Applications of pseudodifferential operators of symbol $\exp S^{\kappa}_{\rho,\delta}$ to the Cauchy problem

Tatsuo Nishitani*

Abstract

In this note we apply the calculus of pseudodifferential operators with symbols of type $\exp(S^{\kappa}_{\rho,\delta})$ given in [8], slightly less precise but much easier to apply than that of [7], to the Cauchy problem for non-effectively hyperbolic operators recovering the results obtained in [2, 3].

1 Preliminaries

Denote the metric defining the class $S_{\rho,\delta}$ by $g_{\rho,\delta}$

$$(g_{\rho,\delta})_X(Y) = \langle\xi\rangle_M^{2\delta}|y|^2 + \langle\xi\rangle_M^{-2\rho}|\eta|^2, \quad X = (x,\xi), Y = (y,\eta) \in \mathbb{R}^n.$$

where $\langle\xi\rangle_M = (M^2 + |\xi|^2)^{1/2}$.

Definition 1.1. A positive function $m(x,\xi)$ is called $g_{\rho,\delta}$ admissible weight if there are positive constants C, N independent of M such that with $g = g_{\rho,\delta}$

$$(1.1)\quad m(X) \le Cm(Y)\big(1 + \max\{g_X(X-Y), g_Y(X-Y)\}\big)^N, \ X, Y \in \mathbb{R}^{2n}.$$

For simplicity denote $g_{1/2,1/2}$ by $g_{1/2}$;

$$(g_{1/2})_X(Y) = \langle\xi\rangle_M|y|^2 + \langle\xi\rangle_M^{-1}|\eta|^2, \quad X = (x,\xi), Y = (y,\eta)$$

and write $S_{1/2}(m) = S_{1/2,1/2}(m)$. In what follows we assume

$$0 \le \delta \le 1/2 \le \rho \le 1 \ \ (\Longrightarrow g_{\rho,\delta} \le g_{1/2}) \ \text{ and } \ \delta < \rho.$$

Let $m > 0$ be $g_{\rho,\delta}$ admissible and $m \in S_{\rho,\delta}(m)$. Since $m^{-1} \in S_{\rho,\delta}(m)$ and m^{-1} is $g_{\rho,\delta}$ admissible we have $m\#m^{-1} = 1 - r$ with $r \in S_{\rho,\delta}(\langle\xi\rangle_M^{-2(\rho-\delta)}) \subset S_{1/2}(M^{-2(\rho-\delta)})$ hence there is $M_0 > 0$ such that $\sum_{j=1}^{\infty} r^{\#j}$ converges to $k \in S_{1/2}(1)$ satisfying $(1-r)\#(1+k) = (1+k)\#(1-r) = 1$ for $M \ge M_0$.

*Department of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka, 560-0043, Osaka, Japan

Lemma 1.1. *Assume that w_α, $\alpha \in \mathbb{N}^{2n}$ are $g_{1/2}$ admissible weights which satisfy $w_\alpha w_\beta \lesssim w_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{N}^{2n}$. Assume that $\partial_X^\alpha r \in S_{1/2}(w_\alpha)$ for $|\alpha| \le N$ then we have $\partial_X^\alpha k \in S_{1/2}(w_\alpha)$ for $|\alpha| \le N$.*

Proof. Note that k satisfies $k = r + r\#k$. Since $r\#k \in S_{1/2}(w_0)$ it is clear that $k \in S_{1/2}(w_0)$. Suppose that $\partial_X^\alpha k \in S_{1/2}(w_\alpha)$ for $|\alpha| \le l < N$. Let $|\beta| = l+1$ then we have

$$\partial_X^\beta k = \partial_X^\beta r + \sum_{\beta'+\beta''=\beta, |\beta''|\le l} C_{\beta'\beta''}(\partial_X^{\beta'} r)\#(\partial_X^{\beta''} k) + r\#\partial_X^\beta k$$

where $\sum \cdots \in S_{1/2}(\sum w_{\beta'} w_{\beta''}) \subset S_{1/2}(w_\beta)$. Thus it follows that

$$(1-r)\#(\partial_X^\beta k) \in S_{1/2}(w_\beta)$$

from which we have $\partial_X^\beta k = (1+k)\#S_{1/2}(w_\beta) \subset S_{1/2}(w_\beta)$. □

Corollary 1.1. *If m is $g_{\rho,\delta}$ admissible weight such that $m \in S_{\rho,\delta}(m)$ there exist $M_0 > 0$ and $k \in S_{\rho,\delta}(M^{-2(\rho-\delta)})$ $(M > M_0)$ such that*

$$m\#m^{-1}\#(1+k) = 1, \ \ (1+k)\#m\#m^{-1} = 1, \ \ m^{-1}\#(1+k)\#m = 1.$$

Proof. Since $r \in S_{\rho,\delta}(M^{-2(\rho-\delta)})$ hence

$$\partial_X^\alpha r \in S_{\rho,\delta}(M^{-2(\rho-\delta)}\langle\xi\rangle_M^{\epsilon(\alpha)}) \subset S_{1/2}(M^{-2(\rho-\delta)}\langle\xi\rangle_M^{\epsilon(\alpha)}), \quad \alpha \in \mathbb{N}^{2n}$$

where $\epsilon(\alpha) = \delta|\alpha_x| - \rho|\alpha_\xi|$ with $\alpha = (\alpha_x, \alpha_\xi) \in N^{2n}$. Thanks to Lemma 1.1 we have $\partial_X^\alpha k \in S_{1/2}(M^{-2(\rho-\delta)}\langle\xi\rangle_M^{\epsilon(\alpha)})$ for all $\alpha \in \mathbb{N}^{2n}$ which implies that $k \in S_{\rho,\delta}(M^{-2(\rho-\delta)})$. □

Lemma 1.2. *Let m_i $(i = 1, 2)$ be $g_{\rho,\delta}$ admissible weights such that $m_i \in S_{\rho,\delta}(m_i)$. If $a \in S_{1/2}(m_1 m_2)$ or $a \in S_{1/2}(m_1)$ there are $C > 0, M_0 > 0$ such that the followings hold for $M > M_0$*

$$\big|(\mathrm{op}(a)u, v)\big| \le C\|\mathrm{op}(m_1)u\|\|\mathrm{op}(m_2)v\|,$$
$$\|\mathrm{op}(a)u\| \le C\|\mathrm{op}(m_1)u\|.$$

Proof. Note that m_i^{-1} are $g_{\rho,\delta}$ admissible. Write

$$a = m_2\#(1+k_2)\#m_2^{-1}\#a\#m_1^{-1}\#(1+k_1)\#m_1 = m_2\#r\#m_1$$

where $r = (1+k_2)\#m_2^{-1}\#a\#m_1^{-1}\#(1+k_1) \in S_{1/2}(1)$ then the proof is clear. For the second assertion it is enough to write $a = a\#m_1^{-1}\#(1+k_1)\#m_1 = r\#m_1$ with $r = a\#(1+k_1)\#m_1^{-1} \in S_{1/2}(1)$. □

Lemma 1.3. *Let $m\in S_{1/2}(m)$ be $g_{1/2}$ admissible weight. If a is $g_{\rho,\delta}$ admissible weight satisfying $a\in S_{\rho,\delta}(a)$ and $a\geq c\,m$ with some $c>0$ then there exist $C>0, M_0>0$ such that*

$$C\|\mathrm{op}(a)u\|\geq \|\mathrm{op}(m)u\|,\quad M\geq M_0.$$

Let $m\in S_{\rho,\delta}(m)$ be $g_{\rho,\delta}$ admissible weight. If $a\in S_{\rho,\delta}(m)$ satisfies $a\geq c\,m$ with some $c>0$ then there exist $C>0, M_0>0$ such that

$$C(\mathrm{op}(a)u,u)\geq \|\mathrm{op}(\sqrt{m})u\|^2,\quad M\geq M_0.$$

Proof. Write $m=m\#a^{-1}\#(1+k)\#a$ where $m\#a^{-1}\#(1+k)\in S_{1/2}(1)$ for a^{-1} is $g_{1/2}$ admissible. This proves the first assertion. Turn to the next assertion. Since $c\,m\leq a\leq Cm$ it is clear that a is $g_{\rho,\delta}$ admissible weight hence so is $\sqrt{a}$. Since $a=\sqrt{a}\#\sqrt{a}+r$ with $r\in S_{\rho,\delta}(M^{-2(\rho-\delta)}a)$ one can write $r=a^{1/2}\#(1+k)\#a^{-1/2}\#r\#a^{-1/2}\#(1+\tilde{k})\#a^{1/2}$ where $(1+k)\#a^{-1/2}\#r\#a^{-1/2}\#(1+\tilde{k})\in S_{\rho,\delta}(M^{-2(\rho-\delta)})$. Thus

$$|(\mathrm{op}(r)u,u)|\leq CM^{-2(\rho-\delta)}\|\mathrm{op}(\sqrt{a})u\|^2.$$

Since $(\mathrm{op}(a)u,u)=\|\mathrm{op}(\sqrt{a})u\|^2+(\mathrm{op}(r)u,u)$ it follows that

$$(\mathrm{op}(a)u,u)\geq(1-CM^{-2(\rho-\delta)})\|\mathrm{op}(\sqrt{a})u\|^2\geq\|\mathrm{op}(\sqrt{m})u\|^2/C,\quad M\geq M_0$$

where the last inequality follows from the first assertion. □

2 Applications to the Cauchy problem

2.1 Some special weights

Let $\phi_1(x,\xi)\in S^{(s)}_{1,0}(1)$ and define $w(x,\xi)$ by

$$w(x,\xi)=\big(\phi_1^{2m}(x,\xi)+\langle\xi\rangle_M^{-l}\big)^{1/2l},\quad l,m\in\mathbb{N},\ \ l\leq m.$$

Let $\phi_2(x)\in G^s(\mathbb{R}^n)$ and define

$$r(x,\xi)=\sqrt{\phi_2^2(x)+w^2(x,\xi)}.$$

Introduce two more metrics. Let

$$\bar{g}_X(Y)=\varrho^{-2}|y|^2+w^{-2l/m}\langle\xi\rangle_M^{-2}|\eta|^2,\quad \varrho^{-1}=r^{-1}+w^{-l/m},$$
$$\underline{g}_X(Y)=w^{-2l/m}|y|^2+w^{-2l/m}\langle\xi\rangle_M^{-2}|\eta|^2,\quad Y=(y,\eta)\in\mathbb{R}^{2n}.$$

Lemma 2.1. *There exist $C>0, A>0$ such that*

$$|\partial_x^\beta \partial_\xi^\alpha w| \le CA^{|\alpha+\beta|}|\alpha+\beta|!^s w w^{-l|\alpha+\beta|/m}\langle\xi\rangle_M^{-|\alpha|} \tag{2.1}$$

that is $w\in S^{(s)}(w,\underline{g})$. In particular we have $w\in S^{(s)}_{\rho,\delta}(w)$ with

$$\rho = 1-l/2m, \quad \delta = l/2m \quad (\textit{hence } \rho+\delta=1). \tag{2.2}$$

Moreover w is $g_{\rho,\delta}$ admissible.

Proof. We only show that w is $g_{\rho,\delta}$ admissible. Thanks to (2.1) we have $|\partial_x^\beta\partial_\xi^\alpha w^{l/m}|\le C\langle\xi\rangle_M^{-|\alpha|}$ for $|\alpha+\beta|=1$. Then

$$|w^{l/m}(X+Y)-w^{l/m}(X)|\le C(|y|+\langle\xi+\theta\eta\rangle_M^{-1}|\eta|) \quad |\theta|<1.$$

Write $g=g_{\rho,\delta}$. If $|\eta|\le\langle\xi\rangle_M/2$ so that $\langle\xi+\theta\eta\rangle_M\approx\langle\xi\rangle_M$ the right-hand side is bounded by $C(|y|+\langle\xi\rangle_M^{-1}|\eta|)\le C\langle\xi\rangle_M^{-\delta}g_X^{1/2}(Y)\le Cw^{l/m}(X)g_X^{1/2}(Y)$. If $|\eta|\ge\langle\xi\rangle_M/2$ then $g_X(Y)\ge\langle\xi\rangle_M^{2\delta}/4$. Therefore $w^{l/m}(X+Y)\le C\le C'\langle\xi\rangle_M^{-\delta}g_X^{1/2}(Y)\le C'w^{l/m}(X)g_X^{1/2}(Y)$ hence $w^{l/m}$ is $g_{\rho,\delta}$ admissible and so is $w=(w^{l/m})^{m/l}$. □

Since $w^{-l/m}\le\langle\xi\rangle_M^{l/2m}=\langle\xi\rangle_M^{\delta}$ and $w^{-l/m}\langle\xi\rangle_M^{-1}\le\langle\xi\rangle_M^{\delta-1}=\langle\xi\rangle_M^{-\rho}$ and $w^{-l/m}\le\varrho^{-1}\lesssim w^{-1}\lesssim\langle\xi\rangle_M^{1/2}$ it is clear that

$$\underline{g}\le\bar{g}, \quad \underline{g}\le g_{\rho,\delta}, \quad \bar{g}\le g_{\rho,1/2}.$$

Lemma 2.2. *We have*

$$|\partial_x^\beta\partial_\xi^\alpha r|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^s r\varrho^{-|\beta|}w^{-l|\alpha|/m}\langle\xi\rangle_M^{-|\alpha|}$$

that is $r\in S^{(s)}(r,\bar{g})$, hence $r\in S^{(s)}_{\rho,1/2}(r)$. Moreover r is $g_{\rho,1/2}$ admissible.

Proof. We only show that r is $g_{\rho,1/2}$ admissible. It suffices to show that $r^2=\phi_2^2(x)+w^2$ is $g_{\rho,1/2}$ admissible. Since w^2 is $g_{\rho,\delta}$ admissible by Lemma 2.1 hence $g_{\rho,1/2}$ admissible because $g_{\rho,\delta}\le g_{\rho,1/2}$. With $g=g_{\rho,1/2}$ note that $|\phi_2(X+Y)-\phi_2(X)|\le C|y|\le C\langle\xi\rangle_M^{-1/2}g_X^{1/2}(Y)\le Cw(X)g_X^{1/2}(Y)$ thus

$$\phi_2^2(X+Y)\le C(\phi_2^2(X)+w^2(X))(1+g_X(Y))\le Cr^2(X)(1+g_X(Y))$$

from which we conclude the assertion. □

Let us define

$$\phi(x,\xi) = i\big\{\log(\phi_2(x) - iw(x,\xi)) - \log(\phi_2(x) + iw(x,\xi))\big\}$$
$$= 2\arg(\phi_2(x) + iw(x,\xi)).$$

Lemma 2.3. *We have* $\phi \in S^{(s)}(\phi,\bar g)$ *hence* $\phi \in S^{(s)}_{\rho,1/2}(\phi)$ *and* ϕ *is* $g_{\rho,1/2}$ *admissible. In particular* $\partial_x^\beta\partial_\xi^\alpha\phi \in S^{(s)}(wr^{-1}\varrho^{-|\beta|}w^{-l|\alpha|/m},\bar g)$ *for* $|\alpha+\beta| = 1$.

Proof. For $|\alpha+\beta| = 1$ one has

$$\partial_x^\beta\partial_\xi^\alpha\phi = -2r^{-2}(x,\xi)[w(x,\xi)\partial_x^\beta\partial_\xi^\alpha\phi_2(x) - \phi_2(x)\partial_x^\beta\partial_\xi^\alpha w(x,\xi)] \tag{2.3}$$

where $\phi_2(x)\partial_x^\beta\partial_\xi^\alpha w \in S^{(s)}(rw^{1-l|\alpha+\beta|/m}\langle\xi\rangle_M^{-|\alpha|},\bar g)$ in view of Lemma 2.1, thus the last assertion is clear from Lemma 2.2. Since there is $c > 0$ such that

$$\phi = 2\arg(\phi_2 + iw) = 2\arctan\frac{w}{r} \geq c\frac{w}{r} \tag{2.4}$$

thanks to Lemmas 2.1 and 2.2 it follows that

$$w/r^2 \in S^{(s)}(w/r^2,\bar g) \subset S^{(s)}(r^{-1}\phi,\bar g),$$
$$\phi_2\partial_x^\beta\partial_\xi^\alpha w/r^2 \in S^{(s)}(w^{1-l|\alpha+\beta|/m}\langle\xi\rangle_M^{-|\alpha|}/r,\bar g) \subset S^{(s)}(w^{-l|\alpha+\beta|/m}\langle\xi\rangle_M^{-|\alpha|}\phi,\bar g)$$

which together with (2.3) shows $\phi \in S^{(s)}(\phi,\bar g)$. Next, we show that ϕ is $g_{\rho,1/2}$ admissible. In view of (2.3) we have

$$|\phi(X+Y) - \phi(X)| \leq C\Big(\frac{w}{r^2} + \frac{w^{1-l/m}}{r}\Big)\Big|_{(X+\theta Y)}|y|$$
$$+C\Big(\frac{w^{1-l/m}}{r}\Big)\Big|_{(X+\theta Y)}\langle\xi+\theta\eta\rangle_M^{-1}|\eta|, \quad |\theta| < 1.$$

Denoting $g = g_{\rho,1/2}$, if $|\eta| \leq \langle\xi\rangle_M/2$ so that $g_X \approx g_{X+\theta Y}$ then recalling that w and r are g admissible one can find N such that

$$w(X+\theta Y)/r^2(X+\theta Y) \leq C(w(X)/r^2(X))(1+g_X(Y))^N,$$
$$w^{1-l/m}(X+\theta Y)/r(X+\theta Y) \leq C(w^{1-l/m}(X)/r(X))(1+g_X(Y))^N$$

from which together with (2.4) it follows that

$$|\phi(X+Y) - \phi(X)| \leq C\phi(X)(\langle\xi\rangle_M^{1/2}|y| + \langle\xi\rangle_M^{-(1-l/2m)}|\eta|)(1+g_X(Y))^N$$
$$\leq C'\phi(X)(1+g_X(Y))^{N+1/2}$$

since $r(X) \geq w(X) \geq \langle\xi\rangle_M^{-1/2}$. If $|\eta| \geq \langle\xi\rangle_M/2$ so that $g_X(Y) \geq \langle\xi\rangle_M^{l/m}/4$ noting that $\phi(X) \geq c\langle\xi\rangle_M^{-1/2}$ in view of (2.4) we have

$$\phi(X+Y) \leq 2\pi \leq C\langle\xi\rangle_M^{-1/2}(1+g_X(Y))^{m/2l} \leq C\phi(X)(1+g_X(Y))^{m/2l}$$

thus the proof is complete. □

Lemma 2.4. *One can write*

$$\partial_x^\beta\partial_\xi^\alpha\phi = A_{\alpha\beta} + \phi_2 B_{\alpha\beta}, \quad |\alpha+\beta| \geq 1,$$
$$A_{\alpha\beta} \in S^{(s)}(wr^{-2}\varrho^{-|\beta|+1}w^{-|\alpha|l/m}\langle\xi\rangle_M^{-|\alpha|}, \bar{g}),$$
$$B_{\alpha\beta} \in S^{(s)}(r^{-2}w^{1-|\alpha+\beta|l/m}\langle\xi\rangle_M^{-|\alpha|}, \bar{g}).$$

Proof. Let $|\alpha'+\beta'| = 1$ and $\alpha = \alpha'+\alpha''$, $\beta = \beta'+\beta''$ then from (2.3) we see

$$\partial_x^\beta\partial_\xi^\alpha\phi = -2\partial_x^{\beta''}\partial_\xi^{\alpha''}(wr^{-2}\partial_x^{\beta'}\partial_\xi^{\alpha'}\phi_2) + 2\partial_x^{\beta''}\partial_\xi^{\alpha''}(\phi_2 r^{-2}\partial_x^{\beta'}\partial_\xi^{\alpha'}w).$$

Since $wr^{-2} \in S^{(s)}(wr^{-2}, \bar{g})$ the first term is $A_{\alpha\beta}$. Consider the second term. Note that $\partial_x^e(\phi_2 r^{-2}) \in S^{(s)}(r^{-2}, \bar{g})$ for $|e| = 1$ and $\partial_x^{\beta'}\partial_\xi^{\alpha'}w \in S^{(s)}(w^{1-l/m}\langle\xi\rangle_M^{-|\alpha'|}, \underline{g})$ then if at least one derivative with respect to x falls on $\phi_2 r^{-2}$ which yields $A_{\alpha\beta}$ otherwise this term will be $\phi_2 B_{\alpha\beta}$. □

2.2 Operators to be considered; non-effectively hyperbolic operators

Consider

$$P = -D_0^2 + 2BD_0 + Q, \quad B = \mathrm{op}(\phi_1\langle\xi\rangle_M), \;\; Q = \mathrm{op}(\phi_2^2\langle\xi\rangle_M^2)$$

where $\phi_i \in S_{1,0}^{(s)}(1)$ are real valued and $\phi_2 = \phi_2(x)$ is independent of ξ. Assume that there exist $c > 0$ and $c_{ij} \in S_{1,0}^{(s)}(1)$ such that

$$\{\xi_0, \phi_i\} = \sum_{j=1}^{2} c_{ij}\phi_j, \quad \langle\xi\rangle_M\{\phi_1, \phi_2\} \geq c > 0. \tag{2.5}$$

This is the general form for the case that $\mathrm{Ker}\, F^2 \cap \mathrm{Im}\, F^2 \neq \{0\}$ on the double characteristic manifold which is assumed to be smooth and of codimension

3 where F denotes the Hamilton map of P. If there exists no bicharacteristics falling on the double characteristic manifold tangentially then the first condition in (2.5) can be strengthened to

$$\{\xi_0, \phi_1\} = \sum_{j=1}^{2} c_{1j}\phi_j, \quad \{\xi_0, \phi_2\} = c_{21}\phi_1^2 + c_{22}\phi_2. \tag{2.6}$$

$P_{mod} = -D_0^2 + 2D_1D_0 + x_1^2 D_2^2$ is the model operator for the case (2.6), which is one of three normal forms of quadratic hyperbolic operators ([5, Section 21.5]). The fundamental solution for P_{mod} is constructed in [6] (see also [4, Chapter 7, p.211]) and proved solvability of the Cauchy problem for $P_{mod} + SD_2$, $S \in \mathbb{C}$ in the Gevrey class 4 using the explicit formulas (although energy estimates giving Gevrey class 4 result was not obtained, see [6, p.159]).

Let

$$w := (\phi_1^{2m} + \langle\xi\rangle_M^{-l})^{1/2l}, \quad r := \sqrt{\phi_2^2(x) + w^2(x,\xi)}$$

be given in Section 2.1 where $w \in S^{(s)}_{\rho,\delta}(w)$ with $\rho = 1 - l/2m$ and $\delta = l/2m$ is $g_{\rho,\delta}$ admissible by Lemma 2.1 and $r \in S^{(s)}_{\rho,1/2}(r)$ is $g_{\rho,1/2}$ admissible by Lemma 2.2. In what follows we consider two cases;

$$(m,l) = (3,2) \text{ in case (2.5)}, \quad (m,l) = (2,1) \text{ in case (2.6)}. \tag{2.7}$$

Note that

$$\rho = 2/3,\ \delta = 1/3 \text{ if } (m,l) = (3,2), \quad \rho = 3/4,\ \delta = 1/4 \text{ if } (m,l) = (2,1).$$

Take κ_1 such that

$$\delta < \kappa_1 < 1/2 \tag{2.8}$$

and consider

$$e^{-\gamma\langle D\rangle_M^{\kappa_1} x_0} P e^{\gamma\langle D\rangle_M^{\kappa_1} x_0}$$

where $\gamma > 0$ is a positive parameter and will be fixed eventually such that

$$\gamma = M^{\epsilon^*}, \quad \epsilon^* > 0,\ \kappa_1 + \epsilon^* < 1/2. \tag{2.9}$$

Since $\gamma\langle\xi\rangle_M^{\kappa_1} \le \langle\xi\rangle_M^{\kappa_1+\epsilon^*}$ one can regard $\gamma\langle\xi\rangle_M^{\kappa_1} \in S^{(s)}_{1,0}(\langle\xi\rangle_M^{\kappa_1+\epsilon^*})$. Since $\langle\xi\rangle_M^{l} \in S^{(s)}_{1,0}(\langle\xi\rangle_M^{l})$ ($l \in \mathbb{R}$) and $\phi_i \in S^{(s)}_{1,0}(1)$ and it is easy to see that

$(\sigma\partial_X)^{\alpha}(\partial_X^{\alpha^0}\phi_i\partial_X^{\alpha^1}\langle\xi\rangle_M^{\kappa_1+\epsilon^*}\cdots\partial_X^{\alpha^k}\langle\xi\rangle_M^{\kappa_1+\epsilon^*}) \in S_{1,0}^{(s)}(\langle\xi\rangle_M^{k(\kappa_1+\epsilon^*)-|\alpha|})$, $|\alpha| \geq k$, thanks to [8, Theorem 2.1] we have

$$e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#(\phi_1\langle\xi\rangle_M)\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0} = \phi_1\langle\xi\rangle_M + i\gamma x_0\{\langle\xi\rangle_M^{\kappa_1},\phi_1\}\langle\xi\rangle_M$$
$$+S_{1,0}^{(s)}(1) + S_{0,0}^{(s)}(e^{-c\langle\xi\rangle_M^{1/s}}) = \phi_1\langle\xi\rangle_M + i\gamma x_0 b_1 + b_2 + r^b$$

with $b_1 = \{\langle\xi\rangle_M^{\kappa_1},\phi_1\}\langle\xi\rangle_M$, $b_2 \in S_{1,0}^{(s)}(1)$ and $r^b \in S_{0,0}^{(s)}(e^{-c\langle\xi\rangle_M^{1/s}})$ and

$$e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#(\phi_2^2\langle\xi\rangle_M^2)\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0} = \phi_2^2\langle\xi\rangle_M^2 + 2i\gamma x_0\{\langle\xi\rangle_M^{\kappa_1},\phi_2\}\phi_2\langle\xi\rangle_M^2$$
$$+S_{1,0}^{(s)}(\langle\xi\rangle_M) + S_{0,0}^{(s)}(e^{-c\langle\xi\rangle_M^{1/s}}) = \phi_2^2\langle\xi\rangle_M^2 + 2i\gamma x_0 q_1 + q_2 + r^q$$

with $q_1 = \{\langle\xi\rangle_M^{\kappa_1},\phi_2\}\phi_2\langle\xi\rangle_M^2$, $q_2 \in S_{1,0}^{(s)}(\langle\xi\rangle_M)$ and $r^q \in S_{0,0}^{(s)}(e^{-c\langle\xi\rangle_M^{1/s}})$. Therefore it follows that

$$e^{-\gamma\langle D\rangle_M^{\kappa_1}x_0}Pe^{\gamma\langle D\rangle_M^{\kappa_1}x_0} = -(D_0 - i\gamma\langle D\rangle_M^{\kappa_1})^2$$
$$+2\mathrm{op}(\phi_1\langle\xi\rangle_M + i\gamma x_0 b_1 + b_2 + r^b)(D_0 - i\gamma\langle D\rangle_M^{\kappa_1})$$
$$+\mathrm{op}(\phi_2^2\langle\xi\rangle_M^2 + 2i\gamma x_0 q_1 + q_2 + r^q).$$

Let us denote

$$\nu = 2 - 2l/m, \quad \psi := 1 - \sqrt{1 - w^{\nu}} = w^{\nu}/\big(1 + \sqrt{1 - w^{\nu}}\,\big) \tag{2.10}$$

where w is assumed to be $|w| \leq c < 1$ without loss of generality. From Lemma 2.1 it follows that

$$|\partial_x^{\beta}\partial_{\xi}^{\alpha}\psi| \leq CA^{|\alpha+\beta|}|\alpha+\beta|!^s w^{\nu} w^{-l|\alpha+\beta|/m}\langle\xi\rangle_M^{-|\alpha|}. \tag{2.11}$$

In particular $\psi \in S_{\rho,\delta}^{(s)}(w^{\nu})$. Noting that $\psi^2 - 2\psi + w^{\nu} = 0$ it is clear that

$$-\xi_0^2 + 2\phi_1\langle\xi\rangle_M\xi_0 + \phi_2^2\langle\xi\rangle_M^2 = -(\xi_0 + \phi_1\psi\langle\xi\rangle_M)(\xi_0 - \phi_1\psi\langle\xi\rangle_M)$$
$$+2\phi_1\langle\xi\rangle_M(\xi_0 - \phi_1\psi\langle\xi\rangle_M) + \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2$$

here we note that $\phi_1^2 w^{\nu} \geq \phi_1^2|\phi_1|^{\nu m/l} = |\phi_1|^{2m/l} = |\phi_1|^{1/\delta}$. Replacing ξ_0 by $\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1}$ we have

Lemma 2.5. *One can write*

$$-(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1})^2 + 2\phi_1\langle\xi\rangle_M(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1}) + \phi_2^2\langle\xi\rangle_M^2$$
$$= -(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} + \phi_1\psi\langle\xi\rangle_M)(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)$$
$$+2\phi_1\langle\xi\rangle_M(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M) + \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2.$$

Since $|\phi_1| \lesssim w^{l/m}$ it is clear from Lemma 2.1

Lemma 2.6. *We have* $\phi_1\psi \in S^{(s)}_{\rho,\delta}(w^{\nu+l/m})$.

In view of Lemma 2.6 and $\phi_1 \in S^{(s)}_{\rho,\delta}(w^{l/m})$ into account, an application of [8, Theorem 2.3] proves

$$
\begin{gathered}
(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} + \phi_1\psi\langle\xi\rangle_M)\#(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)\\
= (\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} + \phi_1\psi\langle\xi\rangle_M)(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)\\
+S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M) + S^{(s/(1-\delta))}_{0,0}(e^{-c\langle\xi\rangle_M^{(1-\delta)/s}}),\\
(\phi_1\langle\xi\rangle_M)\#(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M) = \phi_1\langle\xi\rangle_M(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)\\
+S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M) + S^{(s/(1-\delta))}_{0,0}(e^{-c\langle\xi\rangle_M^{(1-\delta)/s}})
\end{gathered}
$$

where (2.9) is taken into account.

Lemma 2.7. *One can write* $e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#p\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0}$ *as*

$$
\begin{gathered}
-(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} + \phi_1\psi\langle\xi\rangle_M)\#(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)\\
+2(\phi_1\langle\xi\rangle_M + i\gamma x_0 b_1)\#(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M)\\
+\phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^\nu\langle\xi\rangle_M^2 + 2i\gamma x_0 Q_1 + r_1 + \tilde{r}\\
+2(b_2 + r^b)\#(\xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M) + 2(b_2 + r^b)\#(\phi_1\psi\langle\xi\rangle_M)
\end{gathered}
$$

where $Q_1 = q_1 + b_1\phi_1\psi\langle\xi\rangle_M$ *and* $r_1 \in S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M)$, $\tilde{r} \in S^{(s/(1-\delta))}_{0,0}(e^{-c\langle\xi\rangle_M^{(1-\delta)/s}})$.

Proof. From [8, Theorem 2.3] it follows that $\gamma b_1\#(\phi_1\psi\langle\xi\rangle_M) = \gamma b_1\phi_1\psi\langle\xi\rangle_M + S^{(s)}_{\rho,\delta}(\langle\xi\rangle_M) + S^{(s/(1-\delta))}_{0,0}(e^{-c\langle\xi\rangle_M^{s/(1-\delta)}})$ since $b_1 \in S^{(s)}_{1,0}(\langle\xi\rangle_M^{\kappa_1})$. Then it suffices to apply Lemma 2.5. □

Let us denote

$$
\begin{gathered}
\Lambda = \xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \phi_1\psi\langle\xi\rangle_M, \quad M = \xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} + \phi_1\psi\langle\xi\rangle_M,\\
Q = \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^\nu\langle\xi\rangle_M^2 + 2i\gamma x_0 Q_1,\\
R = r_1 + \tilde{r} + 2(b_2 + r^b)\#(\phi_1\psi\langle\xi\rangle_M)
\end{gathered}
$$

such that one has

$$
\begin{gathered}
e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#p\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0} = -M\#\Lambda\\
+2(\phi_1\langle\xi\rangle_M + i\gamma x_0 b_1)\#\Lambda + Q + R + 2(b_2 + r^b)\#\Lambda.
\end{gathered}
$$

Take $\kappa_2 > 0$ such that

$$\kappa_1 + \kappa_2 = 1/2 \tag{2.12}$$

and define ϕ by

$$\phi = -i\langle\xi\rangle_M^{\kappa_2}\big\{\log\left(\phi_2(x) - iw(x,\xi)\right) - \log\left(\phi_2(x) + iw(x,\xi)\right)\big\}. \tag{2.13}$$

Here we remark that $\kappa_2 = 1/2 - \kappa_1 < 1/2 - \delta = \rho - 1/2$ and $\phi \in S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_2})$ by Lemma 2.3 so that one can apply the calculus prepared in [8] if $s > 1$ is enough close to 1. Consider $e^{\phi}\#p\#e^{-\phi}$. In what follows

ϵ, ϵ', ϵ'' denote positive constants which may change line by line.

From [8, Corollary 2.1] it follows that $e^{\phi}\#e^{-\phi} = 1 - r$ with $r \in S_{\rho,1/2}(\langle\xi\rangle_M^{-2\epsilon'})$. Since $r \in S_{1/2}(M^{-2\epsilon'})$ there exists $k \in S_{1/2}(1)$ such that $(1+k)\#(1-r) = (1-r)\#(1+k) = 1$ hence $e^{\phi}\#e^{-\phi}\#(1+k) = (1+k)\#e^{\phi}\#e^{-\phi} = 1$. Since there is $\tilde{k} \in S_{1/2}(1)$ such that $(1+\tilde{k})\#e^{-\phi}\#e^{\phi} = 1$ it follows that

$$e^{-\phi}\#(1+k)\#e^{\phi} = 1. \tag{2.14}$$

Thanks to Corollary 1.1 we have $k \in S_{\rho,1/2}(M^{-2\epsilon'})$

2.3 Conjugation by $\mathrm{op}(e^{\pm\phi})$

Consider $J_1 = e^{\phi}\#(b_2 + r^b)\#\Lambda\#e^{-\phi}\#(1+k)$ with k in (2.14). Then one can write

$$\begin{aligned} J_1 &= e^{\phi}\#(b_2 + r^b)\#e^{-\phi}\#(1+k)\#e^{\phi}\#\Lambda\#e^{-\phi}\#(1+k) \\ &= e^{\phi}\#(b_2 + r^b)\#e^{-\phi}\#(1+k)\#\tilde{\Lambda}, \quad \tilde{\Lambda} = e^{\phi}\#\Lambda\#e^{-\phi}\#(1+k) \end{aligned}$$

Choosing $s > 1$ suitably close to 1 it can be assumed that $1/2 \ge \rho - 1/2 > s\kappa_2$ then [8, Proposition 2.1] and [8, Corollary 2.2] show that

$$\tilde{F}_1 = e^{\phi}\#(b_2 + r^b)\#e^{-\phi}\#(1+k) \in S_{\rho,1/2}(1).$$

Similarly $\tilde{F}_2 = e^{\phi}\#(\phi_1\psi\langle\xi\rangle_M)\#e^{-\phi}\#(1+k) \in S_{\rho,1/2}(\langle\xi\rangle_M)$ and hence

$$e^{\phi}\#((b_2 + r^b)\#(\phi_1\psi\langle\xi\rangle_M))\#e^{-\phi}\#(1+k) = \tilde{F}_1\#\tilde{F}_2 \in S_{\rho,1/2}(\langle\xi\rangle_M).$$

Let $\tilde{s} = s/(1-\delta)$ and $\tilde{\kappa} = (1-\delta)/s$. Noting $\tilde{s}\kappa_2 = s\kappa_2/(1-\delta) < (\rho - 1/2)/(1-\delta) = (1/2-\delta)/(1-\delta) \le 1/2$ and $\tilde{\kappa} = (1-\delta)/s = \rho/s > \kappa_2$ one can apply [8, Proposition 2.1] to obtain $e^{\phi}\#\tilde{r}\#e^{-\phi}\#(1+k) \in S_{\rho,1/2}(\langle\xi\rangle_M^{-N})$. Thus

$$\tilde{R} = e^{\phi}\#R\#e^{-\phi}\#(1+k) \in S_{\rho,1/2}(\langle\xi\rangle_M).$$

We summarize what we have proved as follows:

$$\begin{aligned} e^{\phi}\#e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#p\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#e^{-\phi}\#(1+k) = -\tilde{M}\#\tilde{\Lambda} \\ +2e^{\phi}\#(\phi_1\langle\xi\rangle_M + i\gamma x_0 b_1)\#e^{-\phi}\#(1+k)\#\tilde{\Lambda} \\ +e^{\phi}\#(\phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 + 2i\gamma x_0 Q_1) \\ \#e^{-\phi}\#(1+k) + S_{\rho,1/2}(\langle\xi\rangle_M) + S_{\rho,1/2}(1)\#\tilde{\Lambda}. \end{aligned} \tag{2.15}$$

Lemma 2.8. *Assume $q \in S^{(s)}(\omega, \underline{g})$ with $g_{\rho,\delta}$ admissible ω. Then we have*

$$\begin{aligned} (\sigma\partial_X)^{\alpha}(\partial_X^{\alpha^0}q(X)\partial_X^{\alpha^1}\phi(X)) \in S_{\rho,1/2}(\omega w^{\nu/2}r^{-2}\langle\xi\rangle_M^{-1+\kappa_2-\epsilon'}) \\ +S_{\rho,1/2}(\omega w^{1/2-l/m}r^{-1}\langle\xi\rangle_M^{-1+(\kappa_2+\delta)/2-\epsilon'}) \cap S_{\rho,1/2}(\omega w^{-l/m}r^{-1}\langle\xi\rangle_M^{-1+\kappa_2-\epsilon'}) \end{aligned}$$

for $\alpha = \alpha^0 + \alpha^1$, $|\alpha| \geq 2$ and $|\alpha^1| \geq 1$.

Proof. Thanks to Lemma 2.4 one can write

$$(\sigma\partial_X)^{\alpha}(\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi) = A_{\alpha} + \phi_2 B_{\alpha}, \quad \alpha = \alpha^0 + \alpha^1$$

where A_{α} which is in $S^{(s)}(\omega r^{-2}\varrho^{-|\alpha|+1}w^{1-|\alpha|l/m}\langle\xi\rangle_M^{\kappa_2-|\alpha|}, \bar{g})$ and $\phi_2 B_{\alpha}$ which belongs to $S(\omega r^{-1}w^{1-2|\alpha|l/m}\langle\xi\rangle_M^{\kappa_2-|\alpha|}, \bar{g})$. Note that

$$r^{-2}\varrho^{-|\alpha|+1}w^{1-|\alpha|l/m} \lesssim w^{\nu/2}r^{-2}w^{-(|\alpha|-1)l/m}\sum_{j=0}^{|\alpha|-1} r^{-j}w^{-(|\alpha|-1-j)l/m}$$

$$\lesssim w^{\nu/2}r^{-2}\sum_{j=0}^{|\alpha|-1} w^{-j}w^{-2(|\alpha|-1-j/2)l/m} \leq w^{\nu/2}r^{-2}\sum_{j=0}^{|\alpha|-1}\langle\xi\rangle_M^{2\delta(|\alpha|-1-j/2)+j/2}$$

and $\kappa_2 - |\alpha| + 2\delta(|\alpha| - 1 - j/2) + j/2 = \kappa_2 - 1 - (|\alpha| - j/2 - 1)(\rho - \delta)$ which is less than or equal to $\kappa_2 - 1 - (\rho - \delta)/2$ for $|\alpha| \geq 2$. This proves that $A_{\alpha} \in S_{\rho,1/2}(\omega w^{\nu/2}r^{-2}\langle\xi\rangle_M^{\kappa_2-1-\epsilon'})$. Turn to $\phi_2 B_{\alpha}$. Note that

$$\begin{aligned} r^{-1}w^{1-2|\alpha|l/m} &\leq w^{1/2-l/m}r^{-1}w^{1/2-(2|\alpha|-1)l/m} \\ &\leq w^{1/2-l/m}r^{-1}\langle\xi\rangle_M^{\max\{\delta(2|\alpha|-1)-1/4,0\}} \end{aligned}$$

and $\kappa_2-|\alpha|+\delta(2|\alpha|-1)-1/4 = -1-(|\alpha|-1)(\rho-\delta)+\delta+\kappa_2-1/4 < -1+(\delta+\kappa_2)/2$ for $\delta+\kappa_2 < 1/2$ thus $\phi_2 B_{\alpha} \in S_{\rho,1/2}(\omega w^{1/2-l/m}r^{-1}\langle\xi\rangle_M^{-1+(\delta+\kappa_2)/2-\epsilon'})$. It is also easy to see that $r^{-1}w^{1-2|\alpha|l/m}\le w^{-l/m}r^{-1}\langle\xi\rangle_M^{\max\{\delta(2|\alpha|-1)-1/2,0\}}$ and $\kappa_2-|\alpha|+\delta(2|\alpha|-1)-1/2=-1-(|\alpha|-1)(\rho-\delta)+\delta-1/2+\kappa_2<-1+\kappa_2$ so that $\phi_2B_\alpha\in S_{\rho,1/2}(\omega w^{-l/m}r^{-1}\langle\xi\rangle_M^{-1+\kappa_2-\epsilon'})$. □

Lemma 2.9. *If $q\in S^{(s)}(w^{\mu+l/m}\langle\xi\rangle_M^p,g)$ with $\mu>0$ one can write*

$$e^\phi\#q\#e^{-\phi}=q+i\{q,\phi\}+\tilde q_1+\tilde q_2,$$
$$\tilde q_1\in S_{\rho,1/2}(w^{1+\mu}r^{-2}\langle\xi\rangle_M^{p-1+\kappa_2-\epsilon'}),$$
$$\tilde q_2\in S_{\rho,1/2}(w^{\mu+1/2}r^{-1}\langle\xi\rangle_M^{p-1+(\delta+\kappa_2)/2-\epsilon'})\cap S_{\rho,1/2}(w^{\mu}r^{-1}\langle\xi\rangle_M^{p-1+\kappa_2-\epsilon'}).$$

Proof. Since $q\in S^{(s)}_{\rho,1/2}(w^{\mu+l/m}\langle\xi\rangle_M^p)$ and $\phi\in S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_2})$ one can apply [8, Theorem 2.1]. It leads us to study

$$(\sigma\partial_X)^\alpha\big(\partial_X^{\alpha^0}q(X)\partial_X^{\alpha^1}\phi(X)\cdots\partial_X^{\alpha^k}\phi(X)\big)\tag{2.16}$$

where $\alpha^0+\alpha^1+\cdots+\alpha^k=\alpha$, $|\alpha^j|\ge1$ $(1\le j\le k)$. Write

$$\alpha=\tilde\alpha+\hat\alpha,\quad \tilde\alpha=\alpha^0+\tilde\alpha^1+\cdots+\tilde\alpha^k,\quad \tilde\alpha^j=(\tilde\alpha^j_x,\alpha^j_\xi),\quad |\tilde\alpha^j|=1$$

and $\epsilon(\beta)=1/2|\beta_x|-\rho|\beta_\xi|$ for $\beta=(\beta_x,\beta_\xi)$. It follows from Lemma 2.3 that

$$\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi\cdots\partial_X^{\alpha^k}\phi\in S^{(s)}\big(w^{\mu+l/m+k-(l/m)|\tilde\alpha_\xi|}\varrho^{-|\tilde\alpha_x|}r^{-k}\langle\xi\rangle_M^{p+k\kappa_2-|\tilde\alpha_\xi|+\epsilon(\hat\alpha)},\bar g\big)$$

and hence

$$(\sigma D_X)^\alpha(\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi\cdots\partial_X^{\alpha^k}\phi)\in S^{(s)}(w^{\mu+l/m+k-(l/m)|\tilde\alpha|}r^{-k}\varrho^{-|\tilde\alpha|}$$
$$\times\langle\xi\rangle_M^{p+k\kappa_2-|\tilde\alpha|-(\rho-1/2)|\hat\alpha|},\bar g).$$

We assume $k\ge2$. If $r\le w^{l/m}$ hence $\varrho^{-1}\le 2r^{-1}$ one has (recall $r^{-1}\le w^{-1}$) $w^{-|\alpha^0|l/m}\varrho^{-|\alpha^0|}\langle\xi\rangle_M^{-|\alpha^0|}\le\langle\xi\rangle_M^{-(1/2-\delta)|\alpha^0|}$ and

$$w^{\mu+l/m+k-(l/m)k}r^{-k}\varrho^{-k}\lesssim w^{\mu+1}r^{-2}w^{-1+l/m+k-(l/m)k}r^{-2k+2}$$
$$\le(w^{\mu+1}r^{-2})w^{-(k-1)(1+l/m)}\le(w^{\mu+1}r^{-2})\langle\xi\rangle_M^{(k-1)(1/2+\delta)}.$$

Since $(k-1)(1/2+\delta)-(k-1)+k\kappa_2=-(k-1)(1/2-\delta-\kappa_2)+\kappa_2<\kappa_2$ for $\kappa_2+\delta<1/2$ we have

$$(\sigma D_X)^\alpha(\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi\cdots\partial_X^{\alpha^k}\phi)\in S^{(s)}_{\rho,1/2}(w^{\mu+1}r^{-2}\langle\xi\rangle_M^{p-1+\kappa_2-\epsilon'}),\ k\ge2.\tag{2.17}$$

If $r \geq w^{l/m}$ and hence $\varrho^{-1} \leq 2w^{-l/m}$ we have $w^{-|\alpha^0|l/m}\varrho^{-|\alpha^0|}\langle\xi\rangle_M^{-|\alpha^0|} \leq \langle\xi\rangle_M^{-(1/2-\delta)|\alpha^0|}$ and that

$$\begin{aligned} w^{\mu+l/m+k-(l/m)k}r^{-k}\varrho^{-k} &\lesssim w^{\mu+1}r^{-2}w^{-1+k-3(k-1)(l/m)} \\ &\leq (w^{\mu+1}r^{-2})\langle\xi\rangle_M^{\max\{-(k-1)/2+3(k-1)\delta,0\}}. \end{aligned} \tag{2.18}$$

Note that $-(k-1)/2+3(k-1)\delta+1-k+k\kappa_2 = \kappa_2 - 3(k-1)(\rho-\delta-\kappa_2)/2-(k-1)\kappa_2/2 < \kappa_2$ then one has (2.17) again.

Let $k=1$ and consider $(\sigma\partial_X)^\alpha(\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi)$. Since $\sum_{|\alpha|=1}(\sigma\partial_X)^\alpha\partial_X^\alpha\phi = 0$ the sum over $|\alpha|=1$ yields $i\{q,\phi\}$. Therefore it remains to study the sum over $|\alpha|\geq 2$ to which one can apply Lemma 2.8 with $\omega = w^{\mu+l/m}\langle\xi\rangle_M^p$ to end the proof. □

Lemma 2.10. *One can write*

$$e^{-\phi}\#e^{\phi} = 1-r, \quad r\in S_{\rho,1/2}(w^{\nu/2}r^{-2}\langle\xi\rangle_M^{-1+\kappa_2-\epsilon'}).$$

Proof. Since $\sum_{|\alpha|=l}(\sigma D_X)^\alpha\partial_X^\alpha\phi = 0$ for $l\geq 1$ it is enough to consider the terms corresponding to $k\geq 2$. Then we obtain the assertion from (2.17). □

Corollary 2.1. *We have* $k\in S_{\rho,1/2}(w^{\nu/2}r^{-2}\langle\xi\rangle_M^{-1+\kappa_2-\epsilon'})\cap S_{\rho,1/2}(M^{-2\epsilon'})$.

Proof. The assertion follows immediately from $k = r + r\#k$. □

Lemma 2.11. *We have* $\partial_{x_0}\phi_2 \in S^{(s)}_{\rho,1/2}(w^\nu + r)$ *and* $\partial_{x_0}w\in S^{(s)}_{\rho,\delta}(w+rw^{\nu/2})$. *Moreover we have* $\partial_{x_0}\phi\in S^{(s)}_{\rho,1/2}(r^{-1}w^{\nu/2}\langle\xi\rangle_M^{\kappa_2})\cap S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^{\delta})$.

Proof. Recall that $\partial_{x_0}\phi_2 = \{\xi_0,\phi_2\} = c_{21}\phi_1^{\sigma_{21}} + c_{22}\phi_2$ where $\sigma_{21}=1$ or 2 according to the case $(m,l)=(3,2)$ or $(2,1)$. Since $\phi_2\in S^{(s)}_{\rho,1/2}(r)$ and $\phi_1\in S^{(s)}_{\rho,1/2}(w^{l/m})$ it is clear that $\{\xi_0,\phi_2\}\in S^{(s)}(w^\nu+r)$ because $\sigma_{21}l/m = 2-2l/m = \nu$. Noting that $\{\xi_0,w\} = (m/l)(\phi_1^{2m-1}/w^{2l-1})\{\xi_0,\phi_1\}$ and $\{\xi_0,\phi_1\} = c_{11}\phi_1 + c_{12}\phi_2$ it is clear that $\{\xi_0,w\}\in S^{(s)}_{\rho,1/2}(w+rw^{\nu/2})$. Recall that

$$i\partial_{x_0}\phi = i\{\xi_0,\phi\} = -2\frac{\langle\xi\rangle_M^{\kappa_2}w}{r^2}\{\xi_0,\phi_2\} + 2\frac{\phi_2\langle\xi\rangle_M^{\kappa_2}}{r^2}\{\xi_0,w\}$$

where it is clear

$$wr^{-2}\{\xi_0,\phi_2\}\langle\xi\rangle_M^{\kappa_2}\in S^{(s)}_{\rho,1/2}((r^{-2}w^{1+\nu}+r^{-1}w)\langle\xi\rangle_M^{\kappa_2})\subset S^{(s)}_{\rho,1/2}(r^{-1}w^{\nu}\langle\xi\rangle_M^{\kappa_2})$$

for $\nu \le 1$ which is also in $S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^{\delta})$ because $w^{\nu-1}\langle\xi\rangle_M^{\kappa_2} \le \langle\xi\rangle_M^{2\delta+\kappa_2-1/2} = \langle\xi\rangle_M^{2\delta-\kappa_1} \le \langle\xi\rangle_M^{\delta}$. On the other hand, one has

$$\phi_2 r^{-2}\langle\xi\rangle_M^{\kappa_2}\{\xi_0, w\} \in S^{(s)}_{\rho,1/2}((r^{-1}w + w^{\nu/2})\langle\xi\rangle_M^{\kappa_2})$$

which is also in $S^{(s)}_{\rho,1/2}(r^{-1}w^{\nu/2}\langle\xi\rangle_M^{\kappa_2}) \cap S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^{\delta})$ because $\kappa_2 \le \delta$ (it is clear since $\delta = 1/4$ or $1/3$) from which we conclude the proof. □

Lemma 2.12. *Assume $q \in S^{(s)}(w^2\langle\xi\rangle_M^2, \underline{g})$ and $\partial_{x_0} q \in S^{(s)}(w^{2-l/m}\langle\xi\rangle_M^2, \underline{g})$. Then one can write*

$$e^{-\phi}\#q\#e^{\phi} = q + \tilde{q}, \quad \tilde{q} \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2})$$

where $\partial_{x_0}\tilde{q} \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta-\epsilon'})$.

Proof. The fact $\tilde{q} \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2})$ follows from Lemma 2.9 with $\mu = 2 - l/m$ and Lemma 2.3. Consider

$$(\sigma\partial_X)^{\alpha}\big(\partial_{x_0}(\partial_X^{\alpha^0} q(X)\partial_X^{\alpha^1}\phi(X)\cdots\partial_X^{\alpha^k}\phi(X))\big). \tag{2.19}$$

Note that $\partial_{x_0}\partial_x^{\beta}\partial_\xi^{\alpha}\phi \in S^{(s)}(w^{\nu/2}w^{-|\alpha|l/m}r^{-1}\langle\xi\rangle_M^{\kappa_2-|\alpha|}\varrho^{-|\beta|}, \bar{g})$ by Lemma 2.11 it is clear that (2.19) belongs to

$$S^{(s)}(w^{2+k-(|\tilde{\alpha}|+1)(l/m)}r^{-k}\varrho^{-|\tilde{\alpha}|}\langle\xi\rangle_M^{2+k\kappa_2-|\tilde{\alpha}|}, \bar{g}).$$

If $r \le w^{l/m}$ hence $\varrho^{-1} \le 2r^{-1}$ it follows that $w^{2+k-(k+1)(l/m)}r^{-k}\varrho^{-k} \lesssim r^2 w^{-k-(k+1)l/m} \le r^2\langle\xi\rangle_M^{k/2+(k+1)\delta}$. Since $k/2+(k+1)\delta-k-k\kappa_2 = \delta-k(1-2\delta-2\kappa_2)/2 < \delta$. This proves that (2.19) belongs to $S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta-\epsilon'})$. If $r \ge w^{l/m}$ hence $\varrho^{-1} \le 2w^{-l/m}$ we have $w^{2+k-(k+1)(l/m)}r^{-k}\varrho^{-k} \lesssim r^2 w^{k-(3k+1)l/m} \le r^2\langle\xi\rangle_M^{\max\{(3k+1)\delta-k/2,0\}}$. Since $(3k+1)\delta - k/2 - k + k\kappa_2 = \delta - 3k(1-2\delta - 2\kappa_2/3)/2 < \delta$ we have the same result in this case. □

Lemma 2.13. *Assume $q \in S^{(s)}(r^2\langle\xi\rangle_M^2, \bar{g})$ and that q satisfies $\partial_x^{\beta}\partial_\xi^{\alpha} q \in S^{(s)}(r^{1+|\alpha|}\langle\xi\rangle_M^{2-|\alpha|}, \bar{g})$ for $|\alpha+\beta| = 1$. Then one can write*

$$e^{-\phi}\#q\#e^{\phi} = q + \tilde{q}, \quad \tilde{q} \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}).$$

Moreover if $\partial_{x_0} q \in S^{(s)}(r^{1+\nu}\langle\xi\rangle_M^2, \bar{g})$ then $\partial_{x_0}\tilde{q} \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta-\epsilon'})$.

Proof. From a repetition of the proof of Lemma 2.9 it follows that (2.16) is in $S^{(s)}(r^2w^{k-(l/m)|\tilde\alpha|}r^{-k}\varrho^{-|\tilde\alpha|}\langle\xi\rangle_M^{2+k\kappa_2-|\tilde\alpha|},\bar g)$. If $r\le w^{l/m}$ hence $\varrho^{-1}\le 2r^{-1}$ one has $r^2w^{k-k(l/m)}r^{-2k}\le w^{-k+2-kl/m}\le w^{\nu/2}\langle\xi\rangle_M^{(k-1)\delta+(k-1)/2}$. Since $(k-1)\delta+(k-1)/2-k+1+k\kappa_2=\kappa_2-(k-1)(1/2-\delta-\kappa_2)\le\kappa_2$ we see that (2.16) belongs to $S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2})$. If $r\ge w^{l/m}$ hence $\varrho^{-1}\le 2w^{-l/m}$ and $k\ge 2$ we have $r^2w^{k-k(l/m)}r^{-k}\varrho^{-k}\lesssim w^{\nu/2}w^{k-1-3(k-1)l/m}\le w^{\nu/2}\langle\xi\rangle_M^{\max\{3(k-1)\delta-(k-1)/2,0\}}$. The same arguments as before shows that $3(k-1)\delta-(k-1)/2+1-k+k\kappa_2\le\kappa_2$ then this belongs to $S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2})$. Assume $k=1$ and consider (2.16). Since $\sum_{|\alpha|=l}(\sigma\partial_X)^\alpha\partial^\alpha\phi/\alpha!=0$ for $l\ge 1$ it suffices to consider the case either $|\alpha^0|\neq 0$ or $|\alpha^0|=0$ and at least one derivative falls on q. This shows that for $\partial_X^{\alpha^0+\alpha'}q\partial_X^{\alpha^1+\alpha''}\phi$ we can obtain a better by $w^{l/m}$ estimate if $\alpha^0_\xi+\alpha'_\xi\neq 0$ and we obtain a better by $r^{-1}\varrho$ estimate if $\alpha^0_x+\alpha'_x\neq 0$. This proves, taking $rw^{l/m}\lesssim\varrho$ into account, that

$$(\sigma\partial_X)^\alpha(\partial_X^{\alpha^0}q\partial_X^{\alpha^1}\phi)\in S^{(s)}(w^{1-|\alpha|l/m}\varrho^{-|\alpha|+1}\langle\xi\rangle_M^{\kappa_2+2-|\alpha|},\bar g).$$

Then we conclude the assertion by repeating similar arguments in the proof of Lemma 2.8. The proof of the last assertion is just a repetition of that of Lemma 2.12. □

Thanks to [8, Corollary 2.1] we see that $\partial_{x_0}r=\partial_{x_0}(e^\phi\#e^{-\phi})\in S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^\delta)$ from Lemma 2.11 then applying Corollary 1.1 we have

$$\partial_{x_0}k\in S^{(s)}_{\rho,1/2}(\langle\xi\rangle_M^\delta). \tag{2.20}$$

Lemma 2.14. *One can choose $\epsilon^*>0$ in* (2.9) *such that one has*

$$\tilde\Lambda=\xi_0-i\gamma\langle\xi\rangle_M^{\kappa_1}+S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1}).$$

Proof. Recall $\tilde\Lambda=e^\phi\#(\xi_0-i\gamma\langle\xi\rangle_M^{\kappa_1}-\phi_1\psi\langle\xi\rangle_M)\#e^{-\phi}\#(1+k)$. First consider $e^\phi\#\xi_0\#e^{-\phi}\#(1+k)$ which is

$$\xi_0+e^\phi\#(i(\partial_{x_0}\phi)e^{-\phi})\#(1+k)+e^\phi\#e^{-\phi}\#(-i\partial_{x_0}k).$$

Thanks to Lemma 2.11 an application of [8, Theorem 2.2] proves that the last two terms belong to $S_{\rho,1/2}(\langle\xi\rangle_M^\delta)$. Since $\phi\in S^{(s)}_{\rho,1/2}(\phi)$ in view of Lemma 2.3 it follows from [8, Corollary 2.2] that $e^\phi\#\langle\xi\rangle_M^{\kappa_1}\#e^{-\phi}=\langle\xi\rangle_M^{\kappa_1}+S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1-\epsilon''})$. If $\epsilon^*>0$ is chosen such that $0<\epsilon^*<\epsilon''$ then

$$e^\phi\#\gamma\langle\xi\rangle_M^{\kappa_1}\#e^{-\phi}\#(1+k)=\gamma\langle\xi\rangle_M^{\kappa_1}+S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1}).$$

Next, we apply Lemma 2.9 with $q = \phi_1\psi\langle\xi\rangle_M \in S^{(s)}(w^{\nu+l/m}\langle\xi\rangle_M, \underline{g})$ to obtain $e^{\phi}\#(\phi_1\psi\langle\xi\rangle_M)\#e^{-\phi} = \phi_1\psi\langle\xi\rangle_M + S_{\rho,1/2}(w^{\nu-1}\langle\xi\rangle_M^{\kappa_2})$. Since $w^{\nu-1}\langle\xi\rangle_M^{\kappa_2} \le \langle\xi\rangle_M^{2\delta-1/2+\kappa_2} = \langle\xi\rangle_M^{2\delta-\kappa_1} \le \langle\xi\rangle_M^{\kappa_1}$ we conclude the proof. □

Lemma 2.15. *We have*

$$e^{\phi}\#(\phi_1\langle\xi\rangle_M + i\gamma x_0 b_1)\#e^{-\phi}\#(1+k) = \phi_1\langle\xi\rangle_M + i\{\phi_1\langle\xi\rangle_M, \phi\} + i\gamma x_0 b_1$$
$$+S_{\rho,1/2}(M^{-\epsilon'}wr^{-2}\langle\xi\rangle_M^{\kappa_2}) + S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4}) + S_{\rho,1/2}(M^{-\epsilon'}\langle\xi\rangle_M^{\kappa_1}).$$

Proof. Applying Lemma 2.9 with $q = \phi_1\langle\xi\rangle_M \in S^{(s)}(w^{l/m}\langle\xi\rangle_M, \underline{g})$ we obtain that $e^{\phi}\#(\phi_1\langle\xi\rangle_M)\#e^{-\phi} = \phi_1\langle\xi\rangle_M + i\{\phi_1\langle\xi\rangle_M, \phi\} + S_{\rho,1/2}(M^{-\epsilon'}wr^{-2}\langle\xi\rangle_M^{\kappa_2}) + S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4})$. Since $b_1 \in S^{(s)}(\langle\xi\rangle_M^{\kappa_1}, \underline{g})$ it is clear that $\gamma e^{\phi}\#b_1\#e^{-\phi} = \gamma b_1 + S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1})$. From Corollary 2.1 we conclude that

$$(\phi_1\langle\xi\rangle_M + i\{\phi_1\langle\xi\rangle_M, \phi\})\#k \in S_{\rho,1/2}(M^{-\epsilon'}wr^{-2}\langle\xi\rangle_M^{\kappa_2})$$
$$+S_{\rho,1/2}(M^{-\epsilon'}w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4})$$

for $\{\phi_1\langle\xi\rangle_M, \phi\} \in S_{\rho,1/2}(wr^{-2}\langle\xi\rangle_M^{\kappa_2}) + S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4})$ and $(\phi_1\langle\xi\rangle_M)\#k \in S_{\rho,1/2}(w^{\nu/2}r^{-1}\langle\xi\rangle_M^{\kappa_2-\epsilon'})$ where $w^{\nu/2}r^{-1} \le w^{1/2}r^{-1}\langle\xi\rangle_M^{\delta-1/4+\kappa_2}$ and $\delta - 1/4 + \kappa_2 < 1/4$. This proves the assertion □

Lemma 2.16. *There is $C > 0$ such that*

$$r^2/C \le \phi_2^2 + \phi_1^2 w^{\nu} + \langle\xi\rangle_M^{-1} \le Cr^2.$$

Proof. It suffices to show $w^{2l}/C \le |\phi_1|^{2l}w^{l\nu} + \langle\xi\rangle_M^{-l} \le Cw^{2l}$. Since $|\phi_1|^{2l}\langle\xi\rangle_M^{-l\nu/2} \le C(|\phi_1|^{4l/(2-\nu)} + \langle\xi\rangle_M^{-l})$ and $4l/(2-\nu) = 2m$ and $2l + m\nu = 2m$ the assertion follows. □

Lemma 2.17. *One can write*

$$e^{\phi}\#(\phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 + 2i\gamma x_0 Q_1)\#e^{-\phi}\#(1+k)$$
$$= \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 + 2i\gamma x_0 Q_1 + Q' + Q'',$$
$$Q' \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2),$$
$$Q'' \in S_{\rho,1/2}(M^{-\epsilon'}r\langle\xi\rangle_M^{1+\kappa_1}) \cap S_{\rho,1/2}(M^{-\epsilon''}r^2\langle\xi\rangle_M^2)$$

where $Q_1 = \{\langle\xi\rangle_M^{\kappa_1}, \phi_1\}\phi_1\psi\langle\xi\rangle_M^2 + \{\langle\xi\rangle_M^{\kappa_1}, \phi_2\}\phi_2\langle\xi\rangle_M^2 \in S_{\rho,\delta}(r\langle\xi\rangle_M^{1+\kappa_1})$ is real and $\partial_{x_0}Q'$, $\partial_{x_0}Q'' \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$.

Proof. Applying Lemma 2.13 with $q = \phi_2^2\langle\xi\rangle_M^2 \in S^{(s)}(r^2\langle\xi\rangle_M^2, \bar{g})$ and [8, Corollary 2.2] with $p = \phi_2^2\langle\xi\rangle_M^2$ we obtain $e^{\phi}\#(\phi_2^2\langle\xi\rangle_M^2)\#e^{-\phi} = \phi_2^2\langle\xi\rangle_M^2 + Q'$ with $Q' \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2)$ where $\partial_{x_0}Q' \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$ in view of Lemma 2.13. One can write $e^{\phi}\#(\phi_2^2\langle\xi\rangle_M^2)\#e^{-\phi}\#(1+k)$ in the same form thanks to Corollary 2.1. With $q = \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 \in S^{(s)}(w^2\langle\xi\rangle_M^2, \underline{g})$ we apply Lemma 2.12 and [8, Corollary 2.2] with $p = \phi_1^2 w^{\nu}\langle\xi\rangle_M^2$ hence $e^{\phi}\#(\phi_1^2 w^{\nu}\langle\xi\rangle_M^2)\#e^{-\phi} = \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 + Q'$ with $Q' \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2)$ where $\partial_{x_0}Q' \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$. From this it follows that $e^{\phi}\#(\phi_1^2 w^{\nu}\langle\xi\rangle_M^2)\#e^{-\phi}\#(1+k)$ has the same form as discussed above. Note that $\partial_{x_0}(Q'\#(1+k)) \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$ thanks to (2.20).

Turn to $e^{\phi}\#Q_1\#e^{-\phi}$. Since $Q_1 \in S^{(s)}_{\delta,1/2}(r\langle\xi\rangle_M^{1+\kappa_1}) \cap S^{(s)}_{\delta,1/2}(r^2\langle\xi\rangle_M^{2-\epsilon'})$ for $r^{-1} \le \langle\xi\rangle_M^{1/2}$ then [8, Corollary 2.2] proves that $e^{\phi}\#Q_1\#e^{-\phi} = Q_1 + Q_1''$ where $\gamma Q_1'' \in S_{\rho,1/2}(M^{-\epsilon'}r\langle\xi\rangle_M^{1+\kappa_1}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2)$. On the other hand from Lemma 2.11 we see that $\partial_{x_0}Q_1 \in S^{(s)}_{\rho,1/2}((w^{\nu}+r)\langle\xi\rangle_M^{1+\kappa_1})$ for $\nu + l/m \ge 1$. Since $w^{\nu}\langle\xi\rangle_M^{1+\kappa_1} \le r^{1+\nu}\langle\xi\rangle_M^{3/2+\kappa_1} = r^{1+\nu}\langle\xi\rangle_M^{2-\epsilon'}$ and $r\langle\xi\rangle_M^{1+\kappa_1} \le r^2\langle\xi\rangle_M^{2-\epsilon'}$ we have $\gamma\partial_{x_0}Q_1'' \in S^{(s)}_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$ thanks to Lemma 2.13. □

We summarize what we have proved in

Proposition 2.1. *We have*

$$\begin{gathered}
e^{\phi}\#e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#p\#e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0}\#e^{-\phi}\#(1+k) = -\tilde{M}\#\tilde{\Lambda} \\
+2\Big(\phi_1\langle\xi\rangle_M + i\{\phi_1\langle\xi\rangle_M, \phi\} + i\gamma x_0 b_1 \\
+S_{\rho,1/2}(M^{-\epsilon'}wr^2\langle\xi\rangle_M^{\kappa_2}) + S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4}\Big)\#\tilde{\Lambda} \\
+\phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^{\nu}\langle\xi\rangle_M^2 + i\gamma x_0 Q_1 \\
+S_{\rho,1/2}(M^{-\epsilon'}r\langle\xi\rangle_M^{1+\kappa_1}) \cap S_{\rho,1/2}(M^{-\epsilon''}r^2\langle\xi\rangle_M^2) \\
+S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2) \\
+\big(S_{\rho,1/2}(M^{-\epsilon'}\langle\xi\rangle_M^{\kappa_1}) + S_{\rho,1/2}(1)\big)\#\tilde{\Lambda} + S_{\rho,1/2}(\langle\xi\rangle_M)
\end{gathered}$$

where $\tilde{\Lambda}$, $\tilde{M}$ are given by Lemma 2.14.

2.4 Energy estimates

Recall

$$\tilde{\Lambda} = \xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - \lambda,\ \tilde{M} = \xi_0 - i\gamma\langle\xi\rangle_M^{\kappa_1} - m, \quad \lambda, m \in S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1}) \tag{2.21}$$

and denote

$$\tilde{B} = \phi_1\langle\xi\rangle_M + i\{\phi_1\langle\xi\rangle_M, \phi\} + i\gamma x_0 b_1$$
$$+S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4}) + S_{\rho,1/2}(M^{-\epsilon'}wr^2\langle\xi\rangle_M^{\kappa_2})$$

where $b_1 = \{\langle\xi\rangle_M^{\kappa_1}, \phi_1\}\langle\xi\rangle_M \in S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1})$ and adding $\langle\xi\rangle_M$ to the result in Proposition 2.1 we set

$$\tilde{Q} = \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^\nu\langle\xi\rangle_M^2 + \langle\xi\rangle_M + i\gamma x_0 Q_1$$
$$+S_{\rho,1/2}(M^{-\epsilon'}r\langle\xi\rangle_M^{1+\kappa_1}) \cap S_{\rho,1/2}(M^{-\epsilon''}r^2\langle\xi\rangle_M^2)$$
$$+S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}) \cap S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2)$$

where $Q_1 \in S_{\rho,\delta}(r\langle\xi\rangle_M^{1+\kappa_1})$ is real. We also write

$$R = S_{\rho,1/2}(M^{-\epsilon'}\langle\xi\rangle_M^{\kappa_1}) + S_{\rho,1/2}(1)$$

such that

$$\mathrm{op}(e^{\phi})\mathrm{op}(e^{-\gamma\langle\xi\rangle_M^{\kappa_1}x_0})\mathrm{op}(p)\mathrm{op}(e^{\gamma\langle\xi\rangle_M^{\kappa_1}x_0})\mathrm{op}(e^{-\phi}) + \langle D\rangle_M$$
$$= -\mathrm{op}(\tilde{M})\mathrm{op}(\tilde{\Lambda}) + 2\mathrm{op}(\tilde{B})\mathrm{op}(\tilde{\Lambda}) + \mathrm{op}(\tilde{Q})$$
$$+\mathrm{op}(R)\mathrm{op}(\tilde{\Lambda}) + \mathrm{op}(\langle\xi\rangle_M).$$

Denoting $\mathrm{op}(\tilde{P}) = -\mathrm{op}(\tilde{M})\mathrm{op}(\tilde{\Lambda}) + 2\mathrm{op}(\tilde{B})\mathrm{op}(\tilde{\Lambda}) + \mathrm{op}(\tilde{Q})$ we have

Proposition 2.2. ([1]) *We have*

$$2\mathsf{Im}(\mathrm{op}(\tilde{P})v, \mathrm{op}\tilde{\Lambda})v) = \frac{d}{dx_0}(\|\mathrm{op}(\tilde{\Lambda})v\|^2 + \big(\mathrm{op}(\mathsf{Re}\,\tilde{Q})v, v\big)$$
$$+2\gamma\|\langle D\rangle_M^{\kappa_1/2}\mathrm{op}(\tilde{\Lambda})v\|^2 + 2\gamma\mathsf{Re}(\langle D\rangle_M^{\kappa_1}\mathrm{op}(\mathsf{Re}\,\tilde{Q})v, v)$$
$$+2(\mathrm{op}(\mathsf{Im}\tilde{B})\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\tilde{\Lambda})v) + 2(\mathrm{op}(\mathsf{Im}\,m)\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\tilde{\Lambda})v)$$
$$+2\mathsf{Re}(\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\mathsf{Im}\,\tilde{Q})v) + \mathsf{Im}([D_0 - \mathrm{op}(\mathsf{Re}\,\lambda), \mathrm{op}(\mathsf{Re}\,\tilde{Q})]v, v)$$
$$+2\mathsf{Re}(\mathrm{op}(\mathsf{Re}\,\tilde{Q})v, \mathrm{op}(\mathsf{Im}\,\lambda)v).$$

Since $m \in S_{\rho,\delta}(\langle\xi\rangle_M^{\kappa_1})$ we have by Lemma 1.2 that

$$\big|2(\mathrm{op}(\mathsf{Im}\,m)\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\tilde{\Lambda})v)\big| \le C\|\langle D\rangle_M^{\kappa_1/2}\mathrm{op}(\tilde{\Lambda})v\|^2.$$

Noting that

$$\mathsf{Re}\,\tilde{Q} = \phi_2^2\langle\xi\rangle_M^2 + \phi_1^2 w^\nu\langle\xi\rangle_M^2 + \langle\xi\rangle_M + S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^2)$$

and Lemma 2.16 it is clear that there is $c>0$ and M_0 such that

$$\mathsf{Re}\,\tilde{Q}\geq c\,r^2\langle\xi\rangle_M^2,\quad M\geq M_0.$$

We have $\langle\xi\rangle_M^{\kappa_1}\#\mathsf{Re}\,\tilde{Q}=\langle\xi\rangle_M^{\kappa_1}(\phi_2^2\langle\xi\rangle_M^2+\phi_1^2w^{\nu}\langle\xi\rangle_M^2+\langle\xi\rangle_M)+S_{\rho,1/2}(M^{-\epsilon'}r^2\langle\xi\rangle_M^{2+\kappa_1})$ hence there are $c>0$ and $M_0>0$ such that

$$\mathsf{Re}(\langle\xi\rangle_M^{\kappa_1}\#\mathsf{Re}\,\tilde{Q})\geq c\langle\xi\rangle_M^{\kappa_1+2}r^2,\quad M\geq M_0.$$

Thanks to Lemma 1.3 one has

Lemma 2.18. *There exist $c>0, M_0>0$ such that*

$$\begin{aligned}&(\mathrm{op}(\mathsf{Re}\,\tilde{Q})v,v)\geq c\|\mathrm{op}(r\langle\xi\rangle_M)v\|^2,\quad M\geq M_0,\\ &\mathsf{Re}(\langle D\rangle_M^{\kappa_1}\mathrm{op}(\mathsf{Re}\,\tilde{Q})v,v)\geq c\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2,\quad M\geq M_0.\end{aligned}\tag{2.22}$$

Consider

$$\mathsf{Im}\,\tilde{B}=\{\phi_1\langle\xi\rangle_M,\phi\}+\gamma x_0b_1+S_{\rho,1/2}(w^{1/2}r^{-1}\langle\xi\rangle_M^{1/4})+S_{\rho,1/2}(M^{-\epsilon'}wr^2\langle\xi\rangle_M^{\kappa_2}).$$

Note that, taking Lemma 2.1 into account

$$\begin{aligned}\{\phi_1\langle\xi\rangle_M,\phi\}&=2wr^{-2}\{\phi_1\langle\xi\rangle_M,\phi_2\}\langle\xi\rangle_M^{\kappa_2}-2\phi_2r^{-2}\{\phi_1,w\}\langle\xi\rangle_M^{1+\kappa_2}\\ &+S_{\rho,1/2}(r^{-1}w\langle\xi\rangle_M^{\kappa_2})=2wr^{-2}\{\phi_1,\phi_2\}\langle\xi\rangle_M^{1+\kappa_2}+S_{\rho,1/2}(w^{\nu/2}r^{-1}\langle\xi\rangle_M^{\kappa_2})\\ &=2wr^{-2}\{\phi_1,\phi_2\}\langle\xi\rangle_M^{1+\kappa_2}+S_{\rho,1/2}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{1/4})\end{aligned}$$

for $w^{1/2-l/m}\langle\xi\rangle_M^{\kappa_2}\leq\langle\xi\rangle_M^{(2\delta-1/2)/2+\kappa_2}\leq\langle\xi\rangle_M^{\kappa_1+\kappa_2-1/4}=\langle\xi\rangle_M^{1/4}$. Let $g\in S_{\rho,1/2}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{1/4})$ then from Lemma 1.2 it follows that

$$|(\mathrm{op}(g)w,w)|\leq C\gamma^{1/2}\|\langle D\rangle_M^{\kappa_1/2}w\|^2+C\gamma^{-1/2}\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})w\|^2$$

and if $g\in S_{\rho,1/2}(M^{-\epsilon'}wr^{-2}\langle\xi\rangle_M^{\kappa_2})$ then

$$|(\mathrm{op}(g)w,w)|\leq CM^{-\epsilon'}\|\mathrm{op}(r\sqrt{w}\langle\xi\rangle_M^{\kappa_2/2}w\|^2,\quad M\geq M_0.$$

It is also clear that

$$|(\mathrm{op}(\gamma x_0b_1)w,w)|\leq C\gamma T\|\langle D\rangle_M^{\kappa_1/2}w\|^2,\quad |x_0|\leq T.$$

From the assumption we have $\{\phi_1,\phi_2\}\langle\xi\rangle_M^{1+\kappa_2}\geq c\langle\xi\rangle_M^{\kappa_2}$ with some $c>0$ hence Lemma 1.3 proves that

$$(\mathrm{op}(wr^{-2}\{\phi_1,\phi_2\}\langle\xi\rangle_M^{1+\kappa_2})w,w)\geq c\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})w\|^2$$

which will be applied with $w=\mathrm{op}(\tilde{\Lambda})v$. Summarizing we have proved

Proposition 2.3. *There exist $c > 0, C > 0, M_0 > 0$ such that*

$$2(\mathrm{op}(\mathsf{Im}\,\tilde{B})\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\tilde{\Lambda})v) \geq c\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2}\mathrm{op}(\tilde{\Lambda})v\|^2$$
$$-C\gamma(\gamma^{-1/2}+T)\|\langle D\rangle_M^{\kappa_1/2}\mathrm{op}(\tilde{\Lambda})v\|^2, \quad M \geq M_0.$$

Recall that

$$\mathsf{Im}\,\tilde{Q} = \gamma x_0 Q_1 + S_{\rho,1/2}(M^{-\epsilon'}r\langle\xi\rangle_M^{1+\delta}) + S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2}).$$

Let $f \in S_{\rho,1/2}(r\langle\xi\rangle_M^{1+\kappa_1})$ then Lemma 1.2 gives

$$|(\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(f)v)| \leq C(\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2 + \|\langle D\rangle_M^{\kappa_1/2}\mathrm{op}(\tilde{\Lambda})v\|^2).$$

Let $f \in S_{\rho,1/2}(w^{\nu/2}\langle\xi\rangle_M^{1+\kappa_2})$ then Lemma 1.2 shows $(\nu/2 = 1 - l/m)$

$$|(\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(f)v)| \leq C(M^{-\epsilon''/2}\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2$$
$$+M^{\epsilon''/2}\|\mathrm{op}(rw^{1/2-l/m}\langle\xi\rangle_M^{1+\kappa_2/2})v\|^2)$$
$$\leq CM^{-\epsilon''/2}(\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2 + \|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2)$$

because $w^{1/2-l/m} \leq \langle\xi\rangle_M^{\kappa_1-1/4-\epsilon'}$ and $\kappa_2+\kappa_1 = 1/2$. We summarize

Lemma 2.19. *There exist $C > 0$, $\epsilon > 0$, $T_0 > 0$ such that*

$$|(\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\mathsf{Im}\,\tilde{Q})v)| \leq C(M^{-\epsilon}+\gamma T)\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2$$
$$+CM^{-\epsilon}\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2 + C(M^{-\epsilon}+\gamma T)\|\langle D\rangle_M^{\kappa_1/2}\mathrm{op}(\tilde{\Lambda})v\|^2$$

for $|x_0| \leq T$.

Since $\mathsf{Im}\lambda \in S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1})$ and $\mathsf{Re}\,\tilde{Q} \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^2)$ Lemma 1.2 shows

$$|(\mathrm{op}(\mathsf{Re}\,\tilde{Q})v, \mathrm{op}(\mathsf{Im}\,\lambda)v)| \leq C\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2.$$

Consider $[D_0 - \mathrm{op}(\mathsf{Re}\,\lambda), \mathrm{op}(\mathsf{Re}\,\tilde{Q})] = -i\mathrm{op}(\partial_{x_0}\mathsf{Re}\,\tilde{Q}) - \mathrm{op}((\mathsf{Re}\,\lambda)\#(\mathsf{Re}\,\tilde{Q}) - (\mathsf{Re}\,\tilde{Q})\#(\mathsf{Re}\,\lambda))$. Since $\partial_{x_0}\mathsf{Re}\,\tilde{Q} \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^{2+\delta})$ and $\mathsf{Re}\,\lambda \in S_{\rho,1/2}(\langle\xi\rangle_M^{\kappa_1})$ and $\mathsf{Re}\,\tilde{Q} \in S_{\rho,1/2}(r^2\langle\xi\rangle_M^2)$ it results from Lemma 1.2 that

$$|([D_0 - \mathrm{op}(\mathsf{Re}\,\lambda), \mathrm{op}(\mathsf{Re}\,\tilde{Q})]v, v)| \leq CM^{-\epsilon'}\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2.$$

It remains to estimate the terms

$$|(\mathrm{op}(R)\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\tilde{\Lambda})v)|, \quad |(\mathrm{op}(a)v, \mathrm{op}(\tilde{\Lambda})v)|, \ \ a \in S_{\rho,1/2}(\langle\xi\rangle_M).$$

If $f \in S_{\rho,1/2}(M^{-\epsilon'}\langle\xi\rangle_M^{\kappa_1})$ then $|(\mathrm{op}(f)w,w)| \le CM^{-\epsilon'}\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})w\|^2$ is clear by Lemma 1.2. Let $a \in S_{\rho,1/2}(\langle\xi\rangle_M)$. Write

$$a(x,\xi) = \frac{ra}{\langle\xi\rangle_M^{\kappa_2/2}\sqrt{w}} \cdot \frac{\langle\xi\rangle_M^{\kappa_2/2}\sqrt{w}}{r}$$

where $(ra)/(\langle\xi\rangle_M^{\kappa_2/2}\sqrt{w}) \in S_{\rho,1/2}(r\langle\xi\rangle_M^{1+\kappa_1/2})$ for $(\langle\xi\rangle_M^{\kappa_2/2}\sqrt{w})^{-1} \le \langle\xi\rangle_M^{-\kappa_2/2+1/4}$ and $\kappa_1+\kappa_2 = 1/2$. Thanks to Lemma 1.2 we have

$$\begin{aligned}|(\mathrm{op}(a)v, \mathrm{op}(\tilde{\Lambda})v)| \le C\varepsilon^{-1}\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2 \\ +\varepsilon\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2.\end{aligned}$$

Therefore we have proved that here exist $c>0$, $C>0$ such that

$$\begin{aligned}&2\big|(\mathrm{op}(\tilde{P})v, \mathrm{op}(\tilde{\Lambda})v)\big| \ge \frac{d}{dx_0}(\|\mathrm{op}(\tilde{\Lambda})v\|^2 + \big(\mathrm{op}(\mathsf{Re}\,\tilde{Q})v,v\big) \\ &\ge 2\gamma(1-CT-CM^{-\epsilon}-C\gamma^{-1/2})\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})\mathrm{op}(\tilde{\Lambda})v\|^2 \\ &\quad+2\gamma(c-CT-CM^{-\epsilon}-\gamma^{-1}\varepsilon^{-1}C)\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2 \\ &\quad+(c-M^{-\epsilon}-C\gamma^{-1/2}-\varepsilon)\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2\end{aligned} \tag{2.23}$$

for $|x_0| \le T$ and any $\varepsilon > 0$. Since $\gamma = M^{\epsilon^*}$ as mentioned in (2.9) one can take M_1, $\varepsilon > 0$ and $T > 0$ such that the right-hand side is bounded from below by

$$\begin{aligned}M^{\epsilon^*}\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})\mathrm{op}(\tilde{\Lambda})v\|^2 + c\,M^{\epsilon^*}\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2 \\ +(c/2)\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\|^2, \quad |x_0| \le T,\ M \ge M_1.\end{aligned} \tag{2.24}$$

Denote

$$A = D_0 - i\gamma\langle D\rangle_M^{\kappa_1}.$$

Since $\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2+\kappa_1} = \sqrt{w}r^{-1}\langle\xi\rangle_M^{1/4+\kappa_1/2} \le r\langle\xi\rangle_M^{1+\kappa_1/2}$ Lemma 1.3 and (2.21) proves that there is $C > 0$ such that

$$\begin{aligned}\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde{\Lambda})v\| \ge \|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})Av\| \\ -C\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|.\end{aligned}$$

Since $M^{\kappa_2}\langle\xi\rangle_M^{3\kappa_1/2} \le \langle\xi\rangle_M^{1/2+\kappa_1/2} \le r\langle\xi\rangle_M^{1+\kappa_1/2}$ we have similarly

$$\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})\mathrm{op}(\tilde{\Lambda})v\| \ge \|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})Av\| - CM^{-\kappa_2}\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|.$$

Then there is $M_2 > M_1$ such that (2.24) is bounded from below by

$$\begin{aligned} & cM^{\epsilon^*}\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})Av\|^2 + c\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})Av\|^2 \\ & +cM^{\epsilon^*}\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2, \quad |x_0| \le T,\ M \ge M_2. \end{aligned} \tag{2.25}$$

From similar arguments one has

$$\begin{aligned} \|Av\| + CM^{-\kappa_2}\|\mathrm{op}(r\langle\xi\rangle_M)v\| \ge \|\mathrm{op}(\tilde{\Lambda})v\| \\ \ge \|Av\| - CM^{-\kappa_2}\|\mathrm{op}(r\langle\xi\rangle_M)v\|. \end{aligned} \tag{2.26}$$

Integrating (2.23) from 0 to t it follows that

$$\begin{aligned} 2C\int_0^t \|\mathrm{op}(\tilde{P})v\|\|\mathrm{op}(\tilde{\Lambda})v\|dt + C(\|\mathrm{op}(\tilde{\Lambda})v(0)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(0)\|)^2 \\ \ge (\|\mathrm{op}(\tilde{\Lambda})v(t)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(t)\|)^2. \end{aligned}$$

With $E^2(t) = \sup_{0\le t_1\le t}(\|\mathrm{op}(\tilde{\Lambda})v(t_1)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(t_1)\|)^2$ we have

$$\big(E - C\int_0^t \|\mathrm{op}(\tilde{P})v\|dt\big)^2 \le C(\|\mathrm{op}(\tilde{\Lambda})v(0)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(0)\|)^2$$

from which it follows that

$$E \le C(\|\mathrm{op}(\tilde{\Lambda})v(0)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(0)\|) + C\int_0^t \|\mathrm{op}(\tilde{P})v\|dt.$$

Therefore taking (2.26) into account we conclude

Proposition 2.4. *There exist $M > 0, C > 0, T > 0$ such that*

$$\begin{aligned} C\big\{\int_0^t \|\mathrm{op}(\tilde{P})v\|dt + \|Av(0)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(0)\|\big\} \\ \ge \|Av(t)\| + \|\mathrm{op}(r\langle\xi\rangle_M)v(t)\| \end{aligned}$$

for $0 \le t \le T$.

Corollary 2.2. *There exist $M > 0, C > 0, T > 0$ such that*

$$C\big\{\int_0^t \|\mathrm{op}(\tilde{P})v\|dt + \|Av(0)\| + \|\langle D\rangle_M v(0)\|\big\} \ge \|Av(t)\| + \|\langle D\rangle_M^{1/2} v(t)\|$$

for $0 \le t \le T$.

Proof. From $\langle\xi\rangle_M^{1/2}\le r\langle\xi\rangle_M\le C\langle\xi\rangle_M$ the proof is clear from Lemmas 1.2 and 1.3. □

We now start with

$$
\begin{aligned}
&2\|\mathrm{op}(\langle\xi\rangle_M^{-\kappa_1/2})\mathrm{op}(\tilde P)v\|\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})\mathrm{op}(\tilde\Lambda)v\|\\
&\ge \frac{d}{dx_0}(\|\mathrm{op}(\tilde\Lambda)v\|^2+\big(\mathrm{op}(\mathsf{Re}\,\tilde Q)v,v\big)\ge c\|\mathrm{op}(\langle\xi\rangle_M^{\kappa_1/2})\mathrm{op}(\tilde\Lambda)v\|^2\\
&+c\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2+c\|\mathrm{op}(r^{-1}\sqrt{w}\langle\xi\rangle_M^{\kappa_2/2})\mathrm{op}(\tilde\Lambda)v\|^2.
\end{aligned}
$$

Then integrating (2.23) in t and taking (2.25) into account one has

Proposition 2.5. *There exist $M>0, C>0, T>0$ such that*

$$
\begin{aligned}
&C\{\int_0^t\|\langle D\rangle_M^{-\kappa_1/2}\mathrm{op}(\tilde P)v\|^2dt+\|Av(0)\|^2+\|\mathrm{op}(r\langle\xi\rangle_M)v(0)\|^2\}\\
&\ge\|Av(t)\|^2+\|\mathrm{op}(r\langle\xi\rangle_M)v(t)\|^2+\int_0^t\|\langle D\rangle_M^{\kappa_1/2}Av\|^2dt\\
&+\int_0^t\|\mathrm{op}(\sqrt{w}r^{-1}\langle\xi\rangle_M^{\kappa_2/2})Av\|^2dt+\int_0^t\|\mathrm{op}(r\langle\xi\rangle_M^{1+\kappa_1/2})v\|^2dt
\end{aligned}
$$

for $0\le t\le T$.

Corollary 2.3. *There exist $M>0, C>0, T>0$ such that*

$$
\begin{aligned}
&C\{\int_0^t\|\langle D\rangle_M^{-\kappa_1/2}\mathrm{op}(\tilde P)v\|^2dt+\|Av(0)\|^2+\|\langle D\rangle_Mv(0)\|^2\}\ge\|Av(t)\|^2\\
&+\|\langle D\rangle_M^{1/2}v(t)\|^2+\int_0^t\|\langle D\rangle_M^{\kappa_1/2}Av\|^2dt+\int_0^t\|\langle D\rangle_M^{1/2+\kappa_1/2}v\|^2dt
\end{aligned}
$$

for $0\le t\le T$.

Remark 2.1. Here we remark that one can choose κ_1 ($>\delta$) arbitrarily close to $\delta=1/3$ for the case (2.5) and $\delta=1/4$ for the case (2.6). This proves that the Cauchy problem for P is solvable in the Gevrey class less than 3 for the case (2.5) and the Gevrey class less than 4 for the case (2.5) for arbitrary lower order terms ([2, 3]).

Remark 2.2. Since κ_2 tends to $\rho-1/2$ as $\kappa_1\downarrow\delta$ for $\kappa_2=1/2-\kappa_1<1/2-\delta=\rho-1/2$ the constraint $\rho-1/2>\kappa_2 s$ on s ([8, (2.1)]) implies that s must be enough close to 1 in our arguments. Note that Remark 2.1 is available if the coefficients are real analytic for example.

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