Geometric results for hyperbolic operators with spectral transition of the Hamilton map (Tangent bicharacteristic, Elementary factorization)

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Abstract

1 Tangent bicharacteristics

Let p be of normal form

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \Sigma_{j=2}^{r}\phi_j^2 + \Sigma_{j=r+1}^{d}\phi_j^2$$

and recall that $\theta|_\Sigma$ is conformally invariant. Let $\tilde\theta$ be an extension of $\theta|_\Sigma$ then we say $d\theta$, $d\phi_j$ are linearly independent (resp. dependent) at $\bar\rho$ if $d\tilde\theta$, $d\phi_j$ are linearly independent (resp. dependent) at $\bar\rho$. This is independent of the choice of extensions of $\theta|_\Sigma$. Denote

$$\nu = \{\xi_0 - \phi_1, \{\xi_0 - \phi_1, \theta\}\}(\bar\rho), \quad \kappa = \{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\bar\rho)/\{\phi_1, \phi_2\}(\bar\rho).$$

Then we have

Proposition 1.1. *Assume that p is of normal form up to a term $O^4(\Sigma')$ with $\theta(\bar\rho) = 0$ and*

$$\{\xi_0 - \phi_1, \theta\}(\bar\rho) = 0, \quad \{\phi_j, \theta\}(\bar\rho) = 0, \ \ r+1 \le j \le d. \tag{1.1}$$

If $\kappa^2 - 4\nu > 0$ there exists a bicharacteristic γ of p tangent to Σ at $\bar\rho$. More precisely, parametrizing γ by x_0 with $\gamma(0) = \bar\rho$ we have $\theta(\gamma) = O(x_0^2)$ and $\phi_j(\gamma) = O(x_0^2)$ for $j = 0, \dots, d$ where $\lim_{x_0\to 0}\phi_j(\gamma)/x_0^{1+j} \neq 0$, $j = 1, 2$.

Corollary 1.1. *Assume that p is of normal form up to a term $O^4(\Sigma')$ with $\theta(\bar\rho) = 0$. If $d\theta$, $d\phi_j$ are linearly dependent at $\bar\rho$ and $\kappa^2 - 4\nu > 0$ there exists a bicharacteristic of p tangent to Σ at $\bar\rho$.*

Proof. If $d\theta$, $d\phi_j$ are linearly dependent at $\bar\rho$ it is clear that (1.1) holds. $\Box$

Corollary 1.2. *Assume that p is of normal form up to a term $O^4(\Sigma')$ with $\theta(\bar\rho)=0$ and that $\theta|_\Sigma\geq 0$ or $\theta|_\Sigma\leq 0$ near $\bar\rho$. If $\{\xi_0-\phi_1,\{\xi_0-\phi_1,\theta\}\}(\bar\rho)=0$ and $\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}(\bar\rho)\neq 0$ there exists a bicharacteristic of p tangent to Σ at $\bar\rho$.*

Proof. By extension Lemma we may assume $\theta\geq 0$ or $\theta\leq 0$ in a neighborhood of $\bar\rho$ hence we have $\nabla\theta(\bar\rho)=0$ showing that $d\theta$, $d\phi_j$ are linearly dependent at $\bar\rho$. Thus one can apply Corollary 1.1 with $\nu=0$ and $\kappa\neq 0$. $\Box$

Corollary 1.3. *Assume that p is of normal form up to a term $O^4(\Sigma')$ with $\theta(\bar\rho)=0$ and $\theta|_\Sigma\leq 0$ near $\bar\rho$. Then there exists a bicharacteristic of p tangent to Σ at $\bar\rho$ unless $\{\xi_0-\phi_1,\{\xi_0-\phi_1,\theta\}\}(\bar\rho)=0$ and $\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}(\bar\rho)=0$.*

Proof. Note that (1.1) holds and $\nu\leq 0$. If $\{\xi_0-\phi_1,\{\xi_0-\phi_1,\theta\}\}(\bar\rho)\neq 0$ we have $\nu<0$ hence Proposition 1.1 proves the assertion. $\Box$

Assume that p is of normal form up to a term $O^4(\Sigma')$;

$$p=-(\xi_0+\phi_1)(\xi_0-\phi_1)+\theta\phi_1^2+\Sigma_{j=2}^r\phi_j^2+\Sigma_{j=r+1}^d\phi_j^2+O^4(\Sigma') \tag{1.2}$$

With $\mu=\sqrt{1+\theta}>0$ one can rewrite

$$p=-(\xi_0-\mu\phi_1)(\xi_0+\mu\phi_1)+\Sigma_{j=2}^r\phi_j^2+\Sigma_{j=r+1}^d\phi_j^2+O^4(\Sigma')$$

where denoting $\tilde\phi_1=\mu\phi_1$ one sees

$$\begin{aligned}\{\xi_0-\tilde\phi_1,\phi_j\}&=\{\xi_0-\phi_1+(1-\mu)\phi_1,\phi_j\}\overset{\Sigma'}{=}(1-\mu)\{\phi_1,\phi_j\}\\&\overset{\Sigma'}{=}-\theta/\mu(1+\mu)\{\tilde\phi_1,\phi_j\}\overset{\Sigma'}{=}\hat\theta\{\tilde\phi_1,\phi_j\},\quad 0\leq j\leq d\end{aligned}$$

where we have set

$$\hat\theta=-\theta/(\sqrt{1+\theta}+1+\theta). \tag{1.3}$$

Writing $\mu\phi_1\to\phi_1$ we have

$$p=-(\xi_0-\phi_1)(\xi_0+\phi_1)+\Sigma_{j=2}^r\phi_j^2+\Sigma_{j=r+1}^d\phi_j^2+O^4(\Sigma').$$

We divide the proof of Proposition 1.1 into the case that $d\theta$, $d\phi_j$ are linearly independent or dependent.

1.1 Case that $d\theta$, $d\phi_j$ are linearly independent

In this section, we assume that $d\phi_j$, $d\theta$ are linearly independent at $\bar\rho$. Hence $\Sigma\cap\{\theta=0\}$ is a submanifold of Σ passing through $\bar\rho$ and that p is effectively hyperbolic in the side where $\theta<0$ and noneffectively hyperbolic type I on the other side where $\theta>0$. Taking (1.3) into account we can assume

$$\{\phi_j,\hat\theta\}=O(\Sigma'),\quad 1\leq j\leq r \tag{1.4}$$

thanks to the extension lemma. Recall that the assumption (1.1) implies

$$\{\xi_0-\phi_1,\hat{\theta}\}(\bar{\rho})=0,\quad \{\phi_j,\hat{\theta}\}(\bar{\rho})=0,\;\; r+1\le j\le d. \tag{1.5}$$

Choose a system of symplectic coordinates (X,Ξ) such that $X_0=x_0$ and $\Xi_0=\xi_0-\phi_1$. Writing $(X,\Xi)\to(x,\xi)$ one has

$$p=-\xi_0^2-2\xi_0\phi_1+\Sigma_{j=2}^r\phi_j^2+\Sigma_{j=r+1}^d\phi_j^2+O^4(\Sigma')$$

where

$$\{\phi_i,\phi_j\}\stackrel{\Sigma'}{=}0,\; 0\le i\le d,\;\; j\ge r+1,\; \{\xi_0,\phi_j\}\stackrel{\Sigma'}{=}\hat{\theta}\,\{\phi_1,\phi_j\},\; 1\le j\le d, \tag{1.6}$$

$$\{\phi_1,\phi_2\}(\bar{\rho})\neq 0,\quad \{\phi_2,\phi_j\}\stackrel{\Sigma'}{=}0,\;\; 3\le j\le r,\quad \det(\{\phi_i,\phi_j\})_{3\le i,j\le r}\neq 0 \tag{1.7}$$

and

$$\begin{aligned}\partial_{x_0}^k\hat{\theta}(\bar{\rho})=0,\;\; k=0,1,\quad \partial_{x_0}^2\hat{\theta}(\bar{\rho})=-\partial_{x_0}^2\theta(\bar{\rho})/2=-\nu/2,\\ \{\phi_j,\hat{\theta}\}(\bar{\rho})=0,\quad 1\le j\le d.\end{aligned} \tag{1.8}$$

Lemma 1.1. *We have*

$$\{\phi_2,\{\phi_j,\xi_0\}\}(\bar{\rho})=0,\quad r+1\le j\le d.$$

Proof. Note that $\{\xi_0,\{\phi_2,\phi_j\}\}(\bar{\rho})=0$ for $3\le j\le d$ by (1.6), (1.7) and (1.8). Since $\{\phi_j,\{\xi_0,\phi_2\}\}(\bar{\rho})=\{\phi_j,\hat{\theta}\}(\bar{\rho})\{\phi_1,\phi_2\}(\bar{\rho})=0$ for $r+1\le j\le d$ by (1.6) and (1.8) then Jacobi's identity shows the assertion. □

From (1.6) we see that $d\hat{\theta}$, $dx_0, d\phi_j$, $0\le j\le d$ are linearly independent at $\bar{\rho}$. Take

$$w=(\xi_0,x_0,\hat{\theta},\phi_1,\ldots,\phi_d,\psi_1,\ldots,\psi_k)\quad (d+k=2n-1)$$

to be a system of local coordinates around $\bar{\rho}$ so that $w(\bar{\rho})=0$. Note that we can assume that ψ_j are independent of x_0 taking $\psi_j(0,x',\xi')$ as new ψ_j. Moreover we can assume that $\{\phi_i,\psi_j\}\stackrel{\Sigma'}{=}0$ for $i=1,2$ taking $\psi_j-\{\psi_j,\phi_2\}\phi_1/\{\phi_1,\phi_2\}-\{\psi_j,\phi_1\}\phi_2/\{\phi_2,\phi_1\}$ as new ψ_j. Thus it can be assumed that

$$\{\xi_0,\psi_j\}\equiv 0,\quad \{\phi_i,\psi_j\}\stackrel{\Sigma'}{=}0,\quad i=1,2,\;\; 1\le j\le k. \tag{1.9}$$

Let $\gamma(s)=(x(s),\xi(s))$ be a solution to the Hamilton equation

$$\frac{dx}{ds}=\frac{\partial p}{\partial\xi},\quad \frac{d\xi}{ds}=-\frac{\partial p}{\partial x} \tag{1.10}$$

and recall $df(\gamma(s))/ds=\{p,f\}(\gamma(s))$. We change the parameter from s to t:

$$t=s^{-1}$$

so that we have $d/ds = -tD$ and $D = t(d/dt)$ and hence

$$\frac{d}{ds}(t^p F) = -t^{p+1}(DF + pF).$$

We now introduce new unknowns

$$\begin{cases} \xi_0(\gamma(s)) = t^4 \Xi_0(t), \;\; x_0(\gamma(s)) = tX_0(t), \\ \hat{\theta}(\gamma(s)) = t^2 \Theta(t), \\ \phi_1(\gamma(s)) = t^2 \Phi_1(t), \;\; \phi_2(\gamma(s)) = t^3 \Phi_2(t), \\ \phi_j(\gamma(s)) = t^4 \Phi_j, \quad 3 \le j \le r, \\ \phi_j(\gamma(s)) = t^3 \Phi_j(t), \quad r+1 \le j \le d, \\ \psi_j(\gamma(s)) = t^2 \Psi_j(t), \quad 1 \le j \le k \end{cases} \tag{1.11}$$

and write $W = (\Xi_0, X_0, \Theta, \Phi_1, \ldots, \Phi_d, \Psi_1, \ldots, \Psi_k)$. In what follows $G(t, W)$, which may change from line to line, denotes a smooth function in (t, W) defined near $(0, 0)$ such that $G(t, 0) = 0$. It is clear that

$$\{O^4(\Sigma'), \phi_j\}(\gamma) = t^6 G(t, W), \quad 0 \le j \le d.$$

Denote

$$\{\xi_0, \phi_j\} = \hat{\theta}\{\phi_1, \phi_j\} + \sum_{i=1}^{d} C_i^j \phi_i, \;\; \kappa_j = C_1^j(\bar{\rho}), \;\; \delta = \{\phi_1, \phi_2\}(\bar{\rho}) \ne 0 \tag{1.12}$$

then from Lemma 1.1 we get

$$\kappa_j = 0, \quad j = r+1, \ldots, d. \tag{1.13}$$

Here note that

$$\kappa_2 = C_1^2(\bar{\rho}) = \{\{\xi_0, \phi_2\}, \phi_2\}(\bar{\rho}) / \{\phi_1, \phi_2\}(\bar{\rho}) = \kappa. \tag{1.14}$$

It is clear that

$$\{\xi_0, \phi_j\}(\gamma) = \big(\delta_j \Theta + \kappa_j \Phi_1\big) t^2 + t^3 G(t, W), \quad \delta_j = \{\phi_1, \phi_j\}(\bar{\rho}). \tag{1.15}$$

Note that we have

$$\{\xi_0, \hat{\theta}\}(\gamma) = -\nu X_0 t/2 + t^2 G(t, W).$$

Indeed write $\hat{\theta}(x, \xi') = \Sigma_{k=0}^2 x_0^k \hat{\theta}_k(x', \xi') + O(x_0^3)$. Since $\{\xi_0, \hat{\theta}\}(\bar{\rho}) = 0$ one can write $\hat{\theta}_1(\gamma) = t^2 G(t, W)$ which proves the assertion. The Hamilton equation is reduced to

$$
(1.16)\qquad \begin{cases}
D\Xi_0 = -4\Xi_0 + 2\kappa_2\Phi_1\Phi_2 + 2\delta\Theta\Phi_2 + tG(t,W),\\
DX_0 = -X_0 + 2\Phi_1 + tG(t,W),\\
D\Phi_1 = -2\Phi_1 + 2\delta\Phi_2 + tG(t,W),\\
D\Theta = -2\Theta - \nu X_0\Phi_1 + tG(t,W),\\
D\Phi_2 = -3\Phi_2 + 2\kappa_2\Phi_1^2 + 2\delta\Xi_0 + 2\delta\Phi_1\Theta + tG(t,W),\\
tD\Phi_j = -4t\Phi_j + 2\kappa_j\Phi_1^2 + 2\delta_j\Xi_0 + 2\delta_j\Phi_1\Theta\\
\qquad\qquad -2\Sigma_{k=3}^{r}\{\phi_k,\phi_j\}(\bar\rho)\Phi_k + tG(t,W),\quad 3\le j\le r\\
D\Phi_j = -3\Phi_j + tG(t,W),\ \ r+1\le j\le d,\\
D\Psi_j = -2\Psi_j - 2\sum_{k=r+1}^{d}\{\phi_k,\psi_j\}(\bar\rho)\Phi_k + tG(t,W),\ \ 1\le j\le k.
\end{cases}
$$

We introduce a class of formal power series in $(t,\log t)$

$$
\mathcal{E} = \{\sum_{0\le j\le i} t^i(\log t)^j w_{ij} \mid w_{ij}\in\mathbb{C}^N\}
$$

in which we look for a formal solution to the reduced Hamilton equation (1.16).

Lemma 1.2. *There exists $W = (\Xi_0, X_0, \Theta, \Phi_1, \dots, \Phi_d, \Psi_1, \dots, \Psi_k) \in \mathcal{E}$ such that $\Phi_1(0)\neq 0$ and $X_0(0)\neq 0$ satisfying (1.16) formally.*

Assume that $W = (\Xi_0, X_0, \Theta, \Phi_1, \dots, \Phi_d, \Psi_1, \dots, \Psi_k) \in \mathcal{E}$ satisfies (1.16) formally. Denote

$$
(1.17)\qquad \begin{cases}
X_0 = \sum_{0\le j\le i} t^i(\log 1/t)^j x_{ij}^0,\ \ \Xi_0 = \sum_{0\le j\le i} t^i(\log 1/t)^j \xi_{ij}^0\\
\Theta = \sum_{0\le j\le i} t^i(\log 1/t)^j \theta_{ij},\\
\Phi_\mu = \sum_{0\le j\le i} t^i(\log 1/t)^j \phi_{ij}^\mu,\ \ \Psi_\nu = \sum_{0\le j\le i} t^i(\log 1/t)^j \psi_{ij}^\nu
\end{cases}
$$

and $x_{00}^0 = \bar x_0$, $\xi_{00}^0 = \bar\xi_0$, $\theta_{00} = \bar\theta$, $\phi_{00}^\mu = \bar\phi_\mu$ and $\psi_{00}^\nu = \bar\psi_\nu$. Equating the constant terms of both sides of (1.16) one has (except for the equations for Φ_j with $3\le j\le r$)

$$
\begin{gathered}
-4\bar\xi_0 + 2\kappa\bar\phi_1\bar\phi_2 + 2\delta\bar\theta\bar\phi_2 = 0,\quad -\bar x_0 + 2\bar\phi_1 = 0,\quad -2\bar\phi_1 + 2\delta\bar\phi_2 = 0,\\
-2\bar\theta - \nu\bar x_0\bar\phi_1 = 0,\quad -3\bar\phi_2 + 2\kappa\bar\phi_1^2 + 2\delta\bar\xi_0 + 2\delta\bar\theta\bar\phi_1 = 0,\\
-3\bar\phi_j = 0,\ \ r+1\le j\le d,\quad -2\bar\psi_j - 2\Sigma_{k=r+1}^{d}\{\phi_k,\psi_j\}(\bar\rho)\bar\phi_k = 0.
\end{gathered}
$$

From the third line one has $\bar\phi_j = 0$, $r+1\le j\le d$ and $\bar\psi_j = 0$. Setting $b = \bar\phi_1$ we see

$$
\bar x_0 = 2b,\quad \bar\phi_2 = \delta^{-1}b,\quad \bar\theta = -\nu b^2,\quad 2\bar\xi_0 = \kappa\delta^{-1}b^2 - \nu b^3
$$

hence the second equation on the second line becomes

$$
(1.18)\qquad -3\delta^{-1}b + 3\kappa b^2 - 3\nu\delta b^3 = 3b\big(-1/\delta + \kappa b - \nu\delta b^2\big) = 0.
$$

Note that $\bar\phi_j$, $3\le j\le r$ are uniquely determined since $\det(\{\phi_i,\phi_j\}(\bar\rho))_{3\le i,j\le r}\neq 0$. Let us study

$$
(1.19)\qquad 1/\delta - \kappa b + \nu\delta b^2 = 0.
$$

Since $\delta \neq 0$ we see that (1.19) has a nonzero real root $b = b(\kappa, \delta) \neq 0$ if

$$\kappa^2 - 4\nu > 0.$$

Let us choose one of such $b \neq 0$ when $\nu < 0$ and if $\nu > 0$ we choose $b \neq 0$ such that $\delta\kappa b$ is smaller. Denote

$$\overline{W} = (\bar{\xi}_0, \bar{x}_0, \bar{\theta}, \bar{\phi}_j, \bar{\psi}_i)$$

and look for a formal solution to (1.16) of the form $\overline{W} + W$ with $W \in \mathcal{E}^{\#}$ where

$$\mathcal{E}^{\#} = \{W = \sum_{1 \leq i, 0 \leq j \leq i} t^i (\log t)^j w_{ij} \mid w_{ij} \in \mathbb{C}^{2n+2}\}.$$

To simplify notation we set

$$\begin{cases} W^I = (X_0, \Phi_2, \Xi_0, \Phi_1, \Theta), \quad W^{II} = (\Phi_3, \ldots, \Phi_r) \\ W^{III} = (\Phi_{r+1}, \ldots, \Phi_d), \quad W^{IV} = (\Psi_1, \ldots, \Psi_k) \end{cases}$$

then $W = {}^t(W^I, W^{II}, W^{III}, W^{IV}) = \sum_{1 \leq i, 0 \leq j \leq i} t^i (\log t)^j w_{ij}$ satisfies

$$H\big(i w_{ij} - (j+1) w_{ij+1}\big) = A w_{ij} + \delta_{i1}\delta_{j0} F + G_{ij} \tag{1.20}$$

with $H = I \oplus I \oplus I \oplus O$ where I and O is the identity and zero matrix respectively and

$$A = \begin{bmatrix} A_I & O & O & O \\ B_{II} & A_{II} & O & O \\ O & O & -3I & O \\ O & O & B_{III} & -2I \end{bmatrix}. \tag{1.21}$$

Moreover, F is a constant vector and

$$G_{ij} = G_{ij}(w_{pq} \mid q \leq p \leq i-1), \quad G_{ij} = 0, \;\; i = 0, 1.$$

Here note that $|\lambda - A| = |\lambda - A_I||\lambda - A_{II}||\lambda + 3I||\lambda + 2I|$ and all eigenvalues of A_{II} are pure imaginary. Making a more precise look on A_I we see

$$A_I = \begin{bmatrix} -1 & 0 & 0 & 2 & 0 \\ 0 & -3 & 2\delta & 2(\kappa b + \delta^{-1}) & 2\delta b \\ 0 & 2\delta^{-1} & -4 & 2\kappa\delta^{-1} b & 2b \\ 0 & 2\delta & 0 & -2 & 0 \\ -\nu b & 0 & 0 & -2\nu b & -2 \end{bmatrix} \tag{1.22}$$

where we have used (1.19). To confirm that (1.20) can be solved successively we prove the following

Lemma 1.3. *Assume $\kappa^2 - 4\nu > 0$. Then A_I has an eigenvalue 1 and the other real eigenvalues are negative.*

Proof. Expand $|\lambda - A_I|$ with respect to the last row we see

$$
\begin{aligned}
|\lambda - A_I| &= (\lambda+1)(\lambda+2)\begin{vmatrix} \lambda+3 & -2\delta & -2(\kappa b+\delta^{-1}) \\ -2\delta^{-1} & \lambda+4 & -2\kappa\delta^{-1}b \\ -2\delta & 0 & \lambda+2 \end{vmatrix} \\
&\quad -2\nu b(\lambda+2)\begin{vmatrix} \lambda+3 & -2\delta & -2\delta b \\ -2\delta b & \lambda+4 & -2b \\ -2\delta & 0 & 0 \end{vmatrix} \\
&= (\lambda+1)(\lambda+2)(\lambda+6)(\lambda^2+3\lambda-4\kappa\delta b)+8\nu\delta^2b^2(\lambda+2)(\lambda+6) \\
&= (\lambda-1)(\lambda+2)(\lambda+6)\big(\lambda^2+5\lambda+8-4\kappa\delta b\big)
\end{aligned}
$$

where we have used $\nu\delta^2 b^2 = \kappa b\delta - 1$. Write

$$\lambda^2+5\lambda+8-4\kappa\delta b = (\lambda+4)(\lambda+1)-4\nu\delta^2b^2.$$

Then it is clear that real roots of $(\lambda+4)(\lambda+1) = 4\nu\delta^2b^2$, if exist, are less than or equal to -1 if $\nu \le 0$. Consider the case $\nu > 0$ so that $\kappa^2 - 4\nu > 0$ is satisfied. Note that the roots b of (1.19) are given by

$$b = \frac{\kappa}{2\nu\delta} \pm \frac{\sqrt{\kappa^2-4\nu}}{2\nu|\delta|}$$

from which we have

$$(\lambda+4)(\lambda+1) = 4\nu\delta^2b^2 = 4\kappa\delta b - 4 = 4 + \frac{\sqrt{\kappa^2-4\nu}\,\big(\sqrt{\kappa^2-4\nu}\pm\delta\kappa/|\delta|\big)}{\nu/2}.$$

By our choice of b the right-hand side is less than 4. This proves the assertion in the case $\nu > 0$. □

The rest of the proof of Proposition 1.1 is just the repetition of the arguments in [9, Sections 3.3 and 3.4].

Remark 1.1. Assume $\nu = \{\xi_0, \{\xi_0, \theta\}\}(\bar\rho) = 0$, $\kappa \neq 0$ and that

$$\theta \ge \Sigma_{i=1}^{l} k_i^2 \quad \text{or} \quad -\theta \ge \Sigma_{i=1}^{l} k_i^2$$

where $d\phi_j$, dk_i are linearly independent at $\bar\rho$. Since $\{\xi_0, \{\xi_0, \pm\theta - \Sigma_{i=1}^{l} k_i^2\}\}(\bar\rho) \ge 0$ it follows that $\{\xi_0, k_i\}(\bar\rho) = 0$, $1 \le i \le l$. From this we see easily that

$$dk_i(\gamma(t))/dt = -\{p, k_i\}(\gamma(t))/t^2 = O(t), \quad t \to 0$$

which proves that $k_i(\gamma(x_0)) = O(x_0^2)$. Therefore the bicharacteristic $\gamma(x_0)$ is tangent to the manifold $\Sigma_0 = \Sigma \cap \{k_i = 0, i = 1, \ldots, l\} \supset \Sigma \cap \{\theta = 0\}$.

Remark 1.2. Assume $\theta|_\Sigma \ge 0$ or $\theta|_\Sigma \le 0$ near $\bar\rho$ and $\nu = \{\xi_0, \{\xi_0, \theta\}\}(\bar\rho) \neq 0$. After extending θ thanks to Malgrange preparation theorem one can write

$$\theta = e\big((x_0 - \psi(x', \xi'))^2 + h(x', \xi')\big), \quad h(x', \xi') \ge 0, \quad e(\bar\rho) \neq 0.$$

With $f = x_0 - \psi(x', \xi')$ it is clear that $\{\theta = 0\} \subset \{f = 0\}$. Note that df, $d\phi_j$ are linearly independent at $\bar\rho$ because $\{\xi_0, f\} \neq 0$. Assume $\kappa^2 - 4\nu > 0$ hence there is a bicharacteristic γ tangent to Σ by Proposition 1.1. Note that

$$df(\gamma(t))/dt = -\{p, f\}(\gamma(t))/t^2 \to 2\Phi_1(0)\{\xi_0, f\}(\bar\rho), \quad t \to 0$$

which proves that

$$\begin{aligned}\lim_{x_0\to 0} df(\gamma(x_0))/dx_0 &= \lim_{t\to 0}\big(df(\gamma(t))/dt\big)\big(dt/dx_0\big)\\ &= 2\Phi_1(0)\{\xi_0, f\}(\bar\rho)/X_0(0) = \{\xi_0, f\}(\bar\rho) = 1.\end{aligned}$$

Thus the bicharacterisitic $\gamma(x_0)$ is transversal to the manifold $\Sigma \cap \{f = 0\}$.

1.2 Case that $d\theta$, $d\phi_j$ are linearly dependent

We first note that $d\theta$, $d\phi_j$ are linearly dependent at $\bar\rho$ then

$$\{\xi_0 - \phi_1, \theta\}(\bar\rho) = 0, \quad \{\phi_j, \theta\}(\bar\rho) = 0, \;\; r+1 \le j \le d. \tag{1.23}$$

Repeat the same arguments as in Section 1.1. Choose a system of symplectic coordinates (X, Ξ) such that $X_0 = x_0$ and $\Xi_0 = \xi_0 - \phi_1$. Writing $(X, \Xi) \to (x, \xi)$ one has

$$p = -\xi_0^2 - 2\xi_0\phi_1 + \Sigma_{j=2}^r\phi_j^2 + \Sigma_{j=r+1}^d\phi_j^2 + O^4(\Sigma')$$

where (1.6), (1.7) and (1.8) hold. From (1.6) we see that $dx_0, d\phi_j$, $0 \le j \le d$ are linearly independent at $\bar\rho$. Take

$$w = (\xi_0, x_0, \phi_1, \ldots, \phi_d, \psi_1, \ldots, \psi_k) \quad (d + k = 2n)$$

to be a system of local coordinates around $\bar\rho$ so that $w(\bar\rho) = 0$. Recall that we can assume that (1.9) holds. Let $\gamma(s) = (x(s), \xi(s))$ be a solution to the Hamilton equation as before. We change the parameter from s to $t = s^{-1}$ and introduce new unknowns

$$\begin{cases}\xi_0(\gamma(s)) = t^4\Xi_0(t), \;\; x_0(\gamma(s)) = tX_0(t),\\ \phi_1(\gamma(s)) = t^2\Phi_1(t), \;\; \phi_2(\gamma(s)) = t^3\Phi_2(t),\\ \phi_j(\gamma(s)) = t^4\Phi_j, \quad 3 \le j \le r,\\ \phi_j(\gamma(s)) = t^3\Phi_j(t), \quad r+1 \le j \le d,\\ \psi_j(\gamma(s)) = t^2\Psi_j(t), \quad 1 \le j \le k\end{cases} \tag{1.24}$$

and write $W = (\Xi_0, X_0, \Phi_1, \ldots, \Phi_d, \Psi_1, \ldots, \Psi_k)$ and $G(t, W)$ being as before.

Lemma 1.4. *One can write*

$$\hat\theta(x, \xi') = \ell(\phi_2, \ldots, \phi_d) - \nu x_0^2/4 + \theta_2(w') + \theta_3(w')$$

where ℓ is a linear form in $(\phi_2, \ldots, \phi_d)$ and θ_2 is a quadratic form in $w' = (x_0, \phi_1, \ldots, \phi_d, \psi_1, \ldots, \psi_k)$ containing no such term $c\,x_0^2$ ($c \in \mathbb{R}$) and $\theta_3(w') = O(|w'|^3)$.

Proof. Since $\hat{\theta}(\bar{\rho}) = 0$ the Taylor formula gives $\hat{\theta}(x, \xi') = \hat{\theta}_1(w') + \hat{\theta}_2(w') + O(|w'|^3)$ where $\hat{\theta}_1$ is a linear form in $(\phi_1, \ldots, \phi_d)$ and $\hat{\theta}_2$ is a quadratic form in w' since $d\hat{\theta}$ and $d\phi_j$ are lineraly dependent. Since $\{\phi_2, \hat{\theta}\}(\bar{\rho}) = 0$ we see that $\hat{\theta}_1$ is independent of ϕ_1. The rest of the proof is clear. $\square$

Thanks to this lemma one has

$$\hat{\theta}(\gamma) = -\nu X_0^2 t^2/4 + t^3 G(t, W), \quad \hat{\theta}(\gamma) = t^2 G(t, W). \tag{1.25}$$

Taking (1.12) into account one has

$$\{\xi_0, \phi_j\}(\gamma) = (-\nu X_0^2 \delta_j/4 + \kappa_j \Phi_1)t^2 + t^3 G(t, W) \tag{1.26}$$

where $\delta_j = \{\phi_1, \phi_j\}(\bar{\rho})$. Hence by (1.6) and (1.13) we have

$$\{\phi_j, \xi_0\}(\gamma) = t^3 G(t, W), \quad r+1 \le j \le d. \tag{1.27}$$

Thanks to (1.25), (1.26) and (1.27) the Hamilton equation is reduced to

$$\begin{cases} D\Xi_0 = -4\Xi_0 + 2\kappa\Phi_1\Phi_2 - (\nu\delta/2)X_0^2\Phi_2 + tG(t, W), \\ DX_0 = -X_0 + 2\Phi_1 + tG(t, W), \\ D\Phi_1 = -2\Phi_1 + 2\delta\Phi_2 + tG(t, W), \\ D\Phi_2 = -3\Phi_2 + 2\kappa\Phi_1^2 + 2\delta\Xi_0 - (\nu\delta/2)X_0^2\Phi_1 \\ \qquad +tG(t, W), \\ tD\Phi_j = -4t\Phi_j + 2\kappa_j\Phi_1^2 + 2\delta_j\Xi_0 - (\nu\delta_j/2)X_0^2\Phi_1 \\ \qquad -2\Sigma_{k=3}^r\{\phi_k, \phi_j\}(\bar{\rho})\Phi_k + tG(t, W), \quad 3 \le j \le r \\ D\Phi_j = -3\Phi_j + tG(t, W), \ \ r+1 \le j \le d, \\ D\Psi_j = -2\Psi_j - 2\sum_{k=r+1}^d \{\phi_k, \psi_j\}(\bar{\rho})\Phi_k + tG(t, W), \ \ 1 \le j \le k \end{cases} \tag{1.28}$$

(which is obtained from (1.16) by replacing Θ by $-\nu X_0^2/4$). We look for a formal solution to the reduced Hamilton equation (1.28).

Lemma 1.5. *There exists a formal solution $W \in \mathcal{E}$ satisfying* (1.28) *with $\Phi_1(0) \neq 0$, $X_0(0) \neq 0$.*

Assume that $W = (\Xi_0, X_0, \Phi_1, \ldots, \Phi_d, \Psi_1, \ldots, \Psi_k) \in \mathcal{E}$ satisfies (1.28) formally. Denote X_0, Ξ_0, Φ_j, Ψ_j as (1.17) and $x_{00}^0 = \bar{x}_0$, $\xi_{00}^0 = \bar{\xi}_0$, $\phi_{00}^\mu = \bar{\phi}_\mu$ and $\psi_{00}^\nu = \bar{\psi}_\nu$. Equating the constant terms of both sides of (1.28) one has

$$\begin{gathered} -4\bar{\xi}_0 + 2\kappa\bar{\phi}_1\bar{\phi}_2 - \nu\delta\bar{x}_0^2\bar{\phi}_2/2 = 0, \quad -\bar{x}_0 + 2\bar{\phi}_1 = 0, \\ -2\bar{\phi}_1 + 2\delta\bar{\phi}_2 = 0, \ \ -3\bar{\phi}_2 + 2\kappa\bar{\phi}_1^2 + 2\delta\bar{\xi}_0 - \nu\delta\bar{x}_0^2\bar{\phi}_1/2 = 0, \\ -3\bar{\phi}_j = 0, \ \ r+1 \le j \le d, \quad -2\bar{\psi}_j - 2\Sigma_{k=r+1}^d\{\phi_k, \psi_j\}(\bar{\rho})\bar{\phi}_k = 0. \end{gathered}$$

From the third line one has $\bar{\phi}_j = 0$, $r+1 \le j \le d$ and $\bar{\psi}_j = 0$. Setting $b = \bar{\phi}_1$ as before we see $\bar{x}_0 = 2b$, $\bar{\phi}_2 = \delta^{-1}b$ and $2\bar{\xi}_0 = \kappa\delta^{-1}b^2 - \nu b^3$ hence the second equation on the second line becomes

$$-3\delta^{-1}b + 3\kappa b^2 - 3\nu\delta b^3 = 3b\big(-1/\delta + \kappa b - \nu\delta b^2\big) = 0$$

which is the same as (1.18). We choose the same $b \neq 0$ as in Section 1.1. Denote

$$\overline{W} = (\bar{\xi}_0, \bar{x}_0, \bar{\phi}_j, \bar{\psi}_i)$$

and look for a formal solution to (1.28) of the form $\overline{W} + W$ with $W \in \mathcal{E}^{\#}$. To simplify notation we set

$$\begin{cases} W^I = (X_0, \Phi_2, \Xi_0, \Phi_1), \quad W^{II} = (\Phi_3, \dots, \Phi_r) \\ W^{III} = (\Phi_{r+1}, \dots, \Phi_d), \quad W^{IV} = (\Phi_3, \dots, \Phi_r) \end{cases}$$

then $W = {}^t(W^I, W^{II}, W^{III}, W^{IV}) = \sum_{1 \le i, 0 \le j \le i} t^i (\log t)^j w_{ij}$ satisfies (1.20) with A of the same form as (1.21) where A_I is replaced by

$$A_I = \begin{bmatrix} -1 & 0 & 0 & 2 \\ -2\nu\delta b^2 & -3 & 2\delta & 2(\kappa b + \delta^{-1}) \\ -2\nu b^2 & 2\delta^{-1} & -4 & 2\kappa\delta^{-1} b \\ 0 & 2\delta & 0 & -2 \end{bmatrix} \tag{1.29}$$

where we have used (1.19). To confirm that (1.20) can be solved successively we prove the following

Lemma 1.6. *Assume $\kappa^2 - 4\nu > 0$. Then A_I has an eigenvalue 1 and the other real eigenvalues are negative.*

Proof. Expanding $|\lambda - A_I|$ with respect to the first row we see

$$\begin{aligned} |\lambda - A_I| = (\lambda + 1) &\begin{vmatrix} \lambda + 3 & -2\delta & -2(\kappa b + \delta^{-1}) \\ -2\delta^{-1} & \lambda + 4 & -2\kappa\delta^{-1} b \\ -2\delta & 0 & \lambda + 2 \end{vmatrix} \\ +2\nu b &\begin{vmatrix} 2\delta b & \lambda + 3 & -2\delta \\ 2b & -2\delta^{-1} & \lambda + 4 \\ 0 & -2\delta & 0 \end{vmatrix} \\ = (\lambda + 1)&(\lambda + 6)(\lambda^2 + 3\lambda - 4\kappa\delta b) + 8\nu\delta^2 b^2(\lambda + 6) \\ = (\lambda - 1)&(\lambda + 6)\big(\lambda^2 + 5\lambda + 8 - 4\kappa\delta b\big) \end{aligned}$$

the rest of the proof is just a repetition of the proof of Lemma 1.3. $\square$

At the end of the section, we give a simple example. Consider

$$p = -\xi_0^2 + (\xi_1 + x_0\xi_n)^2 + x_1^2(1 + x_1^k + \varepsilon x_2^l)\xi_n^2, \quad |x_1|, \ |x_2| \ll 1 \tag{1.30}$$

near $(0, e_n)$, $e_n = (0, \dots, 0, 1)$ with $n \ge 3$, $k, l \in \mathbb{N}$ where $\varepsilon = \pm 1$. It is clear that

$$\Sigma = \{\xi_0 = \xi_1 + x_0\xi_n = 0, x_1 = 0\}.$$

Denote

$$\begin{aligned} \varphi_1 &= -x_1(1 + x_1^k + \varepsilon x_2^l)\xi_n / (1 + (1 + k/2)x_1^k + \varepsilon x_2^l) \\ &= -x_1\Big[1 - \frac{k x_1^k / 2}{1 + (1 + k/2)x_1^k + \varepsilon x_2^l}\Big]\xi_n \end{aligned}$$

then one can write

$$
\begin{aligned}
(1.31) \qquad p &= -(\xi_0+\varphi_1)(\xi_0-\varphi_1)+x_1^2(1+x_1+\varepsilon x_2^l)\xi_n^2-\varphi_1^2+(\xi_1+x_0\xi_n)^2\\
&= -(\xi_0+\varphi_1)(\xi_0-\varphi_1)+\theta\varphi_1^2+\varphi_2^2
\end{aligned}
$$

where

$$
(1.32) \qquad \varphi_2=\xi_1+x_0\xi_n, \quad \theta=(1+k)x_1^k+\varepsilon x_2^l+k^2x_1^{2k}(1+x_1^k+\varepsilon x_2^l)^{-1}/4.
$$

which is a normal form. Indeed we see

$$
\{\xi_0-\varphi_1,\varphi_2\}=O(x_1^k)\overset{\Sigma'}{\equiv}0
$$

and it is clear that $\{\xi_0-\varphi_1,\xi_0\}=\{\xi_0-\varphi_1,\varphi_1\}=0$. Note that

$$
\theta|_{\Sigma}=\varepsilon x_2^l, \quad \{\xi_0-\varphi_1,\theta\}\equiv 0.
$$

According to $\varepsilon=\pm 1$ and the parity of l every type of transition occurs. If $k=1$ we have

$$
\{\{\xi_0-\varphi_1,\varphi_2\},\varphi_2\}(0,e_n)=1
$$

hence one can apply Proposition 1.1 to conclude the existence of a tangent bicharacteristic. Some remarks on the case $k\geq 2$ will be given at the end of the next section.

2 Elementary factorization

2.1 Elementary factorization

Consider the principal symbol

$$
(2.1) \qquad p(x,\xi)=-\xi_0^2+A_1(x,\xi')\xi_0+A_2(x,\xi')
$$

where $A_j(x,\xi')\in S^j_{1,0}$ depending smoothly in x_0. We start with the following definition.

Definition 2.1. We say that $p(x,\xi)$ admits a local elementary factorization if there exist real valued $\lambda(x,\xi')$, $\mu(x,\xi')\in S^1_{1,0}$ and $0\leq Q(x,\xi')\in S^2_{1,0}$ such that

$$
p(x,\xi)=-\Lambda(x,\xi)M(x,\xi)+Q(x,\xi')
$$

with $\Lambda(x,\xi)=\xi_0-\lambda(x,\xi')$ and $M(x,\xi)=\xi_0-\mu(x,\xi')$ verifying with some $C>0$

$$
(2.2) \qquad |\{\Lambda(x,\xi),Q(x,\xi')\}|\leq CQ(x,\xi'),
$$

$$
(2.3) \qquad |\{\Lambda(x,\xi),M(x,\xi)\}|\leq C\big(\sqrt{Q(x,\xi')}+|\Lambda(x,\xi')-M(x,\xi')|\big).
$$

If we can find such symbols defined in a conic neighborhood of ρ we say that $p(x,\xi)$ admits a microlocal elementary factorization at ρ.

Of course, elementary factorization is closely related to the classical derivation of energy estimates. To see this note that

$$-\Lambda M = -(\xi_0 - \lambda + ic)\#(\xi_0 - \mu - ic) - i\{\Lambda, M\}/2 - ic(\Lambda - M) + S^0_{1,0}$$
$$= -\tilde{\Lambda}\#\tilde{M} - i(\{\Lambda, M\}/2 - c(\Lambda - M)) + S^0_{1,0}$$

where $c \in S^0_{1,0}$ and $\tilde{\Lambda} = \xi_0 - \lambda + ic$, $\tilde{M} = \xi_0 - \mu - ic$. We also note

$$2\mathsf{Im}(\mathrm{op}(p)v, \mathrm{op}(\tilde{\Lambda})v) = \frac{d}{dx_0}\big(\|\mathrm{op}(\tilde{\Lambda})v\|^2 + (\mathrm{op}(Q)v, v)\big)$$
$$-2\mathsf{Re}(\mathrm{op}(\tilde{\Lambda})v, \mathrm{op}(\{\Lambda, M\}/2 + c(\Lambda - M))v) - \mathsf{Re}(\mathrm{op}(\{\tilde{\Lambda}, Q\})v, v)/2$$

modulo $C(\|\mathrm{op}(\tilde{\Lambda})v\|^2 + \|v\|^2)$. Assuming that one can find $c \in S^0_{1,0}$ such that $|\{\Lambda, M\}/2 + c(\Lambda - M)| \lesssim \sqrt{Q}$ and taking $|\{\tilde{\Lambda}, Q\} - \{\Lambda, Q\}| \lesssim \sqrt{Q}$ into account we will obtain energy estimates.

Lemma 2.1 ([4])**.** *If p admits a microlocal elementary factorization at ρ there is no bichracteristic tangent to Σ at ρ.*

In this section, we restrict ourselves to the case

$$\mathrm{Spec}F_p(\rho) \subset i\mathbb{R} \ \big(\Longleftrightarrow \Pi\lambda_j(\rho) \geq 0 \Longleftrightarrow \theta|_\Sigma \geq 0\big) \quad \rho \in \Sigma.$$

Applying the extension lemma to $\theta\big|_\Sigma$ one can assume from the beginning that

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \Sigma_{j=2}^r \phi_j^2 + \Sigma_{j=r+1}^d \phi_j^2 + R$$

where $R = O(|\phi|^4\langle\xi\rangle_\gamma^{-2}) = O^4(\Sigma')$ with $\phi = (\phi_1, \ldots, \phi_d)$ and that

$$\theta \geq 0, \quad \{\phi_j, \theta\} \overset{\Sigma'}{=} 0, \quad j = 1, \ldots, r. \tag{2.4}$$

Proposition 2.1. *Assume that p is of normal form up to term $O^4(\Sigma')$ and satisfies $\theta|_\Sigma \geq 0$ near $\bar{\rho}$. If*

$$|\{\xi_0 - \phi_1, \theta\}| \leq C(\sqrt{\theta} + |\phi_1| + \sqrt{|\phi'|})^2,$$
$$|\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}| \leq C(\sqrt{\theta} + |\phi_1| + \sqrt{|\phi'|}), \ \ \phi' = (\phi_2, \ldots, \phi_d)$$

holds then p admits a microlocal elementary factorization at $\bar{\rho}$.

Corollary 2.1. *Assume that p is of normal form up to term $O^4(\Sigma')$ and satisfies $\theta|_\Sigma \geq 0$ near $\bar{\rho}$ and $\{\xi_0 - \phi_1, \theta\} \overset{\Sigma'}{=} 0$. If $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} \overset{\Sigma'}{=} 0$ then p admits a microlocal elementary factorization at $\bar{\rho}$ while p does not if $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\bar{\rho}) \neq 0$.*

Proof. Without restrictions we may assume $\theta \geq 0$ near $\bar{\rho}$. From the assumption one can write $\{\xi_0 - \phi_1, \theta\} = \Sigma_{k=1}^d c_d\phi_k$. Note that $\{\xi_0 - \phi_1, \{\phi_2, \theta\}\} \overset{\Sigma'}{=} 0$ and $|\{\theta, \{\xi_0 - \phi_1, \phi_2\}\}| \lesssim \sqrt{\theta}$ for $\theta \geq 0$. It follows from Jacobi's identity we have

$$|\{\phi_2, \{\xi_0 - \phi_1, \theta\}\}| \lesssim |\phi| + \sqrt{\theta}, \quad \phi = (\phi_1, \ldots, \phi_d).$$

This shows that $|\{\xi_0-\phi_1,\theta\}|\lesssim(|\phi|+\sqrt{\theta})|\phi_1|+|\phi'|$ hence the first condition is satisfied. The second condition is obviously satisfied and the assertion follows from Proposition 2.1. Since $\{\xi_0-\phi_1,\{\xi_0-\phi_1,\theta\}\}\overset{\Sigma'}{=}0$ for $\{\xi_0-\phi_1,\theta\}\overset{\Sigma'}{=}0$ if $\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}(\bar\rho)\neq 0$ there is a bicharacteristic tangent to Σ at $\bar\rho$ by Corollary 1.2 hence Lemma 2.1 proves the last assertion. $\square$

Let

$$\Lambda=\phi_1+\ell(\phi'')\phi_1-\lambda\phi_1^3\langle\xi\rangle_\gamma^{-2},\quad \ell(\phi'')=\Sigma_{j=3}^r\beta_j\phi_j,\quad \phi''=(\phi_3,\ldots,\phi_r)$$

where $\lambda>0$ is a parameter and β_j are smooth which are determined later. Write

$$p=-(\xi_0+\Lambda)(\xi_0-\Lambda)+Q$$

where $Q=\theta\phi_1^2+\tilde Q$ with

$$\tilde Q=\Sigma_{j=2}^d\phi_j^2-2\ell\phi_1^2(1+\ell/2)+2\lambda\phi_1^4\langle\xi\rangle_\gamma^{-2}(1+\ell-\lambda\phi_1^2\langle\xi\rangle_\gamma^{-2}/2)+R.$$

Taking $\lambda>0$ large (since we are working in a neighborhood or $\bar\rho$ it can be assumed that $|\phi|\langle\xi\rangle_\gamma^{-1}$ is arbitrarily small) it is clear that there is $c>0$ such that

$$Q\geq c\,(|\phi'|^2+\theta\phi_1^2+\phi_1^4\langle\xi\rangle_\gamma^{-2}),\quad \phi'=(\phi_2,\ldots,\phi_d).$$

Note that

$$\begin{aligned}\{\xi_0-\Lambda,\tilde Q\}=\{\xi_0-\phi_1,|\phi'|^2-2\ell(\phi'')\phi_1^2(1+\ell(\phi'')/2)\}\\ -\{\ell(\phi'')\phi_1,|\phi'|^2\}+O(Q).\end{aligned}$$

Recall that one can write

$$\{\xi_0-\phi_1,\phi_j\}=\Sigma_{k=1}^d\alpha_{jk}\phi_k,\quad 1\leq j\leq d.$$

Moreover since $\{\phi_2,\{\xi_0-\phi_1,\phi_j\}\}\overset{\Sigma'}{=}0$ for $r+1\leq j\leq d$ which follows from Jacobi's identity, one has

$$\alpha_{j1}\overset{\Sigma'}{=}0,\quad r+1\leq j\leq d.$$

Therefore we have

$$\begin{aligned}\{\xi_0-\Lambda,\tilde Q\}=2\sum_{j=2}^d\phi_j\sum_{k=1}^d\alpha_{jk}\phi_k-2\phi_1^2\sum_{j=3}^r\beta_j\sum_{k=1}^d\alpha_{jk}\phi_k(1+\ell(\phi'')/2)\\ -2\phi_1\sum_{j=2}^d\phi_j\sum_{k=3}^r\beta_k\{\phi_k,\phi_j\}+O(Q).\end{aligned}$$

Note that

$$\phi_j\sum_{k=1}^d\alpha_{jk}\phi_k=O(Q),\;\; r+1\leq j\leq d,\quad \phi_1^2\sum_{j=3}^r\beta_j\sum_{k=1}^d\alpha_{jk}\phi_k\ell(\phi'')=O(Q),$$

$$\phi_1\phi_j\sum_{k=3}^r\beta_k\{\phi_k,\phi_j\}=O(Q),\;\; r+1\leq j\leq d$$

we have

$$
\begin{aligned}
\{\xi_0-\Lambda,\tilde{Q}\} &= 2\sum_{j=2}^{r}\phi_j\sum_{k=1}^{d}\alpha_{jk}\phi_k-2\phi_1^2\sum_{j=3}^{r}\beta_j\sum_{k=1}^{d}\alpha_{jk}\phi_k\\
&\quad -2\phi_1\sum_{j=2}^{r}\phi_j\sum_{k=3}^{r}\beta_k\{\phi_k,\phi_j\}+O(Q)\\
&=2\phi_1\sum_{j=2}^{r}\alpha_{j1}\phi_j-2\phi_1^3\sum_{j=3}^{r}\beta_j\alpha_{j1}-2\phi_1\sum_{j=3}^{r}\phi_j\sum_{k=3}^{r}\beta_k\{\phi_k,\phi_j\}+O(Q)\\
&=2\phi_1\sum_{j=3}^{r}\phi_j\Big(\alpha_{j1}-\sum_{k=3}^{r}\{\phi_k,\phi_j\}\beta_k\Big)+2\alpha_{21}\phi_1\phi_2-2\phi_1^3\sum_{j=3}^{r}\beta_j\alpha_{j1}+O(Q)
\end{aligned}
$$

because $\{\phi_k,\phi_2\}\overset{\Sigma'}{=}0$ for $k\geq 3$. Choose β_k such that

$$
\sum_{k=3}^{r}\{\phi_k,\phi_j\}\beta_k=\alpha_{j1},\quad 3\leq j\leq r.
$$

Since $(\{\phi_k,\phi_j\})_{3\leq k,j\leq r}$ is skew symmetric and nonsingular we have

$$
\sum_{j=3}^{r}\beta_j\alpha_{j1}=0.
$$

Therefore we conclude

$$
\{\xi_0-\Lambda,\tilde{Q}\}=2\alpha_{21}\phi_1\phi_2+O(Q).
$$

Since $\alpha_{21}\overset{\Sigma'}{=}\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}/\{\phi_1,\phi_2\}$ we have

$$
|\{\{\xi_0-\phi_1,\phi_2\},\phi_2\}|\leq C(\sqrt{\theta}+|\phi_1|+\sqrt{|\phi'|})\Longrightarrow\{\xi_0-\Lambda,\tilde{Q}\}=O(Q).
$$

It remains to study $\{\xi_0-\Lambda,\theta\phi_1^2\}$. Taking (2.4) into account we have

$$
\begin{aligned}
&\{\xi_0-\Lambda,\theta\phi_1^2\}=\{\xi_0-\phi_1,\theta\phi_1^2\}+O(Q)\\
&=2\theta\phi_1\{\xi_0-\phi_1,\phi_1\}+\phi_1^2\{\xi_0-\phi_1,\theta\}+O(Q)=\phi_1^2\{\xi_0-\phi_1,\theta\}+O(Q).
\end{aligned}
$$

Therefore we conclude that

$$
|\{\xi_0-\phi_1,\theta\}|\leq C(\theta+\sqrt{\theta}|\phi_1|+|\phi'|)\Longrightarrow\{\xi_0-\Lambda,\theta\phi_1^2\}=O(Q).
$$

Consider $\{\xi_0-\Lambda,\xi_0+\Lambda\}=2\{\xi_0,\Lambda\}$ which is

$$
2\{\xi_0,\phi_1+\ell(\phi'')\phi_1-\lambda\phi_1^3\langle\xi\rangle_\gamma^{-2}\}\overset{\Sigma'}{=}0
$$

for $\{\xi_0,\phi_1\}\overset{\Sigma'}{=}0$. Since $|\phi'|\lesssim\sqrt{Q}$ and $|\phi_1|\lesssim|\Lambda|$ we have

$$
|\{\xi_0-\Lambda,\xi_0+\Lambda\}|\lesssim\sqrt{Q}+|\Lambda|
$$

which completes the proof. $\square$

Retake the example (**??**). If $h(x_2) = c\,x_2^l$ with $c > 0$ and even $l \geq 2$ so that $\theta|_\Sigma \geq 0$. It is easy to check that

$$|\{\xi_0 - \varphi_1, \theta\}| \lesssim \theta$$

hence in view of Proposition 2.1 p admits a microlocal elementary factorization at $\bar{\rho}$. Nxet, reconsider p in (1.31) with even l and $\varepsilon = 1$ so that $\theta|_\Sigma \geq 0$. If $k \geq 2$ it is easy to see that

$$\{\{\xi_0 - \varphi_1, \varphi_2\}, \varphi_2\} \overset{\Sigma'}{=} 0$$

then it follows from Corollary 2.1 that p admits a microlocal elementary factorization at $\bar{\rho}$. In particular, there is no tangent bicharacteristic.

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