Notes on tangent bicharacteristics and transition of the spectral type of the Hamilton map

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1 Introduction

In [4, 5, 6] we have studied the Cauchy problem for differential operators P with double characteristics when the spectral type of the Hamilton map F_p changes, more precisely, assuming that the principal symbol $p(x,\xi)$ vanishes exactly of order 2 on a C^∞ manifold Σ with $\operatorname{codim}\Sigma = 3$ and

$$\text{rank}\,\big(\sum_{j=0}^{n} d\xi_j \wedge dx_j\big|_{\Sigma}\big) = \text{constant}, \tag{1.1}$$

$$\left\{\begin{array}{l}\text{the spectral type of } F_p \text{ changes across}\\ \text{a submanifold } S \text{ of } \Sigma \text{ with codimension 1.}\end{array}\right. \tag{1.2}$$

Under these assumptions, after a conjugation with a Fourier integral operator if needed, one can write near any point $\rho \in \Sigma$

$$p(x,\xi) = -\xi_0^2 + \phi_1(x,\xi')^2 + \phi_2(x,\xi')^2, \quad x = (x_0, x') = (x_0, x_1, \dots, x_n)$$

where $d\phi_1$ and $d\phi_2$ are linearly independent at ρ and $\Sigma = \{\xi_0 = 0, \phi_1 = 0, \phi_2 = 0\}$. Since (1.1) is equivalent to that the rank of $(\{\phi_i, \phi_j\})_{i,j=0,1,2}$ is constant on Σ with $\phi_0 = \xi_0$ the rank is either 0 or 2 for the matrix is anti-symmetric. If the rank is 0 then Σ is an involutive manifold and the spectral type is unchanged (see [1], [2] for the Cauchy problem in this case). Under the assumptions $\text{rank}(d\xi \wedge dx) = 2$ on Σ and (1.2), following [4], one can assume without restrictions that

$$\{\xi_0, \phi_2\} > 0, \quad \{\xi_0, \phi_1\} = O(|\phi|) \tag{1.3}$$

near ρ. Here and in what follows $f = O(|\phi|)$, $\phi = (\phi_1, \phi_2)$ means that f is a linear combination of ϕ_1 and ϕ_2 near the reference point. We first recall

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Lemma 1.1. ([5, Lemma 1.2]) *If the spectral structure of F_p changes across S then we have*

$$\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = 0 \text{ on } S.$$

Therefore we have one of the following cases;

(i) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 < 0$ *in* $\Sigma \setminus S$, *that is* p *is noneffectively hyperbolic in* $\Sigma \setminus S$ *with* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 = \{0\}$ *and noneffectively hyperbolic on* S *with* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$,

(ii) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 > 0$ *in* $\Sigma \setminus S$, *that is* p *is effectively hyperbolic in* $\Sigma \setminus S$ *and noneffectively hyperbolic on* S *with* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$,

(iii) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2$ *changes the sign across* S, *that is* p *is effectively hyperbolic in the one side of* $\Sigma \setminus S$, *noneffectively hyperbolic in the other side with* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 = \{0\}$ *and noneffectively hyperbolic on* S *with* $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$.

We have also assumed that the spectral type of F_p changes *simply* across S, that is

$$\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \big(-\theta^2 \textit{ or } \theta^2 \textit{ or } \theta \big) + c_1\phi_1 + c_2\phi_2 \tag{1.4}$$

near $\rho \in S$ according to the case (i), (ii) and (iii) respectively where S is given by $\{\xi_0 = 0, \phi_1 = 0, \phi_2 = 0, \theta = 0\}$ and $d\xi_0, d\phi_1, d\phi_2, d\theta$ are linearly independent. Since $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \{\xi_0 + \phi_1, \phi_2\}\{\xi_0 - \phi_1, \phi_2\}$ we have either $\{\xi_0 + \phi_1, \phi_2\} = 0$ or $\{\xi_0 - \phi_1, \phi_2\} = 0$ on S. Since the arguments are completely parallel we may assume that $\{\xi_0 - \phi_1, \phi_2\} = 0$ on S hence $\{\xi_0, \phi_2\} = \{\phi_1, \phi_2\} > 0$ by (1.3). Now one can write

$$\{\xi_0 - \phi_1, \phi_2\} = \big(-\theta^2 \textit{ or } \theta^2 \textit{ or } \theta \big) + c_1\phi_1 + c_2\phi_2 \tag{1.5}$$

near ρ for the case (i), (ii) and (iii) respectively, if we change the defining function θ, if necessary.

Proposition 1.1. ([4]) *Assume the case* (i).

(1) *If there is a bicharacteristic tangent to* S *at* ρ *then* $\{\xi_0 - \phi_1, \theta\}(\rho) = 0$.

(2) *Assume* $\{\xi_0 - \phi_1, \theta\} = 0$ *near* ρ *on* S. *If* $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\rho) \neq 0$ *there is a bicharacteristic tangent to* S *at* ρ.

Proposition 1.2. ([5, Proposition 2.1]) *Assume the case* (ii).

(1) *If* $\{\xi_0 - \phi_1, \theta\}(\rho) \neq 0$ *there is a bicharacteristic tangent to* Σ *at* ρ.

(2) *Assume* $\{\xi_0 - \phi_1, \theta\} = 0$ *near* ρ *on* S. *If* $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\rho) \neq 0$ *there is a bicharacteristic tangent to* S *at* ρ.

In the proof of Proposition 1.2 in [5] there was a missing case to be examined, so in the next section we simplify slightly the whole proof including the missing case. To exclude the case with bicharacteristics tangent to Σ (or S) in our study of the Cauchy problem we introduce the conditions

$$\{\xi_0 - \phi_1, \theta\}(\rho) \neq 0 \tag{1.6}$$

and

$$\{\xi_0 - \phi_1, \theta\} = 0, \quad \{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = 0 \quad \text{on } S. \tag{1.7}$$

For a coordinates free expression of the condition (1.6) see [4, Lemma 12.3]. In [4] the Cauchy problem for the case (i) is discussed under the condition either (1.6) or (1.7) and we studied the Cauchy problem for the case (ii) in [5] assuming the condition (1.7).

Proposition 1.3. *Assume the case* (iii) *and* $\{\xi_0 - \phi_1, \theta\} = 0$ *near* ρ *on* S. *If* $\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\rho) \neq 0$ *there is a bicharacteristic tangent to* S *at* ρ.

Assuming (1.7) and some additional condition the Cauchy problem for the case (iii) is discussed in [6].

Non existence of tangent bicharacteristics is reflected in

Lemma 1.2. ([5, Lemma 2.2]) *Assume* $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = 0$ *on* S. *Then one can write*

$$\{\xi_0 - \phi_1, \phi_2\} = \big(-\theta^2 \text{ or } \theta^2 \text{ or } \theta \big) + c_0\theta\phi_1 + c_1\phi_1^2 + c_2\phi_2$$

according to the case (i), (ii) *and* (iii) *respectively.*

Proof. Since (1.5) is independent of the choice of defining function θ, replacing θ by

$$\theta + \frac{\{\phi_2, \theta\}}{\{\phi_1, \phi_2\}}\phi_1 - \frac{\{\phi_1, \theta\}}{\{\phi_1, \phi_2\}}\phi_2$$

one can assume that $\{\theta, \phi_j\} = O(|\phi|)$, $j = 1, 2$. Thus

$$\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = c_1\{\phi_1, \phi_2\} + O(|\phi|) = O(|(\theta, \phi)|)$$

which implies that $c_1 = O(|(\theta, \phi)|)$ and hence the result. □

Lemma 1.3. ([5, Lemma 2.3]) *Assume* (1.7). *Then we have*

$$\{\xi_0 - \phi_1, \theta\} = c_0\theta + c_1\phi_1^2 + c_2\phi_2.$$

Proof. Note that $\{\xi_0 - \phi_1, \theta\} = \alpha\theta + \beta\phi_1 + \gamma\phi_2$ where one can assume $\{\theta, \phi_j\} = O(|\phi|)$ as above. Therefore it follows from Lemma 1.2 that

$$\{\theta, \{\xi_0 - \phi_1, \phi_2\}\} = O(|\phi|), \quad \{\xi_0 - \phi_1, \{\theta, \phi_2\}\} = O(|(\theta, \phi)|).$$

Then from the Jacobi identity it follows that $\beta = O(|(\theta, \phi)|)$ and hence

$$\{\xi_0 - \phi_1, \theta\} = \alpha\theta + c_0\theta\phi_1 + c_1\phi_1^2 + c_2\phi_2$$

which proves the assertion. □

Here we give simple examples. Consider

$$(1.8) \quad p_{\pm}(x,\xi) = -\xi_0^2 + \xi_1^2 + (x_0 + x_1 \pm x_1 x_2^2)^2 \xi_n^2 = -\xi_0^2 + \phi_1^2 + \phi_2^2 \quad (n \geq 3)$$

near $(0,\xi) = (0,e_n)$ where $\Sigma = \{\xi_0 = \xi_1 = 0, x_0 + x_1 \pm x_1 x_2^2 = 0\}$. Since $\{\xi_0, \phi_1\} = 0$, $\{\xi_0, \phi_2\} = 1$ and $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \mp 2x_2^2(1 \pm x_2^2/2)\xi_n^2$ we see that the spectral type of $H_{p_{\pm}}$ changes across $S = \Sigma \cap \{x_2 = 0\}$ which corresponds to the case (i) and (ii) in Lemma 1.1. It is clear that $\{\xi_0 - \phi_1, x_2\} = 0$ and $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = \{\mp x_2^2 \xi_n, \phi_2\} = 0$ hence the condition (1.7) is clearly satisfied with $\theta = x_2$. Next consider

$$(1.9) \quad p(x,\xi) = -\xi_0^2 + \xi_1^2 + (x_0 + x_1 - x_1 x_2)^2 \xi_n^2 = -\xi_0^2 + \phi_1^2 + \phi_2^2 \quad (n \geq 3).$$

Since $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = 2x_2(1 - x_2/2)\xi_n^2$ the spectral type changes across $S = \Sigma \cap \{x_2 = 0\}$ which corresponds to the case (iii). It is clear that (1.7) is satisfied with $\theta = x_2$.

2 A proof of Proposition 1.2

Proposition 2.1. *Assume the case* (ii). *If* $\{\xi_0 - \phi_1, \theta\}(\rho) \neq 0$ *then there is a bicharacteristic tangent to* Σ *at* ρ (*which is not tangent to* S).

We follow the arguments given in [3]. Since the assumption is independent of the choice of defining function θ one can assume that $\{\theta, \phi_j\} = O(|\phi|)$, $j = 1, 2$. To simplify notations let us set $\Xi_0 = \xi_0 - \phi_1$, $X_0 = x_0$ and extend them to a full symplectic coordinates (X, Ξ). Switching the notation from (X, Ξ) to (x, ξ) one can write

$$p = -\xi_0(\xi_0 + 2\phi_1) + \phi_2^2$$

where we have from (1.3) that

$$(2.1) \quad \{\xi_0, \phi_1\} = O(|\phi|), \ \ \{\xi_0, \phi_2\} = \theta^2 + O(|\phi|), \ \ \{\theta, \phi_j\} = O(|\phi|), \ j = 1, 2.$$

The assumption is now $\{\xi_0, \theta\}(\rho) \neq 0$ hence one can write $\theta = e(x_0 - \nu(x', \xi'))$ with $e \neq 0$. Thus one can assume $\theta = x_0 + \nu(x', \xi')$. Noting that $d\xi_0$, dx_0, $d\phi_1$, $d\phi_2$ are linearly independent at ρ in view of (2.1) one can take

$$\xi_0, x_0, \phi_1, \phi_2, \psi_1, \ldots, \psi_r, \quad r + 4 = 2(n+1)$$

to be a system of local coordinates around ρ. Thus

$$(2.2) \qquad \nu = O(|(\phi_1, \phi_2, \psi_1, \ldots, \psi_r)|).$$

Note that we can assume that ψ_j are independent of x_0 taking $\psi_j(0, x', \xi')$ as new ψ_j such that $\{\xi_0, \psi_j\} = 0$. Moreover we can assume that

$$(2.3) \qquad \{\psi_j, \phi_k\} = O(|\phi|), \quad k = 1, 2, \ \ j = 1, \ldots, r$$

taking $\psi_j - \{\psi_j,\phi_1\}\phi_2/\{\phi_2,\phi_1\} - \{\psi_j,\phi_2\}\phi_1/\{\phi_1,\phi_2\}$ as new ψ_j. Consider the Hamilton equations

$$dx/ds = \partial p/\partial \xi, \quad d\xi/ds = -\partial p \partial x. \tag{2.4}$$

Let $\gamma(s) = (x(s), \xi(s))$ be a solution to the Hamilton equations and we consider $\xi_0(s)$, $x_0(s)$, $\phi_j(\gamma(s))$, $\theta(\gamma(s))$, $\psi_j(\gamma(s))$ and recall that

$$\frac{d}{ds} f(\gamma(s)) = \{p, f\}(\gamma(s)).$$

Let us change the parameter from s to $t = 1/s$ so that we have

$$d/ds = -tD, \quad D = t(d/dt)$$

and hence $tD(t^pF) = t^{p+1}(DF+pF)$ for $p \in \mathbb{N}$. Let us introduce new unknowns;

$$\begin{cases} \xi_0(s) = t^4\Xi_0(t),\ x_0(s) = tX_0(t),\ \phi_1(\gamma(s)) = t^2\Phi_1(t), \\ \phi_2(\gamma(s)) = t^3\Phi_2(t),\ \psi_j(\gamma(s)) = t^2\Psi_j(t). \end{cases} \tag{2.5}$$

Recall $\delta = \{\phi_1, \phi_2\}(\rho) > 0$ and

$$\{\xi_0, \phi_2\} = \theta^2 + \kappa\phi_1 + C\phi_2 = x_0^2 + 2\nu x_0 + \nu^2 + \kappa\phi_1 + C\phi_2.$$

Let us set

$$V = (X_0, \Phi_1, \Xi_0, \Phi_2, \Psi), \quad \Psi = (\Psi_1, \dots, \Psi_r)$$

then, taking (2.2) and (2.3) into account, it is not difficult to see

$$\begin{cases} DX_0 = -X_0 + 2\Phi_1 + tG(t,V), \\ D\Phi_1 = -2\Phi_1 + 2\delta\Phi_2 + tG(t,V), \\ D\Xi_0 = -4\Xi_0 + 2\kappa\Phi_1\Phi_2 + 2\Phi_2X_0^2 + tG(t,V), \\ D\Phi_2 = -3\Phi_2 + 2\kappa\Phi_1^2 + 2\delta\Xi_0 + 2\Phi_1X_0^2 + tG(t,V), \\ D\Psi_j = -2\Psi_j + tG(t,V) \end{cases} \tag{2.6}$$

where $G(t,V)$ denotes a smooth function in (t,V) such that $G(t,0) = 0$.

Let us define the class of formal series in t and $\log 1/t$

$$\mathcal{E} = \{ \sum_{0 \le j \le i} t^i (\log 1/t)^j V_{ij} \mid V_{ij} \in \mathbb{C}^N \}$$

in which we look for our formal solutions to the reduced Hamilton equations (2.6).

Lemma 2.1. *There exists a formal solution $V \in \mathcal{E}$ satisfying* (2.6) *with* $\Phi_1(0) \neq 0$, $X_0(0) \neq 0$.

Proof. Let us set

$$\begin{cases} X_0 = \sum_{0\le j\le i} t^i(\log 1/t)^j\beta_{ij}^{(0)},\ \ \Xi_0 = \sum_{0\le j\le i} t^i(\log 1/t)^j\alpha_{ij}^{(0)} \\ \Phi_1 = \sum_{0\le j\le i} t^i(\log 1/t)^j\beta_{ij}^{(1)},\ \ \Phi_2 = \sum_{0\le j\le i} t^i(\log 1/t)^j\alpha_{ij}^{(1)} \\ \Psi_k = \sum_{0\le j\le i} t^i(\log 1/t)^j\gamma_{ij}^{(k)}. \end{cases} \tag{2.7}$$

Equating the constant terms of both sides of (2.6) one has

$$\begin{aligned}
-\beta_{00}^{(0)} + 2\beta_{00}^{(1)} &= 0,\\
-2\beta_{00}^{(1)} + 2\delta\alpha_{00}^{(1)} &= 0,\\
-4\alpha_{00}^{(0)} + 2\kappa\beta_{00}^{(1)}\alpha_{00}^{(1)} + 2\alpha_{00}^{(1)}(\beta_{00}^{(0)})^2 &= 0,\\
-3\alpha_{00}^{(1)} + 2\kappa(\beta_{00}^{(1)})^2 + 2\delta\alpha_{00}^{(0)} + 2\beta_{00}^{(1)}(\beta_{00}^{(0)})^2 &= 0,\\
-2\gamma_{00}^{(k)} &= 0.
\end{aligned}$$

Setting $b = \beta_{00}^{(1)}$ we see $\beta_{00}^{(0)} = 2b$, $\alpha_{00}^{(1)} = \delta^{-1}b$ from the first and the second equation. It follows from the third equation that

$$2\alpha_{00}^{(0)} = \kappa\delta^{-1}b^2 + 4\delta^{-1}b^3 = \delta^{-1}b(\kappa b + 4b^2).$$

Inserting these into the fourth equation we have

$$-3\delta^{-1}b + 3\kappa b^2 + 12b^3 = 3b\big(-\frac{1}{\delta} + \kappa b + 4b^2\big) = 0.$$

Let us study

$$-\frac{1}{\delta} + \kappa b + 4b^2 = 0. \tag{2.8}$$

Since $\delta > 0$ it is clear that this equation has nonzero real roots $b = b(\kappa, \delta)$, one is positive and the other one is negative. Let us choose one of such b. Then

$$\overline{V} = \big(\beta_{00}^{(0)}, \beta_{00}^{(1)}, \alpha_{00}^{(0)}, \alpha_{00}^{(1)}, \gamma_{00}^{(k)}\big) = (2b, b, b\delta^{-2}/2, \delta^{-1}b, 0)$$

is uniquely determined. We look for a formal solution to (2.6) in the form $\overline{V}+V$, $V \in \mathcal{E}^{\#}$ where

$$\mathcal{E}^{\#} = \{\sum_{1\le i, 0\le j\le i} t^i(\log 1/t)^j V_{ij} \mid V_{ij} \in \mathbb{C}^N\}.$$

Let us denote

$$V^I = {}^t(X_0, \Phi_1, \Xi_0, \Phi_2), \quad V^{II} = \Psi.$$

Then (2.6) becomes

$$\begin{cases} DV^I = A_I V^I + F_I t + G_I(t, V),\\ DV^{II} = -2V^{II} + F_{II} t + G_{II}(t, V)\end{cases} \tag{2.9}$$

where

$$G_J(t, V) = \sum_{2\le i, 0\le j\le i} G_{J,ij} t^i (\log 1/t)^j, \;\; G_{J,ij} = G_{J,ij}(V_{pq} \mid p \le i-1)$$

and F_J are constant vectors. Making a more precise look on A_I we see

$$A_I = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2\delta \\ 8\delta^{-1}b^2 & 2\kappa\delta^{-1}b & -4 & 2\delta^{-1} \\ 8b^2 & 2(\kappa b+\delta^{-1}) & 2\delta & -3 \end{bmatrix}$$

where we have used (2.8). Then we have

Lemma 2.2. *A_I has real eigenvalues* 1, -6. *Other real eigenvalues of A_I are non positive.*

Proof. We have

$$\begin{aligned} |\lambda - A_I| &= \begin{vmatrix} \lambda+1 & -2 & 0 & 0 \\ 0 & \lambda+2 & 0 & -2\delta \\ -8\delta^{-1}b^2 & -2\kappa\delta^{-1}b & \lambda+4 & -2\delta^{-1} \\ -8b^2 & -2(\kappa b+\delta^{-1}) & -2\delta & \lambda+3 \end{vmatrix} \\ &= (\lambda-1)(\lambda+6)\big(\lambda^2+5\lambda+8-4\kappa\delta b\big) \end{aligned}$$

where we have used $-4b^2\delta = \kappa b\delta - 1$. Noting that

$$1 - \kappa b\delta = 4b^2\delta \geq 0$$

it is clear that real roots of $\lambda^2+5\lambda+8-4\kappa\delta b = 0$, if exist, are less than or equal to -1. □

Proof of Lemma 2.1: Note that (2.9) implies that

$$(iV_{ij} - (j+1)V_{ij+1}) = AV_{ij} + \delta_{i1}\delta_{j0}F + G_{ij}$$

where $G_{ij} = 0$ for $i = 0, 1$. Then we have

$$\begin{cases} (I-A)V_{11} = 0, \\ (I-A)V_{10} = V_{11} + F. \end{cases}$$

Choose $V_{11} \in \mathrm{Ker}\,(I-A)$ so that

$$F + V_{11} \in \mathrm{Im}\,(I-A).$$

Then one can take $V_{10} \neq 0$ so that

$$(I-A)V_{10} = F + V_{11}$$

since $\mathrm{Ker}\,(I-A) \neq \{0\}$ by Lemma 2.2. We turn to the case $i \geq 2$;

$$(iI - A)V_{ij} = (j+1)V_{ij+1} + G_{ij}. \tag{2.10}$$

With $j = i$, (2.10) becomes to

$$(iI - A)V_{ii} = G_{ii}(V_{pq} \mid p \le i-1).$$

Since $iI - A$ is non singular for $i \ge 2$ by Lemma 2.2 one has

$$V_{ii} = (iI - A)^{-1} G_{ii}(V_{pq} \mid p \le i-1).$$

Recurrently one can solve V_{ij} by

$$V_{ij} = (iI - A)^{-1}\big\{(j+1)V_{ij+1} + G_{ij}(V_{pq} \mid p \le i-1)\big\}$$

for $j = i-1,\ i-2, \dots, 0$. This proves the assertion. □

Proof of Proposition 2.1: Repeating a similar but much simpler arguments as in [3] we can conclude that there is a solution to (2.6) which is asymptotic to the formal solution given in Lemma 2.1. Thus we get a solution $(x(s), \xi(s))$ to the Hamilton equations (2.4). From (2.5) we have, with $\xi_0 = \phi_0$, that

$$\frac{d\phi_j}{dx_0}\Big|_{x_0=0} = \Big(\frac{d\phi_j}{dx}\Big/\frac{dx_0}{ds}\Big)\Big|_{x_0=0} = 0, \quad \frac{d\theta}{dx_0} = 1$$

hence it is clear that thus obtained bicharacteristic $(x(s), \xi(x))$ is tangent to Σ but not to S. □

Proposition 2.2. *Assume* $\{\xi_0 - \phi_1, \theta\} = 0$ *near* ρ *on* S*. If* $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\rho) \neq 0$ *there is a bicharacteristic tangent to* S *at* ρ*.*

Proof. As in the proof of Proposition 2.1 setting $\Xi_0 = \xi_0 - \phi_1$, $X_0 = x_0$ we extend them to a full symplectic coordinates (X, Ξ) and then switch the notation from (X, Ξ) to (x, ξ). Now the assumption reads

$$\begin{aligned} &\{\xi_0, \phi_1\} = O(|\phi|), \quad \{\xi_0, \phi_2\} = \{-\theta^2 \textit{ or } \theta^2 \textit{ or } \theta\} + O(|\phi|), \\ &\{\theta, \phi_j\} = O(|\phi|), \quad \{\xi_0, \theta\} = O(|(\theta, \phi)|). \end{aligned} \tag{2.11}$$

Since $d\xi_0$, dx_0, $d\phi_1$, $d\phi_2$, $d\theta$ are linearly independent at ρ in view of (2.11) one can take

$$\xi_0, x_0, \phi_1, \phi_2, \theta, \psi_1, \dots, \psi_r, \quad r + 5 = 2(n+1)$$

to be a system of local coordinates around ρ. After that the proof is a repetition of [5, Proposition 4.3] (similar to the proof of Proposition 2.1). □

Combining Propositions 2.1 and 2.2 we conclude the proof of Proposition 1.2.

References

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