the region where $t(x,\xi)\neq 0$ while this is equivalent to the metric defining the class $S_{1/2,1/2}$ on the set $t(x,\xi)=0$.

Noting that

$$g^\sigma_{(x,\xi)}(y,\eta)=(\phi\psi)^2g_{(x,\xi)}(y,\eta)$$

and $\phi\psi\geq\lambda$, taking λ large compared with M, $T(x,D)^M$ is well controlled in $S_{1/2,1/2}$. But in our case we are forced to take $\lambda\ll M^2$. We explain intuitively why we must take $\lambda\ll M^2$. Let us consider

$$P=p_1p_2,\quad p_i(x,\xi)=\xi_1-q_i(x,\xi')$$

where $q_i(x,\xi')$ are real valued of order 1 and $\xi'=(\xi_2,...,\xi_n)$. Let us assume that

$$|q_1(x,\xi')-q_2(x,\xi')|\geq c|t(x,\xi')||\xi'|$$

and $\langle dp_j, H_t\rangle > 0$, $j = 1, 2$ on $t(x, \xi') = 0$, that is $q_i(x, \xi)$ may coincide on the space-like surface $t(x, \xi') = 0$ where H_t denotes the Hamilton vector field of t. We study P_W with $W = T^M$. Roughly speaking the symbol $P_W(x, \xi)$ has the form

$$P_W(x,\xi) \sim P((x,\xi) + iM\phi^{-1}H_t) \sim \prod_{j=1}^{2}(p_j(x,\xi) + iM\phi^{-1}). \tag{2.5.2}$$

Let us take

$$Q(x,\xi) = \frac{1}{2}\sum_{j=1}^{2}(p_j(x,\xi) + iM\phi^{-1})$$

as a separating operator and consider

$$\mathrm{Im}(P_W u, Qu)_{L^2(\mathbf{R}^n)} = (Su, u)_{L^2(\mathbf{R}^n)} \tag{2.5.3}$$

where $S = (2i)^{-1}(Q^* P_W - P_W^* Q)$. Then we see

$$\begin{aligned} S(x,\xi) &\sim M\phi^{-1}\sum_{j=1}^{2}|p_j(x,\xi) + iM\phi^{-1}|^2 \\ &\geq M\phi^{-1}\sum_{j=1}^{2}p_j(x,\xi)^2 + M^2\phi^{-2} \\ &\geq cM\phi^{-1}[p_1^2 + p_2^2 + (q_1 - q_2)^2 + M^2\phi^{-2}] \\ &\geq cM\phi^{-1}[p_1^2 + p_2^2 + \{t(x,\xi')^2|\xi'|^2 + M^2\phi^{-2}\}] \end{aligned}$$

with some $c > 0$. Here we recall an elementary inequality which is an essence of our arguments: let $\lambda \geq M$ then

$$\begin{aligned} t(x,\xi')^2|\xi'|^2 + M^2\phi^{-2} &\geq M^2\lambda^{-2}(t(x,\xi')^2|\xi'|^2 + \lambda^2\phi^{-2}) \\ \geq M^2\lambda^{-2}\phi^{-2}|\xi'|^2(t^2\phi^2 + \lambda^2|\xi'|^{-2}) &\geq M^2\lambda^{-2}\phi^2|\xi'|^2 = M^2\lambda^{-2}\psi^2. \end{aligned}$$

This estimate from below gives that

$$\left|S^{(\alpha)}_{(\beta)}/S\right| \leq C\lambda M^{-1}\phi^{-|\beta|}\psi^{-|\alpha|}$$

and hence we have $S^t \in S(S^t, (\lambda M^{-1})^2 g)$ for $t \in \mathbf{R}$. Hence we conclude that

$$S \cdot Op(1/S) - 1 \in S(M^{-2}\lambda, (\lambda M^{-1})^2 g).$$

Thus if $\lambda \ll M^2$ then we can construct a E such that $E^*SE \sim 1$. This gives that

$$(Su, u) \succeq \|E^{-1}u\|^2 \succeq \|u\|^2_{H^{-k}(\mathbf{R}^n)}.$$

From (2.5.2) it follows that

$$C\|P_W u\|^2_{H^{k+2}(\mathbf{R}^n)} + C^{-1}\|u\|^2_{H^{-k}(\mathbf{R}^n)} \succeq \|u\|^2_{H^{-k}(\mathbf{R}^n)}$$

and we get an apriori estimate.

Remark: By a symplectic change of coordinates (x,ξ) one may assume that $dt = adx$ at the reference point. Then T^M beongs to $S_{1,1/2}$ and one can avoid the arguments about M, λ. But if we consider several $P^{(j)}$ at the same time:

$$P^{(j)} = p_1^{(j)} p_2^{(j)}, \;\; p_k^{(j)} = \xi_1 - q_k^{(j)}(x,\xi'), \quad k = 1,2, \;\; j = 1,...,\ell$$
$$|q_1^{(j)}(x,\xi') - q_2^{(j)}(x,\xi')| \geq c|t^{(j)}(x,\xi')||\xi'|$$

where $t^{(1)}(x,\xi')$,...,$t^{(\ell)}(x,\xi')$ are *not in involution* then the method of symplectic change of coordinates does not work.

We give another example. Let us consider

$$P = p_1 p_2 p_3, \quad p_j = \xi_1 - q_j(x,\xi').$$

We assume that there are time functions $\alpha(x,\xi')$, $\beta(x,\xi')$, $\gamma(x,\xi')$ such that

$$|q_1 - q_2| \geq c|\alpha(x,\xi')|, \;\; |q_2 - q_3| \geq c|\beta(x,\xi')|, \;\; |q_3 - q_1| \geq c|\gamma(x,\xi')|$$

where α, β, γ are *not in involution.*

To both cases the method employed in this lecture can be applicable.

We now check

$$P((x,\xi) + iM\phi^{-1}H_t) \sim \prod_{j=1}^{2}(p_j(x,\xi) + iM\phi^{-1})$$

in (2.5.2) more carefully assuming that $p(x,\xi)$ is real analytic in (x,ξ) for simplicity. Let us consider

$$p = -\xi_1^2 + q(x,\xi')$$

where $q(x,\xi') \geq 0$ is of degree 2 and $q(\hat{x},\hat{\xi}') = 0$. Let f be a time function at $\hat{z} = (\hat{x},\hat{\xi})$ so that

$$-H_f(\hat{z}) \in \Gamma_{\hat{z}}.$$

Note that

Lemma 2.5.1 ([13]) *We can choose a symplectic coordinates around $\hat{z}$ preserving the x_1 coordinates with which we have*

$$p = -\xi_1^2 + \ell(x,\xi')^2 + b(x,\xi'),$$
$$e(x,\xi')^{-1} f = x_1 - \psi(x',\xi')$$

where $\ell(\hat{z}) = 0$, $b(\hat{z}) = 0$, $b(x,\xi') \geq 0$, $e(\hat{z}) > 0$ and moreover

$$H_f^2 b(\hat{z}) = 0.$$

In this coordinates system it is easy to see that $f = x_1 - \psi(x',\xi')$ is a time function at $\hat{z}$ if and only if

$$\{\ell,\psi\}(\hat{z})^2 < 1. \tag{2.5.4}$$

We now prove

Lemma 2.5.2 *Assume that $f = x_1 - \psi(x', \xi')$ is a time function. Then we have*

$$C^{-1}|p(z + i\rho H_f)| \le |p(z + i\rho(0, \theta))| \le C|p(z + i\rho H_f)| \tag{2.5.5}$$

in a small neighborhood $z \in U$ of $\hat{z}$ and small $0 < \rho \ll 1$.

Proof: Let us denote $g = O^*(\rho^s)$ if for every $\epsilon > 0$ there is a neighborhood U of $\hat{z}$ such that

$$|g(z)/\rho^s| \le \epsilon, \quad z \in U$$

is verified. Note that

$$|p(z + i\rho(0, \theta))| \ge c\rho^2$$

with some $c > 0$ because

$$|p(z + i\rho(0, \theta))| \sim | - \xi_1^2 + \ell^2 + b + \rho^2| + \rho|\xi_1|. \tag{2.5.6}$$

Then to prove Lemma 2.5.2 it is enough to show (2.5.5) modulo $O^*(\rho^2)$. Consider

$$\begin{aligned} p((x,\xi) + i\rho H_f(x,\xi')) = -(\xi_1 + i\rho)^2 + \ell((x,\xi') - i\rho H_\psi) \\ +b((x,\xi') - i\rho H_\psi) = -\xi_1^2 + \rho^2 + [\ell(x,\xi') + i\rho\{\ell,\psi\}(x,\xi') + O(\rho^2)]^2 \\ +b(x,\xi') + i\rho\{b,\psi\} + O^*(\rho^2) \end{aligned}$$

because $H_\psi^2 b(\hat{z}) = 0$. Since $\ell(\hat{z}) = 0$ one has

$$p(z + i\rho H_f) = (-\xi_1^2 + \ell^2 + b + \rho^2) + 2i\rho(\xi_1 + \{\ell,\psi\}\ell + \{b,\psi\}/2) + O^*(\rho^2)$$

and hence

$$|p(z + i\rho H_f)| \sim | - \xi_1^2 + \ell^2 + b + \rho^2| + \rho|\xi_1 + \{\ell,\psi\}\ell + \{b,\psi\}/2|.$$

Assume $|\xi_1| \ge (1 + \delta)|\{\ell,\psi\}\ell + \{b,\psi\}/2|$ with small $\delta > 0$. Then it is clear that

$$C^{-1}|\xi_1| \le |\xi_1 + \{\ell,\psi\}\ell + \{b,\psi\}/2| \le C|\xi_1|$$

and hence the result. Let us now assume that

$$|\xi_1| \le (1 + \delta)|\{\ell,\psi\}\ell + \{b,\psi\}/2|.$$

By the assumption we have

$$|\xi_1| \le (1 + \delta)(1 - \epsilon')|\ell| + (1 + \delta)\epsilon''\sqrt{b}$$

where we may assume that $(1 + \delta)(1 - \epsilon') \le 1 - \epsilon_1$ and one can take ϵ'' as small as we please if we take a neighborhood U small since $b \ge 0$ and $H_\psi^2 b(\hat{z}) = 0$. Hence one has

$$\xi_1^2 \le (1 - \epsilon_1')\ell^2 + \epsilon_2'' b.$$

This proves that

$$| - \xi_1^2 + \ell^2 + b + \rho^2| \ge c(\ell^2 + b + \rho^2)$$

with some $c > 0$. Noting that

$$\begin{aligned}\epsilon\rho|\xi_1| - \epsilon C(\ell^2 + b + \rho^2) \le \rho|\xi_1 + \{\ell, \psi\}\ell + \{b, \psi\}/2| \\ \le \rho|\xi_1| + C(\ell^2 + b + \rho^2)\end{aligned}$$

we get the desired assertion.

The result is shown for hyperbolic polynomials p of order m in §8. To show

$$P_W(x,\xi) \sim P((x,\xi) + iM\phi^{-1}H_t)$$

we need a calculus of pseudodifferential operators with large parameters M, λ which is the subject in subsequent sections.

3 Pseudodifferential operators with large parameters

3.1 Metric associated to a time function

We first introduce a metric with two large parameters which will be used throughout the lecture. Let $t(z)$ be a real valued smooth bounded function in $\mathbf{R}^{2n}$ satisfying

$$|t(z+w) - t(z)| \le C\{|y| + |\xi|^{-1}|\eta|\} \tag{3.1.1}$$

for $z = (x,\xi), w = (y,\eta) \in \mathbf{R}^{2n}$ and $|\eta| \le |\xi|/2$. Let $\Theta(t) \in C^\infty(\mathbf{R})$ be such that $0 \le \Theta(t) \le 1$, $\Theta(t) = 1$ for $|t| \le 1$ and $\Theta(t) = 0$ for $|t| \ge 2$. Set $\Theta_h(\xi) = \theta(|\xi|/h)$ and $t_h(z) = (1 - \Theta_h(\xi))t(z)$. We also denote $\langle\xi\rangle_h^2 = h^2 + |\xi|^2$ where h is a large parameter. Then $t_h(z)$ satisfies

$$|t_h(z+w) - t_h(z)| \le C\{|y| + \langle\xi\rangle_h^{-1}|\eta|\} \tag{3.1.2}$$

for $z, w \in \mathbf{R}^{2n}$. In fact for $|\eta| \le \langle\xi\rangle_h/2$ we have $|\xi| \ge h/3$ on the support of $1 - \Theta_h(\xi+\eta)$ and hence $\langle\xi\rangle_h \approx \langle\xi\rangle$. Moreover on the support of $\Theta'((\xi + s\eta)/h)$, $s \in \mathbf{R}$, $|s| \le 1$ we have $|\xi + s\eta| \approx h + |\xi + s\eta| \approx \langle\xi\rangle_h$ then (3.1.2) follows from (3.1.1) easily. For $|\eta| \ge \langle\xi\rangle_h/2$ we have $|t_h(z+w) - t_h(z)| \le C \le 2C|\eta|\langle\xi\rangle_h^{-1} \le 2C(|y| + \langle\xi\rangle_h^{-1}|\eta|)$.

Let $\lambda \ge 1$ be a large parameter. We set

$$\begin{aligned}\langle\xi\rangle_h^{-1/2}T(z) = (t_h(z)^2 + \lambda\langle\xi\rangle_h^{-1})^{1/2} + t_h(z) = \phi + t_h, \\ \phi(z) = (t_h^2 + \lambda\langle\xi\rangle_h^{-1})^{1/2}, \;\; \psi(z) = \phi(z)\langle\xi\rangle_h, \\ g_z(w) = \phi(z)^{-2}|y|^2 + \psi(z)^{-2}|\eta|^2 = \phi(z)^{-2}(|y|^2 + \langle\xi\rangle_h^{-2}|\eta|^2), \\ g_z^\sigma(w) = \psi(z)^2|y|^2 + \phi(z)^2|\eta|^2 = (\phi\psi)^2 g_z(w).\end{aligned}$$

Since $\phi(z)^{-2} = \langle\xi\rangle_h(t_h(z)^2\langle\xi\rangle_h + \lambda)^{-1}$ the metric is equivalent to the one which defines the class $S_{1,0}$ in the region away from $t(z) = 0$ while this is equivalent to the metric defining the class $S_{1/2,1/2}$ on the set $t(z) = 0$. We always assume that $h \geq \lambda \geq 1$ and hence $\phi(z) = \{t_h(z)^2 + \lambda\langle\xi\rangle_h^{-1}\}^{1/2} \leq C$.

Following Hörmander [5] $g_z(w)$ is said to be slowly varying if there are positive constants c, C such that

$$g_z(w) \leq c \Longrightarrow C^{-1}g_z(X) \leq g_{z+w}(X) \leq Cg_z(X), \quad \forall X \in \mathbf{R}^{2n}.$$

A slowly varying metric $g_z(X)$ is called σ temperate if there exist C, N such that

$$g_w(X) \leq Cg_z(X)(1 + g_w^\sigma(w-z))^N, \quad \forall X \in \mathbf{R}^{2n}.$$

We say that a positive real valued function $m(z)$ defined on $\mathbf{R}^{2n}$ is g continuous if there are positive constants c and C such that

$$C^{-1}m(z) \leq m(z+w) \leq Cm(z) \tag{3.1.3}$$

if $g_z(w) \leq c$ and a g continuous function $m(z)$ is said to be σ, g temperate if there are C and $\ell \in \mathbf{R}$ such that

$$m(w) \leq Cm(z)\left(1 + g_w^\sigma(w-z)\right)^\ell \tag{3.1.4}$$

for $z, w \in \mathbf{R}^{2n}$.

Lemma 3.1.1 *Assume (3.1.1) and $h \geq \lambda \geq 1$. Then $\langle\xi\rangle_h$ and $n(z) = \langle\xi\rangle_h^{1/2}\phi$ are g continuous.*

Proof: Since $g_z(w) \geq \psi^{-2}|\eta|^2 \geq \langle\xi\rangle_h^{-1}n(z)^{-2}|\eta|^2 \geq c'\langle\xi\rangle_h^{-2}|\eta|^2$ for $h \geq \lambda$ we can take c such that $|\eta| \leq \langle\xi\rangle_h/2$ if $g_z(w) \leq c$. Then it is clear that $\langle\xi\rangle_h$ is g continuous because $\langle\xi+\eta\rangle_h \approx \langle\xi\rangle_h$ if $|\eta| \leq \langle\xi\rangle_h/2$. Recall that

$$|n(z+w) - n(z)| = |t_h(z+w)\langle\xi+\eta\rangle_h^{1/2} - t_h(z)\langle\xi\rangle_h^{1/2}|$$
$$\times|t_h(z+w)\langle\xi+\eta\rangle_h^{1/2} + t_h(z)\langle\xi\rangle_h^{1/2}|\left(n(z+w) + n(z)\right)^{-1}.$$

From (3.1.2) and $n(z) \geq |t_h(z)|\langle\xi\rangle_h^{1/2}$ it follows that this is estimated by $C\langle\xi\rangle_h^{1/2}(|y| + \langle\xi\rangle_h^{-1}|\eta|)$ if $\langle\xi+\eta\rangle_h \approx \langle\xi\rangle_h/2$. From the definition we have $n(z)^{-1}\langle\xi\rangle_h^{1/2}\left(|y| + \langle\xi\rangle_h^{-1}|\eta|\right) \leq g_z(w)^{1/2}$ and this implies that

$$|n(z+w)/n(z) - 1| \leq Cg_z(w)^{1/2} \tag{3.1.5}$$

when $\langle\xi+\eta\rangle_h \approx \langle\xi\rangle_h/2$. Hence if c is small $n(z)$ satisfies (3.1.3), that is $n(z)$ is g continuous.

Corollary 3.1.1 *Assume (3.1.1) and $h \geq \lambda \geq 1$. Then ϕ and ψ are g continuous and g is slowly varying.*

Lemma 3.1.2 *Assume (3.1.1) and $h \geq \lambda \geq 1$. Then*

(i) $\phi(z+w)^{-1} \le C\phi(z)^{-1}(1+g_z^\sigma(w))^{1/4}$, $\forall z$, $\forall w \in \mathbf{R}^{2n}$,

(ii) $\phi(z+w) \le C\phi(z)(1+g_z^\sigma(w))^{1/2}$, $\forall z$, $\forall w \in \mathbf{R}^{2n}$,

(iii) ψ *also satisfies (i) and (ii).*

Proof: Since ϕ and ψ are g continuous (i) (ii) and (iii) are valid clearly if $g_z(w) \le c$. To simplify notations we denote $\phi = \phi(z)$, $\phi_1 = \phi(z+w)$, $\psi = \psi(z)$, $\psi_1 = \psi(z+w)$, $g = g_z(w)$, $g_1 = g_{z+w}(w)$, $g^\sigma = g_z^\sigma(w)$ and $g_1^\sigma = g_{z+w}^\sigma(w)$. Asuume that $g \ge c$. Note that $\psi_1\phi_1^{-1} = \langle \xi+\eta\rangle_h \approx \langle\xi\rangle_h = \psi\phi^{-1}$ if $|\eta| \le \langle\xi\rangle_h/2$. Recall that $\phi_1\psi_1 \ge 1$ and $g^\sigma = (\phi\psi)^2 g \ge c(\phi\psi)^2$. Hence we obtain

$$\phi_1^2 = (\phi_1\psi_1)\phi_1\psi_1^{-1} \ge \phi_1\psi_1^{-1} \ge c_1\phi\psi^{-1} \ge c_2\phi^2(g^\sigma)^{-1/2}$$

which implies (i) for $|\eta| \le \langle\xi\rangle_h/2$. For $|\eta| \ge \langle\xi\rangle_h/2$ noting that $\phi^2|\eta|^2 \le g^\sigma$ and $\phi \le C$ it follows that

$$\phi_1^{-1}\phi^{-1/2}(g^\sigma)^{1/4} \le C_2\phi^{-1}(g^\sigma)^{1/4}$$

which proves (i) for $|\eta| \ge \langle\xi\rangle_h/2$. We shall show (ii). When $g_1 \le c$ we have $\phi_1 \approx \phi$ and $\psi_1 \approx \psi$. Therefore we get (ii) easily if $g_1 \le c$. When $g_1 \ge c$ and $|\eta| \le \langle\xi\rangle_h/2$ remarking that $|y|^2 \le \phi^2 g$, $|\eta|^2 \le \psi^2 g$ and $\phi_1\psi_1^{-1} \approx \phi\psi^{-1}$ we have $c(\phi_1\psi_1)^2 \le (\phi_1\psi_1)^2 g_1 = \psi_1^2|y|^2 + \phi_1^2|\eta|^2 \le \psi_1^2\phi^2 g + \phi_1^2\psi^2 g$. Hence $\phi_1^2 \le c^{-1}(\phi^2 + (\phi_1\psi_1^{-1})^2\psi^2)g \le C_1(\phi^2 + (\phi\psi^{-1})^2\psi^2)g \le 2C_1\phi^2 g \le C_2\lambda^{-2}\phi^2 g^\sigma$ which implies (ii) if $g_1 \ge c$ and $|\eta| \le \langle\xi\rangle_h/2$. When $|\eta| \ge \langle\xi\rangle_h/2$ from the fact that $\langle\xi\rangle_h^{-1/2} \le \phi \le C$ and $\phi^2|\eta|^2 \le g^\sigma$ we have $\langle\xi\rangle_h \le 2|\eta| \le C_1\langle\xi\rangle_h^{1/2}(g^\sigma)^{1/2}$. Hence $\phi_1 \le C \le C\langle\xi\rangle_h^{1/2}\phi \le C_1\phi(g^\sigma)^{1/2}$ if $|\eta| \ge \langle\xi\rangle_h/2$. Thus we have proved (i) and (ii) for ϕ. Similar arguments prove (iii).

Corollary 3.1.2 *Assume (3.1.1) and $h \ge \lambda \ge 1$. Then $\phi^\pm$ and $\psi^\pm$ are σ, g temperate and g is σ temperate.*

Lemma 3.1.3 *Assume (3.1.1), $h \ge \lambda \ge 1$. Then $T(z) = n(z) + t_h(z)\langle\xi\rangle_h^{1/2}$ is g continuous and*

$$T(z+w)^{\pm 1} \le CT(z)^{\pm 1}(1+g_z^\sigma(w))$$

for $z, w \in \mathbf{R}^{2n}$. In particular $T(z)$ is σ, g temperate.

Proof: Recall that

$$\begin{aligned} |T(z+w)-T(z)| = |t_h(z+w)\langle\xi+\eta\rangle_h^{1/2} - t_h(z)\langle\xi\rangle_h^{1/2}| \\ \times|T(z+w)+T(z)|\left(n(z+w)+n(z)\right)^{-1}. \end{aligned} \tag{3.1.6}$$

Noting n is g continuous we conclude from the same argument proving (3.1.5) that

$$|T(z+w)/T(z) - 1| \le C\left(T(z+w)/T(z)+1\right)g_z(w)^{1/2}$$

if $g_z(w) \le c$. This proves that T is g continuous. We next prove that T verifies the last assertion. If $|\eta| \ge \langle\xi\rangle_h/2$ then

$$|\eta| \le \phi(z)^{-1} g_z^\sigma(w)^{1/2} \le C g_z^\sigma(w)$$

which implies

$$|T(z+w)/T(z)| \le C|\eta| \le C g_z^\sigma(w)$$

since $|T(z+w)| \le C|\eta|^{1/2}$ and $CT(z) \ge \langle\xi\rangle_h^{-1/2}$. If $|\eta| \le \langle\xi\rangle_h^{1/2}$ then (3.1.6) shows that

$$|T(z+w) - T(z)| \le C\left(T(z+w) + T(z)\right)\left(\langle\xi\rangle_h^{1/2}|y| + \langle\xi\rangle_h^{-1/2}|\eta|\right) \\ \times \left(n(z+w) + n(z)\right)^{-1}.$$

Noting that $\langle\xi\rangle_h|y|^2 + \langle\xi\rangle_h^{-1}|\eta|^2 = (\phi\psi)(z)^{-1} g_z^\sigma(w)$ we obtain

$$|T(x+y,\xi+\eta)/T(x,\xi) - 1| \le C\left(T(z+w)/T(z) + 1\right) \\ \times g_z^\sigma(w)^{1/2}\left(n(z+w)+n(z)\right)^{-1}(\phi\psi)(z)^{-1/2}. \tag{3.1.7}$$

Assume $\left(n(z+w)+n(z)\right)(\phi\psi)^{1/2} \ge 2Cg_z^\sigma(w)^{1/2}$ then it follows from (3.1.7) that $T(z+w) \le 3T(z)$. If $\left(n(z+w)+n(z)\right)(\phi\psi)^{1/2} \le 2Cg_z^\sigma(w)^{1/2}$ we have

$$T(z+w)/T(z) \le 4n(z+w)n(z) \\ \le 8C^2(\phi\psi)^{-1} g_z^\sigma(w) \le 8C^2 g_z^\sigma(w)$$

because $2^{-1}n(z)^{-1} \le T(z) \le 2n(z)$. Thus we have proved the last assertion for T and in particular T is σ, g temperate. Similar argument proves the assertion for T^{-1}.

3.2 Symbols and weights

With $M > 0$, a large parameter, we consider

$$m(z) = T(z)^M. \tag{3.2.1}$$

Since $T(z)$ is g continuous and σ, g temperate we see that

$$m(z)/C^M \le m(z+w) \le C^M m(z) \tag{3.2.2}$$

if $g_z(w) \le c$ and

$$m(w) \le C^M m(z)(1 + g_w^\sigma(w-z))^{MN} \tag{3.2.3}$$

where C, N are independent of h, λ and M.

Motivated by the above observations, we say that $m(z)$ is a σ, g temperate weight function if $m(z)$ verifies (3.2.2) and (3.2.3). We next examine derivatives of $T(z)^M$. Let us denote

$$\omega_\beta^\alpha(z) = T(z)^{-M} D_x^\beta \partial_\xi^\alpha T(z)^M.$$

To control the behavior (with respect to M) of derivatives of $T(z)^M$, we assume that there is $C > 0$ such that for every k we have

$$\begin{aligned} &|\phi|^g_k(z)/\phi(z) \leq C^{k+1}k!^\kappa, \\ &|\psi|^g_k(z)/\psi(z) \leq C^{k+1}k!^\kappa, \\ &|T|^g_k(z)/T(z) \leq C^{k+1}k!^\kappa. \end{aligned} \tag{3.2.4}$$

Lemma 3.2.1 *Assume (3.2.4). Then there is a positive constant C independent of M, λ and h such that*

(i)

$$|\omega^\alpha_\beta|^g_k(z) \leq C^{|\alpha+\beta|+k+1}\phi^{-|\beta|}\psi^{-|\alpha|}\sum_{j=0}^{|\alpha+\beta|}(CM)^{|\alpha+\beta|-j}(k+j)!^\kappa,$$

for $z \in \mathbf{R}^{2n}$. In particular we have

(ii)

$$|T^M|^g_k(z) \leq C^{k+1}\sum_{j=0}^{k}(CM)^{k-j}j!^\kappa T(z)^M.$$

We first show

Lemma 3.2.2 *Let $\kappa \geq 1$ and $M \geq 1$. Then there is $C > 0$ (independent of M) such that*

(i) $j!k! \leq (j+k)! \leq 2^{j+k}j!k!$ *for $j, k = 0, 1, ...,$*

(ii) $j! \leq j^j \leq C^j j!$ *for $j = 0, 1, ...,$*

(iii) $\sum_{\alpha' \leq \alpha}\begin{pmatrix}\alpha \\ \alpha'\end{pmatrix}A_1^{|\alpha'|}|\alpha'|!^\kappa A_2^{|\alpha-\alpha'|}|\alpha-\alpha'|!^\kappa \leq A_1^{1+|\alpha|}(A_1-A_2)^{-1}|\alpha|!^\kappa$ *for $A_1 > A_2$, $|\alpha| = 0, 1, ...,$*

(iv) $\sum_{\alpha' \leq \alpha}\begin{pmatrix}\alpha \\ \alpha'\end{pmatrix}(|\alpha'|^\kappa + M)^{|\alpha'|}(|\alpha-\alpha'|^\kappa + M)^{|\alpha-\alpha'|} \leq 2^{|\alpha|}(|\alpha|^\kappa + M)^{|\alpha|}$ *for $|\alpha| = 0, 1, ...,$*

(v) $((j+k)^\kappa + M)^{j+k} \leq C^{j+k+M}(j^\kappa + M)^j k!^\kappa$ *for $j, k = 0, 1, ...$*

Proof: (i) is obvious and (ii) follows from Stirling's formula. Noting that $\sum_{|\alpha'|=j}\begin{pmatrix}\alpha \\ \alpha'\end{pmatrix} = \begin{pmatrix}|\alpha| \\ j\end{pmatrix}$ we see that

$$\begin{aligned} \sum_{\alpha' \leq \alpha}\begin{pmatrix}\alpha \\ \alpha'\end{pmatrix}A_1^{|\alpha'|}|\alpha'|!^\kappa A_2^{|\alpha-\alpha'|}|\alpha-\alpha'|!^\kappa &\leq \sum_{j=0}^{|\alpha|}\begin{pmatrix}|\alpha| \\ j\end{pmatrix}^{1-\kappa}A_1^j A_2^{|\alpha|-j}|\alpha|!^\kappa \\ &\leq |\alpha|!^\kappa \sum_{j=0}^{|\alpha|}A_1^j A_2^{|\alpha|-j} \end{aligned}$$

which proves (iii). The inequality

$$(|\alpha'|^\kappa + M)^{|\alpha'|}(|\alpha-\alpha'|^\kappa + M)^{|\alpha-\alpha'|} \le \max\{(|\alpha'|^\kappa + M)^{|\alpha|}, (|\alpha-\alpha'|^\kappa + M)^{|\alpha|}\} \le (|\alpha|^\kappa + M)^{|\alpha|}$$

implies (iv). Noticing $(j+k)^\kappa + M \le C_\kappa(j+k+[M^{1/\kappa}])^\kappa$ and setting $l = [M^{1/\kappa}]$, where $[M^{1/\kappa}]$ stands for the integral part of $M^{1/\kappa}$, we get by use of (ii) that

$$(j+k+l)^{j+k} \le C^{j+k+l}(j+k+l)!(j+k+l)^{-l} \le (2C)^{j+k+l}(j+l)!k!(j+k+l)^{-l} \le C_1^{j+k+l}(j+l)^{j+l}(j+k+l)^{-l}k! \le C_2^{j+k+l}(j+l)^j k!.$$

Since $(j+[M^{1/\kappa}])^\kappa \le C_3(j^\kappa + M)$ we obtain (v).

Proof of Lemma 3.2.1: (i) We shall prove the assertion by induction on $|\alpha+\beta|$. For $\alpha=\beta=0$ then (i) is obvious. We assume that $|\alpha+\beta|>0$ and there are $C>0$, $A_1>0$ and $A_2>0$ such that

$$|\omega^{\alpha(\gamma)}_{\beta(\delta)}(z)| \le C(A_1\phi^{-1})^{|\gamma+\delta|}(A_2\phi^{-1})^{|\alpha+\beta|}\langle\xi\rangle_h^{-|\alpha+\gamma|} \times \sum_{j=0}^{|\alpha+\beta|}(CM)^{|\alpha+\beta|-j}(|\gamma+\delta|+j)!^\kappa. \tag{3.2.5}$$

Noting $\psi^{-1} = \phi^{-1}\langle\xi\rangle_h^{-1}$ we get (ii) from (3.2.5) easily. Since $\omega^{\alpha+e_1}_{\beta+e_2} = \omega^{\alpha(e_1)}_{\beta(e_2)} \pm MT^{(e_1)}_{(e_2)}\omega^\alpha_\beta$ for $|e_1+e_2|=1$ using (i) of Lemma 3.2.1 and Lemma 3.2.2 it follows that

$$
\begin{aligned}
|\omega^{\alpha+e_1(\gamma)}_{\beta+e_2(\delta)}| &= \Big|\omega^{\alpha(\gamma+e_1)}_{\beta(\delta+e_2)} + M\sum \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \begin{pmatrix} \delta \\ \delta' \end{pmatrix} T^{(e_1+\gamma')}_{(e_2+\delta')}\omega^{\alpha(\gamma-\gamma')}_{\beta(\delta-\delta')}\Big| \\
&\le C(A_1\phi^{-1})^{|\gamma+\delta|+1}(A_2\phi^{-1})^{|\alpha+\beta|}\langle\xi\rangle_h^{-|\alpha+\gamma+e_1|} \\
&\quad \times \sum_{j=0}^{|\alpha+\beta|}(CM)^{|\alpha+\beta|-j}(|\gamma+\delta|+1+j)!^\kappa \\
&\quad + \sum \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \begin{pmatrix} \delta \\ \delta' \end{pmatrix}(CM)C^{|\gamma'+\delta'|+1}(|\gamma'+\delta'|+1)!^\kappa \\
&\quad \times\phi^{-|e_2+\delta'|}(\phi^{-1}\langle\xi\rangle_h^{-1})^{|e_1+\gamma'|}C(A_1\phi^{-1})^{|\gamma+\delta-\gamma'-\delta'|}(A_2\phi^{-1})^{|\alpha+\beta|}\langle\xi\rangle_h^{-|\gamma-\gamma'+\alpha|} \\
&\quad \times \sum_{j=0}^{|\alpha+\beta|}(CM)^{|\alpha+\beta|-j}(|\gamma-\gamma'+\delta-\delta'|+j)!^\kappa \\
&\le A_1A_2^{-1}C(A_1\phi^{-1})^{|\gamma+\delta|}(A_2\phi^{-1})^{|\alpha+\beta|+1}\langle\xi\rangle_h^{-|\alpha+\gamma+e_1|} \\
&\quad \times \sum_{j=1}^{|\alpha+\beta|+1}(CM)^{|\alpha+\beta|+1-j}(|\gamma+\delta|+j)!^\kappa \\
&\quad +2^\kappa CA_1A_2^{-1}(A_1-2^\kappa C)^{-1}C(A_1\phi^{-1})^{|\gamma+\delta|}(A_2\phi^{-1})^{|\alpha+\beta|+1} \\
&\quad \times\langle\xi\rangle_h^{-|\alpha+\gamma+e_1|}\sum_{j=0}^{|\alpha+\beta|}(CM)^{|\alpha+\beta|+1-j}(|\gamma+\delta|+j)!^\kappa.
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
\sum \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \begin{pmatrix} \delta \\ \delta' \end{pmatrix} & C^{|\gamma'+\delta'|+1}(|\gamma'+\delta'|+1)!^\kappa A_1^{|\gamma+\delta-\gamma'-\delta'|}(|\gamma-\gamma'+\delta-\delta'|+j)!^\kappa \\
&\le \sum \begin{pmatrix} |\gamma+\delta| \\ k \end{pmatrix}(2^\kappa C)^{k+1}k!^\kappa A_1^{|\gamma+\delta|-k}(|\gamma+\delta|-k+j)!^\kappa \\
&\le 2^\kappa CA_1(A_1-2^\kappa C)^{-1}A_1^{|\gamma+\delta|}(|\gamma+\delta|+j)!^\kappa.
\end{aligned}
$$

Thus we get (3.2.5) by induction if we choose A_1 and A_2 so that $A_2^{-1} + 2^\kappa C^2A_1A_2^{-1}(A_1-2^\kappa C)^{-1} \le 1$.

Since $\sum_{j=0}^k M^{k-j}j!^\kappa \le (k^\kappa+M)^k$ it follows from Lemma 3.2.1 that

$$
|T^M|_k^g(z) \le C^{k+1}(k^\kappa+M)^kT(z)^M \tag{3.2.6}
$$

$$
|\omega^\alpha_\beta|_k^g(z) \le C^{|\alpha+\beta|+k}\phi^{-|\beta|}\psi^{-|\alpha|}k!^\kappa(|\alpha+\beta|^\kappa+M)^{|\alpha+\beta|}
$$

for $z \in \mathbf{R}^{2n}$. Motivated by the above estimate of T^M we introduce a class of symbols.

Definition 3.2.1: Let $m(z)$ be a σ, g temperate weight function. We say $p \in S(m,g)$ if p satisfies the estimates

$$
|p|_k^g(z) \le CA^k(k^\kappa+M)^km(z) \tag{3.2.7}
$$

where C, A are independent of M, h and λ.

By the definition we have

$$T(z)^M \in S(T(z)^M, g),$$
$$\omega_\beta^\alpha(z) \in S(C_{\alpha\beta}\phi^{-|\beta|}\psi^{-|\alpha|}(|\alpha+\beta|^\kappa + M)^{|\alpha+\beta|}, g).$$

3.3 A lemma for pseudodifferential calculus

Let $a_i(z) \in S(m_i, g)$. The Weyl-Hörmander calculus associates with $a_i(z)$ the operator

$$a_i^w(x, D)u = \int\int e^{i\langle x-y,\xi\rangle} a_i(\frac{x+y}{2}, \xi)u(y)dyd\xi.$$

Then the symbol of $a_1^w(x, D)a_2^w(x, D)$ is given by $a_1\#a_2$ where

$$a_1\#a_2(z) = \sum(i\sigma(D_z; D_w)/2)^j a_1(z)a_2(w)/j!|_{w=z}.$$

Let us set

$$r_k(z) = a_1\#a_2(z) - \sum_{j<k}(i\sigma(D_z; D_w)/2)^j a_1(z)a_2(w)/j!|_{w=z}$$

and estimate $r_k(z)$. We follow Hörmander [5] line to line, keeping the dependence and independence of large parameters in mind. In what follows constants are independent of h, λ and M if otherwise stated.

Proposition 3.3.1 *Let $0 < \delta < 1$ be $1 < \kappa < 2 - \delta$. Assume that $g/g^\sigma = h(v)^2 \leq C\lambda^{-2}$, $v \in \mathbf{R}^{2n}$. Then for any non negative intergers k and μ there are C_1, C_2 such that*

$$|r_{k+M}|_\mu^g(z) \leq C_2 m_1(z)m_2(z)h(z)^k(C_1 M^{2\kappa-1}\lambda^{-1})^M$$

for $z \in \mathbf{R}^{2n}$ if $h \geq \lambda \geq M^{2-\delta}(\geq 1)$.

Let us set $G = g \oplus g$, $m(z_1, z_2) = m_1(z_1)m_2(z_2)$ and $A(X, Y) = 2\sigma(X, Y) = 2\langle\xi, y\rangle - 2\langle x, \eta\rangle$ with $X = (x, \xi)$, $Y = (y, \eta)$. $G = g \oplus g$ is clearly slowly varying

$$G_w(T)/C \leq G_{w+\tilde{w}}(T) \leq CG_w(T) \quad \text{if} \quad G_w(\tilde{w}) \leq c \tag{3.3.1}$$

for w, $\tilde{w}$, $T \in \mathbf{R}^{4n}$. It is also clear that

$$m(w)/C^M \leq m(w + \tilde{w}) \leq C^M m(w) \quad \text{if} \quad G_w(\tilde{w}) \leq c \tag{3.3.2}$$

for w, $\tilde{w} \in \mathbf{R}^{4n}$. From Proposition 18.4.6 in [5] it follows that G is uniformly A temperate with respect to the diagonal and $m = m_1 \otimes m_2$ is uniformly A, G temperate with respect to the diagonal:

$$G_w(T) \leq CG_{(z,z)}(T)(1 + G_w^A(w - (z, z)))^N \tag{3.3.3}$$

for $w, T \in \mathbf{R}^{4n}$, $z \in \mathbf{R}^{2n}$ and

$$m(w) \leq C^M m(z, z)(1 + G_w^A((z, z) - w))^{MN} \tag{3.3.4}$$

for $z \in \mathbf{R}^{2n}$, $w \in \mathbf{R}^{4n}$. The next lemma follows from Theorem 1.4.10 in [5].

Lemma 3.3.1 *There are a sequence $w_1, w_2, ... \in \mathbf{R}^{4n}$ and $N_0 \in \mathbf{N}$ such that the balls*

$$B_\nu = \{w \in \mathbf{R}^{4n} \mid G_{w_\nu}(w_\nu - w) < R^2\}$$

cover $\mathbf{R}^{4n}$ if $c/16 \le R^2$ and the intersection of more than N_0 balls B_ν is always empty if $R^2 \le c$. Moreover if $c/4 < R^2 < c$ there are $\phi_\nu \in C_0^\infty(B_\nu)$ with $\sum \phi_\nu = 1$ so that for all ν and k

$$|\phi_\nu|_k^G \le (C_{n,\kappa} C N_0/c)^k k!^\kappa.$$

Let us put

$$U_\nu = \{w \in \mathbf{R}^{4n} \mid G_{w_\nu}(w - w_\nu) \le c/2\}$$
$$U'_\nu = \{w \in \mathbf{R}^{4n} \mid G_{w_\nu}(w - w_\nu) \le c\}.$$

Let $d_\nu(w) = \inf_{\tilde{w} \in U_\nu} G^A_{\tilde{w}}(w - \tilde{w})$. Then we have the next lemma (Lemma 18.4.8 in [5])

Lemma 3.3.2 *Assume that $G_{(z,z)} \le G^A_{(z,z)}$. Then there are $C' > 0$ amd L which are independent of z such that*

$$\sum_{\nu=1}^{\infty} (1 + d_\nu(z,z))^{-L} \le C'.$$

We now estimate how small is $d_\nu(z,z)$ when $(z,z) \notin U'_\nu$.

Lemma 3.3.3 *Assume that $h(v)^2 \le C_1/\lambda^2$, $v \in \mathbf{R}^{2n}$. Then there is $C_2 > 0$ such that*

$$(z,z) \notin U'_\nu \Longrightarrow d_\nu(z,z) \ge C_2 \lambda^2.$$

Proof: Let $w = (v_1, v_2) \in U_\nu$. Recall that $G^A_w((z,z) - w) = g^\sigma_{v_1}(z - v_2) + g^\sigma_{v_2}(z - v_1)$. Noting that

$$g^\sigma_{v_1}(z - v_2) + g^\sigma_{v_2}(z - v_1) \ge C_1^{-1} \lambda^2 (g_{v_1}(z - v_2) + g_{v_2}(z - v_1))$$

the assertion is clear if $g_{v_1}(z - v_2) + g_{v_2}(z - v_1) \ge c$. Assume $g_{v_1}(z - v_2) + g_{v_2}(z - v_1) \le c$. This shows that

$$g_{v_j}(X)/C \le g_{v_1 + v_2 - z}(X) \le C g_{v_j}(X), \quad j = 1, 2, \quad X \in \mathbf{R}^{2n}.$$

This gives, for instance, $g_{v_1}(X)/C^2 \le g_{v_2}(X) \le C^2 g_{v_1}(X)$ and hence

$$\begin{aligned} G^A_w((z,z) - w) &\ge C_1^{-1} C^{-2} \lambda^2 (g_{v_1}(z - v_1) + g_{v_2}(z - v_2)) \\ &= C_1^{-1} C^{-2} \lambda^2 G_w((z,z) - w). \end{aligned}$$

To end the proof it is enough to remark that $G_w((z,z) - w) \ge c/8C$. Indeed from

$$G_{w_\nu}((z,z) - w_\nu) \le 4 G_{w_\nu}((z,z) - w) + \frac{1}{2} G_{w_\nu}(w_\nu - w)$$

one has $G_{w_\nu}((z,z) - w) \ge c/8$.

Let $u \in C^\infty(\mathbf{R}^{4n})$ and set $u_\nu = \phi_\nu u$. We are now going to estimate the remainder r_k. We first estimate r_k at $w \in U'_\nu$.

Lemma 3.3.4 *If $w \in U'_\nu$ then we have*

$$|(e^{iA(D)/4}u_\nu)(w) - \sum_{j<k}(iA(D)/4)^j u_\nu(w)/j!|$$
$$\leq C_n(nCH(w))^k \sup_{j\leq 2n+1}\sup_{\tilde w\in U_\nu}|u_\nu|^G_{j+2k}(\tilde w)/k!$$

where $H(w)^2 = \sup G_w/G^A_w$.

Proof: From [5] it follows that

$$|e^{iA(D)/4}u_\nu - \sum_{j<k}(iA(D)/4)^j u_\nu/j!|$$
$$\leq C_n \sup_{j\leq 2n+1}\sup_{\tilde w\in U_\nu}|(A(D)/4)^k u_\nu|^{G_{\tilde w}}_j(\tilde w)/k!. \tag{3.3.5}$$

We may assume that the coordinates are chosen so that $G_{\tilde w}$ is the Euclidean metric and moreover $A(\Xi_1,...,\Xi_{4n}) = \sum_{j=1}^{4n} b_j\Xi_j^2$. It is easy to see that $H(\tilde w) = \sup_{1\leq j\leq 4n}|b_j|$. Since

$$|\sum_{i=1}^{4n} b_i D_i^2 f|^e_j(\tilde w) \leq 4nH(\tilde w)|f|^e_{j+2}$$

repeating the estimate we obtain the desired estimate from (3.3.5) since $H(\tilde w) \leq CH(w)$ for $w,\ \tilde w \in U'_\nu$.

We next study the case $w \notin U_\nu$.

Lemma 3.3.5 *If $w \notin U_\nu$ then we have*

$$|(e^{iA(D)/4}u_\nu)(w)| \leq C_1 B_1^\ell(1+d_\nu(w))^{-\ell/2} \tag{3.3.6}$$
$$\times(B_2H(w_\nu))^k\ell!k!\sum_{\mu=0}^{\ell+2k+2n+1} B_3^\mu \sup_{\tilde w\in B_\nu}|u_\nu|^G_\mu(\tilde w)/\mu!$$

for $\ell, k = 0,1,....$

Proof: Let $w \notin U_\nu$ and take $w_0 \in \mathbf{R}^{4n}$ so that $L(\tilde w) = \langle \tilde w - w, w_0\rangle \neq 0$ for $\tilde w \in U_\nu$. It follows from [5] that

$$|(e^{iA(D)/4}u_\nu)(w)| \leq 2^\ell C_n \sup_{j\leq 2n+1}\sup_{\tilde w\in B_\nu}|(A(D)/4)^k(\langle Aw_0, D\rangle L^{-1})^\ell u_\nu|^{G_{w_\nu}}_j(\tilde w)/k! \tag{3.3.7}$$

for $\ell,\ k = 0,1,...$ where $A(x,\xi,y,\eta) = (-\eta,y,\xi,-x)$. Here we have assumed that $c\leq 4$. Note that we have

$$|(A(D)/4)^k(\langle Aw_0,D\rangle L^{-1})^\ell u_\nu|^{G_{w_\nu}}_j(\tilde w) \leq (nH(w_\nu))^k|(\langle Aw_0,D\rangle L^{-1})^\ell u_\nu|^{G_{w_\nu}}_{j+2k}(\tilde w)$$

for $\tilde{w} \in B_\nu$. We estimate the right-hand side. Recall Lemma 18.4.5 in [5]

$$|L(w_\nu)/L|_k^{G_{w_\nu}}(\tilde{w}) \le 2^{k+1}k!R^{-k}. \tag{3.3.8}$$

Assume that

$$\begin{aligned} &|((\langle Aw_0, D\rangle L^{-1})^\ell u|_k^G \qquad (3.3.9)\\ &\le 8^\ell 2^k R^{-(k+\ell)}(G(Aw_0)^{1/2}/L(w_\nu))^\ell(\ell+k)!\sum_{j=0}^{k+\ell}(2/R)^{-j}|u|_j^G/j! \end{aligned}$$

where $G = G_{w_\nu}$. Then we see

$$\begin{aligned} |((\langle Aw_0, D\rangle L^{-1})^{\ell+1}u|_k^G &= |((\langle Aw_0, D\rangle L^{-1})((\langle Aw_0, D\rangle L^{-1})^\ell u|_k^G \\ &= \frac{1}{|L(w_\nu)|}|\langle Aw_0, D\rangle \frac{L(w_\nu)}{L}((\langle Aw_0, D\rangle L^{-1})^\ell u|_k^G \\ &\le \frac{G(Aw_0)^{1/2}}{|L(w_\nu)|}\sum_{j=0}^{k+1}|((\langle Aw_0, D\rangle L^{-1})^\ell u|_j^G |L(w_\nu)/L|_{k+1-j}^G \\ \le \frac{G(Aw_0)^{1/2}}{|L(w_\nu)|}\sum_{j=0}^{k+1} 8^\ell(\ell+j)!2^{k+1-j+1}&R^{-(k+\ell+1)}(k+1-j)!\sum_{\mu=0}^{\ell+j}(2/R)^{-\mu}|u|_\mu^G/\mu! \\ &= (\frac{G(Aw_0)^{1/2}}{|L(w_\nu)|})^{\ell+1}R^{-(k+\ell+1)}(k+\ell+1)!8^\ell 2^{k+2} \\ &\times \sum_{\mu=0}^{\ell+k+1}(2/R)^{-\mu}|u|_\mu^G/\mu!\sum_{j=0}^{k+1}\frac{(\ell+j)!(k+1-j)!}{(k+\ell+1)!}. \end{aligned}$$

Since $\sum_{j=0}^{k+1}(\ell+j)!(k+1-j)!/(k+\ell+1)! \le 2$ we get (3.3.9) by induction on ℓ. Following [5] one can find $w_0 \in \mathbf{R}^{4n}$ so that $G_{w_\nu}(Aw_0)^{1/2}/|L(w_\nu)| \le (\inf_{\tilde{w}\in U_\nu} G_{w_\nu}^A(w-\tilde{w}))^{-1/2}$ and hence from (3.3.7) and (3.3.8) we obtain the desired assertion.

Let us set $a(z_1, z_2) = a_1(z_1)a_2(z_2)$ and $a_\nu = \phi_\nu a$. Then using the Leibnitz rule and Lemma 3.3.1 we see that

$$|a_\nu|_\mu^G(z_1, z_2) \le (C_{n,\kappa}N_0C^{1/2}/c^{1/2} + 2B)^\mu m_1(z_1)m_2(z_2)(\mu^\kappa + M)^\mu.$$

Let $(z, z) \in U'_\nu$ and estimate the remainder term. Noticing that

$$H(z,z)^2 = \sup\frac{G_{(z,z)}(\tilde{W})}{G_{(z,z)}^A(\tilde{W})} = \sup\frac{g_z(v_1)+g_z(v_2)}{g_z^\sigma(v_2)+g_z^\sigma(v_1)} \le h(z)^2$$

from Lemma 3.3.4 it follows that

$$|\prod_{\ell=1}^{\mu}\langle X_\ell, D\rangle\{(e^{iA(D)/4})a_\nu(z,z)$$
$$-\sum_{j<k}((iA(D)/4)^j a_\nu)(z,z)/j!\}|/\prod_{\ell=1}^{\mu} g_z(X_\ell)^{1/2}$$
$$\leq C_1C_2^kC_3^\mu C_4^M m_1(z)m_2(z)h(z)^k((2n+1+2k+\mu)^\kappa+M)^{2n+1+2k+\mu}/k!$$

where $X_\ell \in \mathbf{R}^{2n}$. In fact $\langle X, D\rangle f(z,z) = \langle (X,X), D\rangle f(w)|_{w=(z,z)}$ and

$$C^{-2}G_{(z_1,z_2)}(X,X) \leq G_{(z,z)}(X,X)/2 = g_z(X)$$
$$m_1(z_1)m_2(z_2) \leq C_2^{4M}m_1(z)m_2(z)$$

for (z,z), $(z_1,z_2) \in U'_\nu$. Thus we get for $(z,z) \in U'_\nu$

$$|(e^{iA(D)/4}a_\nu)(z,z) - \sum_{j<k}((iA(D)/4)^j a_\nu)(z,z)/j!|_\mu^g$$
$$\leq C_1C_2^kC_3^\mu C_4^M m_1(z)m_2(z)h(z)^k((2n+1+2k+\mu)^\kappa+M)^{2n+1+2k+\mu}/k!$$

We turn to the case $(z,z) \notin U'_\nu$. We first remark that

$$H(w)^2 \leq (CC_1)^2h(z)^2(1+d_\nu(z,z))^{2N}$$

for $w \in U_\nu$. Indeed we have $G_w(W) \leq CC_1G_{(z,z)}(W)(1+d_\nu(z,z))^N$ and then this gives that $G^A_{(z,z)}(\tilde{W}) \leq CC_1G^A_w(\tilde{W})(1+d_\nu(z,z))^N$. From this it follows that

$$H(w)^2 \leq (CC_1)^2H(z,z)^2(1+d_\nu(z,z))^{2N}$$

which proves the assertion. We also remark that

$$g_z(X) \geq (2C^2C_1)^{-1}(1+d_\nu(z,z))^{-N}G_w(X,X)$$
$$m(w) \leq C_0^{2M}C_1^M m_1(z)m_2(z)(1+d_\nu(z,z))^{MN_1}.$$

Applying Lemma 3.3.5 we have

$$|(e^{iA(D)/4}a_\nu)(z,z)|_\mu^g \leq C_1C_2^\ell C_3^k C_4^\mu C_5^M m_1(z)m_2(z)h(z)^k$$
$$\times(1+d_\nu(z,z))^{-\ell/2+\mu N/2+MN_1/2+kN}\ell!k!$$
$$\times\sum_{j=0}^{\ell+2k+2n+1}C_6^j((j+\mu)^\kappa+M)^{j+\mu}/j!$$

for $(z,z) \notin U_\nu$ and ℓ, $k = 0, 1, \ldots.$

We now prove Proposition 3.3.1. Let us write

$$r_{k_1+k_2}(z) = \sum_{(z,z)\in U'_\nu}\big[(e^{iA(D)/4}a_\nu)(z,z)$$
$$-\sum_{j<k_1+k_2}((iA(D)/4)^j a_\nu)(z,z)/j!\big] + \sum_{(z,z)\notin U'_\nu}(e^{iA(D)/4}a_\nu)(z,z).$$

From Lemma 3.3.4 and Lemma 3.3.5 we have

$$|r_{k_1+k_2}|^g_\mu(z) \le C_1 m_1(z) m_2(z) h(z)^{k_1}\Big[C_2^{k_1}C_3^{k_2}C_4^{\mu}C_5^{M}\lambda^{-k_2}$$
$$\times((2n+1+2k_1+2k_2+\mu)^\kappa+M)^{2n+1+2k_1+2k_2+\mu}/(k_1+k_2)!$$
$$+C_6^{\ell}C_7^{k_1}C_8^{\mu}C_9^{M}(1+C_{10}\lambda^2)^{-\ell_1}\ell!k_1!$$
$$\times\sum_{j=0}^{\ell+2k_1+2n+1}C_{11}^j((j+\mu)^\kappa+M)^{j+\mu}/j!\Big]$$

where $\ell/2=\ell_1+\mu N/2+MN_1/2+k_1N+L+1$ and $\ell_1>0$. Taking

$$k_2=M,\quad \ell_1=[(N_1\kappa+1)M/(2-\delta-\kappa)]+1,\quad 0<\delta<1$$

and remarking that

$$\ell!\sum_{j=0}^{\ell+2k_1+2n+1}((j+\mu)^\kappa+M)^{j+\mu}/j! \le C\sum_{j=0}^{\tilde\ell}\binom{\tilde\ell}{j}(j^\kappa+M)^j((\tilde\ell-j)^\kappa+M)^{\tilde\ell-j}$$
$$\le C2^{\tilde\ell}(\tilde\ell^\kappa+M)^{\tilde\ell}\le C2^{\tilde\ell}(\tilde\ell+M)^{\kappa\tilde\ell}$$

where $\tilde\ell=2\ell_1+MN_1$, $C=C(\mu,k_1)$ and $\tilde\ell+M\le CM$, $\tilde\ell\le 2(\ell_1+MN_1)$ for $M\ge M_0$ we have

$$|r_{k_1+M}|^g_\mu(z)\le C_1m_1(z)m_2(z)h(z)^{k_1}\big[(C_2M^{2\kappa-1}/\lambda)^M$$
$$+C_3^M(M^{2\kappa(1+(2-\delta)N_1)}/\lambda^{2(N_1\kappa+1)})^{M/(2-\delta-\kappa)}\big]$$

because $\ell\le 2M[1+(2-\delta)N_1]/(2-\delta-\kappa)$. Assume that $\lambda\ge M^{2-\delta}$ and $1<\kappa<2-\delta$. Then it is easy to see that

$$(M^{2\kappa(1+(2-\delta)N_1)}\lambda^{2(N_1\kappa+1)})^{1/(2-\delta-\kappa)}\le M^{2\kappa-1}\lambda^{-1}$$

and hence we get the desired assertion.

Remark: The same argument employed in this section could be applied to obtain a Weyl-Hörmander calculus in the Gevrey classes. A main point is now M would be replaced by $\langle\xi\rangle^\kappa$.

4 Asymptotic formula

4.1 Specified symbols

In this subsection we return to our original metric $g=\phi^{-2}|dx|^2+\psi^{-2}|d\xi|^2$ which is a splitting metric. We study

$$\sum_{j<M}(i\sigma(D_z;D_w)/2)^k/k!a_1(z)a_2(w)|_{w=z}$$

for special $a_i(z)$. We say that $f(z,w)$ is anti-symmetric (resp. symmetric) if $f(z,w)$ verifies $f(w,z)=-f(z,w)$ (resp. $f(w,z)=f(z,w)$)

Lemma 4.1.1 *Assume that $f(z,w)$ is anti-symmetric and $j > k$. Then*

$$\sigma(D_z; D_w)^k f(z,w)^j|_{w=z} = 0.$$

Proof: Let us write $\sigma(f) = \langle D_y, D_\xi\rangle f - \langle D_x, D_\eta\rangle f$ and

$$\sigma[f,g] = \langle D_\xi f, D_y g\rangle - \langle D_x f, D_\eta g\rangle.$$

Then it is clear that $\sigma(f)$ is anti-symmetric (symmetric) if f is symmetric (anti-symmetric) and $\sigma[f,g]$ is anti-symmetric if both f and g are anti-symmetric. It is also easy to see that

$$\sigma(D_z; D_w)(fg) = \sigma(f)g + \sigma(g)f + \sigma[f,g] + \sigma[g,f],$$
$$\sigma[\prod_{i=1}^{p} g_i, f] = \sum_i \sigma[g_i, f] \prod_{\mu\neq i} g_\mu. \tag{4.1.1}$$

We now assume that g_i, $1 \le j \le q$ are anti-symmeric. Then applying (4.1.1) we see that $\sigma(D_z; D_w)\prod_{j=1}^q g_j$ is a finite sum of $q-1$ products of anti-symmetric functions. A repeated application of this arguments proves the assertion.

Corollary 4.1.1 *We have*

$$\sigma(D_z; D_w)^k e^{-\Lambda(z)+\Lambda(w)} f(w)|_{w=z} = \sigma(D_z; D_w)^k \sum_{j=0}^{k} (\Lambda(w) - \Lambda(z))^j f(w)/j!|_{w=z}.$$

Proof: Note that

$$\sigma(D_z; D_w)^k e^{-\Lambda(z)+\Lambda(w)} f(w) = \sum_{j=0} \sigma(D_z; D_w)^k (\Lambda(w) - \Lambda(z))^j f(w)/j!.$$

Then from (4.1.1) and Lemma 4.1.1 we conclude the proof.

Lemma 4.1.2 *Assume that $|\Lambda|_k^g(z)$, $|f|_k^g(z)/m(z) \le C^k k!^\kappa$, $z \in \mathbf{R}^{2n}$, $k = 0, 1, \ldots$. Then there is $C > 0$ such that*

$$\big|[\sigma(D_z; D_w)^k (\Lambda(w) - \Lambda(z))^j f(z)/k!|_{w=z}]_{(\delta)}^{(\gamma)}\big|$$
$$\le C_{\gamma\delta} C^k (2k!)^{\kappa-1} k! \lambda^{-k} \phi^{-|\delta|} \psi^{-|\gamma|} m, \quad 0 \le j \le k.$$

Proof: Note that

$$[\Lambda(w) - \Lambda(z)]^j = \sum \frac{\Lambda_{(\beta_1)}^{(\alpha_1)}(z)\cdots\Lambda_{(\beta_j)}^{(\alpha_j)}(z)}{\alpha_1!\cdots\alpha_j!\beta_1!\cdots\beta_j!}(y-x)^{\beta_1+\cdots+\beta_j}(\eta-\xi)^{\alpha_1+\cdots+\alpha_j} + O(|w-z|^N).$$

Since

$$\sigma(D_z; D_w)^k F/k! = \sum_{|\gamma+\delta|=k} (-1)^{|\gamma|} \frac{\partial_\xi^\gamma \partial_y^\gamma \partial_x^\delta \partial_\eta^\delta F}{\gamma!\delta!}$$

it is easy to see that

$$\sigma(D_z;D_w)^k/k!(\Lambda(w)-\Lambda(z))^j f(z)|_{w=z} \tag{4.1.2}$$
$$= \sum_{|\gamma+\delta|=k}(-1)^{|\gamma|}\sum(-1)^{|\gamma_2|}(-1)^{|\delta_2|}\frac{\partial_x^{\delta_1}\partial_\xi^{\gamma_1}[\Lambda^{(\alpha_1)}_{(\beta_1)}(z)\cdots\Lambda^{(\alpha_j)}_{(\beta_j)}(z)f(z)]\alpha!\beta!}{\gamma_1!\gamma_2!\delta_1!\delta_2!\alpha_1!\cdots\alpha_j!\beta_1!\cdots\beta_j!}$$

where the sum is taken over all α_j, β_j, γ_j and δ_j such that

$$\gamma_1+\gamma_2=\gamma,\ \ \delta_1+\delta_2=\delta,\ \ \gamma_2+\delta=\alpha,\ \ \delta_2+\gamma=\beta$$
$$\alpha_1+\cdots+\alpha_j=\alpha,\ \ \beta_1+\cdots+\beta_j=\beta.$$

The right-hand side of (4.1.2) is estimated by

$$C^k\sum\frac{|\Lambda^{(\alpha_1+\gamma_1)}_{(\beta_1+\delta_1)}(z)\cdots\Lambda^{(\alpha_j+\gamma_j)}_{(\beta_j+\delta_j)}(z)f^{(\gamma_{j+1})}_{(\delta_{j+1})}|k!}{\alpha_1!\cdots\alpha_j!\beta_1!\cdots\beta_j!\gamma_1!\cdots\gamma_{j+1}!\delta_1!\cdots\delta_{j+1}!} \tag{4.1.3}$$

where the sum is taken over all

$$|\alpha_1+\cdots+\alpha_j+\gamma_1+\cdots+\gamma_j+\gamma_{j+1}|=k,$$
$$|\beta_1+\cdots+\beta_j+\delta_1+\cdots+\delta_j+\delta_{j+1}|=k.$$

Recalling that

$$|\Lambda^{(\alpha)}_{(\beta)}|\le C^{|\alpha+\beta|}|\alpha+\beta|!^\kappa\phi^{-|\beta|}\psi^{-|\alpha|},\quad \phi\psi\ge\lambda$$

(4.1.3) is bounded by

$$C_1(2k!)^{\kappa-1}k!\lambda^{-k}$$

and hence the result for the case $|\gamma+\delta|=0$. The estimate for the case $|\gamma+\delta|\ge 1$ is just a repetition.

4.2 Asymptotic formula for $T^{-M}\#T^M$

In this subsection we take $\Lambda(z)=M\log T(z)$ where $T(z)$ is assumed to satisfy

$$|T|^g_k\le T(z)A^kk!^\kappa$$

and hence

$$|(\log T(z))^{(\alpha)}_{(\beta)}|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^\kappa,\quad |\alpha+\beta|\ge 1.$$

Lemma 4.2.1 *Assume that $|f|^g_k\le m(z)A^kk!^\kappa$ where $m(z)$ is σ, g temperate. Then we have*

$$\Big|[\frac{1}{k!}(\frac{i\sigma(D_z;D_w)}{2})^ke^{-\Lambda(z)}e^{\Lambda(w)}f(w)|_{w=z}]^{(\gamma)}_{(\delta)}\Big|$$
$$\le C_{\gamma\delta}m(z)M^{(2\kappa-1)k}\lambda^{-k}\phi^{-|\delta|}\psi^{-|\gamma|}$$

for $k\le M$.

Proof: From Corollary 4.1.1 it follows that

$$\sigma(D_z;D_w)^k e^{-\Lambda(z)}e^{\Lambda(w)}f(w)|_{w=z}$$
$$=\sigma(D_z;D_w)^k\sum_{j=0}^{k}M^j(\log T(w)-\log T(z))^j f(w)/j!|_{w=z}.$$

From the proof of Lemma 4.1.2 we conclude that

$$|\sigma(D_z;D_w)^k/k!e^{-\Lambda(z)}e^{\Lambda(w)}f(w)|_{w=z}|$$
$$\le C^k\sum_{j=0}^{k}\frac{(2k)!^{\kappa-1}k!}{j!}M^j\lambda^{-k}\le C_1^k\sum_{j=0}^{k}\binom{k}{j}k^{2k(\kappa-1)}(k-j)^{k-j}M^j\lambda^{-k}$$
$$\le C_2^k\sum_{j=0}^{k}\binom{k}{j}k^{2k(\kappa-1)}k^{k-j}M^j\lambda^{-k}\le C_3^k k^{2k(\kappa-1)}M^k\lambda^{-k}\le C_3^k M^{(2\kappa-1)k}\lambda^{-k}$$

since $\kappa>1$. The estimate for the case $|\gamma+\delta|\ge 1$ is similar.

Proposition 4.2.1 *Assume $M^{2\kappa-1}\lambda^{-1}\ll 1$ and $\ell\le M$. Assume also that $m(z)$ is σ, g temperate and $|f|_k^g\le m(z)A^k k!^\kappa$, $k=0,1,\ldots.$ Let $k(z)=T^M(z)f(z)$ and $q(z)=T^{-M}\#k$. Then we have for any ℓ*

$$\Big|\Big(\big[q(z)-\sum_{k<\ell}\frac{1}{k!}(\frac{i\sigma}{2})^k e^{-\Lambda(z)}e^{\Lambda(w)}f(w)\big]\Big)_{(\delta)}^{(\gamma)}\Big|$$
$$\le C_{k\gamma\delta}m(z)(M^{2\kappa-1}\lambda^{-1})^\ell\phi^{-|\delta|}\psi^{-|\gamma|}.$$

Proof: Note $T^M(z)f(z)\in S(mT^M,g)$. Set

$$r_M(z)=q(z)-\sum_{k<M}\frac{1}{k!}(\frac{i\sigma}{2})^k T(z)^{-M}T(w)^M f(w)|_{w=z}.$$

From Proposition 3.3.1 we see that

$$|r_M|_k^g\le C_2(C_1M^{2\kappa-1}\lambda^{-1})^M m(z).$$

On the other hand from Lemma 4.3.1 it follows that

$$\Big|\Big(\sum_{k=\ell}^{M}\frac{1}{k!}(\frac{i\sigma}{2})^k T(z)^{-M}T(w)^M f(w)|_{w=z}\Big)_{(\delta)}^{(\gamma)}\Big|$$
$$\le C_{\gamma\delta}m(z)\phi^{-|\delta|}\psi^{-|\gamma|}\sum_{k=\ell}^{M}(M^{2\kappa-1}\lambda^{-1})^k.$$

Since $M^{2\kappa-1}\lambda^{-1}\le 1/2$ and $\ell\le M$ we conclude the assertion.

4.3 Regularization of time function T

As we observed in §§3.3, there are a sequence $z_1, z_2, ... \in \mathbf{R}^{2n}$ and $N_0 \in \mathbf{N}$ such that the balls

$$B_\nu = \{z \in \mathbf{R}^{2n} \mid g_{z_\nu}(z - z_\nu) < R^2\}$$

cover $\mathbf{R}^{2n}$ and the intersection of more than N_0 balls B_ν is empty and there are $\chi_\nu \in C_0^\infty(B_\nu)$ with $\sum \chi_\nu = 1$ so that for all ν and k

$$|\chi_\nu|_k^g \le (C_{n,\kappa} C N_0/c)^k k!^\kappa$$

where $c/4 < R^2 < c$. Let us define

$$\tilde{\phi}(z) = \sum \phi(z_\nu)\chi_\nu(z), \quad \tilde{\psi}(z) = \sum \psi(z_\nu)\chi_\nu(z).$$

Then it is easy to check that with some $C > 0$

$$C^{-1}\phi(z) \le \tilde{\phi}(z) \le C\phi(z), \quad C^{-1}\psi(z) \le \tilde{\psi}(z) \le C\psi(z) \tag{4.3.1}$$

for $z \in \mathbf{R}^{2n}$ and

$$|\tilde{\phi}|_k^{\tilde{g}}(z)|/\tilde{\phi}(z) + |\tilde{\psi}|_k^{\tilde{g}}(z)|/\tilde{\psi}(z) \le C^{k+1} k!^\kappa \tag{4.3.2}$$

for $z \in \mathbf{R}^{2n}$ where $\tilde{g}_z(w) = \tilde{\phi}(z)^{-2}|y|^2 + \tilde{\psi}(z)^{-2}|\eta|^2$. Hereafter we write ϕ, ψ and $\tilde{g}$ instead of $\tilde{\phi}$, $\tilde{\psi}$, $\tilde{g}$ so that

$$\phi \in S(\phi, g), \quad \psi \in S(\psi, g).$$

Let $t(z)$ be a time function of p at z^0 with respect to $\Gamma(p_{z^0}, (0, \theta))$, that is $t(z^0) = 0$ and

$$-H_t(z^0) \in \Gamma(p_{z^0}, (0, \theta)). \tag{4.3.3}$$

We want to regularize $T(x, \xi)$ keeping (4.3.3) ($t(z)$ replaced by $T(z)$). Let $\rho(s)$ be a function in $C_0^\infty(\mathbf{R})$ satisfying $\rho(s) \ge 0$, $\rho(s) = 0$ for $|s| \ge c^2$, $\int \rho(|w|^2)dw = 1$ and

$$|\rho^{(k)}(s)| \le C A^k k!^\kappa \tag{4.3.4}$$

for $s \in \mathbf{R}$ where $\kappa > 1$ and $A > 0$. Let us define

$$\tilde{T}(z) = \int\int \rho(g_z(z - w))|g_z|^{-1/2} T(w) dw. \tag{4.3.5}$$

Here we have

Lemma 4.3.1 *There is a positive constant C such that*

$$C^{-1}T(z) \le \tilde{T}(z) \le CT(z) \tag{4.3.6}$$

for $z \in \mathbf{R}^{2n}$ and $\tilde{T} \in S(\tilde{T}(z), g)$. Moreover there are a conic neighborhood U of z^0, a compact set $K \subset \Gamma(p_{z^0}, (0, \theta))$ and $\nu > 0$ such that

$$-\phi(z)(\langle \xi \rangle_h \nabla_\xi \tilde{T}, -\nabla_x \tilde{T})/\tilde{T}(z) \in K, \quad z \in U$$

when $|\xi| \ge \nu h$ and $c \ll 1$.

Proof: It is clear that

$$\tilde{T}(z) = \int\int \rho(|w|^2)T(x-\phi(z)y, \xi-\psi(z)\eta)dw.$$

Since $T(z)$ is g continuous and $g_z(\phi(z)y, \psi(z)\eta) \le c^2$ if $\rho(|w|^2) \neq 0$ we get (4.3.6). Recall

$$\tilde{T}^{(\alpha)}_{(\beta)}(z) = \int\int D_x^\beta \partial_\xi^\alpha \{\rho(g_z(z-w))(\phi\psi)(z)^{-n}T(w)\}dw.$$

It is sufficient to prove that there are $C > 0$ and $C_2 > 0$ such that

$$|D_x^\beta \partial_\xi^\alpha \{\rho(g_z(z-w))\}| \le CC_2^{|\alpha+\beta|}|\alpha+\beta|!^\kappa \phi(z)^{-|\beta|}\psi(z)^{-|\alpha|}. \tag{4.3.7}$$

Note that

$$|D_x^\beta \partial_\xi^\alpha g_z(z-w)| \le C_1^{|\alpha+\beta|+1}|\alpha+\beta|!^\kappa \phi(z)^{-|\beta|}\psi(z)^{-|\alpha|} \tag{4.3.8}$$

if $g_z(z-w) \le c^2$. It is easy to see that

$$D_x^\beta \partial_\xi^\alpha \rho(g_z(z-w))$$
$$= \sum C_{\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_s}\rho^{(s)}D_x^{\beta_1}\partial_\xi^{\alpha_1}g_z(z-w)\cdots D_x^{\beta_s}\partial_\xi^{\alpha_s}g_z(z-w)$$

where $\sum C_{\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_s}$ is estimated by $C^{|\alpha+\beta|}$. Noting that

$$s!^\kappa|\alpha_1+\beta_1|!^\kappa \cdots |\alpha_s+\beta_s|!^\kappa \le C|\alpha+\beta|!^\kappa$$

we obtain (4.3.7) from (4.3.8) and hence $\tilde{T} \in S(\tilde{T}, g)$.

We turn to the second assertion. We write $\nabla_\xi T = T_\xi$, $\nabla_x T = T_x$. Note that

$$(\langle\xi\rangle_h \tilde{T}_\xi(z), -\tilde{T}_x(z))/\tilde{T}(z)$$
$$= \tilde{T}^{-1}\int \rho(|w|^2)(\langle\xi\rangle_h T_\xi(x-y\phi, \xi-\eta\psi), -T_x(x-y\phi, \xi-\eta\psi))dw + r(z)$$

with $\phi = \phi(z)$, $\psi = \psi(z)$ where

$$r(z) = \tilde{T}^{-1}\int \rho(|w|^2)(\langle\xi\rangle_h(-y\phi_\xi)T_x + \langle\xi\rangle_h(-\eta\psi_\xi)T_\xi,$$
$$(y\phi_x)T_x + (\eta\psi_x)T_\xi)dw$$

and $T_x = T_x(x-y\phi, \xi-\eta\psi)$, $T_\xi = T_\xi(x-y\phi, \xi-\eta\psi)$. Noting that

$$T_\xi(z) = n(z)^{-1}\nabla_\xi(t_h(z)\langle\xi\rangle_h^{1/2})T(z),$$
$$T_x(z) = n(z)^{-1}\nabla_x(t_h(z)\langle\xi\rangle_h^{1/2})T(z)$$

we have

$$(\langle\xi\rangle_h T_\eta(w), -T_y(w))$$
$$= T(w)(\langle\xi\rangle_h\langle\eta\rangle_h^{-1/2}\nabla_\eta(t_h(w)\langle\eta\rangle_h^{1/2}), -\nabla_y t_h(w))\phi(w)^{-1}.$$

Then we have

$$\tilde{T}^{-1}\int \rho(|w|^2)(\langle\xi\rangle_h T_\xi(x-y\phi,\xi-\eta\psi),-T_x(x-y\phi,\xi-\eta\psi))dw$$
$$=\int \rho(g_z(z-w))\tilde{T}(z)^{-1}(\langle\xi\rangle_h T_\eta(w),-T_y(w))(\phi\psi)^{-n}(z)dw$$
$$=\int \rho(g_z(z-w))T(w)\tilde{T}(z)^{-1}$$
$$\times(\langle\xi\rangle_h\nabla_\eta(t_h(w)\langle\eta\rangle_h^{1/2},-\nabla_y t_h(w)\langle\eta\rangle_h^{1/2})n^{-1}(\phi\psi)^{-n}dw$$

Note that we can choose a conic neighborhood U of z^0 and a convex compact set $K\subset\subset\Gamma(p_{z^0},(0,\theta))$ and $\nu>2$ such that

$$-\phi(z)(\langle\xi\rangle_h\nabla_\eta t_h(w),-\nabla_y t_h(w))\phi(w)^{-1}\in K$$

for $z\in U$, $|\xi|\geq\nu h\gg\lambda$ and w with $g_z(z-w)\leq c^2$ where $c\ll 1$ is taken sufficiently small. Indeed since $\phi(z)\leq C$ and $\psi(z)=\langle\xi\rangle_h\phi(z)$ the assertion follows from the assumption (4.3.3) and g continuity of $\phi(z)$. Note that $\tilde{T}(z)$, $\langle\xi\rangle_h$ are g continuous, $\phi(z)^{-1}=\langle\xi\rangle_h^{1/2}n(z)^{-1}$, $t_h(x^0,\xi^0|\xi|)=0$ for $|\xi|\geq\nu h$ and that

$$C^{-1}\leq\int \rho(g_z(z-w))T(w)\tilde{T}(z)^{-1}(\phi\psi)^{-n}dw\leq C.$$

Then we conclude that

$$\phi(z)\int \rho(g_z(z-w))T(w)\tilde{T}(z)^{-1} \qquad (4.3.9)$$
$$\times(\langle\xi\rangle_h\nabla_\eta(t_h(w)\langle\eta\rangle_h^{1/2}),-\nabla_y t_h(w)\langle\eta\rangle^{1/2})n(w)^{-1}(\phi\psi)^{-n}dw\in K$$

for $z\in U$ with $|\xi|\geq\nu h$, modifying K if necessary. We now estimate $r(z)$. Remarking that $g_z(y\phi,\eta\psi)\leq c$ if $\rho(|w|^2)\neq 0$ we have from (4.3.6) that $|T_x(x-y\phi,\xi-\eta\psi)/\tilde{T}(z)|\leq C\phi(z)^{-1}$, $|T_\xi(x-y\phi,\xi-\eta\psi)/\tilde{T}(z)|\leq C\phi(z)^{-1}\langle\xi\rangle_h^{-1}$. On the other hand since

$$\langle\xi\rangle_h|\phi_\xi|,\ \langle\xi\rangle_h|\psi_\xi|,\ |\phi_x|,\ |\psi_x|\leq C$$

we conclude that

$$|r(z)|\leq Cc\phi(z)^{-1}. \qquad (4.3.10)$$

Then combining (4.3.9) and (4.3.10) we get the desired assertion, modifying $K\subset\subset\Gamma(p_{z^0},(0,\theta))$ if necessary.

5 Asymptotic expansion of P_{T^M}

5.1 Composition formula

We first show

Lemma 5.1.1 *Let $p \in S(m_1, g)$, $q \in S(m_2, \tilde{g})$ where $\tilde{g} = mg$ with $m \geq 1$ and assume that $g = \phi^{-2}|dx|^2 + \psi^{-2}|d\xi|^2$. Then we have*

$$p(z)q(z) - \sum_{|\alpha+\beta|<k} \frac{(-1)^{|\alpha|}}{2^{|\alpha+\beta|}} \sigma((p^{(\alpha)}_{(\beta)})^w (q^{(\beta)}_{(\alpha)})^w)/\alpha!\beta! \in S(m_1 m_2 H^k, \tilde{g})$$

where

$$H^2 = \sup g/\tilde{g}^\sigma = \sup \tilde{g}/g^\sigma.$$

Proof: Note that $\tilde{g}/2 \leq (g+\tilde{g})/2 \leq mg = \tilde{g}$. Then we have

$$\sum_{|\alpha+\beta|<k} \frac{(-1)^{|\alpha|}}{2^{|\alpha+\beta|}} \sigma((p^{(\alpha)}_{(\beta)})^w (q^{(\beta)}_{(\alpha)})^w)/\alpha!\beta!$$

$$- \sum_{|\alpha+\beta|<k, |\gamma+\delta|<k} \frac{(-1)^{|\alpha+\delta|}}{2^{|\alpha+\beta+\gamma+\delta|}} p^{(\alpha+\gamma)}_{(\beta+\delta)} q^{(\beta+\delta)}_{(\alpha+\gamma)}/\alpha!\beta!\gamma!\delta! \in S(m_1 m_2 H^k, \tilde{g}).$$

Remark that

$$\sum_{|\alpha+\beta|<k, |\gamma+\delta|<k, |\alpha+\beta+\gamma+\delta|\geq k} \frac{(-1)^{|\alpha+\delta|}}{2^{|\alpha+\beta+\gamma+\delta|}} p^{(\alpha+\gamma)}_{(\beta+\delta)} q^{(\beta+\delta)}_{(\alpha+\gamma)}/\alpha!\beta!\gamma!\delta! \in S(m_1 m_2 H^k, \tilde{g}).$$

In fact since g has the form

$$g = \phi^{-2}|dx|^2 + \psi^{-2}|d\xi|^2$$

we have $h = (\phi\psi)^{-1}$ and

$$\begin{aligned} \big|p^{(\alpha+\gamma)}_{(\beta+\delta)} q^{(\beta+\delta)}_{(\alpha+\gamma)}\big| &\leq C(\sqrt{m}\phi^{-1})^{\alpha+\gamma|}(\sqrt{m}\psi^{-1})^{|\beta+\delta|} \\ &\leq C\sqrt{m}^{|\alpha+\beta+\gamma+\delta|}(\phi\psi)^{-|\alpha+\beta+\gamma+\delta|} \leq CH^{|\alpha+\beta+\gamma+\delta|} \end{aligned}$$

because $H = \sqrt{m}h$. Note that

$$\sum_{|\alpha+\beta+\gamma+\delta|<k} \frac{(-1)^{|\alpha+\delta|}}{2^{|\alpha+\beta+\gamma+\delta|}} p^{(\alpha+\gamma)}_{(\beta+\delta)} q^{(\beta+\delta)}_{(\alpha+\gamma)}/\alpha!\beta!\gamma!\delta! = p(z)q(z)$$

which proves the assertion.

Lemma 5.1.2 *Under the same assumption as in Lemma 5.1.1 we have*

$$p^w q^w - \sum_{|\alpha+\beta|<k} \frac{(-1)^{|\beta|}}{2^{|\alpha+\beta|}\alpha!\beta!} (q^{(\beta)}_{(\alpha)})^w (p^{(\alpha)}_{(\beta)})^w \in S(m_1 m_2 H^k, \tilde{g}).$$

Proof: Recall

$$p\#q - \sum_{|\alpha+\beta|<k} \frac{(-1)^{|\beta|}}{2^{|\alpha+\beta|}\alpha!\beta!} p^{(\alpha)}_{(\beta)} q^{(\beta)}_{(\alpha)} \in S(m_1 m_2 H^k, \tilde{g}).$$

From Lemma 5.1.1 we have

$$p^{(\alpha)}_{(\beta)} q^{(\beta)}_{(\alpha)} = \sum_{|\gamma+\delta|<k} \frac{(-1)^{|\gamma|}}{2^{|\gamma+\delta|}\gamma!\delta!} \sigma\big((q^{(\beta+\gamma)}_{(\alpha+\delta)})^w (p^{(\alpha+\delta)}_{(\beta+\gamma)})^w\big) + S(m_1 m_2 H^k, \tilde{g}).$$

We insert this expression into the above formula and divide the sum over $|\alpha + \beta + \gamma + \delta| < k$ and $|\alpha + \beta + \gamma + \delta| \geq k$. Since

$$q^{(\beta+\gamma)}_{(\alpha+\delta)} \# p^{(\alpha+\delta)}_{(\beta+\gamma)} \in S(m_1 m_2 H^k, \tilde{g})$$

if $|\alpha + \beta + \gamma + \delta| \geq k$ we get the assertion.

Lemma 5.1.3 *Let $p \in S(m_1, g)$, $q \in S(m_2, \tilde{g})$, $\tilde{q} \in S(m_2^{-1}, \tilde{g})$ where $\tilde{g} = mg$, $m \geq 1$ and $g = \phi^{-2}|dx|^2 + \psi^{-2}|d\xi|^2$. Then we have*

$$\tilde{q}\#p\#q - \sum_{|\alpha+\beta|<k} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!} p^{(\alpha)}_{(\beta)} w^{\beta}_{\alpha} \in S(m_1 H^k, \tilde{g})$$

where

$$w^{\beta}_{\alpha} = \sum \binom{\alpha}{\tilde{\alpha}} \binom{\beta}{\tilde{\beta}} (\tilde{w}^{\tilde{\beta}}_{\tilde{\alpha}})^{(\beta-\tilde{\beta})}_{(\alpha-\tilde{\alpha})}$$

$$\tilde{w}^{\beta}_{\alpha} = (-1)^{|\beta|} \tilde{q} \# q^{(\beta)}_{(\alpha)}. \tag{5.1.1}$$

Proof: Set

$$(\tilde{w}^{\beta}_{\alpha})^w = (-1)^{|\beta|} \tilde{q}^w (q^{(\beta)}_{(\alpha)})^w$$

then we have

$$\tilde{q}^w p^w q^w - \sum_{|\alpha+\beta|<k} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!} (\tilde{w}^{\beta}_{\alpha})^w (p^{(\alpha)}_{(\beta)})^w \in S(m_1 H^k, \tilde{g}).$$

Note that

$$\begin{aligned} \sigma[(\tilde{w}^{\beta}_{\alpha})^w (p^{(\alpha)}_{(\beta)})^w] - \sum_{|\gamma+\delta|<\ell} \frac{(-1)^{|\delta|}}{2^{|\gamma+\delta|}\gamma!\delta!} (\tilde{w}^{\beta}_{\alpha})^{(\gamma)}_{(\delta)} p^{(\alpha+\delta)}_{(\beta+\gamma)} \\ \in S(m_1(\sqrt{m}\phi^{-1})^{|\alpha|}(\sqrt{m}\psi^{-1})^{|\beta|}\phi^{-|\beta|}\psi^{-|\alpha|}H^{\ell}, \tilde{g}) \\ = S(m_1 H^{|\alpha+\beta|+\ell}, \tilde{g}) \end{aligned}$$

because $h = (\phi\psi)^{-1}$, $H = \sqrt{m}h$. Take $\ell = k - |\alpha + \beta|$ so that

$$\sigma[(\tilde{w}^{\beta}_{\alpha})^w (p^{(\alpha)}_{(\beta)})^w] - \sum_{|\alpha+\beta+\gamma+\delta|<k} \frac{(-1)^{|\delta|}}{2^{|\gamma+\delta|}\gamma!\delta!} (\tilde{w}^{\beta}_{\alpha})^{(\gamma)}_{(\delta)} p^{(\alpha+\delta)}_{(\beta+\gamma)} \in S(m_1 H^k, \tilde{g})$$

and hence

$$\tilde{q}\#p\#q-\sum_{|\alpha+\beta+\gamma+\delta|<k}\frac{(-1)^{|\delta|}}{2^{|\alpha+\beta+\gamma+\delta|}\alpha!\beta!\gamma!\delta!}(\tilde{w}_{\alpha}^{\beta})_{(\delta)}^{(\gamma)}p_{(\beta+\gamma)}^{(\alpha+\delta)}\in S(m_1H^k,\tilde{g}).$$

Note that

$$\sum_{|\alpha+\beta+\gamma+\delta|<k}\frac{(-1)^{|\delta|}}{2^{|\alpha+\beta+\gamma+\delta|}\alpha!\beta!\gamma!\delta!}(\tilde{w}_{\alpha}^{\beta})_{(\delta)}^{(\delta)}p_{(\beta+\gamma)}^{(\alpha+\delta)}$$
$$=\sum_{|\tilde{\alpha}+\tilde{\beta}|<k}\frac{1}{2^{|\tilde{\alpha}+\tilde{\beta}|}\tilde{\alpha}!\tilde{\beta}!}\Big[\sum(-1)^{|\tilde{\alpha}-\alpha|}\binom{\tilde{\alpha}}{\alpha}\binom{\tilde{\beta}}{\beta}(\tilde{w}_{\alpha}^{\beta})_{(\tilde{\alpha}-\alpha)}^{(\tilde{\beta}-\beta)}\Big]p_{(\tilde{\beta})}^{(\tilde{\alpha})}.$$

Setting

$$w_{\tilde{\alpha}}^{\tilde{\beta}}=\sum(-1)^{|\tilde{\alpha}-\alpha|}\binom{\tilde{\alpha}}{\alpha}\binom{\tilde{\beta}}{\beta}(\tilde{w}_{\alpha}^{\beta})_{(\tilde{\alpha}-\alpha)}^{(\tilde{\beta}-\beta)}$$

we get the assertion.

5.2 Composition $\langle\xi\rangle_{\gamma}^{-a\rho}\#e^{-\gamma\zeta(x)}\#P\#e^{\gamma\zeta(x)}\#\langle\xi\rangle_{\gamma}^{a\rho}$

Let us recall

$$P(x,\xi;\gamma)=e^{-\gamma\zeta(x)}\#P(x,\xi)\#e^{\gamma\zeta(x)}.$$

Then it is clear that

$$P(x,\xi;\gamma)=\sum_{|\alpha|\le m}\frac{1}{2^{|\alpha|}\alpha!}P^{(\alpha)}(x,\xi)\tilde{\omega}_{\alpha},$$
$$\tilde{\omega}_{\alpha}-(-i\gamma\nabla\zeta(x))^{\alpha}=O(\gamma^{|\alpha|-1}),$$
$$P(x,\xi;\gamma)\in S(\langle\xi\rangle_{\gamma}^{m},g).$$

We apply Lemma 5.1.3 with $q=\langle\xi\rangle_h^{a\rho}$, $\tilde{q}=\langle\xi\rangle_h^{-a\rho}$ and

$$\tilde{g}=(a\log\langle\xi\rangle_h)^2(|dx|^2+\langle\xi\rangle_h^{-2}|d\xi|^2)=(a\log\langle\xi\rangle_h)^2g$$

so that $H=(a\log\langle\xi\rangle_h)\langle\xi\rangle_h^{-1}$. Note that $q\in S(q,\tilde{g})$, $\tilde{q}\in S(\tilde{q},\tilde{g})$. It is clear that

$$\tilde{w}_{\alpha}^{\beta}\in S((a\log\langle\xi\rangle_h)^{|\alpha+\beta|}\langle\xi\rangle_h^{-|\beta|},\tilde{g}),$$
$$\tilde{w}_{\alpha}^{\beta}=(-1)^{|\alpha+\beta|}(a\log\langle\xi\rangle_h)^{|\alpha+\beta|}\rho_{\xi}^{\beta}(-i\rho_x)^{\alpha}$$
$$+S((a\log\langle\xi\rangle_h)^{|\alpha+\beta|-1}\langle\xi\rangle_h^{-|\beta|},\tilde{g}).$$

It is not difficult to check that

$$w_{\alpha}^{\beta}=(-1)^{|\alpha|}(a\log\langle\xi\rangle_h)^{|\alpha+\beta|}(-\rho_{\xi})^{\beta}(-i\rho_x)^{\alpha}$$
$$+S((a\log\langle\xi\rangle_h)^{|\alpha+\beta|-1}\langle\xi\rangle_h^{-|\beta|},\tilde{g}).$$

We denote $(-1)^{|\alpha|}w_\alpha^\beta$ by w_α^β again we get

$$\begin{aligned}&\langle\xi\rangle_\gamma^{-a\rho}\#P(x,\xi;\gamma)\#\langle\xi\rangle_\gamma^{a\rho}\\ &=\sum_{|\alpha+\beta|<k}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P_{(\beta)}^{(\alpha)}(x,\xi;\gamma)w_\alpha^\beta+S(\langle\xi\rangle_\gamma H^k,\tilde{g}).\end{aligned}$$

We insert

$$\Big[\sum_{|\delta|\le m}\frac{1}{2^{|\delta|}\delta!}P^{(\delta)}(x,\xi)\tilde{w}_\delta\Big]_{(\beta)}^{(\alpha)}$$

into $P_{(\beta)}^{(\alpha)}(x,\xi;\gamma)$. Consider

$$\sum_{|\alpha+\beta|<k}\sum_{|\delta|\le m}\frac{1}{2^{|\alpha+\beta+\delta|}\alpha!\beta_1!\delta!\beta_2!}P_{(\beta_1)}^{(\delta+\alpha)}(\tilde{w}_\delta)_{\beta_2}w_\alpha^{\beta_1+\beta_2}.$$

Note that the sum over $|\alpha+\beta+\delta|\ge k$ belongs to $S(\langle\xi\rangle_h^m[(\gamma+a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}]^k,\tilde{g})$ because $|(\tilde{\omega})_{(\beta_2)}|\le C(\gamma+a\log\langle\xi\rangle_\gamma)^{|\delta|}$. Study the sum over $|\alpha+\beta+\delta|<k$ which is

$$\begin{aligned}&\sum\frac{1}{2^{|\tilde{\alpha}+\beta_1|}\tilde{\alpha}!\beta_1!}P_{(\beta_1)}^{(\tilde{\alpha})}\sum_{\beta\ge\beta_1}\frac{1}{2^{|\beta-\beta_1|}(\beta-\beta_1)!}\binom{\tilde{\alpha}}{\delta}(\tilde{\omega}_\delta)_{(\beta-\beta_1)}w_{\tilde{\alpha}-\delta}^{\beta_1+\beta_2}\\ &=\sum\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P_{(\beta)}^{(\alpha)}\Big(\sum_{\tilde{\beta}\ge\beta}\frac{1}{2^{|\tilde{\beta}-\beta|}(\tilde{\beta}-\beta)!}\binom{\alpha}{\delta}(\tilde{\omega}_\delta)_{(\tilde{\beta}-\beta)}w_{\alpha-\delta}^{\tilde{\beta}}\Big)\\ &=\sum\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P_{(\beta)}^{(\alpha)}\tilde{w}_\alpha^\beta\end{aligned}$$

where

$$\tilde{w}_\alpha^\beta=\sum_{\tilde{\beta}\ge\beta,|\tilde{\beta}|<k-|\alpha|}\frac{1}{2^{|\tilde{\beta}-\beta|}(\tilde{\beta}-\beta)!}\binom{\alpha}{\delta}(\tilde{\omega}_\delta)_{(\tilde{\beta}-\beta)}w_{\alpha-\delta}^{\tilde{\beta}}.$$

If $\tilde{\beta}>\beta$ then it follows that

$$\begin{aligned}(\tilde{\omega}_\delta)_{(\tilde{\beta}-\beta)}w_{\alpha-\delta}^{\tilde{\beta}}&\in S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\tilde{\beta}+\alpha|}\langle\xi\rangle_\gamma^{-|\tilde{\beta}|},\tilde{g})\\ &\subset S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g})\end{aligned}$$

and then one has

$$\tilde{w}_\alpha^\beta-\sum\binom{\alpha}{\delta}\tilde{w}_\delta w_{\alpha-\delta}^\beta\in S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g}).$$

On the other hand from

$$\begin{aligned}\tilde{\omega}_\delta&=(-i\gamma\zeta_x)^\delta+S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\delta|-1},g),\\ w_{\alpha-\delta}^\beta&=(a\log\langle\xi\rangle_h)^{|\alpha+\beta-\delta|}(-\rho_\xi)^\beta(-i\rho_x)^{\alpha-\delta}\\ &\quad+S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta-\delta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g})\end{aligned}$$

it follows that

$$\tilde{\omega}_\delta w^\beta_{\alpha-\delta} = (-i\gamma\zeta_x)^\delta(-a\log\langle\xi\rangle_\gamma\rho_\xi)^\beta(-ia\log\langle\xi\rangle_\gamma\rho_x)^{\alpha-\delta}$$
$$+S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g}).$$

Hence we conclude that

$$\sum\binom{\alpha}{\delta}\tilde{\omega}_\delta w^\beta_{\alpha-\delta} = (-i\gamma\zeta_x - ia\log\langle\xi\rangle_\gamma\rho_x)^\alpha(-a\log\langle\xi\rangle_\gamma\rho_\xi)^\beta$$
$$+S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g}).$$

Thus we have proved

Proposition 5.2.1 *We have*

$$\begin{aligned}&\langle\xi\rangle_\gamma^{-a\rho}\#e^{-\gamma\zeta(x)}\#P(x,\xi)\#e^{\gamma\zeta(x)}\#\langle\xi\rangle_\gamma^{a\rho}\\ &=\sum_{|\alpha+\beta|<k}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P^{(\alpha)}_{(\beta)}(x,\xi)\tilde{w}^\beta_\alpha\\ &\quad+S(\big[(\gamma+a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}\big]^k,\tilde{g})\end{aligned}$$

where

$$\tilde{w}^\beta_\alpha = (-i\tilde{\Lambda}_x)^\alpha(-\tilde{\Lambda}_\xi)^\beta + S((\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|},\tilde{g})$$

with $\tilde{\Lambda} = \gamma\zeta(x) + a\rho(z)\log\langle\xi\rangle_\gamma$.

Remark: Note that

$$\begin{aligned}&(\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-|\beta|}\\ &=(\gamma+a\log\langle\xi\rangle_\gamma)^{|\alpha+\beta|-1}\phi^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}\\ &=(\gamma+a\log\langle\xi\rangle_\gamma)^{-1}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)]^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}.\end{aligned}$$

Taking γ so that $\gamma + a\log\gamma \geq \lambda M^{-(2\kappa-1)}$ the right-hand side is estimated by

$$\lambda^{-1}M^{2\kappa-1}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)]^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}.$$

5.3 $T^{-M}\#P\#T^M = \tilde{P}_{T^M}$

We now apply Lemma 5.1.3 with $q = T^M$, $\tilde{q} = T^{-M}$. Let us set

$$\tilde{P}^w_{T^M} = (T^{-M})^w P^w (T^M)^w$$

then

Proposition 5.3.1 *We have*

$$\tilde{P}_{T^M} - \sum_{|\alpha+\beta|<k}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P^{(\alpha)}_{(\beta)}(z)w^\beta_\alpha(z) \in S(C(k,M)(a\log\langle\xi\rangle_\gamma)^k\langle\xi\rangle_\gamma^{m-k/2},\hat{g})$$

where $w_0^0 = 1$ *and*

$$w_\alpha^\beta - (-MT_\xi/T)^\beta(-iMT_x/T)^\alpha \in S(C_{\alpha\beta}M^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}M^{2\kappa-1}\lambda^{-1}, \hat{g})$$

and $\hat{g} = (g + \tilde{g})/2$ *for* $\gamma \geq \gamma_0(a, M, \lambda)$.

Proof: Recall that

$$H = (a\log\langle\xi\rangle_h)\langle\xi\rangle_h^{-1}\phi^{-1} = (a\log\langle\xi\rangle_h)\psi^{-1} \leq \langle\xi\rangle_h^{-1/2}(a\log\langle\xi\rangle_h).$$

From Lemma 5.1.3 the first assertion follows immediately. Recall that $\tilde{w}_\alpha^\beta = (-1)^{|\beta|}T^{-M}\#(T^M)_{(\alpha)}^{(\beta)}$. Let us denote

$$(T^M)_{(\alpha)}^{(\beta)} = \Omega_\alpha^\beta T^M.$$

Lemma 5.3.1 *We have*

$$T^{-M}\#(T^M)_{(\alpha)}^{(\beta)} = \sum_{j=0}^{\ell-1}\frac{1}{j!}\sum_{k=j}^{\ell-1}\frac{1}{k!}(\frac{i\sigma}{2})^k M^j(\log T(w) - \log T(z))^j\Omega_\alpha^\beta(w)|_{w=z}$$
$$+S(C_{\alpha\beta}(|\alpha+\beta|^\kappa + M)^{|\alpha+\beta|}(M^{2\kappa-1}\lambda^{-1})^\ell\phi^{-|\alpha|}\psi^{-|\beta|}, g).$$

Proof: Recall

$$T^{-M}\#(T^M)_{(\alpha)}^{(\beta)} = \sum\frac{1}{k!}(\frac{i\sigma}{2})^k T^{-M}(z)T^M(w)\Omega_\alpha^\beta(w)|_{w=z}.$$

Writing

$$T^{-M}(z)T^M(w) = \sum_{j=0}^{\infty}\frac{1}{j!}(M\log T(w) - M\log T(z))^j$$

it follows from Lemma 4.1.1 that

$$(\frac{i\sigma}{2})^k T^{-M}(z)T^M(w)\Omega_\alpha^\beta(w)|_{w=z}$$
$$= (\frac{i\sigma}{2})^k\sum_{j=0}^{k}\frac{1}{j!}(M\log T(w) - M\log T(z))^j\Omega_\alpha^\beta|_{w=z}$$

which together with Proposition 3.3.1 proves the assertion.

In Lemma 5.3.1 taking $\ell = 1$ we have

$$\tilde{w}_\alpha^\beta = (-1)^{|\beta|}\Omega_\alpha^\beta + S(C_{\alpha\beta}M^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}(M^{2\kappa-1}\lambda^{-1}), g),$$
$$\tilde{w}_\alpha^\beta \in S(C_{\alpha\beta}M^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}, g). \tag{5.3.1}$$

Lemma 5.3.2 *We have*

$$(-1)^{|\beta|}\Omega_\alpha^\beta = (-MT_\xi/T)^\beta(-iMT_x/T)^\alpha \in S(M^{|\alpha+\beta|-1}\phi^{-|\alpha|}\psi^{-|\beta|}, g).$$

Proof: We proceed by induction on $|\alpha+\beta|$. For $|\alpha+\beta|=1$ we see that $\Omega^{e_j}=MT_{\xi_j}/T$ and $\Omega_{e_j}=-iMT_{x_j}/T$ and hence the result. Let $|e+f|=1$. Then one has

$$(-1)^{|\beta+f|}\Omega^{\beta+f}_{\alpha+e}=(-1)^{|\beta+f|}D^e_x\partial^f_\xi\Omega^\beta_\alpha+(-1)^{|\beta+f|}M\Omega^\beta_\alpha D^e_x\partial^f_\xi T/T.$$

Since $D^e_x\partial^f_\xi\Omega^\beta_\alpha\in S(M^{|\alpha+\beta|}\phi^{-|\alpha+e|}\psi^{-|\beta+f|},g)$ we get the desired assertion.

Note that

$$(\tilde{w}^{\tilde{\beta}}_{\tilde{\alpha}})^{(\beta-\tilde{\beta})}_{(\alpha-\tilde{\alpha})}\in S(C_{\alpha\beta}M^{|\tilde{\alpha}+\tilde{\beta}|}\phi^{-|\alpha|}\psi^{-|\beta|},g)\subset S(C_{\alpha\beta}M^{|\alpha+\beta|-1}\phi^{-|\alpha|}\psi^{-|\beta|},g)$$

if $|\tilde{\alpha}+\tilde{\beta}|<|\alpha+\beta|$. This gives that

$$w^\beta_\alpha=\tilde{w}^\beta_\alpha+S(C_{\alpha\beta}M^{|\alpha+\beta|-1}\phi^{-|\alpha|}\psi^{-|\beta|},g)$$

and hence

$$\begin{aligned}w^\beta_\alpha&=(-1)^{|\beta|}\Omega^\beta_\alpha+S(C_{\alpha\beta}M^{|\alpha+\beta|-1}\phi^{-|\alpha|}\psi^{-|\beta|},g)\\&=(-MT_\xi/T)^\beta(-iMT_x/T)^\alpha+S(C_{\alpha\beta}M^{|\alpha+\beta|-1}\phi^{-|\alpha|}\psi^{-|\beta|},g).\end{aligned}$$

Since $M^{-1}\le M^{2\kappa-1}\lambda^{-1}$ we conclude the result.

5.4 Asymptotic expansion of P_{T^M}

Let us denote

$$P_{T^M}=T^{-M}\#\langle\xi\rangle^{-a\rho}_\gamma\#P(z;\gamma)\#\langle\xi\rangle^{a\rho}_\gamma\#T^M$$

and show

Proposition 5.4.1 *We have*

$$P_{T^M}(z)-\sum_{|\alpha+\beta|<\ell}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}P^{(\alpha)}_{(\beta)}w^\beta_\alpha\in S(C(M,a)\langle\xi\rangle^{m-\ell/2}_\gamma(\log\langle\xi\rangle_\gamma)^\ell,\hat{g})$$

for all $\ell\ge 0$ where $w^0_0=1$ and

$$\begin{aligned}&w^\beta_\alpha-(-i\Lambda_x)^\alpha(-\Lambda_\xi)^\beta\\&\in S(C_{\alpha\beta}M^{2\kappa-1}\lambda^{-1}((\gamma+a\log\langle\xi\rangle_\gamma)\phi+M)^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|},\hat{g})\end{aligned}$$

with $\Lambda=\gamma\zeta(x)+a\rho(z)\log\langle\xi\rangle_\gamma+M\log T(z)$.

Proof: Set

$$\tilde{P}=\langle\xi\rangle^{-a\rho}_\gamma\#P(z;\gamma)\#\langle\xi\rangle^{a\rho}_\gamma\in S(\langle\xi\rangle^m_\gamma,g)$$

and recall that $T^M\in S(T^M,\tilde{g})$. Note that

$$T^{-M}\#\tilde{P}\#T^M-\sum_{|\gamma+\delta|<k}\frac{1}{2^{|\gamma+\delta|}\gamma!\delta!}\tilde{P}^{(\gamma)}_{(\delta)}w^\delta_\gamma\in S(C(k,M)(a\log\langle\xi\rangle_\gamma)^k\langle\xi\rangle^{m-k/2}_\gamma,\hat{g}).$$

Insert the expression

$$\tilde{P} = \sum_{|\alpha+\beta|<\ell} \frac{1}{2^{|\alpha+\beta|}\alpha!\beta!} P^{(\alpha)}_{(\beta)} \tilde{w}^{\beta}_{\alpha} + S([(\gamma + a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}]^\ell, g)$$

with $\ell = k - |\gamma+\delta|$. Note that

$$\begin{aligned} &S(\langle\xi\rangle_\gamma^m [(\gamma + a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}]^{k-|\gamma+\delta|}, g)^{(\gamma)}_{(\delta)} w^{\delta}_{\gamma} \\ \in\ &S(\langle\xi\rangle_\gamma^m [(\gamma + a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}]^{k-|\gamma+\delta|} (a\log\langle\xi\rangle_\gamma)^{|\gamma+\delta|}\langle\xi\rangle_\gamma^{-|\gamma|} M^{|\gamma+\delta|}\phi^{-|\gamma|}\psi^{-|\delta|}, \hat{g}). \end{aligned}$$

Since $\psi^{-1} = \phi^{-1}\langle\xi\rangle_\gamma^{-1}$ the right-hand side belongs to

$$S(\langle\xi\rangle_\gamma^m [(\gamma + a\log\langle\xi\rangle_\gamma)\langle\xi\rangle_\gamma^{-1}]^k M^{|\delta+\gamma|}\phi^{-|\gamma+\delta|}, \hat{g}).$$

In view of $\phi^{-1} \le \lambda^{-1/2}\langle\xi\rangle_\gamma^{1/2}$ the term is in $S(C(k,M)\langle\xi\rangle_\gamma^{m-k/2}(\gamma + a\log\langle\xi\rangle_\gamma)^k, \hat{g})$.

We now study the sum over $|\alpha+\beta+\gamma+\delta| < k$:

$$\begin{aligned} &\sum_{|\alpha+\beta+\gamma+\delta|<k} \frac{1}{2^{|\alpha+\beta+\gamma+\delta|}\alpha!\beta!\gamma!\delta!} (P^{(\alpha)}_{(\beta)}\tilde{w}^{\beta}_{\alpha})^{(\gamma)}_{(\delta)} \\ =&\sum \frac{1}{2^{|\tilde{\alpha}+\tilde{\beta}|}\tilde{\alpha}!\tilde{\beta}!} P^{(\tilde{\alpha})}_{(\tilde{\beta})} \sum \frac{1}{2^{|\gamma-\gamma'+\delta-\delta'|}(\gamma-\gamma')!(\delta-\delta')!} \binom{\tilde{\alpha}}{\gamma'}\binom{\tilde{\beta}}{\delta'} (\tilde{w}^{\tilde{\beta}-\delta'}_{\tilde{\alpha}-\gamma'})^{(\gamma-\gamma')}_{(\delta-\delta')} w^{\delta}_{\gamma}. \end{aligned}$$

Let us set

$$\hat{w}^{\tilde{\beta}}_{\tilde{\alpha}} = \sum \frac{1}{2^{|\gamma-\gamma'+\delta-\delta'|}(\gamma-\gamma')!(\delta-\delta')!} \binom{\tilde{\alpha}}{\gamma'}\binom{\tilde{\beta}}{\delta'} (\tilde{w}^{\tilde{\beta}-\delta'}_{\tilde{\alpha}-\gamma'})^{(\gamma-\gamma')}_{(\delta-\delta')} w^{\delta}_{\gamma}$$

and write

$$\begin{aligned} &\hat{w}^{\tilde{\beta}}_{\tilde{\alpha}} = \sum \binom{\tilde{\alpha}}{\gamma}\binom{\tilde{\beta}}{\delta} \tilde{w}^{\tilde{\beta}-\delta}_{\tilde{\alpha}-\gamma} w^{\delta}_{\gamma} \\ +&\sum_{|\gamma-\gamma'+\delta-\delta'|\ge 1} \frac{1}{2^{|\gamma-\gamma'+\delta-\delta'|}(\gamma-\gamma')!(\delta-\delta')!} \binom{\tilde{\alpha}}{\gamma'}\binom{\tilde{\beta}}{\delta'} (\tilde{w}^{\tilde{\beta}-\delta'}_{\tilde{\alpha}-\gamma'})^{(\gamma-\gamma')}_{(\delta-\delta')} w^{\delta}_{\gamma}. \end{aligned}$$

We show that the second term in the right-hand side is in

$$S([\phi(\gamma + a\log\langle\xi\rangle_\gamma) + M]^{|\tilde{\alpha}+\tilde{\beta}|}(M\lambda^{-1})\phi^{-|\tilde{\alpha}|}\psi^{-|\tilde{\beta}|}, \hat{g}).$$

Indeed $(\tilde{w}^{\tilde{\beta}-\delta'}_{\tilde{\alpha}-\gamma'})^{(\gamma-\gamma')}_{(\delta-\delta')} w^{\delta}_{\gamma}$ is in

$$\begin{aligned} &S([\phi(\gamma + a\log\langle\xi\rangle_\gamma)]^{|\tilde{\alpha}+\tilde{\beta}-\gamma'-\delta'|}\phi^{-|\tilde{\alpha}-\gamma'+\delta-\delta'|}\psi^{-|\tilde{\beta}-\delta'+\gamma-\gamma'|}, g) \\ &\times S(M^{|\delta+\gamma|}\phi^{-|\gamma|}\psi^{-|\delta|}, \hat{g}) \\ \subset\ &S([\phi(\gamma + a\log\langle\xi\rangle_\gamma) + M]^{|\tilde{\alpha}+\tilde{\beta}|}\phi^{-|\tilde{\alpha}|}\psi^{-|\tilde{\beta}|}(M\phi^{-1}\psi^{-1})^{|\gamma-\gamma'+\delta-\delta'|}, \hat{g}) \end{aligned}$$

and hence the assertion because $|\gamma-\gamma'+\delta-\delta'|\geq 1$. We turn to

$$\sum\binom{\tilde\alpha}{\gamma}\binom{\tilde\beta}{\delta}\tilde w^{\tilde\beta-\delta}_{\tilde\alpha-\gamma}w^\delta_\gamma$$

$$=\sum\binom{\tilde\alpha}{\gamma}\binom{\tilde\beta}{\delta}\Big[(-i\gamma\zeta_x-ia\log\langle\xi\rangle_\gamma\rho_x)^{\tilde\alpha-\gamma}(-a\log\langle\xi\rangle_\gamma\rho_\xi)^{\tilde\beta-\delta} \tag{5.4.1}$$
$$+S((\gamma+a\log\langle\xi\rangle_\gamma)^{-1}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)]^{|\tilde\alpha+\tilde\beta-\gamma-\delta|}\phi^{-|\tilde\alpha-\gamma|}\psi^{-|\tilde\beta-\delta|},g)\Big]$$
$$\times\Big[(-iMT_x/T)^\gamma(-MT_\xi/T)^\delta+S(M^{|\gamma+\delta|-1}\phi^{-|\gamma|}\psi^{-|\delta|},\hat g)\Big].$$

Note that

$$(-i\gamma\zeta_x-ia\log\langle\xi\rangle_\gamma\rho_x)^{\tilde\alpha-\gamma}(-a\log\langle\xi\rangle_\gamma\rho_\xi)^{\tilde\beta-\delta}$$
$$\in S([\phi(\gamma+a\log\langle\xi\rangle_\gamma)]^{|\tilde\alpha+\tilde\beta-\gamma-\delta|}\phi^{-|\tilde\alpha-\gamma|}\psi^{-|\tilde\beta-\delta|},g),$$
$$(MT_x/T)^\gamma(MT_\xi/T)^\delta\in S(M^{|\gamma+\delta|}\phi^{-|\gamma|}\psi^{-|\delta|},\hat g)$$

because $\langle\xi\rangle_\gamma^{-|\tilde\beta-\delta|}=\phi^{|\tilde\alpha+\tilde\beta-\gamma-\delta|}\phi^{-|\tilde\alpha-\gamma|}\psi^{-|\tilde\beta-\delta|}$. Then (5.4.1) turns to

$$(-i\gamma\zeta_x-ia\log\langle\xi\rangle_\gamma\rho_x-iMT_x/T)^{\tilde\alpha}(-a\log\langle\xi\rangle_\gamma\rho_\xi-MT_\xi/T)^{\tilde\beta}$$
$$+S(M^{2\kappa-1}\lambda^{-1}[\phi(\gamma+a\log\langle\xi\rangle_\gamma+M]^{|\tilde\alpha+\tilde\beta|}\phi^{-|\tilde\alpha|}\psi^{-|\tilde\beta|},\hat g)$$

choosing γ so that $(\gamma+a\log\gamma)^{-1}\leq M^{2\kappa-1}\lambda^{-1}$ and $M^{-1}\leq M^{2\kappa-1}\lambda^{-1}$. This proves the assertion.

6 Studies on the principal symbol of P_{T^M}

6.1 Definition of $p(z;\zeta)$, $Q(z)$

We set

$$p(z;(y,\eta))=\sum_{|\alpha+\beta|<m+2}\frac{1}{\alpha!\beta!}\partial_x^\beta\partial_\xi^\alpha p(z)(i\eta)^\alpha(iy)^\beta.$$

Let $H_\Lambda=(\Lambda_\xi,-\Lambda_x)$ and hence

$$p(z;H_\Lambda/2)=\sum_{|\alpha+\beta|<m+2}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta p(z)(i\Lambda_\xi)^\beta(-i\Lambda_x)^\alpha$$
$$=\sum_{|\alpha+\beta|<m+2}\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}p^{(\alpha)}_{(\beta)}(z)(-\Lambda_\xi)^\beta(-i\Lambda_x)^\alpha. \tag{6.1.1}$$

Note that one can write $p(z;\rho\zeta)$ as

$$p(z;\rho\zeta)=\sum_{j=0}^{m+1}(i\rho\frac{\partial}{\partial t})^j p(z+t\zeta)/j!|_{t=0}.$$

Define $Q(z)$ by

$$Q(z) = (\langle\xi\rangle_\gamma^2|\Lambda_\xi|^2 + |\Lambda_x|^2)^{-1/2} H_\Lambda p(z;\zeta)|_{\zeta=H_\Lambda/2}$$

where $H_\Lambda = \Lambda_\xi\partial/\partial x - \Lambda_x\partial/\partial\xi$. Let us set $\tilde{H}_\Lambda = (\langle\xi\rangle_\gamma\Lambda_\xi, -\Lambda_x)$. Then we have

$$Q(z) = 2|\tilde{H}_\Lambda|^{-1}\sum_{j=0}^{m+1}(\frac{\partial}{\partial t})(i\frac{\partial}{\partial t})^j p(z+tH_\Lambda/2)/j!|_{t=0}.$$

Indeed we see

$$\sum_{j=0}^{m+1}\frac{\partial}{\partial t}(i\frac{\partial}{\partial t})^j p(z+tH_\Lambda/2)/j!|_{t=0}$$

$$= \frac{1}{2}\sum_{j=0}^{m+1}\sum_{|\alpha+\beta|=j}\Big[(\Lambda_\xi\frac{\partial}{\partial x} - \Lambda_x\frac{\partial}{\partial\xi})\frac{1}{2^{|\alpha+\beta|}\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta p(z)\Big](i\Lambda_\xi)^\beta(-i\Lambda_x)^\alpha$$

$$= \frac{1}{2}(\Lambda_\xi\frac{\partial}{\partial x} - \Lambda_x\frac{\partial}{\partial\xi})p(z;\zeta)|_{\zeta=H_\Lambda/2} = \frac{1}{2}(H_\Lambda p)(z;H_\Lambda/2).$$

6.2 More about $p(z;\zeta)$

We prove

Proposition 6.2.1 *Let $K \subset\subset \Gamma(p_{z^0},(0,\theta))$ be compact convex where $z^0 = (x^0,\xi^0)$ with $|\xi^0| = 1$. Then there is a neighborhood V of z^0 and $C > 0$ such that*

$$C^{-1} \le |p(z;\rho\zeta)/p(x,\xi - i\rho\theta)| \le C$$

for $z \in V$, $-\zeta \in K$, $0 < \rho \ll 1$.

Proof: By Proposition 8.3.1 one can find a neighborhood V of z^0 and

$$e(z,\zeta,t) \in C^\infty(V)\times(-K)\times\{|t| < t_0\}$$

so that

$$p(z+t\zeta) = e(z,\zeta,t)\prod_{j=1}^{r}(t-\mu_j(\zeta;z)) \qquad (6.2.1)$$

for $z \in V$, $-\zeta \in K$, where $\mu_j(\zeta,z)$ are real and

$$C^{-1}|\mu_j((0,\theta);z)| \le |\mu_j(\zeta;z)| \le C|\mu_j((0,\theta);z)|$$

for $z \in V$, $-\zeta \in K$, $1 \le j \le r$ (note that $\mu_j((0,\theta);z^0) = 0$ for $1 \le j \le r$). Write

$$\prod_{j=1}^{r}(t-\mu_j) = \sum_{\ell=0}^{r} p_\ell t^\ell.$$

Then note that

$$\sum_{j=0}^{m+1}\frac{1}{j!}(\rho\frac{\partial}{\partial t})^j e\prod_{j=0}^{r}(t-\mu_j)|_{t=0}$$

$$=\sum_{j=0}^{m+1}\sum_{k+\ell=j}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e\frac{1}{\ell!}(\rho\frac{\partial}{\partial t})^\ell\prod_{j=1}^{r}(t-\mu_j)|_{t=0}$$

$$=\sum_{\ell=0}^{r}\sum_{k=0}^{m+1-\ell}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e|_{t=0}p_\ell\rho^\ell$$

$$=\sum_{\ell=0}^{r}\Big(\sum_{k=0}^{m+1}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e|_{t=0}-\sum_{k\geq m-\ell+2}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e|_{t=0}\Big)p_\ell\rho^\ell$$

$$=\sum_{\ell=0}^{r}(\sum_{k=0}^{m+1}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e)|_{t=0}p_\ell\rho^\ell+O(\rho^{m+2})$$

$$=\sum_{k=0}^{m+1}\frac{1}{k!}(\rho\frac{\partial}{\partial t})^k e|_{t=0}\prod_{j=1}^{r}(\rho-\mu_j)+O(\rho^{m+2}).$$

This proves that

$$p(z;\rho\zeta)=\sum_{j=0}^{m+1}(i\rho\frac{\partial}{\partial t})^j p(z+t\zeta)/j!|_{t=0}$$

$$=e_1(z,\zeta,\rho)\prod_{k=0}^{r}(i\rho-\mu_j(\zeta;z))+O(\rho^{r+1}) \tag{6.2.2}$$

where

$$e_1=\sum_{k=0}^{m+1}(i\rho\frac{\partial}{\partial t})^k e(z;\zeta,t)/k!|_{t=0}.$$

Note that $e_1(z,\zeta,0)\neq 0$. Recalling

$$p(x,\xi-i\rho\theta)=\prod_{j=1}^{m}(i\rho-\mu_j((0,\theta);z))$$

where $\mu_j((0,\theta);z^0)\neq 0$ for $j=r+1,...,m$ we conclude the assertion.

6.3 Estimate of Λ

Recall that

$$\Lambda(z)=\gamma\zeta(x)+a\rho(z)\log\langle\xi\rangle_\gamma+M\log T(z).$$

Since $p(x,a\xi;y,a\eta)=a^m p(x,\xi;y,\eta)$ for $a>0$ one can rewrite

$$p(z;H_\Lambda/2)=\langle\xi\rangle_\gamma^m p(\tilde{z};\langle\xi\rangle_\gamma^{-1}\tilde{H}_\Lambda/2)=\langle\xi\rangle_\gamma^m p(\tilde{z};\lambda\tilde{H}_\Lambda/|\tilde{H}_\Lambda|)$$

where

$$\lambda = |\tilde{H}_\Lambda|\langle\xi\rangle_\gamma^{-1}/2, \quad \tilde{z} = (x, \xi\langle\xi\rangle_\gamma^{-1}).$$

Lemma 6.3.1 *There is a conic neighborhood U of z^0 such that for $z \in U$, $h = \gamma \geq \gamma_0(M, \lambda, a)$, $M \gg 1$ we have*

$$C^{-1} \leq |p(z; H_\Lambda)/p(x, \xi - i\langle\xi\rangle_\gamma \lambda(x,\xi)\theta)| \leq C.$$

To prove this we first show

Lemma 6.3.2 *Let $\tilde{H}_\Lambda = (\langle\xi\rangle_\gamma \Lambda_\xi, -\Lambda_x)$. Then we have*

(i) *there are a conic neighborhood U of z^0 and a compact convex set $K \subset \Gamma(p_{z^0}, (0,\theta))$ such that*

$$-\tilde{H}_\Lambda/|\tilde{H}_\Lambda| \in K$$

for $z \in U$, $h = \gamma \geq \gamma_0(M, a)$, $M \gg 1$,

(ii) *there are a conic neighborhood U and $C > 0$ such that*

$$C^{-1} \leq |\tilde{H}_\Lambda|/(\gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1}) \leq C.$$

Proof: Recall $\zeta(x) = \langle x - x^0, \tilde{\theta}\rangle + k|x - x^0|^2$ near x^0 where $(0, \tilde{\theta}) \in \Gamma(p_{z^0}, (0,\theta))$. This shows

$$|(0, \gamma\zeta_x(x)) - (0, \gamma\tilde{\theta})| \leq \gamma C|x - x^0| \tag{6.3.1}$$

near x^0. Since $\rho(z)\log\langle\xi\rangle_\gamma$ coincides with $(x_1 - x_1^0 + |x - x^0|^2 + |\xi/|\xi| - \xi^0|^2)\log\langle\xi\rangle_\gamma$ near z^0 we may assume that

$$|\tilde{H}_{a\rho\log\langle\xi\rangle_\gamma} - (0, a\log\langle\xi\rangle_\gamma\theta)| \leq Ca\log\langle\xi\rangle_\gamma(|x - x^0| + |\xi/|\xi| - \xi^0|) \tag{6.3.2}$$

near z^0. From Lemma 4.2.1 it follows that

$$-\phi(\langle\xi\rangle_\gamma T_\xi, -T_x)/T \in K$$

for $z \in U$, $|\xi| \geq \nu h$ with a compact $K \subset \Gamma(p_{z^0}, (0,\theta))$. This shows that

$$C^{-1} \leq \phi(z)|(\langle\xi\rangle_\gamma T_\xi, -T_x)|/T \leq C \tag{6.3.3}$$

for $z \in U$, $|\xi| \geq \nu h \geq \lambda$. From (6.3.1) and (6.3.2) it follows that

$$-\tilde{H}_{\gamma\zeta(x) + a\rho\log\langle\xi\rangle_\gamma}/(\gamma + a\log\langle\xi\rangle_\gamma) \in K$$

for $z \in U$ since we may assume that $\theta, \tilde{\theta}$ are contained in the interior of K. From (6.3.3) we have

$$-\tilde{H}_{M\log T}/M\phi^{-1} \in K.$$

Then we conclude that

$$-\tilde{H}_\Lambda/(\gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1}) \in K.$$

This proves (ii) when $|\xi| \geq \nu h$. Writing

$$-\tilde{H}_\Lambda/|\tilde{H}_\Lambda| = -\tilde{H}_\Lambda(\gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1})^{-1}\Big[\frac{\gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1}}{|\tilde{H}_\Lambda|}\Big]$$

modifying K if necessary, we have (i) for $|\xi| \geq \nu h$ in view of (ii).

Consider the case $z \in U$ and $|\xi| \leq \nu h$. Note that

$$|\tilde{H}_{a\rho\log\langle\xi\rangle_\gamma + M\log T}| \leq a\log h + CMh^{1/2}\lambda^{-1/2} \leq Ch^{1/2}(a+M)$$

because $|\tilde{H}_{\log T}| \leq C\phi^{-1} \leq Ch^{1/2}\lambda^{-1/2}$ if $|\xi| \leq \nu h$. On the other hand we have $\gamma|\zeta_x(x)| \geq c\gamma$ with some $c > 0$ if $|x - x^0|$ is sufficiently small. Then taking $\gamma = h \geq \gamma_0(M, a)$ large enough and U small we may assume that $-\tilde{H}_\Lambda/|\tilde{H}_\Lambda|$ is enough close to $(0, \tilde{\theta}) \in \Gamma(p_{z^0}, (0, \theta))$ and $C^{-1}\gamma \leq |\tilde{H}_\Lambda| \leq C\gamma$. This proves (i) and (ii) when $|\xi| \leq \nu h$.

Proof of Lemma 6.3.1: Recall

$$p(z; H_\Lambda/2) = \langle\xi\rangle_\gamma^m p(\tilde{z}; \lambda(z)\tilde{H}_\Lambda/|\tilde{H}_\Lambda|),$$
$$p(x, \xi - i\lambda(z)\langle\xi\rangle_\gamma\theta) = \langle\xi\rangle_\gamma^m p(x, \xi\langle\xi\rangle_\gamma^{-1} - i\lambda(z)\theta).$$

By Lemma 6.3.2 we have $-\tilde{H}_\Lambda/|\tilde{H}_\Lambda| \in K$ when $z \in U$, $\gamma = h \geq \gamma_0$. Assume $|\xi| \geq \nu h$ and hence we may assume that $(x, \xi/\langle\xi\rangle_\gamma) \in V$ if $z \in U$ then from Proposition 6.2.1 it follows that

$$C^{-1} \leq |p(\tilde{z}; \lambda(z)\tilde{H}_\Lambda/|\tilde{H}_\Lambda|)/p(x, \xi\langle\xi\rangle_\gamma^{-1} - i\lambda(z)\theta)| \leq C$$

if $z \in U$, $0 < \lambda(z) \ll 1$. On the other hand (ii) of Lemma 6.3.2 shows that

$$C^{-1} \leq |\tilde{H}_\Lambda|/(\gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1}) \leq C$$

and then we make $\lambda(z)$ as small as we please when $|\xi| \geq \nu h$ taking ν large. If $|\xi| \leq \nu h$ and $h = \gamma$ we have

$$C^{-1}\gamma^m \leq |p(x, \xi - i\langle\xi\rangle_\gamma\lambda(z)\theta)| \leq C\gamma^m.$$

Indeed we have $\gamma \leq \langle\xi\rangle_\gamma \leq C\gamma$ and $C^{-1} \leq \lambda(z) \leq C$ if $z \in U$ and $|\xi| \leq \nu\gamma$. We turn to $p(z; H_\Lambda/2)$. Write

$$p(z; H_\Lambda/2) = (\rho^{-1}\gamma)^m p((x, \rho\gamma^{-1}\xi); \rho(\rho^{-1}\Lambda_\xi, -\gamma^{-1}\Lambda_x)/2).$$

Taking ρ_0 small so that $|\rho_0\gamma^{-1}\xi|$ is small if $|\xi| \leq \nu\gamma$. Note that $|\Lambda_\xi| \leq C\gamma^{-1/2}$ and $\gamma^{-1}\Lambda_x$ is close to $\tilde{\theta}$ if γ is large. Since $(0, \tilde{\theta}) \in \Gamma_{(x^0,0)} = \mathbf{R}^n \times \Gamma(p(x^0, \cdot), \theta)$ one can apply Proposition 8.3.1 to conclude that

$$p(\tilde{w}; \rho\zeta) = e(\tilde{w}, \zeta, \rho)\prod_{j=1}^{m}(i\rho - \mu_j(\zeta; \tilde{w})) + O(\rho^{m+1})$$

where $\tilde{w} = (x, \rho\gamma^{-1}\xi)$, $\zeta = (\rho^{-1}\Lambda_\xi, -\gamma^{-1}\Lambda_x)/2$. This proves

$$C^{-1}\gamma^m \leq p(z; H_\Lambda) \leq C\gamma^m \tag{6.3.4}$$

and hence the assertion.

6.4 $Q(z)$ separates $p(z;H_\Lambda)$

We show that

Lemma 6.4.1 *Let $\tilde z\in V$, $\tilde z=(x,\xi\langle\xi\rangle_\gamma^{-1})$, $\lambda(z)=|\tilde H_\Lambda|\langle\xi\rangle_\gamma^{-1}/2$, $\omega=\tilde H_\Lambda/|\tilde H_\Lambda|$. Then we have*

$$Q(z)=\langle\xi\rangle_\gamma^{m-1}\big\{-i\partial_\lambda e_1(\tilde z,\omega,\lambda)\prod_{j=1}^r(i\lambda-\mu_j(\omega,\tilde z))$$
$$+e_1(\tilde z,\omega,\lambda)\sum_{1\le j_1<\cdots<j_{r-1}\le r}(i\lambda-\mu_1(\omega,\tilde z))\cdots(i\lambda-\mu_{j_{r-1}}(\omega,\tilde z))+O(\lambda^r)\big\}.$$

Proof: Recall

$$\begin{aligned}
Q(z)&=2|\tilde H_\Lambda|^{-1}\sum_{j=0}^{m+1}(\frac{\partial}{\partial t})(i\frac{\partial}{\partial t})^j p(z+tH_\Lambda/2)/j!|_{t=0}\\
&=2|\tilde H_\Lambda|^{-1}\langle\xi\rangle_\gamma^m\sum_{j=0}^{m+1}(\frac{\partial}{\partial t})(i\frac{\partial}{\partial t})^j p(\tilde z+t\lambda(z)\omega)/j!|_{t=0}\\
&=\langle\xi\rangle_\gamma^{m-1}\sum_{j=0}^{m+1}\lambda(z)^j(\frac{\partial}{\partial t})(i\frac{\partial}{\partial t})^j p(\tilde z+t\omega)/j!|_{t=0} \qquad (6.4.1)\\
&=\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial\rho}\sum_{j=0}^{m+1}\frac{\rho^{j+1}}{(j+1)!}(\frac{\partial}{\partial t})(i\frac{\partial}{\partial t})^j p(\tilde z+t\omega)|_{t=0,\rho=\lambda(z)}\\
&=\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial\rho}\frac{1}{i}\sum_{j=0}^{m+1}\frac{1}{(j+1)!}(i\rho\frac{\partial}{\partial t})^{j+1}p(\tilde z+t\omega)|_{t=0,\rho=\lambda(z)}\\
&=\frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial\rho}\Big\{p(\tilde z;\rho\omega)+O(\rho^{m+2})\Big\}_{\rho=\lambda(z)}\\
&=\frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\Big\{\frac{\partial}{\partial\rho}p(\tilde z;\rho\omega)|_{\rho=\lambda(z)}+O(\lambda^{m+1})\Big\}.
\end{aligned}$$

Here we have used (6.2.2) and $\partial p(\tilde z+t\omega)/\partial\rho=0$. Since

$$p(\tilde z;\rho\omega)=e_1(\tilde z,\omega,\rho)\prod_{j=1}^r(i\rho-\mu_j(\omega;\tilde z))+O(\rho^{r+1})$$

the right-hand side of (6.4.1) is

$$\frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\Big[\frac{\partial}{\partial\rho}(e_1(\tilde z,\omega,\rho)\prod_{j=1}^r(i\rho-\mu_j(\omega;\tilde z)))|_{\rho=\lambda}+O(\lambda^r)\Big]. \qquad (6.4.2)$$

This proves the assertion.

Lemma 6.4.2 *Let $w(z) = \gamma + a\log\langle\xi\rangle_\gamma + M\phi(z)^{-1}$ and set*

$$S_0(z) = \mathrm{Im}(p(z; H_\Lambda/2)\overline{Q(z)}).$$

Then we have

$$S_0(z) \sim w(z)h_{m-1}(x, \xi - iw(z)\theta)$$

for $\tilde{z} \in V$, $h = \gamma \geq \gamma_0(M, \lambda, a)$.

Proof: Recall that

$$p(\tilde{z}; \lambda(z)\omega) = e_1(\tilde{z}, \omega, \rho)\prod_{j=1}^{r}(i\rho - \mu_j(\omega, \tilde{z})) + O(\rho^{r+1})|_{\rho=\lambda},$$
$$p(z; H_\Lambda/2) = \langle\xi\rangle_\gamma^m p(\tilde{z}; \lambda\omega).$$

Then we have

$$p(z; H_\Lambda/2)\overline{Q(z)} = i\langle\xi\rangle_\gamma^{2m-1}\Big[e_1(\tilde{z}, \omega, \lambda)\prod_{j=1}^{r}(i\lambda - \mu_j(\omega; \tilde{z})) + O(\lambda^{r+1})\Big]$$
$$\times\Big[\frac{\partial}{\partial\lambda}(\overline{e_1(\tilde{z}, \omega, \rho)}\prod_{j=1}^{r}(-i\lambda - \mu_j(\omega, \tilde{z})) + O(\lambda^r)\Big]$$
$$= i\langle\xi\rangle_\gamma^{2m-1}|e_1|^2\lambda\Big\{\sum_{1\leq\ell_1<\cdots<\ell_{r-1}\leq r}(\lambda^2 + \mu_{\ell_1}^2)\cdots(\lambda^2 + \mu_{\ell_{r-1}}^2)(1 + O(\lambda + \sum_{j=1}^{r}|\mu_j|))\Big\}.$$

Indeed writing $e_1(\tilde{z}, \omega, \lambda) = e(\tilde{z}, \omega, 0) + i\lambda\partial e/\partial\lambda(\tilde{z}, \omega, 0) + O(\lambda^2)$ we see

$$|e_1|^2 = |e(\tilde{z}, \omega, 0)|^2 + O(\lambda^2)$$

and hence

$$\mathrm{Re}\frac{\partial}{\partial\lambda}\overline{e_1}e_1 = \frac{1}{2}\frac{\partial}{\partial\lambda}|e_1|^2 = O(\lambda).$$

Since $\mu_j(\omega; \tilde{z}^0) = 0$ for $j = 1, ..., r$ we see that

$$S_0(z) \sim \langle\xi\rangle_\gamma^{2m-1}\lambda\sum_{1\leq\ell_1<\cdots<\ell_{r-1}\leq r}\prod(\lambda^2 + \mu_{\ell_k}(\omega, \tilde{z})^2).$$

On the other hand we have

$$h_{m-1}(x, \xi - iw(z)\theta) \sim |\xi|^{2m-1}\sum_{1\leq\ell_1<\cdots<\ell_{r-1}\leq r}\prod(w(z)^2|\xi|^{-2} + \mu_{\ell_k}((0, \theta), (x, \xi/|\xi|))^2).$$

Noting that $\lambda(z) \sim w(z)\langle\xi\rangle_\gamma^{-1}$ we get the assertion.

7 Microlocal hyperbolic apriori estimate

7.1 Estimate of P_j

Let us recall

$$\Lambda(z) = \gamma\zeta(x) + a\rho(z)\log\langle\xi\rangle_\gamma + M\log T(z).$$

Let us set

$$\begin{aligned}\rho_\alpha^\beta(z) &= w_\alpha^\beta(z) - (-i\Lambda_x)^\alpha(-\Lambda_\xi)^\beta \\ &\in S(C_{\alpha\beta}M^{2\kappa-1}\lambda^{-1}[\phi(\gamma + a\log\langle\xi\rangle_\gamma) + M]^{|\alpha+\beta|}\phi^{-|\alpha|}\psi^{-|\beta|}, g)\end{aligned}$$

for $|\alpha+\beta| \ge 1$. Then from Proposition 5.4.1 it follows that

$$\begin{aligned}P_{T^M}(z) = p(z; H_\Lambda/2) &+ \sum_{0<|\alpha+\beta|<m+2} p_{(\beta)}^{(\alpha)}\rho_\alpha^\beta/2^{|\alpha+\beta|}\alpha!\beta! \\ &+ \sum_{j=0}^{m-1}\sum_{|\alpha+\beta|<m+2} P_{j(\beta)}^{(\alpha)}w_\alpha^\beta/2^{|\alpha+\beta|}\alpha!\beta! + r(z)\end{aligned} \tag{7.1.1}$$

where $r(z) \in S(C(M,a)\langle\xi\rangle_\gamma^{(m-2)/2}(\log\langle\xi\rangle_\gamma)^{m+2}, g)$. Note that in the sum only $p_{(\beta)}^{(\alpha)}$, $|\alpha+\beta| \ge 1$ occurs.

In what follows we assume

$$r = 2. \tag{7.1.2}$$

Note that

$$\begin{aligned}h_{m-1}(z) = \sum_{1\le\ell_1<\cdots<\ell_{m-1}\le m} &|q_{\ell_1}(z)|^2\cdots|q_{\ell_{m-1}}(z)|^2 \\ &\ge c|\xi|^{2m-4}\big(|q_1(z)|^2 + q_2(z)|^2\big)\end{aligned}$$

for $z \in U$. This gives

$$\begin{aligned}h_{m-1}(x, \xi - iw(z)\theta) = \sum_{1\le\ell_1<\cdots<\ell_{m-1}\le m} &(w(z)^2 + |q_{\ell_1}(z)|^2)\cdots(w(z)^2 + |q_{\ell_{m-1}}(z)|^2) \\ &\ge c\langle\xi\rangle_\gamma^{2m-4}\big(w(z)^2 + q_1(z)^2 + q_2(z)^2\big)\end{aligned}$$

for $h = \gamma \ge \gamma_0$ because $w(z) \ge \gamma$.

Lemma 7.1.1 *Let $\tau(z) = \gamma + a\log\langle\xi\rangle_\gamma + M\lambda^{-1}\psi$. Then we have*

$$c(\tau(z) + w(z))^2\langle\xi\rangle_\gamma^{2m-4} \le h_{m-1}(x, \xi - iw(z)\theta)$$

for $z \in U$, $h = \gamma \ge \gamma_0$.

Proof: It suffices to show that

$$w(z) + |q_1(z)| + |q_2(z)| \ge cM\lambda^{-1}\psi$$

with some $c > 0$. Let $\lambda \geq M$ then we see

$$t_h(z)^2\langle\xi\rangle_\gamma^2 + M^2\phi^{-2} \geq M^2\lambda^{-2}\big[t_h(z)^2\langle\xi\rangle_\gamma^2 + \lambda^2\phi^{-2}\big]$$
$$= M^2\lambda^{-2}\phi^{-2}\langle\xi\rangle_\gamma^2\big[t_h(z)^2\phi^2 + \lambda^2\langle\xi\rangle_\gamma^{-2}\big] \geq M^2\lambda^{-2}\phi^2\langle\xi\rangle_\gamma^2 = M^2\lambda^{-2}\psi^2.$$

Since $2(q_1^2 + q_2^2) \geq (q_1 - q_2)^2 \geq ct_h(z)^2\langle\xi\rangle_\gamma^2$ we get the desired assertion.

We now show

Lemma 7.1.2 *There is a conic neighborhood U of z^0 such that*

$$|P_{j(\beta)}^{(\alpha)}| \leq C_{\alpha\beta}\tau(z)^{-2(m-j)-|\alpha+\beta|+1}\langle\xi\rangle_\gamma^{m-j+|\beta|}\sqrt{m_1}$$

for $m - j + |\alpha + \beta| \geq 1$, $z \in U$, $h = \gamma \geq \gamma_0(M, a)$ where

$$m_1(z) = h_{m-1}(x, \xi - iw(z)\theta).$$

Proof: By Lemma 8.3.2 we have for $|\alpha + \beta| = 1$

$$|p_{(\beta)}^{(\alpha)}| \leq Ch_{m-1}(z)^{1/2}|\xi|^{|\beta|} \leq Ch_{m-1}(x, \xi - iw(z)\theta)\langle\xi\rangle_\gamma^{|\beta|} = C\sqrt{m_1}\langle\xi\rangle_\gamma^{|\beta|}.$$

When $|\alpha + \beta| \geq 2$ then we have

$$|p_{(\beta)}^{(\alpha)}| \leq C_{\alpha\beta}\langle\xi\rangle_\gamma^{m-|\alpha|} \leq C_{\alpha\beta}\langle\xi\rangle_\gamma^{m-2}\langle\xi\rangle_\gamma^{2-|\alpha|} \leq C_{\alpha\beta}\tau^{-1}\sqrt{m_1}\langle\xi\rangle_\gamma^{2-|\alpha|} \qquad (7.1.3)$$

since $\langle\xi\rangle_\gamma^{m-2} \leq C\tau^{-1}h_{m-1}(x, \xi - iw(z)\theta)^{1/2}$ by Lemma 7.1.1. Since $\tau(z) \leq C\langle\xi\rangle_\gamma$ we get the assertion.

Study P_j with $j < m$. Note

$$|P_{j(\beta)}^{(\alpha)}| \leq C_{\alpha\beta}\langle\xi\rangle_\gamma^{j-|\alpha|} \leq C_{\alpha\beta}\langle\xi\rangle_\gamma^{m-2}\langle\xi\rangle_\gamma^{j-m-|\alpha|+2}$$
$$\leq C_{\alpha\beta}\tau^{-1}\sqrt{m_1}\langle\xi\rangle_\gamma^{m-j+|\beta|}\langle\xi\rangle_\gamma^{-2(m-j)-|\alpha+\beta|+2}.$$

Since $-2(m - j) - |\alpha + \beta| + 2 \leq 0$ we get the desired assertion.

7.2 Derivatives of P_{T^M}

We show

Lemma 7.2.1 *We have*

$$|P_{T^M(\delta)}^{(\rho)}| \leq C_{\rho\delta}\tau\sqrt{m_1}(\lambda M^{-1})^{|\rho+\delta|}\phi^{-|\delta|}\psi^{-|\rho|}$$

for $|\rho + \delta| \geq 1$, $z \in U$, $\gamma = h \geq \gamma_0(M, \lambda, a)$, $\lambda = M^{2-\epsilon} \gg M$.

Proof: Recall

$$P_{T^M} - \sum_{j=0}^{m} \sum_{|\alpha+\beta|<k} P_{j(\beta)}^{(\alpha)} w_\alpha^\beta / 2^{|\alpha+\beta|} \alpha! \beta! \in S(C(k,M)(a\log\langle\xi\rangle_\gamma)^k \langle\xi\rangle_\gamma^{m-k/2}, g)$$

and hence

$$P_{T^M(\delta)}^{(\rho)} - \sum_{j=0}^{m} \sum_{|\alpha+\beta|<m+2} \sum \binom{\rho}{\rho'}\binom{\delta}{\delta'} P_{j(\beta+\delta')}^{(\alpha+\rho')} w_{\alpha(\delta-\delta')}^{\beta(\rho-\rho')} / 2^{|\alpha+\beta|}\alpha!\beta!$$
$$\in S(C(M,a)(\log\langle\xi\rangle_\gamma)^{m+2}\langle\xi\rangle_\gamma^{(m-2)/2}\phi^{-|\delta|}\psi^{-|\rho|}, g).$$

This shows that

$$|P_{T^M(\delta)}^{(\rho)}| \le \sum \tau^{-2(m-j)-|\alpha+\beta+\rho'+\delta'|+1}\langle\xi\rangle_\gamma^{m-j+|\beta+\delta'|}\sqrt{m_1}$$
$$\times[\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]^{|\alpha+\beta|}\phi^{-|\alpha+\delta-\delta'|}\psi^{-|\beta+\rho-\rho'|}$$
$$+C_{\rho\delta}C(M,a)\phi^{-|\delta|}\psi^{-|\rho|}\langle\xi\rangle_\gamma^{(m-2)/2}(\log\langle\xi\rangle_\gamma)^{m+2}$$

since $w_{\alpha(\delta-\delta')}^{\beta(\rho-\rho')} = 0$ if $|\alpha+\rho'+\beta+\delta'| = 1$ and $|\rho+\delta| \ge 0$ (Lemma 7.1.2 holds for $m - j + |\alpha+\beta| \ge 1$). Noting

$$\phi^{-|\alpha-\delta'|}\psi^{-|\beta-\rho'|} = \phi^{-|\alpha+\beta-\delta'-\rho'|}\langle\xi\rangle_\gamma^{-|\beta-\rho'|}$$

we have

$$\tau^{-2(m-j)-|\alpha+\beta+\rho'+\delta'|+1}\langle\xi\rangle_\gamma^{m-j+|\beta+\delta'|}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]^{|\alpha+\beta|}$$
$$= \tau^{-2(m-j)}\langle\xi\rangle_\gamma^{m-j}\big([\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]\phi^{-1}\tau^{-1}\big)^{|\alpha+\beta|} \qquad (7.2.1)$$
$$\times\tau^{-|\rho'+\delta'|+1}\psi^{|\delta'+\rho'|}.$$

Remark that

$$\tau^{-2}\langle\xi\rangle_\gamma \le C\lambda M^{-2} \le C,$$
$$\psi\tau^{-1} \le \lambda M^{-1}, \quad [\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]\phi^{-1}\tau^{-1} \le C$$

for $\phi\tau \ge \phi(\gamma+a\log\langle\xi\rangle_\gamma)+M$ and hence the right-hand side of (7.2.1) is bounded by

$$\tau(\lambda M^{-2})^{m-j}(\lambda M^{-1})^{|\rho'+\delta'|} \le \tau(\lambda M^{-1})^{|\rho+\delta|}.$$

This gives the desired estimate. On the other hand we have

$$C(M,a)\langle\xi\rangle_\gamma^{(m-2)/2}(\log\langle\xi\rangle_\gamma)^{m+2} \le C(M,a)\sqrt{m_1}\tau^{-1}(\log\langle\xi\rangle_\gamma)^{m+2}$$
$$\le C(M,a)\tau\sqrt{m_1}(\log\langle\xi\rangle_\gamma)^{m+2}\langle\xi\rangle_\gamma^{-1/2} \le C_{\rho\delta}\tau\sqrt{m_1}(\lambda M^{-1})^{|\rho+\delta|}$$

if $h = \gamma \ge \gamma_0(M,\lambda,a)$. This proves the assertion.

Let us define

$$g_1 = (\lambda M^{-1})^2 g$$

then

Corollary 7.2.1 *Let* $\psi(x,\xi)\in S(1,g_1)$ *with* $\mathrm{supp}\psi\subset U$ *then we have*

$$\psi P^{(\alpha)}_{T^M(\beta)}\in S(\tau\sqrt{m_1}(\lambda M^{-1})\phi^{-|\beta|}\psi^{-|\alpha|},g_1)$$

for $|\alpha+\beta|=1$.

Lemma 7.2.2 *Let* $\psi(x,\xi)\in S(1,g_1)$ *with* $\mathrm{supp}\psi\subset U$*. Then we have*

$$\psi(P_{T^M}-p(z;H_\Lambda/2))\in S((M^{2\kappa-1}\lambda^{-1}+\lambda M^{-2})w\sqrt{m_1},g_1)$$

for $h=\gamma\geq\gamma_0(M,\lambda,a)$.

Proof: Recall that

$$\begin{aligned}P_{T^M}-p(z;H_\Lambda/2)&=\sum_{0<|\alpha+\beta|<m+2}p^{(\alpha)}_{(\beta)}\rho^\beta_\alpha/2^{|\alpha+\beta|}\alpha!\beta!\\&+\sum_{j=0}^{m-1}\sum_{|\alpha+\beta|<m+2}P^{(\alpha)}_{j(\beta)}w^\beta_\alpha/2^{|\alpha+\beta|}\alpha!\beta!+r\end{aligned}$$

where

$$\rho^\beta_\alpha=w^\beta_\alpha-(-i\Lambda_x)^\alpha(-\Lambda_\xi)^\beta$$

and $r\in S(C(M,a)\langle\xi\rangle_\gamma^{(m-2)/2}(\log\langle\xi\rangle_\gamma)^{m+2},g)$. Then we see that

$$\begin{aligned}&[P_{T^M}-p(z;H_\Lambda/2)]^{(\nu)}_{(\delta)}\\&=\sum_{1\leq|\alpha+\beta|<m+2}\binom{\nu}{\nu'}\binom{\delta}{\delta'}p^{(\alpha+\nu')}_{(\beta+\delta')}\rho^{\beta(\nu-\nu')}_{\alpha(\delta-\delta')}/2^{|\alpha+\beta|}\alpha!\beta!\qquad(7.2.2)\\&+\sum_{j=0}^{m-1}\sum_{|\alpha+\beta|<m+2}\binom{\nu}{\nu'}\binom{\delta}{\delta'}P^{(\alpha+\nu')}_{j(\beta+\delta')}w^{\beta(\nu-\nu')}_{\alpha(\delta-\delta')}/2^{|\alpha+\beta|}\alpha!\beta!+r^{(\nu)}_{(\delta)}.\end{aligned}$$

We estimate the first term in the right-hand side of (7.2.2).

$$\begin{aligned}&\sum_{1\leq|\alpha+\beta|<m+2}\tau^{-|\alpha+\beta+\nu'+\delta'|+1}\langle\xi\rangle_\gamma^{|\beta+\delta'|}\sqrt{m_1}M^{2\kappa-1}\lambda^{-1}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]^{|\alpha+\beta|}\\&\qquad\times\phi^{-|\alpha+\delta-\delta'|}\psi^{-|\beta+\nu-\nu'|}\\&\leq M^{2\kappa-1}\lambda^{-1}\phi^{-|\delta|}\psi^{-|\nu|}([\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]\phi^{-1}\tau^{-1})\tau\sqrt{m_1}\tau^{-|\nu'+\delta'|}\psi^{|\nu'+\delta'|}\\&\leq M^{2\kappa-1}\lambda^{-1}[(\gamma+a\log\langle\xi\rangle_\gamma)+M\phi^{-1}](\lambda M^{-1})^{|\nu+\delta|}\phi^{-|\delta|}\psi^{-|\nu|}\end{aligned}$$

since $|\alpha+\beta|\geq 1$ and $\tau^{-1}\psi\leq\lambda M^{-1}$. Recalling $w=\gamma+a\log\langle\xi\rangle_\gamma+M\phi^{-1}$ the assertion is clear.

We estimate the second term in the right-hand side of (7.2.2).

$$\sum_{j=0}^{m-1} \tau^{-2(m-j)-|\alpha+\beta+\nu'+\delta'|+1}\langle\xi\rangle_\gamma^{m-j+|\beta+\delta'|}[\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]^{|\alpha+\beta|}$$
$$\times\phi^{-|\alpha+\delta-\delta'|}\psi^{-|\beta+\nu-\nu'|}$$
$$\leq \sum_{j=0}^{m-1}([\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]\phi^{-1}\tau^{-1})^{|\alpha+\beta|}\tau\sqrt{m_1}\phi^{-|\delta|}\psi^{-|\nu|}(\tau^{-1}\psi)^{|\delta'+\nu'|}(\tau^{-2}\langle\xi\rangle_\gamma)^{m-j}$$
$$\leq (\lambda M^{-1})^{|\nu+\delta|}\phi^{-|\delta|}\psi^{-|\nu|}\tau^{-1}\langle\xi\rangle_\gamma\sqrt{m_1}$$

because $m-j\geq 1$ and $\tau^{-2}\langle\xi\rangle_\gamma\leq C\lambda M^{-2}\leq C$. Since

$$\tau^{-1}\langle\xi\rangle_\gamma\leq\langle\xi\rangle_\gamma(M\lambda^{-1}\psi)^{-1}=\lambda M^{-1}\phi^{-1}\leq\lambda M^{-2}w$$

the assertion is clear.

We estimate the third term of the right-hand side of (7.2.2). Note that

$$|r_{(\delta)}^{(\nu)}|\leq C(M,a)\tau^{-1}\sqrt{m_1}(\log\langle\xi\rangle_\gamma)^{m+2}\phi^{-|\delta|}\psi^{-|\nu|}$$
$$\leq C(M,a)\sqrt{m_1}\lambda M^{-2}w\langle\xi\rangle_\gamma^{-1}(\log\langle\xi\rangle_\gamma)^{m+2}\phi^{-|\delta|}\psi^{-|\nu|}$$
$$\leq C(\lambda M^{-2})w\sqrt{m_1}\phi^{-|\delta|}\psi^{-|\nu|}$$

which proves the assertion.

7.3 Estimate of $Q(z)$

We show

Lemma 7.3.1 *Let $\psi(x,\xi)\in S(1,g_1)$ with $\mathrm{supp}\psi\subset U$. Then we have*

$$\psi Q(z)\in S(\sqrt{m_1},g_1)$$

for $h=\gamma\geq\gamma_0(M,\lambda,a)$.

Proof: Recall that

$$Q=|\tilde{H}_\Lambda|^{-1}\sum_{j=0}^{m+1}\sum_{|\alpha+\beta|=j}\Big[(\Lambda_\xi\frac{\partial}{\partial x}-\Lambda_x\frac{\partial}{\partial\xi})p_{(\beta)}^{(\alpha)}(z)\Big](-\Lambda_\xi)^\beta(-i\Lambda_x)^\alpha/2^{|\alpha+\beta|}\alpha!\beta!.$$

A general term is

$$p_{(\beta)}^{(\alpha)}(z)\Lambda_\xi^\beta\Lambda_x^\alpha/(\langle\xi\rangle_\gamma^2|\Lambda_\xi|^2+|\Lambda_x|^2)^{1/2},\quad 1\leq|\alpha+\beta|\leq m+2.$$

Note that $(\langle\xi\rangle_\gamma^2|\Lambda_\xi|^2+|\Lambda_x|^2)^{1/2}\in S(w,g)$ by Lemma 6.3.2 and hence from Lemma 5.3.3 we have

$$\Lambda_\xi^\beta\Lambda_x^\alpha(\langle\xi\rangle_\gamma^2|\Lambda_\xi|^2+|\Lambda_x|^2)^{-1/2}\in S([\phi(\gamma+a\log\langle\xi\rangle_\gamma)+M]^{|\alpha+\beta|}w^{-1}\phi^{-|\alpha|}\psi^{-|\beta|},g).$$

Then denoting $V_\alpha^\beta = \Lambda_\xi^\beta \Lambda_x^\alpha (\langle\xi\rangle_\gamma^2 |\Lambda_\xi|^2 + |\Lambda_x|^2)^{-1/2}$ we have

$$\Big|\Big(\sum_{1\le|\alpha+\beta|\le m+2} p_{(\beta)}^{(\alpha)} V_\alpha^\beta\Big)_{(\delta)}^{(\gamma)}\Big| \le C \sum_{1\le|\alpha+\beta|\le m+2} |p_{(\beta+\delta')}^{(\alpha+\gamma')} V_{\alpha(\delta-\delta')}^{\beta(\gamma-\gamma')}|$$

$$\le C \sum_{1\le|\alpha+\beta|\le m+2} \tau^{-|\alpha+\beta+\gamma'+\delta'|+1} \langle\xi\rangle_\gamma^{|\beta+\delta'|} \sqrt{m_1}[\phi(\gamma + a\log\langle\xi\rangle_\gamma) + M]^{|\alpha+\beta|}$$

$$\times w^{-1}\phi^{-|\alpha+\delta-\delta'|}\psi^{-|\beta+\gamma-\gamma'|}$$

$$\le C\sum \phi^{-|\delta|}\psi^{-|\gamma|}([\phi(\gamma + a\log\langle\xi\rangle_\gamma) + M]\phi^{-1}\tau^{-1})^{|\alpha+\beta|}\psi^{|\gamma'+\delta'|}\tau^{-|\gamma'+\delta'|+1}\sqrt{m_1}w^{-1}$$

$$\le C(\lambda M^{-1})^{|\gamma+\delta|}\sqrt{m_1}\phi^{-|\delta|}\psi^{-|\gamma|}$$

since $|\alpha+\beta| \ge 1$ and $w = \gamma + a\log\langle\xi\rangle_\gamma + M\phi^{-1}$. This proves the assertiion.

7.4 Estimate of $S(z) = (\bar{Q}\# P_{T^M} - \overline{T_{T^M}}\# Q)/2i$

We start with

Lemma 7.4.1 *Let $g_1 = (\lambda M^{-1})^2 g$. Assume that $M \le \lambda \le M^\ell$ with some $\ell \ge 2$. Then if $a > 0$ is σ, g temperate then a is σ, g_1 temperate.*

Proof: Since $g \le g_1$ it is obvious that a is g_1 continuous. Note that $g_z^\sigma = (\lambda M^{-1})^2 g_{1z}^\sigma$ and

$$a(w) \le Ca(z)(1 + g_w^\sigma(w-z))^N.$$

If $g_z(w-z) \le c_0$ then $a(w) \le Ca(z) \le Ca(z)(1+g_{1w}^\sigma(w-z))$. If $g_z(w-z) \ge c_0$ then

$$g_{1z}^\sigma(w-z) = \lambda^{-2}M^2 g_z^\sigma(w-z) \ge M^2 g_z(w-z) \ge c_0 M^2.$$

This proves that

$$g_z^\sigma(w-z) \le M^{2\ell}M^{-2}g_{1z}^\sigma(w-z)$$
$$\le M^{2\ell-2}g_{1z}^\sigma(w-z) \le Cg_{1z}^\sigma(w-z)^\ell$$

which proves the assertion.

Lemma 7.4.2 *Let $M \le \lambda \le M^\ell$ and $h = \gamma \ge \gamma_0(M)$. Then $w(z)$, $\tau(z)$, $m(z)$ and $m_1(z)$ are σ, g_1 temperate.*

Proof: By Lemma 7.4.1 we see that ϕ, ψ, $\langle\xi\rangle_h$, τ are σ, g_1 temperate. Study

$$m_1(z) = h_{m-1}(x, \xi - iw\theta).$$

Recall

$$m_1 = \sum_{1\le\ell_1<\cdots<\ell_{m-1}\le m} (q_{\ell_1}(z)^2 + w^2)\cdots(q_{\ell_{m-1}}(z)^2 + w^2)$$

and $|q_j(x+y,\xi+\eta)| \leq |q_j(x,\xi)| + C(|\xi+\eta||y| + |\eta|)$ because $q_j(x,\xi)$ is Lipschitz continuous near z^0. This proves

$$m_1(x+y,\xi+\eta) \leq C \sum_{j=0}^{m-1} h_j(x,\xi)(|\xi+\eta|^2|y|^2 + |\eta|^2 + w(x+y,\xi+\eta)^2)^{m-1-j}.$$

Note that

$$h_j(x,\xi) = \sum_{1\leq \ell_1 < \cdots \ell_j \leq m} |q_{\ell_1}(z)|^2 \cdots |q_{\ell_j}(z)|^2 \leq C\langle\xi\rangle_\gamma^{2j}$$

$$(\tau+w)^{2m-2j-2} h_j(x,\xi) \leq C\langle\xi\rangle_\gamma^{2j}(\tau+w)^{2m-2j-2} \leq m_1$$

because $m_1 \geq c\langle\xi\rangle_\gamma^{2m-4}(\tau+w)^2 \geq c\langle\xi\rangle_\gamma^{2j}(\tau+w)^{2m-2j-2}$ by Lemma 7.1.1 for $j \leq m-2$. This shows that

$$h_j(z)/m_1(z) \leq C(\tau(z)+w(z))^{-2(m-j-1)}$$

for $j \leq m-1$. Thus one gets

$$\begin{aligned} m_1(x+y,\xi+\eta)/m_1(x,\xi) &\leq C \sum_{j=0}^{m-1} (\tau+w)^{-2(m-j-1)} \\ &\quad \times (|\xi+\eta|^2|y|^2 + |\eta|^2 + w(x+y,\xi+\eta)^2)^{m-1-j} \\ \leq C \sum_{j=0}^{m-1} \big((|\xi+\eta|^2|y|^2 &+ |\eta|^2)\tau^{-2} + w(x+y,\xi+\eta)^2 w(x,\xi)^{-2}\big)^{m-1-j}. \end{aligned}$$

Recall

$$\begin{aligned} g_{1z}(w) &= (\lambda M^{-1})^2(\phi^{-2}(z)|y|^2 + \psi(z)^{-2}|\eta|^2), \\ \tau^{-1} &\leq (\lambda M^{-1})\psi^{-1} = (\lambda M^{-1})\langle\xi\rangle_\gamma^{-1}\phi^{-1} \end{aligned}$$

and hence

$$|\eta|^2\tau^{-2} \leq g_{1z}(w), \quad |\xi|^2|y|^2\tau^{-2} \leq g_{1z}(w).$$

Thus one has

$$\begin{aligned} &m_1(x+y,\xi+\eta)/m_1(x,\xi) \\ &\leq C \sum_{j=0}^{m-1} (g_{1(x,\xi)}(y,\eta) + |\eta|^2|y|^2\tau^{-2} + w(x+y,\xi+\eta)^2 w(x,\xi)^{-2})^{m-1-j}. \end{aligned}$$

We first examine that m_1 is g_1 continuous. If $g_1 \leq c_0$ we have $|y| \leq c_0^{1/2} M\lambda^{-1}\phi \leq C$ $(\lambda \geq M)$ and $|\eta| \leq c_0^{1/2} M\lambda^{-1}\psi \leq c_0^{1/2}\tau$ and this shows that

$$m_1(x+y,\xi+\eta)/m_1(x,\xi) \leq C.$$

Recalling that $|y|^2 \le (M^{-1}\lambda)^2\psi^{-2}g_1^\sigma$ and $|\eta|^2 \le (\lambda M^{-1})^2\phi^{-2}g_1^\sigma$ and noting that $\lambda^{-1}M\psi \le \tau$, $\lambda \le \phi\psi$, $\lambda \le \psi$ we have

$$|\eta|^2|y|^2\tau^{-2} \le (\lambda M^{-1})^4(\phi\psi)^{-2}(g_1^\sigma)^2\tau^{-2}$$
$$\le (\lambda M^{-1})^4\lambda^{-2}(\lambda M^{-1}\psi^{-1})^2(g_1^\sigma)^2 \le (\lambda M^{-1})^6(g_1^\sigma)^2.$$

This shows that m_1 is σ, g_1 temperate.

7.5 Bound of $S^w(x, D)$

Let us take $\psi \in C^\infty(T^*\mathbf{R}^n \setminus 0)$ which is homogeneous of degree 0 in ξ such that supp$\psi \in U$. We denote by the same ψ which is cut off by $\chi(|\xi|/h)$. Note that

$$\sigma(P_{T^M}^w\psi^w) - P_{T^M}\psi \in S(\tau\sqrt{m_1}(\lambda M^{-1})^2(\phi\psi)^{-1}, g_1),$$
$$\sigma(Q^w\psi^w) - Q\psi \in S(\sqrt{m_1}(\lambda M^{-1})^2(\phi\psi)^{-1}, g_1).$$

In fact we have

$$\sup \frac{g_1}{g_1^\sigma} = (\lambda M^{-1})^4(\phi\psi)^{-2}.$$

Recall that

$$S^w = \frac{1}{2i}((Q^w)^*P_{T^M}^w - (P_{T^M}^w)^*Q^w)$$

so that

$$S = \frac{1}{2i}(\bar{Q}\#P_{T^M} - \overline{P_{T^M}}\#Q).$$

Consider

$$\psi\#S\#\psi = \frac{1}{2i}(\psi\#\bar{Q}\#P_{T^M}\#\psi - \psi\#\overline{P_{T^M}}\#Q\#\psi).$$

By Corollary 7.2.1 and Lemma 7.3.1 we have

$$\psi\Big[(\bar{Q}\#P_{T^M} - \overline{P_{T^M}}\#Q) - (\bar{Q}P_{T^M} - \overline{P_{T^M}}Q)\Big]$$
$$\in S(\tau m_1(\lambda M^{-1})^2(\phi\psi)^{-1}, g_1).$$

Thus one has

$$\psi\#S\#\psi = \frac{1}{2i}(P_{T^M}\bar{Q} - \overline{P_{T^M}}Q)\psi^2 + S(\tau m_1(\lambda M^{-1})^2(\phi\psi)^{-1}, g_1).$$

Since $\lambda\tau(\phi\psi)^{-1} \le w$, with

$$m = wm_1$$

we have

$$\psi\#S\#\psi = \frac{1}{2i}(P_{T^M}\bar{Q} - \overline{P_{T^M}}Q)\psi^2 + S(m(\lambda M^{-2}), g_1). \tag{7.5.1}$$

Let us set

$$F = \mathrm{Im}(P_{T^M}\bar{Q})\psi^2$$

which is in $S(m, g_1)$ by Lemma 6.4.2. Recall

$$\psi(P_{T^M} - p(z; H_\Lambda/2)) \in S((M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2})w\sqrt{m_1}, g_1)$$

from which we conclude

$$F(z) - S_0(z)\psi^2 \in S((M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2})m, g_1)$$

where $S_0 = \text{Im}p(z; H_\Lambda/2)\bar{Q}$ and hene

$$\psi \# S \# \psi = S_0\psi^2 + S(m(M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2}), g_1).$$

We localize S_0 near z^0. Let $\chi \in C_0^\infty(\mathbf{R})$ which is equal to 1 in $|t| \le \delta$ and $\chi = 0$ in $|t| \ge 2\delta$ so that when $|\xi| \ge \nu h$ one has

$$(x, \xi) \in \mathbf{R}^{2n} \Longrightarrow (X(x), \Xi(\xi)) \in U,$$
$$(x, \xi) \in U_1 \Longrightarrow (X(x), \Xi(\xi)) = (x, \xi),$$

where

$$X(x) = \chi(|x - x^0|)(x - x^0) + x^0 \Xi(\xi) = \chi(|\xi/|\xi| - \xi^0|)(\xi - \langle\xi\rangle_\gamma\xi^0) + \langle\xi\rangle_\gamma\xi^0.$$

Set

$$\tilde{m}(z) = m(X(x), \Xi(\xi)), \quad \tilde{m}_1(z) = m_1(X(x), \Xi(\xi)),$$
$$\tilde{g}_{1(x,\xi)} = g_{1(X(x),\Xi(\xi))}.$$

Lemma 7.5.1 *For any ℓ_1, ℓ_2 there are $S_1 \in S(\tilde{m}(M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2}), \tilde{g}_1)$ and r such that*

$$\psi \# S \# \psi = S_0\psi^2 + S_1 + r, \tag{7.5.2}$$
$$|r|_k^{\tilde{g}_1} \le C_k\langle\xi\rangle_\gamma^{-\ell_1}, \quad k = 0, 1, ..., \ell_2.$$

Proof: We first note that for any ℓ there are $\tilde{S}$, $\tilde{r}$ such that

$$\psi \# S \# \psi = S_0\psi^2 + \tilde{S} + \tilde{r}$$

where $\tilde{S} \in S(m(M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2}), g_1)$, $\text{supp}\tilde{S} \subset U_1$, $\tilde{r} \in S(\langle\xi\rangle_\gamma^{-\ell}, g_1)$. It is clear that $\tilde{S} \in S(\tilde{m}(M^{2\kappa-1}\lambda^{-1} + \lambda M^{-2}), \tilde{g}_1)$. Then taking ℓ enough large it is clear that (7.5.2) holds.

Let us set

$$E(z) = S_0(X(x), \Xi(\xi))^{-1/2}$$

then $E \in S(\tilde{m}^{-1/2}, \tilde{g}_1)$ and $E^{-1} \in S(\tilde{m}^{1/2}, \tilde{g}_1)$. Then taking (7.5.2) into account we have

$$E \# \psi \# S \# \psi \# E = \psi \# \psi + \tilde{S}_1$$

where $|\tilde{S}_1|_k^{\tilde{g}_1} \leq C_k(M^{2\kappa-1}\lambda^{-1}+\lambda M^{-2})$ for $k=0,1,...,\ell_2$ (ℓ_2 is preassigned) and hence

$$\begin{aligned}(S^w\psi^w E^w u, \psi^w E^w u) &\geq \|\psi^w u\|_{L^2}^2 - C(M^{2\kappa-1}\lambda^{-1}+\lambda M^{-2})\|u\|_{L^2}^2\\ &\geq \frac{1}{4}\|u\|_{L^2}^2 - \|(1-\psi^w)u\|_{L^2}^2\end{aligned}$$

taking $M^{2\kappa-1}\lambda^{-1}+\lambda M^{-2}$ small. Note that

$$E\#E^{-1}-1 \in S(\lambda M^{-2}, \tilde{g}_1)$$

because

$$\frac{g_1}{g_1^\sigma} = (\lambda M^{-1})^4\frac{g}{g^\sigma} \leq (\lambda M^{-1})^4\lambda^{-2} = (\lambda M^{-2})^{-2}.$$

Taking $\lambda M^{-2} \ll 1$ there exists $\tilde{E} \in S(\tilde{m}^{1/2}, \tilde{g}_1)$ such that $E\#\tilde{E}=1$. Noting that

$$C\langle\xi\rangle_\gamma^{2m-1} \geq m = wm_1 \geq c\langle\xi\rangle_\gamma^{2m-4}\tau^2 w \geq c\langle\xi\rangle_\gamma^{2m-3}$$

one has

$$C^{-1}\langle\xi\rangle_\gamma^{m-3/2} \leq m^{1/2} \leq C\langle\xi\rangle_\gamma^{m-1/2}$$

and hence we conclude that

$$C^{-1}\|\langle D\rangle_\gamma^{m-3/2}v\| \leq \|\tilde{E}v\| \leq C\|\langle D\rangle_\gamma^{m-1/2}u\|.$$

Thus with $E^w u = v$ ($\tilde{E}^w v = u$) we have

$$(S^w\psi^w v, \psi^w v) \geq \|\tilde{E}^w v\|^2 - \|(1-\psi^w)\tilde{E}^w v\|.$$

Since $(1-\psi^w)\tilde{E}^w = \tilde{E}^w(1-\psi^w) - [\psi^w, \tilde{E}^w]$ and

$$\|[\psi^w, \tilde{E}^w]v\| = \|[\psi^w, \tilde{E}^w]E^w\tilde{E}^w v\| \leq C\lambda M^{-2}\|\tilde{E}^w v\|$$

it follows that

$$\begin{aligned}(S^w\psi^w v, \psi^w v) &\geq \frac{1}{4}\|\tilde{E}^w v\|^2 - \|\tilde{E}^w(1-\psi^w)v\|^2 - \|[\psi^w, \tilde{E}^w]v\|^2\\ &\geq c\|\langle D\rangle_\gamma^{m-3/2}v\|^2 - C\|\langle D\rangle_\gamma^{m-1/2}(1-\psi^w)v\|^2. \qquad (7.5.3)\end{aligned}$$

We now derive the microlocal hyperbolic apriori estimate. Note that

$$\begin{aligned}(S^w\psi^w v, \psi^w) &= \mathrm{Im}(P_{T^M}^w\psi^w v, Q^w\psi^w)\\ &\leq \|\langle D\rangle_\gamma^{(m-1)/2}P_{T^M}^w\psi^w v\|\|\langle D\rangle_\gamma^{-(m-1)/2}Q^w\psi^w v\|\\ &\leq C\|\langle D\rangle_\gamma^{(m-1)/2}P_{T^M}^w\psi^w v\|^2 + \frac{c}{2}\|\langle D\rangle_\gamma^{(m-1)/2}\psi^w v\|^2.\end{aligned}$$

We obtain from (7.5.3) that (note $m-3/2 \geq (m-1)/2$)

$$\begin{aligned}\|\langle D\rangle_\gamma^{(m-1)/2}v\| &\leq C\|\langle D\rangle_\gamma^{(m-1)/2}P_{T^M}^w\psi^w v\|\\ &\quad + C\|\langle D\rangle_\gamma^{m-1/2}(1-\psi^w)v\|.\end{aligned}$$

Recall that $P^w_{T^M}=(T^{-M})^w(\langle\xi\rangle^{-a\rho})^wP(z;\gamma)^w(\langle\xi\rangle^{a\rho})^w(T^M)^w$ where $P(z;\gamma)^w=e^{-\gamma\zeta(x)}P^we^{\gamma\zeta(x)}$. Since $T^{\pm M}\in S(T^{\pm M},g)\subset S(\langle\xi\rangle_\gamma^{CM},g)$, putting $v=[(T^M)^w]^{-1}u$ we get the estimate

$$\|\langle D\rangle_\gamma^{-CM+(m-1)/2}u\|\le C_M\|\langle D\rangle_\gamma^{(m-1)/2}v\|.$$

Noting that

$$\begin{aligned}(T^M)^w\psi^w[(T^M)^w]^{-1}u&=u+(T^M)^w(\psi^w-1)[(T^M)^w]^{-1}u\\ =u+(T^M)^w(\psi^w-1)[(T^M)^w]^{-1}&(1-\psi_1^w)u+(T^M)^w(\psi^w-1)[(T^M)^w]^{-1}\psi_1^wu\end{aligned}$$

we have

$$\begin{aligned}&\|\langle D\rangle_\gamma^{(m-1)/2}(T^{-M})^w(\langle\xi\rangle^{-a\rho})^wP(z;\gamma)^w(\langle\xi\rangle^{a\rho})^w(T^M)^w\psi^w[(T^M)^w]^{-1}u\|\\ &\quad\le C_M\|\langle D\rangle_\gamma^{CM+(m-1)/2}P^w_\gamma u\|+C_M\|\langle D\rangle_\gamma^{m+(m-1)/2+CM}(1-\psi_1^w)u\|\\ &\qquad\qquad+C_M\gamma^{-1}\|\langle D\rangle_\gamma^{(m-1)/2-CM}u\|.\end{aligned}$$

Similarly one has

$$\begin{aligned}&\|\langle D\rangle_\gamma^{m-1/2}(1-\psi^w)[(T^M)^w]^{-1}u\|\\ &\le C_M\|\langle D\rangle_\gamma^{m-1/2+CM}(1-\psi_1^w)u\|\\ &\quad+C_M\gamma^{-1}\|\langle D\rangle_\gamma^{-CM+(m-1)/2}u\|\end{aligned}$$

where ψ_1 is chosen so that $\mathrm{supp}(1-\psi)\cap\mathrm{supp}\psi_1=\emptyset$. Taking γ large enough we have the desired *microlocal hyperbolic apriori estimate* (2.4.2):

$$\begin{aligned}&\|\langle D\rangle_\gamma^{-CM+(m-1)/2}u\|\\ &\le C_M\big\{\|\langle D\rangle_\gamma^{CM+(m-1)/2}(\langle\xi\rangle^{-a\rho})^we^{-\gamma\zeta(x)}P^we^{\gamma\zeta(x)}(\langle\xi\rangle^{a\rho})^wu\|\\ &\qquad+\|\langle D\rangle_\gamma^{m+(m-1)/2}(1-\psi_1^w)u\|\big\}.\end{aligned}$$

8 Hyperbolic polynomials

In this section we collect several facts about hyperbolic polynomials which are used in this lecture. Although one can find the proofs in [2], [15] and [10] we give the proofs for this note to be self-contained.

8.1 Hyperbolic polynomials

In this subsection we follow [2]. Let $P(\xi)$ be a polynomial in $\xi\in\mathbf{R}^n$ of degree m and write

$$P(\xi)=P_m(\xi)+P_{m-1}(\xi)+\cdots$$

where $P_j(\xi)$ is the homogeneous part of degree j in ξ.

DEFINITION 8.1.1: P is said to be hyperbolic with respect to $\theta \in \mathbf{R}^n$ if there is C such that

$$P_m(\theta) \neq 0, \quad P(\xi + t\theta) \neq 0, \quad \forall \xi \in \mathbf{R}^n, \ \mathrm{Im}t < C. \tag{8.1.1}$$

Let us denote $p = P_m$. It is clear that $p(\xi + t\theta) \neq 0$ if $\mathrm{Im}t \neq 0$ and hence one can write

$$p(\xi + s\theta) = p(\theta) \prod_{k=1}^{m} (s + \lambda_k(\theta, \xi)) \tag{8.1.2}$$

where $\lambda_k(\theta, \xi)$ is real and homogeneous of degree 1 with respect to ξ.

DEFINITION 8.1.2: We define the hyperbolic cone $\Gamma(p, \theta)$ as the connected component of

$$\{\xi \mid p(\xi) \neq 0\}$$

containing θ.

Since $\lambda_k(\theta, \xi)$ is continuous with respect to ξ we see that

$$\xi \in \Gamma(p, \theta) \Longleftrightarrow \lambda_k(\theta, \xi) > 0, \ k = 1, ..., m.$$

Lemma 8.1.1 *Let P be hyperbolic with respect to $\theta \in \mathbf{R}^n$ and let $\eta \in \Gamma(p, \theta)$. Then there is C such that*

$$\xi \in \mathbf{R}^n, \ \mathrm{Im}t \leq 0, \ \mathrm{Im}s < C \Longrightarrow P(\xi + t\eta + s\theta) \neq 0.$$

Proof: Since (8.1.1) holds, the polynomial

$$t \mapsto \lambda^{-m} P(\xi + t\lambda\eta + (s + (i(1 - \lambda))\theta), \ \mathrm{Im}s < C, \ \xi \in \mathbf{R}^n, \ \lambda \geq 1 \tag{8.1.3}$$

has no real zeros. Hence (8.1.3) has a constant number of zeros in the lower half plane. As $\lambda \to \infty$ the polynomial (8.1.3) goes to

$$p(t\eta - i\theta) = \prod(-i + t\lambda_k(\theta, \eta)).$$

Since $\eta \in \Gamma(p, \theta)$ and hence $\lambda_k(\theta, \eta)$ are all positive, we conclude that all zeros of the polynomial (8.1.3) lie in the upper half plane. Hence, setting $\lambda = 1$, we get the desired assertion.

Corollary 8.1.1 *Let p be hyperbolic with respect to θ and $\eta \in \Gamma(p, \theta)$. Then we have*

$$\xi \in \mathbf{R}^n, \ \mathrm{Im}t \leq 0, \ \mathrm{Im}s \leq 0, \ \mathrm{Im}(t + s) < 0 \Longrightarrow p(\xi + t\eta + s\theta) \neq 0.$$

In particular p is hyperbolic with respect to η.

Proof: If Im$s < 0$ the same proof as in Lemma 8.1.1 shows the result. If Im$s = 0$ then since $\eta - \varepsilon\theta \in \Gamma(p,\theta)$ for small ε, to conclude the proof it is enough to note

$$p(\xi + t(\eta - \varepsilon\theta) + t\varepsilon\theta) \neq 0.$$

Lemma 8.1.2 *Let p be a homogeneous hyperbolic polynomial with respect to θ. Then $\Gamma(p,\theta)$ is convex.*

Proof: Let $\Delta_\theta = \{\xi \in \mathbf{R}^n \mid p(\xi + t\theta) = 0 \Longrightarrow t < 0\}$ and we first show that $\Gamma(p,\theta) = \Delta_\theta$. It is clear that Δ_θ is open and $\theta \in \Delta_\theta$. Let $\xi \in \overline{\Delta_\theta}$ then we have

$$p(\xi + t\theta) = 0 \Longrightarrow t \leq 0.$$

Hence $\xi \in \Delta_\theta$ if $p(\xi) \neq 0$, that is Δ_θ is closed in $\{\xi \mid p(\xi) \neq 0\}$. This shows that $\Gamma(p,\theta) \subset \Delta_\theta$. Let $\xi \in \Delta_\theta$ then

$$p(\epsilon\xi + (1-\epsilon)\theta) = \epsilon^m p(\xi + (1-\epsilon)\epsilon^{-1}\theta) \neq 0$$

if $0 < \epsilon \leq 1$. Hence $p \neq 0$ on the line segment between ξ and θ. This shows $\Delta_\theta \subset \Gamma(p,\theta)$ and hence $\Gamma(p,\theta) = \Delta_\theta$.

We now show that $\Gamma(p,\theta)$ is convex. Let $\tilde{\eta}$, $\eta \in \Gamma(p,\theta)$. Recall that p is hyperbolic with respect to $\eta \in \Gamma(p,\theta)$ by Corollary 8.1.1. Since the component of η in $\{\xi \mid p(\xi) \neq 0\}$ is $\Gamma(p,\theta)$, repeating the same argument as above, we conclude that

$$\{\xi \mid p(\xi + t\eta) = 0 \Longrightarrow t < 0\} = \Gamma(p,\theta).$$

This shows that $p(s\tilde{\eta} + (1-s)\eta + t\eta) = 0$, $0 \leq s \leq 1$ implies that $1 - s + t < 0$ and hence $t < -(1-s) \leq 0$. This proves $s\tilde{\eta} + (1-s)\eta \in \Gamma(p,\theta)$ and hence the result.

Lemma 8.1.3 *Let p be a homogeneous hyperbolic polynomial with respect to θ and ξ, $\eta \in \mathbf{R}^n$. Then the polynomial*

$$s, t \mapsto p(\xi + t\eta + s\theta)$$

can be factorized as

$$p(\xi + t\eta + s\theta) = p(\theta)\prod_{k=1}^{m}(s + \lambda_k(\theta, \xi + t\eta))$$

where the function

$$t \in \mathbf{R}, \ \ t \mapsto \lambda_k(\theta, \xi + t\eta)$$

is real and analytic with simple pole at ∞ so that

$$\lambda_k(\theta, \xi + t\eta) = t\lambda_k(\theta, \eta) + O(1), \ t \to \infty.$$

When $\eta \in \Gamma(p, \theta)$ *then*

$$\frac{d}{dt}\lambda_k(\theta, t\eta)|_{t=0} > 0$$

and if $\eta \in \overline{\Gamma}(p, \theta)$ *then*

$$\frac{d}{dt}\lambda_k(\theta, t\eta)|_{t=0} \geq 0.$$

Proof: Fix $t_0 \in \mathbf{R}$. Then near t_0, $\lambda_k(\theta, \xi + t\eta)$ can be developped in a convergent Puiseux series

$$\lambda_k(\theta, \xi + t\eta) = \lambda_k(\theta, \xi + t_0\eta) + c_k(t - t_0)^{r_k}(1 + o(1)) \tag{8.1.4}$$

where $c_k \neq 0$ and $r_k > 0$ is rational. Since $\lambda_k(\theta, \xi + t\eta)$ is real, it follows that r_k is an integer and the series is a power series in $t - t_0$. That is $\lambda_k(\theta, \xi + t\eta)$ is analytic near t_0. Varying t_0 and making analytic continuations we conclude that $\lambda_k(\theta, \xi + t\eta)$ is analytic in t. Taking t large and noting that

$$t^{-m}p(\xi + t\eta + st\theta) \to p(\eta + s\theta) = p(\theta)\prod(s + \lambda_k(\theta, \eta))$$

as $t \to \infty$ we see that

$$\lambda_k(\theta, \xi + t\eta) = \lambda_k(\theta, \eta)t + o(t), \quad t \to \infty.$$

This proves the first assertion. Let $\eta \in \Gamma(p, \theta)$. Note that

$$\begin{aligned} p(\xi + t\eta + s\theta) &= p(\theta)\prod(s + \lambda_k(\theta, \xi + t\eta)) \\ &= p(\xi + \mathrm{Re}(t\eta + s\theta) + i\mathrm{Im}(t\eta + s\theta)) \neq 0 \end{aligned}$$

if $\mathrm{Im}t \geq 0$, $\mathrm{Im}s \geq 0$ and $\mathrm{Im}(t + s) > 0$. This shows that

$$\mathrm{Im}t > 0 \Longrightarrow \mathrm{Im}\lambda_k(\theta, \xi + t\eta) > 0.$$

This is possible only if $r_k = 1$ and $c_k > 0$ in (8.1.4) ($t_0 = 0$). A slight modification of this argument shows that $c_k \geq 0$ if $\eta \in \overline{\Gamma}(p, \theta)$.

Definition 8.1.3: Let P be hyperbolic with respect to θ and $\xi \in \mathbf{R}^n$. We define P_ξ by

$$\mu^m P(\mu^{-1}\xi + \eta) = \mu^r\left[P_\xi(\eta) + O(\mu)\right], \quad \mu \to 0.$$

$P_\xi(\eta)$ is called the localization of P at ξ and $r = m_\xi(P)$ is called the multiplicity of ξ relative to P.

Lemma 8.1.4 *For any* $\xi \in \mathbf{R}^n$, p_ξ *is hyperbolic with respect to* θ *and*

$$\Gamma(p, \theta) \subset \Gamma(p_\xi, \theta).$$

Proof: By Lemma 8.1.2 we have

$$p(\xi+t\eta)=p(\theta)\prod_{k=1}^{m}\lambda_k(\theta,\xi+t\eta)=p(\theta)\prod_{k=1}^{m}\left\{\lambda_k(\theta,\xi)+tc_k+O(t^2)\right\}$$

when $\eta\in\Gamma(p,\theta)$. Comparing this with

$$p(\xi+t\eta)=t^r\left[p_\xi(\eta)+O(t)\right],\ \ t\to 0$$

we conclude that

$$p_\xi(\eta)=p(\theta)\left[\prod_{\lambda_k(\theta,\xi)\neq 0}\lambda_k(\theta,\xi)\right]\prod_{\lambda_k(\theta,\xi)=0}c_k.$$

This proves that

$$\eta\in\Gamma(p,\theta)\Longrightarrow p_\xi(\eta)\neq 0$$

and hence $\Gamma(p,\theta)\subset\Gamma(p_\xi,\theta)$. From the definition it follows that

$$t^{m-r}p(t^{-1}\xi+\eta+s\theta)\to p_\xi(\eta+s\theta),\ \ t\to 0.$$

Since $s\mapsto t^{m-r}p(t^{-1}\xi+\eta+s\theta)$ has no zeros with $\mathrm{Im}s<0$ when η is real we conclude that $s\mapsto p_\xi(\eta+s\theta)$ has no zeros with $\mathrm{Im}s<0$. This proves that $p_\xi(\cdot)$ is hyperbolic with respect to θ because $p_\xi(\theta)\neq 0$.

8.2 Semi-continuity of hyperbolic cones

In this subsection we follow [15].

Theorem 8.2.1 ([15]) *Let $P(t,x)=t^m+a_1(x)t^{m-1}+\cdots+a_m(x)$ be a hyperbolic polynomial and $a_j(x)\in C^{2r+3}(\{|x|<\delta\})$. Assume that $t=0$ is a root of $P(t,0)=0$ of multiplicity r. Let a compact $K\subset\Gamma_{(0,0)}$ be given. Then there is a $\delta_0>0$ such that we have*

$$K\subset\Gamma_{(t,x)}\quad\text{for}\quad |t|<\delta_0,\ |x|<\delta_0.$$

We first explain the idea of the proof. Take the convex hull of K and denote it by $\hat K$ and show that $\hat K\subset\Gamma_{(t,x)}$. Suppose that $\hat K\not\subset\Gamma_{(t,x)}$. Take $(\hat\tau,\hat\xi)\in\hat K$ and $(\hat\tau,\hat\xi)\notin\Gamma_{(t,x)}$ and consider

$$P_{(t,x)}((1-s)\theta+s(\hat\tau,\hat\xi))\tag{8.2.1}$$

which is positive at $s=0$. This can not be positive in $0<s\le 1$ by assumption, then there is a $\hat s$, $0<\hat s\le 1$ such that (8.2.1) beomes zero. We denote by $(\hat\tau,\hat\xi)$

the point $(1-\hat{s})\theta + \hat{s}(\hat{\tau}, \hat{\xi})$ again. Recall that

$$P(t+\lambda\tau, x+\lambda\xi) = \lambda^\nu\{P_{(t,x)}(\tau,\xi) + O(\lambda)\}$$

$$= \sum_{l=0}^{r}\left(\sum_{j+|\alpha|=l}\frac{1}{j!\alpha!}\partial_t^j\partial_\xi^\alpha P(t,x)\tau^j\xi^\alpha\right)\lambda^l + O(\lambda^{r+1})$$

$$= \sum_{l=0}^{r}\lambda^l p_l(t,x;\tau,\xi) + O(\lambda^{r+1}).$$

This shows that $p_l(t,x;\tau,\xi) = 0$, $l = 0,1,...,\nu-1$ and $p_\nu(t,x;\tau,\xi) = P_{(t,x)}(\tau,\xi)$. Take $\xi = \hat{\xi}$ and formally we replace λ by $i\sigma$ in the above equality to get

$$P(t+i\sigma\tau, x+i\sigma\hat{\xi}) = (i\sigma)^\nu\{P_{(t,x)}(\tau,\hat{\xi}) + \sum_{l=\nu+1}^{r}(i\sigma)^{l-\nu}p_l(t,x;\tau,\hat{\xi})\} + O(|\sigma|^{r+1}).$$

Since $P_{(t,x)}(\hat{\tau},\hat{\xi}) = 0$, there is $\tau(\sigma)$ with $\tau(0) = \hat{\tau}$ which kills the right-hand side. On the other hand "*if $P(t+\lambda\tau, x+\lambda\xi)$ is hyperbolic with respect to $(\hat{\tau},\hat{\xi})$* " one has

$$|P((t,x) + i\sigma(\hat{\tau},\hat{\xi}))| \geq c|\sigma|^r$$

with some $c > 0$ which is a contradiction.

We go to the proof. Recall that we may assume that we are working with

$$P(t,x) = t^r + a_1(x)t^{r-1} + \cdots + a_r(x), \quad a_j(0) = 0$$

where $a_j(x) \in C^{2r+3}(\{|x| < \delta\})$. Let us set

$$g(t,x,\tau,\xi,\lambda) = P(t+\lambda\tau, x+\lambda\xi)$$

which is defined in $|\lambda| < \bar{\lambda}$, $|t| < \bar{t}$, $|x| < \bar{x}$, $|\tau|, |\xi| < M$, where we note that we can take M as large as we please taking $\bar{\lambda}$ small. Let a compact $L \subset \Gamma_{(0,0)}$ be given. We take $\bar{\lambda}$, M so that L is contained in $\{(t,\xi) \mid |\tau|, |\xi| \leq M\}$. Note that

$$g(0,0,\tau,\xi,\lambda) = \lambda^r P_{(0,0)}(\tau,\xi) + O(\lambda^{r+1}),\ \lambda \to 0$$

and $P_{(0,0)}(\tau,\xi) \neq 0$ for $(\tau,\xi) \in L$. Then it follows

$$\partial_\lambda^j g(0,0,\tau,\xi,0) = 0,\ 0 \leq j \leq r-1,\ \ \partial_\lambda^r g(0,0,\tau,\xi,0) \neq 0$$

and hence one can apply Malgrange's preparation theorem. Therefore we have

$$g(t,x,\tau,\xi,\lambda) = c(t,x,\tau,\xi,\lambda)p(t,x,\tau,\xi,\lambda)$$

holds in $|t| \leq \delta_0$, $|x| \leq \delta_0$ and $(\tau,\xi) \in W$ where $L \subset W$ and

$$p(t,x,\tau,\xi,\lambda) = \lambda^r + a_1(t,x,\tau,\xi)\lambda^{r-1} + \cdots + a_r(t,x,\tau,\xi)$$

and $c(0,0,\tau,\xi,0) \neq 0$, $a_j(0,0,\tau,\xi) = 0$, $(\tau,\xi) \in W$. Moreover c and a_j are holomorphic in (t,τ). We now prove

Lemma 8.2.1 *We have with some* $\delta_1 > 0$

$$p(t,x,\tau,\xi,\lambda) \neq 0 \quad \text{if} \quad \mathrm{Im} t \leq 0,\ (\tau,\xi) \in L,\ \mathrm{Im}\lambda < 0 \tag{8.2.2}$$

for $|t| \leq \delta_1,\ |x| \leq \delta_1$.

An immediate corollary is

Corollary 8.2.1 *We have*

$$|p(t,x,\tau,\xi,i\sigma)| \geq c|\sigma|^r$$

for $\sigma < 0,\ |t| \leq \delta_0,\ |x| \leq \delta_0,\ (\tau,\xi) \in L$.

Proof: Since one can write

$$p(t,x,\tau,\xi,\lambda) = \prod_{j=1}^{r}(\lambda - \lambda_j(t,x,\tau,\xi))$$

with $\mathrm{Im}\lambda_j(t,x,\tau,\xi) \geq 0$ by Lemma 8.2.1, the assertion follows.

Proof of Lemma 8.2.1: Let us introduce

$$Q(t,x,\tau,\xi,\lambda;z) = \sum_{j=0}^{r} \frac{1}{j!}\partial_\lambda^j c(t,x,\tau,\xi,\lambda) z^j, \quad z \in \mathbf{C}$$

and put

$$G(t,x,\tau,\xi,\lambda;i\sigma) = Q(t,x,\tau,\xi,\lambda;i\sigma)p(t,x,\tau,\xi,\lambda+i\sigma).$$

We first show

$$|p(t,x,1,0,\lambda+i\sigma)| \geq c|\sigma|^r,\ \mathrm{Im}t \leq 0,\ \sigma < 0. \tag{8.2.3}$$

Note that

$$P(t+\lambda+i\sigma,x) = \sum_{j=0}^{r}\frac{1}{j!}\partial_\lambda^j P(t+\lambda,x)(i\sigma)^j + O(|\sigma|^{r+1})$$

and

$$\begin{aligned}\partial_\lambda^j P(t+\lambda,x) &= \partial_\lambda^j[c(t,x,1,0,\lambda)p(t,x,1,0,\lambda)]\\ &= (\frac{1}{i}\frac{\partial}{\partial\sigma})^j[Q(t,x,1,0,\lambda;i\sigma)p(t,x,1,0,\lambda+i\sigma)]|_{\sigma=0}\end{aligned}$$

for $0 \leq j \leq r$ and hence

$$P(t+\lambda+i\sigma,x) = G(t,x,1,0,\lambda;i\sigma) + O(|\sigma|^{r+1}). \tag{8.2.4}$$

On the other hand it is clear that

$$|P(t+\lambda+i\sigma,x)|\geq|\mathrm{Im}t+\sigma|^r$$

for $\mathrm{Im}t\leq 0$, $\sigma<0$. Then from (8.2.4) one gets $|G(t,x,1,0,\lambda;i\sigma)|\geq c|\sigma|^r$ with some $c>0$ and hence (8.2.3) since $Q(t,x,1,0,\lambda;0)=c(t,x,1,0,\lambda)$.

Now suppose that (8.2.2) were not true so that there is $(\hat{\tau},\hat{\xi})\in L$ such that $p(t,x,\hat{\tau},\hat{\xi},\lambda)=0$, $\mathrm{Im}t\leq 0$ has a root $\mathrm{Im}\lambda<0$. Moving t little bit, we may suppose that $p(t,x,\hat{\tau},\hat{\xi},\lambda)=0$, $\mathrm{Im}t<0$ has a root $\mathrm{Im}\lambda<0$. Let $(\tau(s),\xi(s))$ be a curve connecting θ and $(\hat{\tau},\hat{\xi})$. With $\Lambda(s)=\min_j\mathrm{Im}\lambda_j(s)$ where $\lambda_j(s)$ are the roots of $p(t,x,\tau(s),\xi(s),\lambda)=0$, it is clear that $\Lambda(s)$ is continuous and $\Lambda(0)\geq 0$ and $\Lambda(1)<0$ by hypothesis. Then there is a $\hat{s}$ such that

$$p(t,x,\tau(\hat{s}),\xi(\hat{s}),\hat{\lambda})=0,\ \mathrm{Im}\hat{\lambda}=0.$$

Since $\mathrm{Im}t<0$ this contradicts the hyperbolicity of P. This ends the proof.

We next prove

Lemma 8.2.2 *We have*

$$\sum_{l=0}^{r}(i\sigma)^l p_l(t,x,\tau,\xi)=G(t-\sigma\mathrm{Im}\tau,x,\mathrm{Re}\tau,\xi;i\sigma)+O(|\sigma|^{r+1}). \tag{8.2.5}$$

Proof: Recall that

$$\partial_\lambda^j G(t,x,\tau,\xi,0;0)=\sum_{k+|\alpha|=j}\frac{j!}{k!\alpha!}\partial_t^k\partial_x^\alpha P(t,x)\tau^k\xi^\alpha,\ 0\leq j\leq r.$$

This gives that

$$\sum_{\mu+\nu=l}\frac{1}{\mu!\nu!}\partial_t^\nu\partial_\lambda^\mu G(t,x,\tau,\xi,0;0)\zeta^\nu=\sum_{j+|\alpha|=l}\frac{1}{j!\alpha!}\partial_t^j\partial_x^\alpha P(t,x)(\tau+\zeta)^j\xi^\alpha.$$

The left-hand side is equal to

$$\frac{1}{l!}\left(\frac{1}{i}\frac{\partial}{\partial\sigma}\right)^l G(t+i\sigma\zeta,x,\tau,\xi,0;i\sigma)|_{\sigma=0}.$$

After multiplying $(i\sigma)^l$ to both sides we sum up over $l=0,1,...,r$ to get

$$\sum_{l=0}^{r}(i\sigma)^l p_l(t,x,\tau+\zeta,\xi)=G(t+i\sigma\zeta,x,\tau,\xi;i\sigma)+O(|\sigma|^{r+1}).$$

Plugging $i\mathrm{Im}\tau$, $\mathrm{Re}\tau$ into ζ and τ respectively we get the desired assertion.

Proof of Theorem 8.2.1: As noted before we can take $\tau(\sigma)$ so that the left-hand side of (8.2.5) is $O(|\sigma|^{r+1})$ with $\xi=\hat{\xi}$. On the other hand from Corollary 8.2.1 we see that the right-hand side is bounded from below by positive constant times $|\sigma|^r$ which is a contradiction.

8.3 Applications of semi-continuity

We show

Proposition 8.3.1 ([10]) *Let* $P(t,x) = t^m + a_1(x)t^{m-1} + \cdots + a_m(x)$ *be a hyperbolic polynomial where* $a_j(x) \in C^{2r+3}(|x| < \delta)$*. Assume that* $t = 0$ *is a root of* $P(t,0) = 0$ *of multiplicity* r*. Let* L *be a convex compact set in* $\Gamma_{(0,0)}$*. Then there is* $\delta > 0$ *such that we can write*

$$P(t+\lambda\tau, x+\lambda\xi) = c(t,x,\tau,\xi,\lambda)\prod_{j=1}^{r}(\lambda - \mu_j(t,x,\tau,\xi))$$

for $|(t,x)| < \delta$, $(\tau,\xi) \in L$ *where* $c(t,x,\tau,\xi,\lambda) \neq 0$ *for* $|(t,x)| < \delta$, $(\tau,\xi) \in L$ *and* $\mu_j(t,x,\tau,\xi)$ *are real valued continuous function in* (t,x,τ,ξ) *with*

$$\mu_1(t,x,\tau,\xi) \le \mu_2(t,x,\tau,\xi) \le \cdots \le \mu_r(t,x,\tau,\xi).$$

Moreover there exists $C > 0$ *such that*

$$|\mu_j(t,x,\tau,\xi)| \le C|\mu_j(t,x,\tau',\xi')|$$

for every $|(t,x)| < \delta$, (τ,ξ), $(\tau',\xi') \in L$, $1 \le j \le r$*. In particular, assuming* $(1,0) \in L$*, we have*

$$|\mu_j(t,x,1,0)|/C \le |\mu_j(t,x,\tau,\xi)| \le C|\mu_j(t,x,1,0)|.$$

Proof: To simplify notations we write $z = (t,x)$ and $\theta = (\tau,\xi)$. Since

$$\partial_\lambda^j P(z+\lambda\theta)|_{z=0,\lambda=0} = 0, \quad j < r,$$
$$\partial_\lambda^r P(z+\lambda\theta)|_{z=0,\lambda=0} = P_{(0,0)}(\theta) \neq 0, \quad \theta \in L$$

thanks to Malgranges' preparation theorem one can write

$$\begin{aligned} P(z+\lambda\theta) &= c(z,\theta,\lambda)(\lambda^r + a_1(z,\theta)\lambda^{r-1} + \cdots + a_r(z,\theta)) \\ &= c(z,\theta,\lambda)p(z,\theta,\lambda) \end{aligned}$$

where $c(z,\theta,\lambda) \neq 0$ for $|z| < \delta$, $|\lambda| < \delta$, $\theta \in L$ and $a_j(0,\theta) = 0$, $\theta \in L$. From Lemma 8.2.1 we see that $p(z,\theta,\lambda) \neq 0$ for $|z| \le \delta_1$, $\theta \in L$ if $\mathrm{Im}\lambda < 0$. Since $a_j(z,\theta)$ are real valued then $p(z,\theta,\lambda) = 0$ has only real roots for any $|z| \le \delta_1$, $\theta \in L$. This proves the first assertion.

Let K be a closed convex cone in $\mathbf{R}^{n+1} \setminus \{0\}$ such that $L \subset\subset K \subset \Gamma_{(0,0)}$. By Theorem 8.2.1 we may assume that $K \subset \Gamma_z$ if $|z| < \delta_1$. Choose $0 < \delta_2 < \delta_1$ so that $|z + \mu_j(z,\theta)\theta| < \delta_1$ if $|z| < \delta_2$ and $\theta \in L$ which is possible because $\mu_j(0,\theta) = 0$ for all $\theta \in L$. Put $c_0 = \inf_{\theta\in L} \mathrm{dist}(\theta/|\theta|, K^c)$ where K^c stands for the complement of K. Fix $1 \le j \le r$, θ', $\theta'' \in L$, z and consider

$$Z(s) = z + \mu_j(z,\theta(s))\theta(s), \quad s \in [0,1]$$

where $\theta(s) = (1-s)\theta' + s\theta''$. We show that $\mu_j(z,\theta(s))$ vanishes identically in s if $\mu_j(z,\theta(s))$ vanishes at some $s \in [0,1]$. Suppose otherwise so that $Z(s') = z$

and $Z(s'') \neq z$ for some $s', s'' \in [0,1]$. We may assume that $s' < s''$. Let $\tilde{s} = \sup\{s \mid s < s'', Z(s) = z\}$. Then we have $\tilde{s} < s''$, $\mu_j(z, \theta(\tilde{s})) = 0$ and $\mu_j(z, \theta(s)) \neq 0$ for $\tilde{s} < s \leq s''$. Then we see that

$$\begin{aligned} 0 = P(Z(s)) &= P(z + \mu_j(z, \theta(s))\theta(s)) \\ &= \mu_j(z, \theta(s))^{\nu}(P_z(\theta(s)) + o(1)), \quad s \downarrow \tilde{s} \end{aligned}$$

which contradicts $P_z(\theta(s)) \neq 0$ for $\theta(s) \in K \subset \Gamma_z$.

We next show that

$$Z(1) - Z(0) \notin \pm K. \tag{8.3.1}$$

If $\mu_j(z, \theta(s)) = 0$ at some s then $Z(1) = Z(0) = z$ and hence $0 = Z(1) - Z(0) \notin \pm K$ so that we may assume that $\mu_j(z, \theta(s)) \neq 0$, $s \in [0,1]$. For definiteness we assume $\mu_j(z, \theta(s)) > 0$ in $[0,1]$. If θ' and θ'' are linearly dependent, say $\theta' = \alpha\theta''$ with some $\alpha > 0$ then it is easy to see $\alpha\mu_j(z, \theta') = \mu_j(z, \theta'')$ so that $Z(1) = Z(0)$. Thus the conclusion. Therefore we may assume that θ' and θ'' are linearly independent. Recall that

$$P(Z(s) + t\delta z) = t^{\nu(s)}(P_{Z(s)}(\delta z) + o(1)), \quad t \downarrow 0$$

where $o(1)$ goes to zero uniformly in $\{\delta z \mid |\delta z| \leq 1\}$. It is clear that there is $c(s) > 0$ such that $|P_{Z(s)}(\delta z)| \geq c(s)$ for $\delta z \in \pm K$ with $|\delta z| = 1$ since $K \subset \Gamma_{Z(s)}$. Hence there exists $\delta(s) > 0$ such that

$$Z(s') - Z(s) \notin \pm K \quad \text{if } |s' - s| \leq \delta(s).$$

Let us set

$$H = \{a\theta' + b\theta'' \mid a\theta' + b\theta'' \notin \pm K, b \geq 0, a \in \mathbf{R}\}.$$

With the two dimensional subspace $\pi = \langle \theta', \theta'' \rangle$ we see that

$$H = \big(\pi \setminus ((K \cap \pi) \cup ((-K) \cap \pi))\big) \cap h$$

with the half space $h = \{a\theta' + b\theta'' \mid a \in \mathbf{R}, b \geq 0\}$ in π and hence it is clear that H is a convex cone. We now consider

$$\begin{aligned} &Z(s') - Z(s) \\ &= [\{\mu_j(z, \theta(s')) - \mu_j(z, \theta(s))\}s + \mu_j(z, \theta(s'))(s' - s)]\theta'' \\ &+ [\{\mu_j(z, \theta(s')) - \mu_j(z, \theta(s))\}(1 - s) - \mu_j(z, \theta(s'))(s' - s)]\theta'. \end{aligned}$$

This shows that

$$\{\mu_j(z, \theta(s')) - \mu_j(z, \theta(s))\}s + \mu_j(z, \theta(s'))(s' - s) \geq 0. \tag{8.3.2}$$

Indeed if not and hence $\{\mu_j(z, \theta(s')) - \mu_j(z, \theta(s))\}s < 0$ for $s' - s > 0$ we must have

$$\{\mu_j(z, \theta(s')) - \mu_j(z, \theta(s))\}(1 - s) - \mu_j(z, \theta(s'))(s' - s) > 0$$

because $Z(s') - Z(s) \notin \pm K$ for $|s' - s| < \delta(s)$ which would give a contradiction. Thus we have proved

$$0 < s' - s < \delta(s) \Longrightarrow Z(s') - Z(s) \in H.$$

Analogous arguments show that

$$0 < s - s' < \delta(s) \Longrightarrow Z(s) - Z(s') \in H.$$

Since $[0,1]$ is compact one can find $0 = s_0 < s_1 < \cdots < s_N = 1$ such that $Z(s_i) - Z(s_{i-1}) \in H$ for $i = 0, ..., N-1$ which gives (8.3.1). We now conclude the proof. If $\mu_j(z, \theta') = 0$ then the assertion is clear so that we may assume $\mu_j(z, \theta') \neq 0$. From (8.3.1) we have

$$\frac{\mu_j(z, \theta'')\theta''}{\mu_j(z, \theta')|\theta'|} - \frac{\theta'}{|\theta'|} \notin K$$

which gives that

$$\frac{|\mu_j(z, \theta'')|}{|\mu_j(z, \theta')|} \geq c_0 \frac{|\theta'|}{|\theta''|}$$

and hence the assertion.

We now estimate $|\partial_z^\alpha P(z)|$ applying Proposition 8.3.1. Let us recall

$$P(t, x) = \prod_{j=1}^{m} (t - \lambda_j(x)) = \prod_{j=1}^{m} q_j(z), \quad q_j(z) = t - \lambda_j(x).$$

Since $t = 0$ is a root of $P(t, 0) = 0$ of multiplicity r we may assume that $\lambda_1(0) = \cdots \lambda_r(0) = 0$ and $\lambda_j(0)$, $j \geq r+1$ are different from zero and hence $q_1(0) = \cdots = q_r(0) = 0$, $q_j(0) \neq 0$ for $j \geq r+1$. Let us write

$$|P(z - is(1,0))|^2 = \prod_{j=1}^{m} |-is + q_j(z)|^2 = \sum_{j=0}^{m} s^{2(m-j)} h_j(z)$$

so that

$$h_m(z) = |P(z)|^2, \;\; h_j(z) = \sum_{1 \leq \ell_1 < \cdots < \ell_j \leq m} \prod_{k=1}^{j} |q_{\ell_k}(z)|^2.$$

Then we have

$$P(z + \lambda(1,0)) = c(z, \lambda) \prod_{j=1}^{r} (\lambda - \mu_j(z, (1,0)))$$

where $c(0,0) \neq 0$ and this shows that

$$\{\mu_1(z, (1,0)), ..., \mu_r(z, (1,0))\} = \{-q_1(z), ..., -q_r(z)\}$$

so that

$$\sum_{1\le\ell_1<\cdots<\ell_j\le r}\prod_{k=1}^{j}|\mu_{j_k}(z,(1,0))|^2\le Ch_j(z)$$

for $|z|\le\delta$ with some $C>0$ since $|q_j(z)|\ge c>0$ for $j\ge r+1$, $|z|\le\delta$.

Proposition 8.3.2 *We have*

$$|\partial_z^\alpha P(z)|\le Ch_{m-|\alpha|}(z)^{1/2}$$

for $|z|\le\delta$, $1\le|\alpha|\le m-1$.

Proof: Take $L=\{\theta\in\mathbf{R}^{n+1}\mid|\theta-(1,0)|\le\delta\}$ with some $\delta>0$. It is easy to see

$$\left|\partial_\lambda^j P(z+\lambda\theta)|_{\lambda=0}\right|\le C\left\{\sum_{1\le\ell_1<\cdots<\ell_{r-j}\le r}\prod_{k=1}^{r-j}|\mu_{j_k}(z,\theta)|^2\right\}^{1/2}$$

for $|z|+|\theta-(1,0)|\le\delta$ because $|\mu_j(z,\theta)|\le C_1$ there. Applying Proposition 8.3.1 one has

$$\left|\sum_{|\alpha|=j}\frac{j!}{\alpha!}\partial_z^\alpha P(z)\theta^\alpha\right|\le Ch_{r-j}(z)^{1/2}.$$

Since θ is arbitrary provided $|\theta-(1,0)|\le\delta$ one concludes the assertion.

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