On pseudodifferential operators of symbol exp $S^{\kappa}_{\rho,\delta}$

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Abstract

In this note we give a proof of composition formula of pseudodifferential operators with symbols of type $\exp(S^{\kappa}_{\rho,\delta})$ acting on Gevrey spaces without of use of almost analytic extension.

1 A lemma

Definition 1.1. We say that $f(x) \in C^{\infty}(\mathbb{R}^n)$ belongs to $G^s(\mathbb{R}^n)$, the (global) Gevrey class of order s, if there exist $C>0$, $A>0$ such that

$$|D^{\alpha}f(x)| \le CA^{|\alpha|}|\alpha|!^s, \quad x\in\mathbb{R}^n, \ \alpha\in\mathbb{N}^n.$$

Let us denote $\langle\xi\rangle_M = (M^2+|\xi|^2)^{1/2}$ where $M\ge 1$ is a positive parameter.

Definition 1.2. Let $m = m(x,\xi;M) > 0$ be a positive function. We define $S^{(s)}_{\rho,\delta}(m)$ to be the set of all $a(x,\xi;M)\in C^{\infty}(\mathbb{R}^n\times\mathbb{R}^n)$ such that we have

$$|\partial_x^{\beta}\partial_{\xi}^{\alpha}a(x,\xi;M)| \le CA^{|\alpha+\beta|}|\alpha+\beta|!^s m(x,\xi,M)\langle\xi\rangle_M^{\delta|\beta|-\rho|\alpha|} \tag{1.1}$$

for any α, $\beta\in\mathbb{N}^n$ with some $C>0$, $A>0$ independent of $M\ge 1$ and $S_{\rho,\delta}(m)$ to be the set of all $a(x,\xi,M)$ satisfying (1.1) with $C_{\alpha\beta}$ instead of $CA^{|\alpha+\beta|}|\alpha+\beta|!^s$ which may depend on α,β but not on M. We often write just $a(x,\xi)$ or $m(x,\xi)$ dropping M.

Lemma 1.1. *Let $m = m(x,\xi;M) > 0$ be a positive function and $f\in S^{(s)}_{\rho,\delta}(m)$. Denote $\omega^{\alpha}_{\beta} = e^{-f}\partial_x^{\beta}\partial_{\xi}^{\alpha}e^f$ then there exist $A>0, C>0$ such that the following holds.*

$$\begin{aligned}\big|\partial_x^{\nu}\partial_{\xi}^{\mu}\omega^{\alpha}_{\beta}\big| &\le CA^{|\nu+\mu+\alpha+\beta|}\langle\xi\rangle_M^{\delta|\beta+\nu|-\rho|\alpha+\mu|}\\ &\quad\times\sum_{j=0}^{|\alpha+\beta|}m^{|\alpha+\beta|-j}(|\mu+\nu|+j)!^s.\end{aligned} \tag{1.2}$$

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Corollary 1.1. *There are $A > 0, C > 0$ such that*

$$|\partial_x^\beta \partial_\xi^\alpha e^f| \le Ce^{|f|}A^{|\alpha+\beta|}\langle\xi\rangle_M^{\delta|\beta|-\rho|\alpha|}(m + |\alpha+\beta|^s)^{|\alpha+\beta|}, \quad \alpha, \beta \in \mathbb{N}^n.$$

In particular $e^{f(x,\xi)} \in S_{\rho,\delta}^{(s)}(e^{|f|+sm^{1/s}})$.

Proof. Taking $\mu = \nu = 0$ in (1.2) gives

$$|\partial_x^\beta \partial_\xi^\alpha e^f| \le Ce^{|f|}A^{|\alpha+\beta|}\langle\xi\rangle_M^{\delta|\beta|-\rho|\alpha|}\sum_{j=0}^{|\alpha+\beta|} m^{|\alpha+\beta|-j}j!^s$$

which proves the first inequality. Noting that $m^N \le N!^s e^{sm^{1/s}}$ $(s > 0)$ for any $N \in \mathbb{N}$ one can find $C > 0$ independent of $s > 1$ such that

$$\sum_{j=0}^{|\alpha+\beta|} m^{|\alpha+\beta|-j}j!^s \le e^{sm^{1/s}}\sum_{j=0}^{|\alpha+\beta|}(|\alpha+\beta|-j)!^s j!^s \le Ce^{sm^{1/s}}|\alpha+\beta|!^s$$

which proves the second assertion. □

1.1 Pseudodifferential operators of type $S_{\rho,\delta}$ in the Gevrey classes

We introduce a symbol class for which we define oscillatory integral.

Definition 1.3. Let $m = m(x,\xi,M)$ be a positive function and $0 \le \delta < 1$, $1 < s$. We say that $a(x,\xi,y) \in C^\infty(\mathbb{R}^{3n})$ belongs to $\mathcal{A}_\delta^{(s)}(m)$ if there are $C > 0, A > 0$ such that

$$(1.3) \qquad |\partial_{x,y}^\beta \partial_\xi^\alpha a(x,\xi,y)| \le CA^{|\alpha+\beta|}|\alpha+\beta|!^s(|\beta|^{\delta s/(1-\delta)} + \langle\xi\rangle_M^\delta)^{|\beta|}m(x,\xi)$$

for all $\alpha, \beta \in \mathbb{N}^n$. By abuse of notation we denote by the same $\mathcal{A}_\delta^{(s)}(m)$ the set of all $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying (1.3).

Assume that

$$(1.4) \qquad a(x,\xi,y) \in \mathcal{A}_\delta^{(s)}(e^{c\langle\xi\rangle_M^\kappa}) \;\; (c > 0), \quad 1 - \delta > s\kappa.$$

Let $\chi(t) \in G_0^s(\mathbb{R}^n)$ be such that $\chi(t) = 1$ in some neighborhood of 0 and set $\chi_\epsilon(y) = \chi(\epsilon y)$, $\chi_\epsilon(\eta) = \chi(\epsilon\eta)$. Let $\rho(t) \in G^s(\mathbb{R})$ be such that $\rho(t) = 0$ for $|t| \le 1/2$ and $\rho(t) = 1$ for $|t| \ge 1$ and set $\rho_M(\eta) = \rho(M^{-1}\eta)$, $\rho_M^c(\eta) =$

$1-\rho_M(\eta)$. For $a(x,\xi,y)\in\mathcal{A}_\delta^{(s)}(m)$ we define $\mathcal{O}p(a)u(x)$ for $u\in G^{s/(1-\delta)}(\mathbb{R}^n)$ by the oscillatory integral

$$
\begin{aligned}
&(2\pi)^{-n}\lim_{\epsilon\to 0}\int e^{i(x-y)\eta}\chi_\epsilon(x-y)\chi_\epsilon(\eta)a(x,\eta,y)u(y)dyd\eta\\
&=(2\pi)^{-n}\lim_{\epsilon\to 0}\int e^{-iy\eta}\chi_\epsilon(y)\chi_\epsilon(\eta)a(x,\eta,y+x)u(y+x)dyd\eta.
\end{aligned}
\tag{1.5}
$$

Noting that $\langle\eta\rangle^{-2N}\langle D_y\rangle^{2N}e^{-iy\eta}=e^{-iy\eta}$ and $\langle y\rangle^{-2\ell}\langle D_\eta\rangle^{2\ell}e^{-iy\eta}=e^{-iy\eta}$ after integration by parts $\mathcal{O}p(\rho_M a)u(x)$ yields

$$
\int e^{-iy\eta}\langle D_y\rangle^{2N}\langle\eta\rangle^{-2N}\langle D_\eta\rangle^{2\ell}\langle y\rangle^{-2\ell}\chi_\epsilon(y)\chi_\epsilon(\eta)\rho_M a(x,\eta,y+x)u(y+x)dyd\eta.
$$

Since $s+s\delta/(1-\delta)=s/(1-\delta)$ and $\langle\eta\rangle_M\le 3\langle\eta\rangle$ if $\rho_M\neq 0$ the integrand is bounded uniformly in $\epsilon>0$ by (C,A may change line by line but not depend on N)

$$
\begin{aligned}
&CA^{2N}(2N)!^s((2N)^{s\delta/(1-\delta)}+\langle\eta\rangle_M^\delta)^{2N}\langle y\rangle^{-2\ell}\langle\eta\rangle^{-2N}e^{c\langle\eta\rangle_M^\kappa}\\
&\le CA^{2N}N^{2Ns}(N^{s\delta/(1-\delta)}+\langle\eta\rangle_M^\delta)^{2N}\langle y\rangle^{-2\ell}\langle\eta\rangle^{-2N}e^{c\langle\eta\rangle_M^\kappa}\\
&\le C\langle y\rangle^{-2\ell}\Big(\frac{rAN^s}{\langle\eta\rangle^{1-\delta}}\Big)^{2N}\Big(\frac{N^{s\delta/(1-\delta)}}{r\langle\eta\rangle^\delta}+\frac{3^\delta}{r}\Big)^{2N}e^{c\langle\eta\rangle_M^\kappa}
\end{aligned}
$$

with $r>0$. Choose r such that $(1/rA)^{\delta/(1-\delta)}+3^\delta/r\le 1$ and the maximal $N=N(\eta)\in\mathbb{N}$ such that $N^s\le\langle\eta\rangle^{1-\delta}/(4e^2A)$ one can find $c'>0$ so that

$$
\Big(\frac{rAN^s}{\langle\eta\rangle^{1-\delta}}\Big)^{2N}\Big(\frac{N^{s\delta/(1-\delta)}}{r\langle\eta\rangle^\delta}+\frac{1}{r}\Big)^{2N}\le Ce^{-c'\langle\eta\rangle^{(1-\delta)/s}}.
\tag{1.6}
$$

Since $\kappa<(1-\delta)/s$ the integrand is bounded by $C\langle y\rangle^{-2\ell}e^{-c''\langle\eta\rangle^{(1-\delta)/s}}$. Noting that $\partial_{y,\eta}^\alpha\chi_\epsilon\to 0$ as $\epsilon\to 0$ if $|\alpha|\ge 1$ we conclude that $\mathcal{O}p(\rho_M a)u(x)$ is

$$
\int e^{-iy\eta}\langle D_y\rangle^{2N}\langle\eta\rangle^{-2N}\langle D_\eta\rangle^{2\ell}\langle y\rangle^{-2\ell}\rho_M(\eta)a(x,\eta,y+x)u(y+x)dyd\eta
\tag{1.7}
$$

On the other hand, it is clear that

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int e^{-iy\eta}\chi_\epsilon(y)\chi_\epsilon(\eta)\rho_M^c a(x,\eta,y+x)u(y+x)dyd\eta\\
&=\int e^{-iy\eta}\langle D_y\rangle^{2N}\langle\eta\rangle^{-2N}\langle D_\eta^{2\ell}\rangle\langle y\rangle^{-2\ell}\rho_M^c a(x,\eta,y+x)u(y+x)dyd\eta
\end{aligned}
$$

and hence (1.5) is equal to

$$\int e^{-iy\eta}\langle D_y\rangle^{2N}\langle\eta\rangle^{-2N}\langle D_\eta^{2\ell}\rangle\langle y\rangle^{-2\ell}a(x,\eta,y+x)u(y+x)dyd\eta$$

which is independent of the choice of χ. Next, consider $\partial_x^\beta \mathcal{O}p(a)u(x)$;

$$\int e^{-iy\eta}\langle D_y\rangle^{2N}\langle\eta\rangle^{-2N}\langle D_\eta\rangle^{2\ell}\langle y\rangle^{-2\ell}\chi_\epsilon(y)\chi_\epsilon(\eta)\partial_x^\beta(a(x,\eta,y+x)u(y+x))dyd\eta.$$

Here we remark the following easy lemma.

Lemma 1.2. *Let $A, B \geq 0$. Then there exists $C > 0$ independent of $n, m \in \mathbb{N}$, A, B such that*

$$(A+(n+m)^s B)^{n+m} \leq C^{n+m}(A+n^s B)^n (A+m^s B)^m.$$

Taking Lemma 1.2 into account the integrand is bounded by

$$\begin{aligned} CA^{2N+|\beta|}(2N+|\beta|)!^s((2N+|\beta|)^{s\delta/(1-\delta)}+\langle\eta\rangle_M^\delta)^{2N+|\beta|}\langle y\rangle^{-2\ell}\langle\eta\rangle^{-2N}e^{c\langle\eta\rangle_M^\kappa} \\ \leq CA^{2N+|\beta|}|\beta|!^s(|\beta|^{s\delta/(1-\delta)}+\langle\eta\rangle_M^\delta)^{|\beta|} \\ \times N^{2Ns}(N^{s\delta/(1-\delta)}+\langle\eta\rangle_M^\delta)^{2N}\langle y\rangle^{-2\ell}\langle\eta\rangle^{-2N}e^{c\langle\eta\rangle_M^\kappa}. \end{aligned}$$

Noting that for any $\epsilon > 0$ there are $C > 0, A > 0$ such that

$$\langle\eta\rangle_M^{\delta|\beta|} \leq CA^{|\beta|}|\beta|!^{s\delta/(1-\delta)}e^{\epsilon\langle\eta\rangle_M^{(1-\delta)/s}} \tag{1.8}$$

and applying (1.6) we obtain the following

Lemma 1.3. *We have $\mathcal{O}p(a)\big(G^{s/(1-\delta)}(\mathbb{R}^n)\big) \subset G^{s/(1-\delta)}(\mathbb{R}^n)$ if $a(x,\xi,y) \in \mathcal{A}_\delta^{(s)}(e^{c\langle\xi\rangle_M^\kappa})$ and $1-\delta > \kappa s$.*

For $a(x,\xi) \in S_{\rho,\delta}^{(s)}(m)$ and $0 \leq t \leq 1$ we define $\mathrm{op}^t(a)$ by

$$\mathrm{op}^t(a)u(x) = \mathcal{O}p(\tilde{a})u(x), \;\; \tilde{a}(x,\xi,y) = a(ty+(1-t)x,\xi) \in \mathcal{A}_\delta^{(s)}(m). \tag{1.9}$$

Definition 1.4. $\mathrm{op}^{1/2}(a)$ is called the Wyle quantization of a and denoted by $\mathrm{op}(a)$ dropping $1/2$.

Let $\tilde{a}_i(x,\xi,y) \in \mathcal{A}_\delta^{(s)}(e^{c\langle\xi\rangle_M^\kappa})$ with $1-\delta > \kappa s$ and consider $\mathrm{op}(\tilde{a}_1)\mathrm{op}(\tilde{a}_2)$. Suppose that

$$\begin{aligned} (2\pi)^{-n}\int e^{i(x-y)\eta+i(y-z)\zeta}\chi_{\epsilon_1}(x-y)\chi_{\epsilon_1}(\eta)\chi_{\epsilon_2}(y-z)\chi_{\epsilon_2}(\zeta) \\ \times\tilde{a}_1(x,\eta,y)\tilde{a}_2(y,\zeta,z)u(z)dyd\xi dzd\eta \end{aligned}$$

is equal to

$$(2\pi)^{-n}\int e^{i(x-z)\theta}b_\epsilon((x+z)/2,\theta)u(z)dzd\theta,\quad \epsilon=(\epsilon_1,\epsilon_2)$$

for any $u\in G_0^{s/(1-\delta)}(\mathbb{R}^n)$. This implies that $\int e^{i(x-z)\theta}b_\epsilon((x+z)/2,\theta)d\theta$ is equal to

$$(2\pi)^{-n}\int e^{i(x-y)\eta+i(y-z)\zeta}\chi_{\epsilon_1}(x-y)\chi_{\epsilon_1}(\eta)\chi_{\epsilon_2}(y-z)\chi_{\epsilon_2}(\zeta)$$
$$\times\tilde{a}_1(x,\eta,y)\tilde{a}_2(y,\zeta,z)dyd\eta d\zeta.$$

Making the change of variables $x+z=2\tilde{x}$, $z-x=2\tilde{z}$, $y=\tilde{y}$, $\eta+\zeta=2\tilde{\eta}$, $\eta-\zeta=2\tilde{\zeta}$ the integral $\int e^{-2i\tilde{z}\theta}b_\epsilon(\tilde{x},\theta)d\theta=(\mathcal{F}b_\epsilon)(2\tilde{z})$ (where $\mathcal{F}b_\epsilon$ denotes the Fourier transform of b_ϵ) yields

$$\pi^{-n}\int e^{2i(\tilde{x}-\tilde{y})\tilde{\zeta}-2i\tilde{z}\tilde{\eta}}\chi_{\epsilon_1}(\tilde{x}-\tilde{z}-\tilde{y})\chi_{\epsilon_1}(\tilde{\eta}+\tilde{\zeta})\chi_{\epsilon_2}(\tilde{y}-\tilde{x}-\tilde{z})\chi_{\epsilon_2}(\tilde{\eta}-\tilde{\zeta})$$
$$\times\tilde{a}_1(\tilde{x}-\tilde{z},\tilde{\eta}+\tilde{\zeta},\tilde{y})\tilde{a}_2(\tilde{y},\tilde{\eta}-\tilde{\zeta},\tilde{x}+\tilde{z})d\tilde{y}d\tilde{\zeta}d\tilde{\eta}.$$

From the Fourier inversion formula one has

$$b_\epsilon(\tilde{x},\theta)=\pi^{-2n}\int e^{2i(\tilde{x}-\tilde{y})\tilde{\zeta}-2i\tilde{z}(\tilde{\eta}-\theta)}\chi_{\epsilon_1}(\tilde{x}-\tilde{z}-\tilde{y})\chi_{\epsilon_1}(\tilde{\eta}+\tilde{\zeta})\chi_{\epsilon_2}(\tilde{y}-\tilde{x}-\tilde{z})$$
$$\times\chi_{\epsilon_2}(\tilde{\eta}-\tilde{\zeta})\tilde{a}_1(\tilde{x}-\tilde{z},\tilde{\eta}+\tilde{\zeta},\tilde{y})\tilde{a}_2(\tilde{y},\tilde{\eta}-\tilde{\zeta},\tilde{x}+\tilde{z})d\tilde{y}d\tilde{\zeta}d\tilde{\eta}d\tilde{z}.$$

After the translation $\tilde{\eta}\to\tilde{\eta}+\theta$, $\tilde{y}\to\tilde{y}+\tilde{x}$ the right-hand is

$$\pi^{-2n}\int e^{-2i\tilde{y}\tilde{\zeta}-2i\tilde{z}\tilde{\eta}}\chi_{\epsilon_1}(-\tilde{z}-\tilde{y})\chi_{\epsilon_1}(\tilde{\eta}+\tilde{\zeta}+\theta)\chi_{\epsilon_2}(\tilde{y}-\tilde{z})\chi_{\epsilon_2}(\tilde{\eta}-\tilde{\zeta}+\theta)$$
$$\times\tilde{a}_1(\tilde{x}-\tilde{z},\tilde{\eta}+\tilde{\zeta}+\theta,\tilde{y}+\tilde{x})\tilde{a}_2(\tilde{y}+\tilde{x},\tilde{\eta}-\tilde{\zeta}+\theta,\tilde{x}+\tilde{z})d\tilde{y}d\tilde{\zeta}d\tilde{\eta}d\tilde{z}.$$

Making the change of variables $\tilde{\eta}+\tilde{\zeta}=\eta$, $\tilde{\eta}-\tilde{\zeta}=\zeta$, $\tilde{y}-\tilde{z}=2y$, $\tilde{y}+\tilde{z}=2z$ one concludes

$$b_\epsilon(\tilde{x},\theta)=\pi^{-2n}\int e^{-2i(z\eta-y\zeta)}\chi_{\epsilon_1}(-2z)\chi_{\epsilon_1}(\eta+\theta)\chi_{\epsilon_2}(2y)\chi_{\epsilon_2}(\zeta+\theta)$$
$$\times\tilde{a}_1(\tilde{x}+y-z,\theta+\eta,\tilde{x}+y+z)\tilde{a}_2(\tilde{x}+y+z,\theta+\zeta,\tilde{x}-y+z)dyd\zeta d\eta dz.$$

Here we note that if $\tilde{a}_i(x,\xi,y)=a_i((x+y)/2,\xi)$ this shows that

$$b_\epsilon(\tilde{x},\theta)=\pi^{-2n}\int e^{-2i(z\eta-y\zeta)}\chi_{\epsilon_1}(-2z)\chi_{\epsilon_1}(\eta+\theta)\chi_{\epsilon_2}(2y)\chi_{\epsilon_2}(\zeta+\theta)$$
$$\times a_1(\tilde{x}+y,\theta+\eta)a_2(\tilde{x}+z,\theta+\zeta)dyd\zeta d\eta dz. \tag{1.10}$$

Letting $\epsilon_i \to 0$ it follows from the definition of the oscillatory integral we have $\mathrm{op}(a_1)\mathrm{op}(a_2) = \mathrm{op}(b)$ where ($\chi(x+\theta) = 1$ near $x = 0$ can be assumed)

$$b(x,\xi) = \pi^{-2n} \int e^{-2i(z\eta - y\zeta)} a_1(x+y, \xi+\eta) a_2(x+z, \xi+\zeta) dy d\zeta d\eta dz.$$

Return to $\tilde{a}_i$. Denoting $\tilde{b}(x,\theta,\tilde{x}) = b((x+\tilde{x})/2, \theta)$ and letting $\epsilon_i \to 0$ we conclude $\mathcal{O}p(\tilde{b}) = \mathcal{O}p(\tilde{a}_1)\mathcal{O}p(\tilde{a}_2)$ where $\tilde{b}(x,\theta,\tilde{x})$ is given by the oscillatory integral

$$\pi^{-2n} \int e^{-2i(z\eta - y\zeta)} \tilde{a}_1((x+\tilde{x})/2 + y - z, \theta+\eta, (x+\tilde{x})/2 + y + z)$$
$$\times \tilde{a}_2((x+\tilde{x})/2 + y + z, \theta+\zeta, (x+\tilde{x})/2 - y + z) dy d\zeta d\eta dz.$$

In what follows we write $X = (x,\xi)$, $Y = (y,\eta)$, $Z = (z,\zeta)$ and $\sigma(Y,Z) = \eta z - y\zeta = \langle \sigma Y, Z\rangle$.

Proposition 1.1. *If $\tilde{a}_i(x,\xi,y) \in \mathcal{A}_\delta^{(s)}(e^{c_i\langle\xi\rangle_M^\kappa})$ with $c_i > 0$, $1-\delta > \kappa s$ there exists $\tilde{b} \in \mathcal{A}_\delta^{(s)}(e^{c_3\langle\xi\rangle_M^\kappa})$ ($c_3 > 0$) such that $\mathcal{O}p(\tilde{b}) = \mathcal{O}p(\tilde{a}_1)\mathcal{O}p(\tilde{a}_2)$.*

Proof. It remains to show $\tilde{b}(x,\xi,\tilde{x}) \in \mathcal{A}_\delta^{(s)}(e^{c_3\langle\xi\rangle_M^\kappa})$. Denoting

$$F(X,Y,Z) = \tilde{a}_1(x+y-z, \xi+\eta, x+y+z)\tilde{a}_2(x+y+z, \xi+\zeta, x-y+z)$$

we estimate

$$\partial_x^\beta \partial_\xi^\alpha \int e^{-2i\sigma(Y,Z)} F(X,Y,Z) dY dZ.$$

Let $\chi(x) \in G^s(\mathbb{R})$ be 1 in $|x| \le 1/5$ and 0 outside $|x| \le 1/4$ and denote

$$\bar{\chi}(\eta,\zeta) = \chi(\langle\eta\rangle\langle\xi\rangle_M^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_M^{-1}), \quad \bar{\chi}^c = 1 - \bar{\chi}. \tag{1.11}$$

Write

$$\int e^{-2i\sigma(Y,Z)} F(X,Y,Z)\bar{\chi} dY dZ + \int e^{-2i\sigma(Y,Z)} F(X,Y,Z)\bar{\chi}^c dY dZ = I + II.$$

Since $|\partial_{\xi,\eta,\zeta}^\alpha(\bar{\chi},\bar{\chi}^c)| \le A^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{-|\alpha|}$ and $\langle\xi+\eta\rangle_M \approx \langle\xi\rangle_M$, $\langle\xi+\zeta\rangle_M \approx \langle\xi\rangle_M$ if $\bar{\chi} \ne 0$ it is easy to see

$$|\partial_{x,y,z}^\beta \partial_{\xi,\eta,\zeta}^\alpha F| \le CA^{|\alpha+\beta|}|\alpha+\beta|!^s(|\beta|^{\delta s/(1-\delta)} + \langle\xi\rangle_M^\delta)^{|\beta|} e^{c\langle\xi\rangle_M^\kappa}, \quad \bar{\chi} \ne 0.$$

Making integration by parts we see

$$\partial_x^\beta \partial_\xi^\alpha I = \int e^{-2i\sigma(Y,Z)} \langle D_y\rangle^{2\ell}\langle\zeta\rangle^{-2\ell}\langle D_z\rangle^{2\ell}\langle\eta\rangle^{-2\ell}$$
$$\times \langle D_\zeta\rangle^{2\ell}\langle z\rangle^{-2\ell}\langle D_y\rangle^{2\ell}\langle\zeta\rangle^{-2\ell}\partial_x^\beta\partial_\xi^\alpha(F\bar{\chi}) dY dZ$$

and here the integrand is bounded by

$$CA^{|\alpha+\beta|}|\alpha+\beta|!^s(|\beta|^{s\delta/(1-\delta)}+\langle\xi\rangle_M^{\delta})^{|\beta|}\langle\zeta\rangle^{-2\ell}\langle\eta\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle\zeta\rangle^{-2\ell}\langle\xi\rangle_M^{4\delta\ell}e^{c\langle\xi\rangle_M^{\kappa}}.$$

Since $\langle\xi\rangle_M^{4\delta\ell}$ can be absorbed in $e^{c\langle\xi\rangle_M^{\kappa}}$ changing c we have $I\in\mathcal{A}_\delta^{(s)}(e^{c\langle\xi\rangle_M^{\kappa}})$. Next, consider II. Write

$$\begin{aligned}\bar\chi^c(\eta,\zeta)=\chi^c(\langle\eta\rangle\langle\xi\rangle_M^{-1})\chi^c(\langle\zeta\rangle\langle\xi\rangle_M^{-1})+\chi^c(\langle\eta\rangle\langle\xi\rangle_M^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_M^{-1})\\ +\chi^c(\langle\zeta\rangle\langle\xi\rangle_M^{-1})\chi(\langle\eta\rangle\langle\xi\rangle_M^{-1})=\varphi_1+\varphi_2+\varphi_3\end{aligned}\tag{1.12}$$

and consider $\int e^{-2i\sigma(Y,Z)}\partial_X^\alpha(F\varphi_i)dYdZ$. Let $\chi_0(t)\in G^s(\mathbb{R})$ be 1 in $|t|<1$ and 0 outside $|t|\le 2$ and study

$$\begin{aligned}\int e^{-2i\sigma(Y,Z)}\langle\eta\rangle^{-2N_2}\langle\zeta\rangle^{-2N_1}\langle D_z\rangle^{2N_2}\langle D_y\rangle^{2N_1}\\ \times\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}(\partial_x^\beta\partial_\xi^\alpha F\varphi_1)(\chi_*+\chi_*^c)dYdZ\end{aligned}\tag{1.13}$$

where $\chi_*=\chi_0(\langle\zeta\rangle\langle\eta\rangle^{-1})$ and $\chi_*^c=1-\chi_*$. Consider

$$\big|\langle\eta\rangle^{-2N_2}\langle\zeta\rangle^{-2N_1}\langle D_z\rangle^{2N_2}\langle D_y\rangle^{2N_1}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}(\partial_X^\alpha F\varphi_1)\chi_*\big|\tag{1.14}$$

Choosing $N_1=\ell$, $N_2=N$ and noting $\langle\xi\rangle_M\le C\langle\eta\rangle$, $\langle\xi+\eta\rangle_M\le C\langle\eta\rangle$, $\langle\xi+\zeta\rangle_M\le C\langle\eta\rangle$ if $\varphi_1\chi_*\neq 0$ it is not difficult to see that this is bounded by

$$\begin{aligned}CA^{2N+|\alpha+\beta|}\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle\eta\rangle^{2\delta\ell}(2N)!^s|\alpha+\beta|!^s\\ \times(N^{\delta s/(1-\delta)}+\langle\eta\rangle^{\delta})^{2N}(|\beta|^{s\delta/(1-\delta)}+\langle\eta\rangle^{\delta})^{|\beta|}e^{c\langle\eta\rangle^{\kappa}}\end{aligned}\tag{1.15}$$

where C,A may depend on ℓ but not on N, α,β. Here writing

$$A^{2N}N^{2sN}\langle\eta\rangle^{-2N}\big(N^{s\delta/(1-\delta)}+\langle\eta\rangle^{\delta}\big)^{2N}=\Big(\frac{AN^{s/(1-\delta)}}{\langle\eta\rangle}+\frac{AN^s}{\langle\eta\rangle^{1-\delta}}\Big)^{2N}$$

we choose the maximal $N\in\mathbb{N}$ such that $AN^s\le c_1\langle\eta\rangle^{(1-\delta)}$ with small $c_1>0$ so that the right-hand side is bounded by $Ce^{-c\langle\eta\rangle^{(1-\delta)/s}}$ with some $c>0$. Recalling (1.8) and $1-\delta>\kappa s$ one sees that (1.15) is estimated by

$$C_\ell A_\ell^{|\alpha+\beta|}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}|\alpha+\beta|!^s|\beta|^{s\delta|\beta|/(1-\delta)}e^{-c\langle\eta\rangle^{(1-\delta)/s}}\tag{1.16}$$

which proves $\int e^{-2i\sigma(Y,Z)}F(X,Y,Z)\varphi_1\chi_*dYdZ\in\mathcal{A}_\delta^{(s)}(1)$. Similarly for the case χ_*^c choosing $N_1=N$, $N_2=\ell$ it is proved that (1.15) is bounded by

$$C_\ell A_\ell^{2N}\langle\eta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}|\alpha+\beta|!^s|\beta|^{s\delta|\beta|/(1-\delta)}e^{-c\langle\zeta\rangle^{(1-\delta)/s}}$$

which together with (1.13) shows $\int e^{-2i\sigma(Y,Z)}F(X,Y,Z)\varphi_1 dYdZ \in \mathcal{A}_\delta^{(s)}(1)$. Turn to $\int e^{-2i\sigma(Y,Z)}\partial_x^\beta\partial_\xi^\alpha(F\varphi_2)dYdZ$. Consider

$$\int e^{-2i\sigma(Y,Z)}\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle D_z\rangle^{2N}\langle D_y\rangle^{2\ell}$$
$$\times\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}\partial_x^\beta\partial_\xi^\alpha F\varphi_2 dYdZ.$$

Since $\langle\xi+\eta\rangle_M \leq C\langle\eta\rangle$, $\langle\xi+\zeta\rangle_M \approx \langle\xi\rangle_M \leq C\langle\eta\rangle$ if $\varphi_2 \neq 0$ this is bounded by (1.15). The rest of the argument is the same as for the case $\varphi\chi_*$. The case φ_3 is similar to the case $\varphi_1\chi_*^c$. Thus we obtain $II \in \mathcal{A}_\delta^{(s)}(1)$ which completes the proof. $\square$

The next lemma is a special case of Proposition 1.1.

Lemma 1.4. *Let* $a_i(x,\xi) \in \mathcal{A}_\delta^{(s)}(e^{c_i\langle\xi\rangle_M^\kappa})$ *with* $1-\delta > \kappa s$. *If we set*

$$b(X) = \pi^{-2n}\int e^{-2i\sigma(Y,Z)}a_1(X+Y)a_2(X+Z)dYdZ = (a_1\#a_2)(X)$$

then $b(X) \in \mathcal{A}_\delta^{(s)}(e^{c'\langle\xi\rangle_M^\kappa})$ $(c' > 0)$ *and verifies* $\mathrm{op}(a_1)\mathrm{op}(a_2) = \mathrm{op}(b)$.

Let $a_i(x,\xi) \in \mathcal{A}_\delta^{(s)}(e^{c_i\langle\xi\rangle_M^\kappa})$ with $1-\delta > \kappa s$. Consider $a_1\#a_2\#a_3$. Recall

$$(a_2\#a_3)(Y) = \pi^{-2n}\int e^{-2i\sigma(S,T)}a_2(X+S)a_3(X+T)dSdT$$

and then

$$a_1\#a_2\#a_3 = \pi^{-4n}\int e^{-2i\sigma(Y,Z)-2i\sigma(S,T)}$$
$$\times a_1(X+Y)a_2(X+Z+S)a_3(X+Z+T)dYdZdSdT.$$

It is easily seen from the definition of the oscillatory integral that linear change of variables can be done freely. Making the change of variables $Z \to Z-T$, $S \to S+Y$ and after that $S \to S+T$ again, the above integral turns to be

$$\pi^{-4n}\int e^{-2i\sigma(Y,Z)-2i\sigma(S,T)}$$
$$\times a_1(X+Y)a_2(X+Y+Z+S)a_3(X+Z)dYdZdSdT.$$

Noting that $\int e^{-2i\sigma(S,T)}dT = \pi^{2n}\delta(S)$ we have

$$\begin{aligned} &a_1\#a_2\#a_3 \\ &= \pi^{-2n}\int e^{-2i\sigma(Y,Z)}a_1(X+Y)a_2(X+Y+Z)a_3(X+Z)dYdZ. \end{aligned} \tag{1.17}$$

2 Composition formula

Let $\phi(x,\xi) \in S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M^{\kappa})$ and in what follows we assume

$$0 \le \delta < \rho \le 1, \quad \rho - \delta > \kappa \ge 0, \quad s > 1, \tag{2.1}$$

$$\bar\epsilon := \frac{\rho-\delta}{s} - \kappa - \frac{s-1}{s}\max\{\delta, 1-\rho\} > 0. \tag{2.2}$$

Since $\rho-\delta > \kappa$ by (2.1) the assumption (2.2) is always satisfied for any $s>1$ sufficiently close to 1. If $s=1$ (2.2) reduces to $\rho-\delta>\kappa$ with $\bar\epsilon = \rho-\delta-\kappa$. If $\rho=1$, $\delta=0$ (2.2) reduces to $s\kappa<1$. Denote the metric defining the class $S_{\rho,\delta}$ by g;

$$g_X(Y) = \langle\xi\rangle_M^{2\delta}|y|^2 + \langle\xi\rangle_M^{-2\rho}|\eta|^2, \quad X=(x,\xi),\ Y=(y,\eta)\in\mathbb{R}^n.$$

Definition 2.1. ([2]) A positive function $m(x,\xi;M)$ is called $S_{\rho,\delta}$ admissible weight if there are positive constants C, N such that

$$m(X) \le Cm(Y)\big(1+\max\{g_X(X-Y), g_Y(X-Y)\}\big)^N,\ X,Y\in\mathbb{R}^{2n}. \tag{2.3}$$

Theorem 2.1. *Let $p(x,\xi)\in S^{(s)}_{\rho,\delta}(w)$ and w be $S_{\rho,\delta}$ admissible weight. Then there exists $c>0$ such that for any $l,m\in\mathbb{N}$ we have*

$$e^{\phi}\#p\#e^{-\phi} = \sum_{k=0}^{m}\frac{(-1)^k}{k!}\sum_{\substack{\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha\\ |\alpha^0|\le l-1, 1\le|\alpha^j|\le l, 1\le j\le k}}\frac{1}{\alpha^0!\alpha^1!\cdots\alpha^k!}(\sigma D_Y/2)^{\alpha}$$
$$\times(\partial_X^{\alpha^0}p(X+2Y)\partial_X^{\alpha^1}\phi(X+Y)\cdots\partial_X^{\alpha^k}\phi(X+Y))\big|_{Y=0}$$
$$+S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-l(\rho-\delta)}) + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-\varepsilon-\bar\epsilon(m+1)}) + S^{(s/(1-\delta))}_{0,0}(we^{-c\langle\xi\rangle_M^{(1-\delta)/s}})$$

where $\varepsilon=\rho-\delta-\kappa$ and $\sigma D_Y = (D_\eta, -D_y)$.

Corollary 2.1. *Let $p(x,\xi)\in S^{(s)}_{\rho,\delta}(w)$ and w be $S_{\rho,\delta}$ admissible. Then for any $N\in\mathbb{N}$ one can find $l,m\in\mathbb{N}$ such that*

$$e^{\phi}\#p\#e^{-\phi} = \sum_{k=0}^{m}\frac{(-1)^k}{k!}\sum_{\substack{\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha\\ |\alpha^0|\le l-1, 1\le|\alpha^j|\le l, 1\le j\le k}}\frac{1}{\alpha^0!\alpha^1!\cdots\alpha^k!}(\sigma D_Y/2)^{\alpha}$$
$$\times(\partial_X^{\alpha^0}p(X+2Y)\partial_X^{\alpha^1}\phi(X+Y)\cdots\partial_X^{\alpha^k}\phi(X+Y))\big|_{Y=0} + S_{\rho,\delta}(w\langle\xi\rangle_M^{-N}).$$

In particular, for any $N \in \mathbb{N}$ there are $l, m \in \mathbb{N}$ such that

$$e^{\phi} \# e^{-\phi} = 1 + \sum_{k=2}^{m} \frac{(-1)^k}{k!} \sum_{\substack{\alpha^1+\cdots+\alpha^k=\alpha \\ 1\le|\alpha^j|\le l, 1\le j\le k}} \frac{1}{(2i)^{|\alpha|}\alpha^1!\cdots\alpha^k!} (\sigma\partial_X)^{\alpha}$$
$$\times (\partial_X^{\alpha^1}\phi(X)\cdots\partial_X^{\alpha^k}\phi(X)) + S_{\rho,\delta}(\langle\xi\rangle_M^{-N}).$$

Proof. It suffices to note that for any α there is C_α such that $e^{-c\langle\xi\rangle_M^{\rho/s}}$, $e^{-c\langle\xi\rangle_M^{(1-\delta)/s}} \le C_\alpha \langle\xi\rangle_M^{-\rho|\alpha|}$ and $\sum_{|\alpha|=l}(\sigma\partial_X)^{\alpha}(\partial_X\phi)^{\alpha}/\alpha! = 0$ for $l \ge 1$. □

Corollary 2.2. *Let $p(x,\xi) \in S^{(s)}_{\rho,\delta}(w)$ and w be $S_{\rho,\delta}$ admissible weight. Then for any $m, N \in \mathbb{N}$ we have*

$$e^{\phi}\#p\#e^{-\phi} = \sum_{|\alpha|\le m} \frac{(-1)^{|\alpha|}}{(2i)^{|\alpha|}\alpha!}(\sigma\partial_Y)^{\alpha}\big(p(X+2Y)(\nabla_X\phi(X+Y))^{\alpha}\big)\big|_{Y=0}$$
$$+S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-(\rho-\delta)}) + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-\varepsilon-\bar{\epsilon}(m+1)}) + S_{\rho,\delta}(w\langle\xi\rangle_M^{-N})$$

where $\nabla_X\phi = (\partial_x\phi, \partial_\xi\phi)$. In particular, $e^{\phi}\#p\#e^{-\phi} = p + S_{\rho,\delta}(w\langle\xi\rangle_M^{-\varepsilon})$.

Proof. We choose $l = 1$ in Theorem 2.1. □

Assume that $s > 1$ satisfies $\bar{\epsilon} \ge 2\varepsilon/3$ which is possible if s is enough close to 1. Choosing $m = 2$ in Corollary 2.2 and compute

$$\sum_{|\alpha|\le 2} \frac{1}{(2i)^{|\alpha|}\alpha!}(\sigma\partial_Y)^{\alpha}\big(p(X+2Y)(\nabla_X\phi(X+Y))^{\alpha}\big)\big|_{Y=0}$$

explicitly. Then we see that there is $c > 0$ such that for any $N \in \mathbb{N}$

$$e^{\phi}\#p\#e^{-\phi} = p(1-E(\phi)) + i\{p,\phi\} - ((\mathrm{Hess}\,p)H_\phi, H_\phi) + ((\mathrm{Hess}\,\phi)H_p, H_\phi)$$
$$+S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-3\varepsilon}) + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-(\rho-\delta)}) + S_{\rho,\delta}(w\langle\xi\rangle_M^{-N})$$

where $\mathrm{Hess}\,p$ and H_p are the Hessian and the Hamilton vector field of p respectively and

$$E(\phi) = \frac{1}{2}\sum_{i,j=1}^{n}\Big(\frac{\partial^2\phi}{\partial x_i\partial\xi_j}\frac{\partial^2\phi}{\partial\xi_i\partial x_j} - \frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}\Big).$$

Note that one can write

$$i\{p,\phi\} - ((\mathrm{Hess}\,p)H_\phi, H_\phi) + ((\mathrm{Hess}\,\phi)H_p, H_\phi)$$
$$= -i\sigma(H_p, H_\phi) + 2\sigma(F_pH_\phi, H_\phi) - 2\sigma(F_\phi H_p, H_\phi)$$

where F_p is the fundamental matrix of p;

$$F_p = \frac{1}{2}\begin{pmatrix} \partial^2 p/\partial x\partial\xi & \partial^2 p/\partial\xi\partial\xi \\ -\partial^2 p/\partial x\partial x & \partial^2 p/\partial\xi\partial x \end{pmatrix}.$$

Theorem 2.2. *Let $a_i(x,\xi) \in S^{(s)}_{\rho,\delta}(w_i)$ and w_i be $S_{\rho,\delta}$ admissible weights. Then there exists $c > 0$ such that for any $l, m \in \mathbb{N}$ we have*

$$(a_1 e^{\phi})\#(a_2 e^{-\phi}) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{\substack{\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha \\ |\alpha^0|\leq l-1, 1\leq|\alpha^j|\leq l, 1\leq j\leq k}} \frac{1}{(2i)^{|\alpha|}\alpha^0!\alpha^1!\cdots\alpha^k!}(\sigma\partial_X)^{\alpha}$$
$$\times (a_1(X)\partial_X^{\alpha^0} a_2(X)\partial_X^{\alpha^1}\phi(X)\cdots\partial_X^{\alpha^k}\phi(X)) + S^{(s)}_{\rho,\delta}(w_1 w_2 \langle\xi\rangle_M^{-l(\rho-\delta)})$$
$$+ S^{(s)}_{\rho,\delta}(w_1 w_2 \langle\xi\rangle_M^{-\varepsilon-\bar{\epsilon}(m+1)}) + S^{(s/(1-\delta))}_{0,0}(w_1 w_2 e^{-c\langle\xi\rangle_M^{(1-\delta)/s}}).$$

Theorem 2.3. *Let $a_i(x,\xi) \in S^{(s)}_{\rho,\delta}(w_i)$ and w_i be $S_{\rho,\delta}$ admissible weights. Then there is $c > 0$ such that for any $l \in \mathbb{N}$ we have*

$$a_1\#a_2 = \sum_{|\alpha|\leq l-1} \frac{1}{(2i)^{|\alpha|}\alpha!}\{(\sigma\partial_X)^{\alpha} a_1(X)\}\partial_X^{\alpha} a_2(X)$$
$$+ S^{(s)}_{\rho,\delta}(w_1 w_2 \langle\xi\rangle_M^{-l(\rho-\delta)}) + S^{(s/(1-\delta))}_{0,0}(w_1 w_2 e^{-c\langle\xi\rangle_M^{(1-\delta)/s}}).$$

Proposition 2.1. *Let $\tilde{\kappa} > \kappa$ and $p \in S^{(\tilde{s})}_{0,0}(w e^{-c\langle\xi\rangle_M^{\tilde{\kappa}}})$ with some $c > 0$ and $S_{\rho,\delta}$ admissible weight w. Assume $1-\delta > \tilde{s}\kappa$ then for any $N \in \mathbb{N}$ we have $e^{\phi}\#p\#e^{-\phi} \in S_{\rho,\delta}(w\langle\xi\rangle_M^{-N})$.*

When $\tilde{s} = s/(1-\delta)$ we have $1-\delta > \tilde{s}\kappa$ if $(1-\delta)^2 \geq \rho - \delta$ which is always verified if $\delta = 1-\rho$.

2.1 Proof of Theorem 2.1

Let $p \in S^{(s)}_{\rho,\delta}(w)$ where w is $S_{\rho,\delta}$ admissible weight and consider

$$e^{\phi}\#p\#e^{-\phi} = \pi^{-2n}\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\phi(X+Y)-\phi(X+Z)} dY dZ.$$

In the same way as (1.11) we define

$$\underline{\chi}(y,z) = \chi(\langle y\rangle)\chi(\langle z\rangle), \quad \underline{\chi}^c = 1 - \underline{\chi}$$

and write, disregarding the factor π^{-2n}

$$\begin{aligned}
&\int e^{-2i\sigma(Y,Z)}p(X+Y+Z)e^{\phi(X+Y)-\phi(X+Z)}\underline{\chi}\bar{\chi}dYdZ\\
(2.4)\qquad &+\int e^{-2i\sigma(Y,Z)}p(X+Y+Z)e^{\phi(X+Y)-\phi(X+Z)}\underline{\chi}^c\bar{\chi}dYdZ\\
&+\int e^{-2i\sigma(Y,Z)}p(X+Y+Z)e^{\phi(X+Y)-\phi(X+Z)}\bar{\chi}^c dYdZ.
\end{aligned}$$

After the change of variables : $Z\to Z+Y$, the first integral turns to be

$$(2.5)\qquad \int e^{-2i\sigma(Y,Z)}p(X+2Y+Z)e^{\phi(X+Y)-\phi(X+Y+Z)}\varphi(X,Y,Z)dYdZ$$

where $\varphi(X,Y,Z)=\underline{\chi}(y,y+z)\bar{\chi}(\eta,\eta+\zeta)$. Since

$$(2.6)\quad \bar{\chi}(\eta,\eta+\zeta)\neq 0\Longrightarrow \langle\xi+\eta\rangle_M\approx\langle\xi\rangle_M,\ \langle\xi+\eta+\theta\zeta\rangle_M\approx\langle\xi\rangle_M,\ \ |\theta|\leq 1$$

it is clear that

$$(2.7)\qquad |\partial^\beta_{x,y,z}\partial^\alpha_{\xi,\eta,\zeta}\varphi(X,Y,Z)|\leq CA^{|\alpha+\beta|}|\alpha+\beta|!^s\langle\xi\rangle_M^{-|\alpha|}.$$

Let us denote

$$\psi(X,Y,Z)=\phi(X+Y)-\phi(X+Y+Z).$$

To simplify notation we denote

$$(2.8)\qquad \epsilon(\alpha)=\delta|\alpha_x|-\rho|\alpha_\xi|,\quad \sigma\alpha=(\alpha_\xi,-\alpha_x),\quad \alpha=(\alpha_x,\alpha_\xi)\in\mathbb{N}^{2n}$$

so that $\epsilon(\alpha)+\epsilon(\sigma\alpha)=-(\rho-\delta)|\alpha|$.

Lemma 2.1. *On the support of $\bar{\chi}(\eta,\eta+\zeta)$ one has*

$$\big|\partial^\alpha_{X,Y}\psi(X,Y,Z)\big|\leq CA^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{\epsilon(\alpha)}\langle\xi\rangle_M^\kappa g_X^{1/2}(Z),$$
$$|\partial^\alpha_{X,Y}e^{\psi(X,Y,Z)}|\leq CA^{|\alpha|}\langle\xi\rangle_M^{\epsilon(\alpha)}\big(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+|\alpha|^s\big)^{|\alpha|}e^{|\psi(X,Y,Z)|}.$$

Proof. Write

$$\partial^\alpha_{X,Y}\psi(X,Y,Z)=Z\cdot\int_0^1\nabla_X\partial^\alpha_{X,Y}\phi(X+Y+\theta Z)d\theta$$

which together with (2.6) proves

$$\begin{aligned}
\big|\partial^\alpha_{X,Y}\psi(X,Y,Z)\big|&\leq CA^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{\epsilon(\alpha)}\langle\xi\rangle_M^\kappa(|z|\langle\xi\rangle_M^\delta+|\zeta|\langle\xi\rangle_M^{-\rho})\\
&\leq CA^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{\epsilon(\alpha)}\langle\xi\rangle_M^\kappa g_X^{1/2}(Z),\quad \bar{\chi}(\eta,\eta+\zeta)\neq 0.
\end{aligned}$$

Applying Corollary 1.1 with $m=\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)$ we conclude the assertion. □

By the Taylor formula, one can write

$$\psi = -\sum_{1\le|\alpha|\le l}\frac{1}{\alpha!}\partial_X^\alpha\phi(X+Y)Z^\alpha + \sum_{|\mu|=l+1}\tilde r_{l\mu}(X,Y,Z)Z^\mu,$$
$$\tilde r_{l\mu}(X,Y,Z) = \frac{l+1}{\mu!}\int_0^1(1-\theta)^l\partial_X^\mu\phi(X+Y+\theta Z)d\theta \tag{2.9}$$

where one has

$$\left|\partial^\beta_{X,Y,Z}\tilde r_{l\mu}\right| \le C_l A_l^{|\beta|}|\beta|!^s\langle\xi\rangle_M^{\kappa+\epsilon(\mu)+\epsilon(\beta)},\quad \bar\chi(\eta,\eta+\zeta)\neq 0. \tag{2.10}$$

Write

$$e^\psi = \sum_{k=0}^m\frac{\psi^k}{k!} + \frac{\psi^{m+1}}{m!}\int_0^1(1-\theta)^m e^{\theta\psi}d\theta = \sum_{k=0}^m\frac{\psi^k}{k!} + R_m. \tag{2.11}$$

Corollary 2.3. *On the support of* $\bar\chi(\eta,\eta+\zeta)$ *we have*

$$|(\langle\xi\rangle_M^{-\rho}\partial_z)^\gamma(\langle\xi\rangle_M^\delta\partial_\zeta)^\beta\partial^\alpha_{X,Y}R_m| \le C_{\beta,\gamma}A^{|\alpha|}_{\beta,\gamma}\langle\xi\rangle_M^{\epsilon(\alpha)}$$
$$\times\sum_{j=0}^{m+1}\{\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)\}^{m+1-j}\langle\xi\rangle_M^{-\varepsilon j}(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+|\alpha|^s)^{|\alpha|}e^{|\psi|}.$$

Proof. Note that one can write

$$(\langle\xi\rangle_M^{-\rho}\partial_z)^\gamma(\langle\xi\rangle_M^\delta\partial_\zeta)^\beta\big(\psi^{m+1}e^{\theta\psi}\big) = e^{\theta\psi}\sum_{j=0}^{m+1}\psi^{m+1-j}q_j^{(\beta,\gamma)}$$

where $q_j^{(\beta,\gamma)}(X,Y,Z)$ satisfies

$$|\partial^\alpha_{X,Y}q_j^{(\beta,\gamma)}| \le CA^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{-\varepsilon j+\epsilon(\alpha)},\quad \bar\chi(\eta,\eta+\zeta)\neq 0.$$

Then the assertion follows from Lemma 2.1. □

Lemma 2.2. *One can write*

$$\psi^k = (-1)^k\Big(\sum_{1\le|\alpha|\le l}\frac{1}{\alpha!}\partial_X^\alpha\phi(X+Y)Z^\alpha\Big)^k + r^\psi_{lk}(X,Y,Z)$$

where $r^\psi_{l0}=0$. *In particular when* $l=1$ *we have*

$$\frac{\psi^k}{k!} = \sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}(\nabla_X\phi(X+Y))^\alpha Z^\alpha + r^\psi_{1k}(X,Y,Z)$$

where $r^{\psi}_{lk}=\sum_{l+k\le|\mu|\le k(l+1)} r_{lk\mu}(X,Y,Z)Z^{\mu}$ *and* $r_{lk\mu}$ *satisfy*

$$|\partial^{\alpha}_{X,Y,Z}r_{lk\mu}|\le C_{lk}A_{lk}^{|\alpha|}|\alpha|!^{s}\langle\xi\rangle_M^{k\kappa+\epsilon(\mu)+\epsilon(\alpha)},\quad \bar\chi(\eta,\eta+\zeta)\ne 0.$$

Proof. Since $|\partial^{\alpha}_{X,Y}(\partial^{\mu}_{X}\phi(X+Y))|\le CA^{|\alpha|}|\alpha|!^{s}\langle\xi\rangle_M^{\kappa+\epsilon(\mu)+\epsilon(\alpha)}$ if $\bar\chi(\eta,\eta+\zeta)\ne 0$ the assertion follows from (2.9) and (2.10). □

Lemma 2.3. *If* w *is* $S_{\rho,\delta}$ *admissible there are* $C>0, N>0$ *such that*

$$w(X+2Y+Z)\le Cw(X)(1+g_X(Y))^N(1+g_X(Z))^N,\quad \bar\chi(\eta,\eta+\zeta)\ne 0.$$

Proof. Note that $g_{X+Y}\approx g_X$ and $g_{X+2Y+Z}\approx g_X$ if $\bar\chi(\eta,\eta+\zeta)\ne 0$. From definition one has

$$\begin{gathered}w(X+2Y+Z)\le Cw(X+Y)\\ \times(1+\max\{g_{X+Y}(Y+Z),g_{X+2Y+Z}(Y+Z)\})^{N_1}\\ \le C_1w(X+Y)(1+g_X(Y+Z))^{N_1}\le C_2w(X)(1+g_X(Y))^{N_2}\\ \times(1+g_X(Y+Z))^{N_1}.\end{gathered}$$

Since $g_X(Y+Z)\le 2(g_X(Y)+g_X(Z))$ the proof is complete. □

Denote $q(X,Y,Z)=p(X+2Y+Z)\varphi$ and write

$$\begin{aligned}q(X,Y,Z)&=\sum_{|\alpha|\le l-1}\frac{1}{\alpha!}\partial^{\alpha}_{Z}q(X,Y,0)Z^{\alpha}+\sum_{|\mu|=l}r^{q}_{l\mu}(X,Y,Z)Z^{\mu},\\ r^{q}_{l\mu}&=\frac{l}{\mu!}\int_0^1(1-\theta)^{l-1}(\partial^{\mu}_{Z}q)(X,Y,\theta Z)d\theta.\end{aligned}\tag{2.12}$$

Lemma 2.4. *There is* N *such that*

$$|\partial^{\alpha}_{X,Y,Z}r^{q}_{l\mu}|\le CA^{|\alpha|}|\alpha|!^{s}w(X)(1+g_X(Y)+g_X(Z))^N\langle\xi\rangle_M^{\epsilon(\alpha)+\epsilon(\mu)}$$

when $\bar\chi(\eta,\eta+\zeta)\ne 0$.

Proof. In view of Lemma 2.3 it suffices to note (2.7) and $\langle\xi+2\eta+\theta\zeta\rangle\approx\langle\xi\rangle_M$ if $\bar\chi(\eta,\eta+\zeta)\ne 0$. □

Recalling that

$$\begin{gathered}e^{\psi}=\sum_{k=0}^{m}\frac{(-1)^k}{k!}\Big(\sum_{1\le|\alpha|\le l}\frac{1}{\alpha!}\partial^{\alpha}_{X}\phi(X+Y)Z^{\alpha}\Big)^k+\sum_{k=1}^{m}\frac{1}{k!}r^{\psi}_{lk}+R_m,\\ q=\sum_{|\alpha|\le l-1}\frac{1}{\alpha!}\partial^{\alpha}_{Z}q(X,Y,0)Z^{\alpha}+\sum_{|\mu|=l}r^{q}_{l\mu}Z^{\mu}\end{gathered}$$

we first consider

$$
\int e^{-2i\sigma(Y,Z)}q(X,Y,Z)R_m(X,Y,Z)dYdZ, \tag{2.13}
$$
$$
\int e^{-2i\sigma(Y,Z)}e^{\psi}r^q_{l\mu}(X,Y,Z)Z^{\mu}dYdZ
$$

where R_m and $r^q_{l\mu}$ are given in (2.11) and (2.12). Introduce the following differential operators and symbols

$$
\begin{cases} L = 1+4^{-1}\langle\xi\rangle_M^{2\rho}|D_\eta|^2+4^{-1}\langle\xi\rangle_M^{-2\delta}|D_y|^2 = 1+g_X^\sigma(\sigma D_Y)/4, \\ M = 1+4^{-1}\langle\xi\rangle_M^{2\delta}|D_\zeta|^2+4^{-1}\langle\xi\rangle_M^{-2\rho}|D_z|^2 = 1+g_X(\sigma D_Z)/4, \\ \Phi = 1+\langle\xi\rangle_M^{2\rho}|z|^2+\langle\xi\rangle_M^{-2\delta}|\zeta|^2 = 1+g_X^\sigma(Z), \\ \Psi = 1+\langle\xi\rangle_M^{2\delta}|y|^2+\langle\xi\rangle_M^{-2\rho}|\eta|^2 = 1+g_X(Y) \end{cases}
$$

so that $\Phi^{-N}L^Ne^{-2i\sigma(Y,Z)} = e^{-2i\sigma(Y,Z)}$ and $\Psi^{-\ell}M^\ell e^{-2i\sigma(Y,Z)} = e^{-2i\sigma(Y,Z)}$. Using these relations we make integration by parts in (2.5). Let $F = q(X,Y,Z)R_m(X,Y,Z)$ and consider

$$
\int e^{-2i\sigma(Y,Z)}\Phi^{-N}L^N\Psi^{-\ell}M^\ell(\partial_X^\alpha F)dYdZ. \tag{2.14}
$$

Here note that

$$
|(\langle\xi\rangle_M^{-\delta}\partial_y)^\beta(\langle\xi\rangle_M^{\rho}\partial_\eta)^\alpha\Psi^{-\ell}| \le C_\ell A_\ell^{|\alpha+\beta|}|\alpha+\beta|!\Psi^{-\ell}, \quad \alpha,\beta\in\mathbb{N}^n,
$$
$$
|\partial^\alpha_{X,Y,Z}q(X,Y,Z)| \le CA^{|\alpha|}|\alpha|!^s w(X+2Y+Z)\langle\xi\rangle_M^{\epsilon(\alpha)}.
$$

Applying Corollary 2.3 and Lemma 2.3 one can estimate the integrand of the right-hand side of (2.14) such as

$$
\begin{aligned} |\Phi^{-N}L^N\Psi^{-\ell}M^\ell(\partial_X^\alpha F)| &\le C_\ell A_\ell^{2N+|\alpha|}\langle\xi\rangle_M^{\epsilon(\alpha)}\Phi^{-N}\Psi^{-\ell} \\ \times\sum_{j=0}^{m+1}\{\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)\}^{m+1-j}\langle\xi\rangle_M^{-\varepsilon j}&\big(\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)+(2N+|\alpha|)^s\big)^{2N+|\alpha|} \\ &\times w(X)(1+g_X(Y))^{N_1}(1+g_X(Z))^{N_1}e^{c\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)}. \end{aligned} \tag{2.15}
$$

The right-hand side of (2.15) can be bounded by

$$
\begin{aligned} CA^{2N+|\alpha|}\Phi^{-N}\Psi^{-\ell+N_1}\langle\xi\rangle_M^{\epsilon(\alpha)}\sum_{j=0}^{m+1}\{\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)\}^{m+1-j}\langle\xi\rangle_M^{-\varepsilon j} \\ \times(\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)+N^s)^{2N}(\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)+|\alpha|^s)^{|\alpha|} \\ \times w(X)(1+g_X(Z))^{N_1}e^{c\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)}. \end{aligned}
$$

Writing

$$A^{2N}\Phi^{-N}(\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)+N^s)^{2N}=\Big(\frac{A\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)}{\Phi^{1/2}}+\frac{AN^s}{\Phi^{1/2}}\Big)^{2N}$$

we choose the maximal $N=N(Z,\xi)\in\mathbb{N}$ such that $AN^s\le\bar{c}\,\Phi^{1/2}$ with a suitably chosen $\bar{c}>0$. Then noting that $\Phi^{1/2}=\langle\xi\rangle_M^{\rho-\delta}g_X^{1/2}(Z)$ and hence $\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)\Phi^{-1/2}=\langle\xi\rangle_M^{-\varepsilon}\le M^{-\varepsilon}$ we have

$$\text{(2.16)}\quad \Big(\frac{A\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)}{\Phi^{1/2}}+\frac{AN^s}{\Phi^{1/2}}\Big)^{2N}\le Ce^{-c_1\Phi^{1/2s}}=Ce^{-c_1\langle\xi\rangle_M^{(\rho-\delta)/s}g_X^{1/2s}(Z)}$$

choosing $\bar{c}$ small and $M\ge M_0$ large. Since $|z|\le C$ and $|\eta|\le C\langle\xi\rangle_M$ on the support of φ one sees

$$\begin{aligned}\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)&=\langle\xi\rangle_M^{\kappa-(\rho-\delta)/s}g_X^{(s-1)/2s}(Z)\big(\langle\xi\rangle_M^{(\rho-\delta)/s}g_X^{1/2s}(Z)\big)\\ \le C\langle\xi\rangle_M^{\kappa-(\rho-\delta)/s}&\langle\xi\rangle_M^{\max\{\delta,1-\rho\}(s-1)/s}\big(\langle\xi\rangle_M^{(\rho-\delta)/s}g_X^{1/2s}(Z)\big)\le\langle\xi\rangle_M^{-\bar{\epsilon}}\Phi^{1/2s}.\end{aligned}$$

Noting $(\Phi^{1/2s})^{|\alpha|}\le\epsilon^{-|\alpha|}|\alpha|!e^{\epsilon\Phi^{1/2s}}$ for any $\epsilon>0$ it follows that (recall $\varepsilon\ge\bar{\epsilon}$)

$$\begin{aligned}\langle\xi\rangle_M^{-\varepsilon j}\{\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)\}^{m+1-j}(1+g_X(Z))^{N_1}(\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)+|\alpha|^s)^{|\alpha|}\\ \times e^{c\langle\xi\rangle_M^{\kappa}g_X^{1/2}(Z)}e^{-c_1\,\Phi^{1/2s}}\le C_{m,\ell}A_{m,\ell}^{|\alpha|}|\alpha|^{s|\alpha|}\langle\xi\rangle_M^{-(m+1)\bar{\epsilon}}e^{-c'\Phi^{1/2s}}.\end{aligned}$$

Therefore choosing ℓ such that $\ell-N_1>n/2$ we have

(2.17)

$$|\Phi^{-N}L^N\Psi^{-\ell}M^{\ell}(\partial_X^{\alpha}F)|\le CA^{|\alpha|}|\alpha|^{s|\alpha|}\langle\xi\rangle_M^{\epsilon(\alpha)}\langle\xi\rangle_M^{-\bar{\epsilon}(m+1)}w(X)\Psi^{-\ell'}\Phi^{-\ell'}$$

where $\ell'>n/2$. Since $\int\Theta^{-\ell'}\Phi^{-\ell'}dY\,dZ=C$ we conclude

Lemma 2.5. *We have*

$$\Big|\partial_X^{\alpha}\int e^{-2i\sigma(Y,Z)}qR_m dY\,dZ\Big|\le\;CA^{|\alpha|}|\alpha|!^s\langle\xi\rangle_M^{\epsilon(\alpha)}\langle\xi\rangle_M^{-\bar{\epsilon}(m+1)}w(X).$$

Since $Z^{\alpha}e^{-2i\sigma(Y,Z)}=(-\sigma D_Y/2)^{\alpha}e^{-2i\sigma(Y,Z)}$ the second integral in (2.13) is

$$\int e^{-2i\sigma(Y,Z)}(\sigma D_Y/2)^{\mu}\{r_{l\mu}^{q}(X,Y,Z)e^{\psi}\}dY\,dZ.$$

With $F = r^q_{l\mu}(X,Y,Z)e^{\psi}$, after integration by parts, one obtains

$$\partial_X^\alpha \int e^{-2i\sigma(Y,Z)}(\sigma D_Y/2)^\mu F(X,Y,Z)dYdZ$$
$$= \int e^{-2i\sigma(Y,Z)}\Phi^{-N}L^N\Psi^{-\ell}M^\ell(\sigma D_Y/2)^\mu \partial_X^\alpha F dYdZ.$$

Since M produce no positive power of $\langle\xi\rangle_M$ because $\kappa+\delta-\rho<0$, it follows from Lemma 2.4 and Corollary 1.1 that

$$|\Phi^{-N}L^N\Psi^{-\ell}M^\ell(\sigma D_Y/2)^\mu\partial_X^\alpha F| \le C_{\ell\mu}A_{\ell\mu}^{2N+|\alpha|}\langle\xi\rangle_M^{\epsilon(\alpha)+\epsilon(\mu)+\epsilon(\sigma\mu)}$$
$$\times\Phi^{-N}\Psi^{-\ell}\big(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+(2N+|\alpha|)^s\big)^{2N+|\alpha|+|\mu|}$$
$$\times w(X)(1+g_X(Y))^{N_1}(1+g_X(Z))^{N_1}e^{c\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)}.$$

Since $\epsilon(\mu)+\epsilon(\sigma\mu) = -(\rho-\delta)|\mu|$ and $|\mu| = l$ the right-hand is bounded by

$$CA^{2N+|\alpha|}\Phi^{-N}\Psi^{-\ell+N_1}\langle\xi\rangle_M^{\epsilon(\alpha)-l(\rho-\delta)}(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+|\alpha|^s)^{|\alpha|}$$
$$\times(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+N^s)^{2N}(1+\langle\xi\rangle_M^\kappa g_X^{1/2}(Z))^l$$
$$\times w(X)(1+g_X(Z))^{N_1}e^{c\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)}.$$

Repeating the same arguments as before one has (2.16) and hence

$$(1+\langle\xi\rangle_M^\kappa g_X^{1/2}(Z))^l(1+g_X(Z))^{N_1}(\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)+|\alpha|^s)^{|\alpha|}$$
$$\times e^{c\langle\xi\rangle_M^\kappa g_X^{1/2}(Z)}e^{-c_1\Phi^{1/2s}} \le CA^{|\alpha|}|\alpha|^{s|\alpha|}e^{-c'\Phi^{1/2s}}$$

with $c'>0$. Therefore we have

Lemma 2.6. *For $|\mu| = l$ we have*

$$\int e^{-2i\sigma(Y,Z)}e^\psi r^q_{l\mu}(X,Y,Z)Z^\mu dYdZ \in S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M^{-l(\rho-\delta)}w).$$

We consider

$$J = \sum_{k=1}^m \int e^{-2i\sigma(Y,Z)}q(X,Y,Z)r^\psi_{kl}dYdZ$$
$$= \sum_{k=1}^m \sum_{l+k\le|\mu|\le k(l+1)} \int e^{-2i\sigma(Y,Z)}q(X,Y,Z)r_{lk\mu}(X,Y,Z)Z^\mu dYdZ.$$

Denoting $R_{lk\mu} = q(X,Y,Z)r_{lk\mu}$ the right-hand is

$$\sum_{k=1}^{m} \sum_{l+k \le |\mu| \le (l+1)k} \int e^{-2i\sigma(Y,Z)} (\sigma D_Y/2)^{\mu} R_{lk\mu}(X,Y,Z) dY dZ.$$

Consider

$$\begin{aligned} &\partial_X^{\alpha} \int e^{-2i\sigma(Y,Z)} (\sigma D_Y/2)^{\mu} R_{lk\mu}(X,Y,Z) dY dZ \\ &= \int e^{-2i\sigma(Y,Z)} \Phi^{-\ell} L^{\ell} \Psi^{-\ell} M^{\ell} (\sigma D_Y/2)^{\mu} \partial_X^{\alpha} R_{lk\mu} dY dZ. \end{aligned} \tag{2.18}$$

It follows from Lemmas 2.2 and 2.3 that

$$|\partial_{X,Y,Z}^{\alpha} R_{lk\mu}| \le C A^{|\alpha|} |\alpha|!^s \omega(X) (1+g_X(Y))^{N_1} (1+g_X(Z))^{N_1} \langle \xi \rangle_M^{\epsilon(\alpha)+k\kappa+\epsilon(\mu)}.$$

and hence the integrand is bounded as

$$\begin{aligned} |\Phi^{-\ell} L^{\ell} \Psi^{-\ell} M^{\ell} (\sigma D_Y/2)^{\mu} \partial_X^{\alpha} R_{lk\mu}| \le C A^{|\alpha|} w(X) |\alpha|!^s \\ \times \langle \xi \rangle_M^{\epsilon(\alpha)+k\kappa+\epsilon(\mu)+\epsilon(\sigma\mu)} \Phi^{-\ell+N_1} \Psi^{-\ell+N_1}. \end{aligned}$$

Since $\epsilon(\mu) + \epsilon(\sigma\mu) = -(\rho-\delta)|\mu|$ and $|\mu| \ge k+l$ which is bounded by

$$C A^{|\alpha|} |\alpha|!^s \langle \xi \rangle_M^{\epsilon(\alpha)-\varepsilon k - l(\rho-\delta)} \Phi^{-\ell+N_1} \Psi^{-\ell+N_1}.$$

Choosing ℓ such that $\ell - N_1 > n/2$ we conclude that

$$J \in S_{\rho,\delta}^{(s)}(w \langle \xi \rangle_M^{-\varepsilon - l(\rho-\delta)}). \tag{2.19}$$

It remains to consider

$$\begin{aligned} \sum_{|\alpha| \le l-1} \frac{1}{\alpha!} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \int e^{-2i\sigma(Y,Z)} \partial_Z^{\alpha} q(X,Y,0) Z^{\alpha} \\ \times \Big(\sum_{1 \le |\beta| \le l} \frac{1}{\beta!} \partial_X^{\beta} \phi(X+Y) Z^{\beta} \Big)^k dY dZ \end{aligned}$$

which is

$$\begin{aligned} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \int e^{-2i\sigma(Y,Z)} \Big(\tilde{\sum} \frac{1}{\alpha^0! \alpha^1! \cdots \alpha^k!} \partial_Z^{\alpha^0} q(X,Y,0) \partial_X^{\alpha^1} \phi(X+Y) \\ \cdots \partial_X^{\alpha^k} \phi(X+Y) \Big) Z^{\alpha} dY dZ \end{aligned}$$

where the sum $\tilde{\sum}$ is taken over all $\alpha^0 + \alpha^1 + \cdots + \alpha^k = \alpha, |\alpha^0| \le l-1, 1 \le |\alpha^j| \le l,\ 1 \le j \le k$. Recalling $Z^\alpha e^{-2i\sigma(Y,Z)} = (-\sigma D_Y/2)^\alpha e^{-2i\sigma(Y,Z)}$ and noting that

$$\int e^{-2i\sigma(Y,Z)} dZ = \pi^{2n}\delta(Y)$$

and $\varphi(X,0,0)=1$, $\partial^\alpha_{Y,Z}\varphi(X,Y,Z)|_{Y=Z=0} = 0$ for $|\alpha| \ge 1$ it yields

$$\sum_{k=0}^m \frac{(-1)^k}{k!}(\sigma D_Y/2)^\alpha \big(\tilde{\sum} \frac{1}{\alpha^0!\alpha^1!\cdots\alpha^k!}\partial_X^{\alpha^0} p(X+2Y)\partial_X^{\alpha^1}\phi(X+Y) \cdots \partial_X^{\alpha^k}\phi(X+Y)\big)\big|_{Y=0}.$$

Combining Lemmas 2.5, 2.6 and (2.19) we have

Lemma 2.7. *The following holds.*

$$\begin{aligned}
&\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\phi(X+Y)-\phi(X+Z)} \underline{\chi}\bar{\chi} dY dZ \\
&= \pi^{2n}\sum_{k=0}^m \frac{(-1)^k}{k!}\Big(\sum_{\substack{\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha \\ |\alpha^0|\le l-1, 1\le |\alpha^j| \le l, 1\le j\le k}} \frac{1}{\alpha^0!\alpha^1!\cdots\alpha^k!}(\sigma D_Y/2)^\alpha \\
&\quad \times (\partial_X^{\alpha^0} p(X+2Y)\partial_X^{\alpha^1}\phi(X+Y)\cdots\partial_X^{\alpha^k}\phi(X+Y))\big|_{Y=0} \\
&\quad + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-l(\rho-\delta)}) + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-\varepsilon-\bar{\epsilon}(m+1)}).
\end{aligned}$$

Choosing $l=1$ we have

Corollary 2.4. *The following holds.*

$$\begin{aligned}
&\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\phi(X+Y)-\phi(X+Z)} \underline{\chi}\bar{\chi} dY dZ \\
&= \pi^{2n}\sum_{|\alpha|\le m} \frac{(-1)^{|\alpha|}}{\alpha!}(\sigma D_Y/2)^\alpha \big(p(X+2Y)(\nabla_X\phi(X+Y))^\alpha\big)\big|_{Y=0} \\
&\quad + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-(\rho-\delta)}) + S^{(s)}_{\rho,\delta}(w\langle\xi\rangle_M^{-\varepsilon-\bar{\epsilon}(m+1)}).
\end{aligned}$$

Turn to the second term of (2.4). After integration by parts we have

$$\int e^{-2i\sigma(Y,Z)}(|y|^2+|z|^2)^{-N}(|D_\zeta|^2+|D_\eta|^2)^N F dY dZ$$

where $F = p(X+Y+Z)e^{\psi}\tilde{\varphi}(X,Y,Z)$ and $\tilde{\varphi} = \underline{\chi}^c(y,z)\bar{\chi}(\eta,\zeta)$. Since $|\psi| \le C\langle\xi\rangle_M^{\kappa}$ and $\langle\xi+\eta\rangle_M \approx \langle\xi\rangle_M$, $\langle\xi+\zeta\rangle_M \approx \langle\xi\rangle_M$, $\langle\xi+\eta+\zeta\rangle_M \approx \langle\xi\rangle_M$ if $\tilde{\varphi} \ne 0$, thanks to Corollary 1.1 it is not difficult to show

$$\begin{aligned}\big|(|D_\zeta|^2+|D_\eta|^2)^N(|y|^2+|z|^2)^{-N}\partial_X^{\alpha}F\big| \le CA^{2N+|\alpha|}w(X+Y+Z)\langle\xi\rangle_M^{\epsilon(\alpha)}\\ \times(\langle\xi\rangle_M^{\kappa}+|\alpha|^s)^{|\alpha|}\langle\xi\rangle_M^{-2\rho N}(\langle\xi\rangle_M^{\kappa}+N^s)^{2N}(|y|^2+|z|^2)^{-N}e^{c\langle\xi\rangle_M^{\kappa}}.\end{aligned}$$

Choose the maximal $N \in \mathbb{N}$ such that $AN^s \le c_1\langle\xi\rangle_M^{\rho}$ with small $c_1 > 0$ and repeating similar arguments as before one obtains (recall $\rho > \kappa$)

$$A^{2N}\langle\xi\rangle_M^{-2\rho N}\big(\langle\xi\rangle_M^{\kappa}+N^s\big)^{2N} \le Ce^{-c\langle\xi\rangle_M^{\rho/s}}.$$

Since $g_X \approx g_{X+Y+Z}$ if $\bar{\chi}(\eta,\zeta) \ne 0$ a repetition of the proof of Lemma 2.3 shows $w(X+Y+Z) \le Cw(X)(1+g_X(Y)+g_X(Z))^{N_2}$. Note that

$$\begin{aligned}g_X(Y)+g_X(Z) &\le \langle\xi\rangle_M^{2\delta}(|y|^2+|z|^2)+2\langle\xi\rangle_M^{2(1-\rho)}\\ &\le C\langle\xi\rangle_M^{2\max\{\delta,1-\rho\}}(1+|y|^2+|z|^2)\end{aligned}$$

and for any $\epsilon > 0$ there are $C, A > 0$ such that (recall $\kappa < \rho$)

$$\langle\xi\rangle_M^{\kappa|\alpha|} \le CA^{|\alpha|}|\alpha|^{s|\alpha|}e^{\epsilon\langle\xi\rangle_M^{\rho/s}}.$$

Since $\langle\xi\rangle_M^{-2(n+1)}\int(|y|^2+|z|^2)^{-n-1}\tilde{\varphi}dYdZ \le C$ and $\rho/s > \kappa$ we obtain

Lemma 2.8. *There is $c > 0$ such that*

$$\int e^{-2i\sigma(Y,Z)}p(X+Y+Z)e^{\phi(X+Y)-\phi(X+Z)}\underline{\chi}^c\bar{\chi}dYdZ \in S_{\rho,\delta}^{(s)}(we^{-c\langle\xi\rangle_M^{\rho/s}})$$

hence clearly belongs to $S_{\rho,\delta}^{(s)}(w\langle\xi\rangle_M^{-\varepsilon-\bar{\epsilon}(m+1)})$.

To estimate the third term of (2.4) it suffices to repeat the same arguments that estimate (1.13). Write $\bar{\chi}^c(\eta,\zeta)$ as (1.12) and study (1.13). Since $\langle\xi+\eta+\zeta\rangle_M \le C\langle\eta\rangle$ and $\langle\xi+\eta\rangle_M \le C\langle\eta\rangle$, $\langle\xi+\zeta\rangle_M \le C\langle\eta\rangle$ if $\varphi_1\chi_* \ne 0$ it is not difficult to see that (1.14) with $F = p(X+Y+Z)e^{\psi}$, $\psi = \phi(X+Y)-\phi(X+Z)$ is bounded by

$$\begin{aligned}CA^{2N+|\alpha|}\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}w(X+Y+Z)\langle\eta\rangle^{2\delta\ell}\langle\eta\rangle^{6\ell\kappa}\\ \times(\langle\eta\rangle^{\kappa}+N^s)^{2N}\langle\eta\rangle^{2\delta N}(\langle\eta\rangle^{\kappa}+|\alpha|^s)^{|\alpha|}\langle\eta\rangle^{\delta|\alpha|}e^{|\psi|}.\end{aligned} \tag{2.20}$$

Here writing

$$A^{2N}\langle\eta\rangle^{-2N+2\delta N}\big(\langle\eta\rangle^{\kappa}+N^s\big)^{2N} = \Big(\frac{A\langle\eta\rangle^{\kappa}}{\langle\eta\rangle^{1-\delta}}+\frac{AN^s}{\langle\eta\rangle^{1-\delta}}\Big)^{2N}$$

we choose the maximal $N \in \mathbb{N}$ such that $AN^s \le c_1 \langle \eta \rangle^{1-\delta}$ with small $c_1 > 0$ so that the right-hand side is bounded by $Ce^{-c\langle \eta \rangle^{(1-\delta)/s}}$ with some $c > 0$. Noting that $\langle \eta \rangle^{(1-\delta)N/s} \le \epsilon^{-N} N^N e^{\epsilon \langle \eta \rangle^{(1-\delta)/s}}$ it is clear that for any $\epsilon > 0$ there are $A > 0$, $C > 0$ such that (recall $\kappa + \delta < \rho \le 1$)

$$\langle \eta \rangle^{(\kappa+\delta)|\alpha|} \le CA^{|\alpha|} |\alpha|^{s|\alpha|/(1-\delta)} e^{\epsilon \langle \eta \rangle^{(1-\delta)/s}},$$
$$|\alpha|^{s|\alpha|} \langle \eta \rangle^{\delta|\alpha|} \le CA^{|\alpha|} |\alpha|^{s|\alpha|/(1-\delta)} e^{\epsilon \langle \eta \rangle^{(1-\delta)/s}}.$$

Consider $w(X+Y+Z)$. Since $g_Y(Y+Z) \le C(\langle \eta \rangle^{2\delta}(|y|^2 + |z|^2) + |\eta|^2) \le C\langle \eta \rangle^2 \langle y \rangle^2 \langle z \rangle^2$ and $\langle \xi \rangle_M$, $\langle \xi + \eta + \zeta \rangle_M \le C\langle \eta \rangle$ if $\varphi_1 \chi_* \neq 0$ it follows that

$$\begin{aligned} w(X+Y+Z) &\le Cw(X)(1 + \max\{g_X(Y+Z), g_{X+Y+Z}(Y+Z)\})^{N_1} \\ &\le Cw(X)\langle \eta \rangle^{2N_1} \langle y \rangle^{2N_1} \langle z \rangle^{2N_1}. \end{aligned}$$

Recalling that $1 - \delta \ge \rho - \delta > \kappa s$ one sees that (2.20) is bounded by

$$CA^{|\alpha|} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell + 2N_1} \langle z \rangle^{-2\ell + 2N_1} w(X) |\alpha|^{s|\alpha|/(1-\delta)} e^{-c\langle \eta \rangle^{(1-\delta)/s}}.$$

Noting $e^{-c\langle \eta \rangle^{(1-\delta)/s}} \le e^{-c_1 \langle \xi \rangle_M^{(1-\delta)/s}} e^{-c_2 \langle \eta \rangle^{(1-\delta)/s}}$ with some $c_i > 0$ we conclude

$$\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\psi(X,Y,Z)} \varphi_1 \chi_* dY dZ \in S_{0,0}^{(s/(1-\delta))}(we^{-c\langle \xi \rangle_M^{(1-\delta)/s}}).$$

Similarly if the case χ_*^c is chosen, choosing $N_1 = N$, $N_2 = \ell$ it is proved that (2.20) is estimated by

$$CA^{2N} \langle \eta \rangle^{-2\ell} \langle y \rangle^{-2\ell + 2N_1} \langle z \rangle^{-2\ell + 2N_1} w(X) |\alpha|^{s|\alpha|/(1-\delta)} e^{-c_1 \langle \xi \rangle_M^{(1-\delta)/s}} e^{-c_2 \langle \zeta \rangle^{(1-\delta)/s}}.$$

Thus one can find $c > 0$ such that

$$\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\psi(X,Y,Z)} \varphi_1 dY dZ \in S_{0,0}^{(s/(1-\delta))}(we^{-c\langle \xi \rangle_M^{(1-\delta)/s}}).$$

Turn to $\int e^{-2i\sigma(Y,Z)} \partial_X^\alpha (F\varphi_2) dY dZ$. Consider

$$\begin{aligned} &\int e^{-2i\sigma(Y,Z)} \langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle D_z \rangle^{2N} \langle D_y \rangle^{2\ell} \\ &\times \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\zeta \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_X^\alpha F \varphi_2 dY dZ. \end{aligned}$$

Since $\langle \xi + \eta + \zeta \rangle_M \le C\langle \eta \rangle$ and $\langle \xi + \eta \rangle_M \le C\langle \eta \rangle$, $\langle \xi + \zeta \rangle_M \approx \langle \xi \rangle_M \le C\langle \eta \rangle$ if $\varphi_2 \neq 0$ this is bounded by (2.20). The rest of the argument is the same as the case that $\varphi \chi_*$ is chosen. The case of φ_3 is similar to the case that $\varphi_1 \chi_*^c$ is chosen. We summarize what we have proved in

Lemma 2.9. *There is $c > 0$ such that*

$$\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\phi(X+Y)-\phi(X+Z)} \bar{\chi}^c dY dZ$$
$$\in S_{0,0}^{(s/(1-\delta))}(we^{-c\langle\xi\rangle_M^{(1-\delta)/s}}).$$

Combining Lemmas 2.7, 2.8 and 2.9 we end the proof of Theorem 2.1.

2.2 Proof of Theorems 2.2, 2.3 and Proposition 2.1

In view of Lemma 1.4, disregarding the factor π^{-2n} and denoting $\psi = \phi(X+Y) - \phi(X+Z)$, we write $(a_1 e^{\phi})\#(a_2 e^{-\phi})$ as

$$\int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) e^{\psi(X,Y,Z)} \underline{\chi}\bar{\chi} dY dZ$$
$$+ \int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) e^{\psi(X,Y,Z)} \underline{\chi}^c \bar{\chi} dY dZ$$
$$+ \int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) e^{\psi(X,Y,Z)} \bar{\chi}^c dY dZ.$$

To study the first term, after the change of variables : $Z \to Z+Y$ denoting $q(X,Y,Z) = a_1(X+Y)a_2(X+Y+Z)$, it suffices to repeat the proof of Theorem 2.1. Since $\partial_Z^{\alpha^0} q(X,Y,0) = a_1(X+Y)\partial_X^{\alpha^0} a_2(X+Y)$ we have

Lemma 2.10. *The following holds.*

$$\int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) e^{\psi(X,Y,Z)} \underline{\chi}\bar{\chi} dY dZ$$
$$= \pi^{2n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \Big(\sum_{\substack{\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha \\ |\alpha^0|\le l-1, 1\le|\alpha^j|\le l, 1\le j\le k}} \frac{1}{\alpha^0!\alpha^1!\cdots\alpha^k!} (\sigma D_X/2)^{\alpha}$$
$$\times \big(a_1(X)\partial_X^{\alpha^0} a_2(X) \partial_X^{\alpha^1}\phi(X)\cdots\partial_X^{\alpha^k}\phi(X)\big)$$
$$+ S_{\rho,\delta}^{(s)}(w_1 w_2 \langle\xi\rangle_M^{-l(\rho-\delta)}) + S_{\rho,\delta}^{(s)}(w_1 w_2 \langle\xi\rangle_M^{-\varepsilon-\bar{\varepsilon}(m+1)}).$$

The rest of the proof is the same as for Theorem 2.1 except for obvious modifications for estimation about $w_1(X+Y)$ and $w_2(X+Z)$. □

Turn to the proof of Theorem 2.3. Using Lemma 1.4 we write

$$a_1\#a_2 = \int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) \bar{\chi} dY dZ$$
$$+ \int e^{-2i\sigma(Y,Z)} a_1(X+Y) a_2(X+Z) \bar{\chi}^c dY dZ = J_1 + J_2.$$

Write $a_2(X+Z) = \sum_{|\alpha|\le l-1} \partial_X^\alpha a_2(X) Z^\alpha/\alpha! + \sum_{|\mu|=l} r_{l\mu}^{a_2}(X,Z)Z^\mu$ and repeat the same argument that estimates (2.18) with $R_{lk\mu} = r_{l\mu}^{a_2}(X,Z)$. Then we have

$$J_1 = \sum_{|\alpha|\le l-1} \frac{1}{\alpha!}\{(\sigma D_X/2)^\alpha a_1(X)\}\partial_X^\alpha a_2(X) + S_{\rho,\delta}^{(s)}(w_1 w_2 \langle\xi\rangle_M^{-l(\rho-\delta)})$$

where we have used

$$\sum_{|\alpha|=l} \frac{1}{\alpha!}(\sigma D_X)^\alpha \big(a_1(X)\partial_X^\alpha a_2(X)\big) = \sum_{|\alpha|=l} \frac{1}{\alpha!}\{(\sigma D_X/2)^\alpha a_1(X)\}\partial_X^\alpha a_2(X).$$

As for J_2 it is enough to repeat the same arguments that estimate the third term of (2.4).

We proceed to the proof of Proposition 2.1. Suppose that $N \in \mathbb{N}$ is given. We first note that for any $\alpha, \beta \in \mathbb{N}^n$ there is $C_{\alpha\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle_M^{-N-\rho|\alpha|} e^{-(c/2)\langle\xi\rangle_M^{\tilde\kappa}}.$$

Write

$$\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\psi(X,Y,Z)} \bar\chi \, dY dZ$$
$$+ \int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\psi(X,Y,Z)} \bar\chi^c \, dY dZ.$$

Denoting $F = p(X+Y+Z)e^{\psi(X,Y,Z)}\bar\chi$ it is easy to see

$$|\partial_{X,Y,Z}^\alpha F| \le C_\alpha \langle\xi\rangle_M^{-N+\epsilon(\alpha)} \langle\xi\rangle_M^{\kappa|\alpha|} w(X)$$
$$\times (1+g_X(Y))^{N_1}(1+g_X(Z))^{N_1} e^{-(c/2)\langle\xi\rangle_M^{\tilde\kappa} + C\langle\xi\rangle_M^\kappa}.$$

Thus the integrand of $\int e^{-2i\sigma(Y,Z)}\Phi^{-\ell}L^\ell \Psi^{-\ell}M^\ell(\partial_X^\alpha F)dYdZ$ is bounded by

$$C_{\alpha,\ell}\Phi^{-\ell+N_1}\Psi^{-\ell+N_1}\langle\xi\rangle_M^{-N+\epsilon(\alpha)} w(X) \langle\xi\rangle_M^{(4\ell+|\alpha|)\kappa} e^{-(c/2)\langle\xi\rangle_M^{\tilde\kappa} + C\langle\xi\rangle_M^\kappa}.$$

Since $\tilde\kappa > \kappa$, choosing ℓ such that $\ell - N_1 > n/2$ we conclude that the first integral belongs to $S_{\rho,\delta}(w\langle\xi\rangle_M^{-N})$. To estimate the second integral it suffices to repeat the same arguments for estimating (1.13). Without restrictions we may assume that $1-\delta > \tilde s\kappa$ and $\phi \in S_{\rho,\delta}^{(\tilde s)}(\langle\xi\rangle_M^\kappa)$. Then a repetition of the same arguments as before proves that

$$\int e^{-2i\sigma(Y,Z)} p(X+Y+Z) e^{\psi(X,Y,Z)} \bar\chi^c \, dY dZ \in S_{0,0}^{(\tilde s/(1-\delta))}(w e^{-c\langle\xi\rangle_M^{(1-\delta)/\tilde s}})$$

with some $c > 0$ which clearly belongs to $S_{\rho,\delta}(w\langle\xi\rangle_M^{-N})$ for any $N \in \mathbb{N}$.

References

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[2] T.Nishitani; *A more direct way to the Cauchy problem for effectively hyperbolic operators*, preprint, 2023.