Geometric results for hyperbolic operators with spectral transition of the Hamilton map (Normal form, Extension lemma)

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Abstract

1 Normal form of the principal symbol

1.1 Some notations

Assume that the set $\Sigma = \{p = 0, \nabla p = 0\}$ of critical points of $p = 0$ is a smooth manifold of codimension $d + 1$ and that

(1.1) $\quad p$ vanishes exactly to second order on Σ.

From this assumption without restrictions one can assume that for any $\rho \in \Sigma$ there is a neighborhood of ρ where we can write

$$p = -\xi_0^2 + \sum_{j=1}^{d} \phi_j^2.$$

Here $d\phi_j$ ($\phi_0 = \xi_0$) are linearly independent at ρ and Σ is given by $\phi_j = 0$, $0 \le j \le d$. We assume that

(1.2) $$\operatorname{rank} \sigma\big|_{\Sigma} = \text{constant}$$

where $\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$ is the symplectic 2-form. In [6] we find detailed discussions on the Cauchy problem for $\mathrm{op}(p)$ under the assumptions (1.1), (1.2) and $\mathrm{Spec}F_p(\rho) \subset i\mathbb{R}$, $\rho \in \Sigma$ assuming further that *there is no spectral transition of the Hamilton map* F_p, that is

For any $\rho \in \Sigma$ there is a conic neighborhood V of ρ such that either $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 = \{0\}$ or $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 \neq \{0\}$ holds throughout $V \cap \Sigma$.

Our main concern in this note is to derive normal forms of p under assumptions (1.1) and (1.2) in the presence of a spectral transition. Let us denote

$$W(\rho) = \mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho), \quad \rho \in \Sigma.$$

We sometimes denote $\xi_0 = \phi_0$. Since

$$\mathrm{dim}T_\rho\Sigma + \mathrm{rank}\,\sigma|_{(T_\rho\Sigma)^\sigma} = \mathrm{dim}(T_\rho\Sigma)^\sigma + \mathrm{rank}\,\sigma|_{T_\rho\Sigma}$$

and $\mathrm{rank}\,\sigma|_{(T_\rho\Sigma)^\sigma} = \mathrm{rank}(\{\phi_i, \phi_j\}(\rho))$ we see that (1.2) is equivalent to

$$\mathrm{rank}\big(\{\phi_i, \phi_j\}\big) = \mathrm{constant} = r \; (= 2d + \mathrm{rank}\,\sigma|_{T_\rho\Sigma} - 2n) \tag{1.3}$$

where $\big(\{\phi_i, \phi_j\}\big)$ denotes the $(d+1)\times(d+1)$ matrix with (i,j)-th entry $\{\phi_{i-1}, \phi_{j-1}\}$.

Lemma 1.1. *Assume* (1.2). *If the spectral transition occurs at $\bar{\rho}$ then $W(\bar{\rho}) \neq \{0\}$.*

Proof. Assume $W(\bar{\rho}) = \{0\}$. Thanks to [3, Theorem 1.4.6] (or [2, Theorem 21.5.3]) one can choose a symplectic basis such that

$$\text{either} \quad p_{\bar{\rho}} = -\xi_0^2 + \Sigma_{j=1}^{l_1}\mu_j(x_j^2 + \xi_j^2) + \Sigma_{j=l_1+1}^{l_2}\xi_j^2, \quad \mu_j > 0,$$
$$\text{or} \;\; p_{\bar{\rho}} = e(x_0^2 - \xi_0^2) + \Sigma_{j=1}^{l_1-1}\mu_j(x_j^2 + \xi_j^2) + \Sigma_{j=l_1+1}^{l_2}\xi_j^2, \quad \mu_j > 0, \;\; e > 0$$

where $\pm e$, $\pm i\mu_j$ are non-zero eigenvalues of $F_p(\bar{\rho})$. Here we note that $2l_1 = r$ for (1.3). From the continuity of the eigenvalues of $F_p(\rho)$ with respect to $\rho \in \Sigma$, $F_p(\rho)$ has at least r non-zero eigenvalues near $\bar{\rho}$ (with counting multiplicity). On the other hand, if $W(\rho) \neq \{0\}$ again thanks to [3, Theorem 1.4.6] (or [2, Theorem 21.5.3]) in a suitable symplectic basis

$$p_\rho = -\xi_0^2 + 2\xi_0\xi_1 + x_1^2 + \sum_{j=2}^{l_1'}\mu_j'(x_j^2 + \xi_j^2) + \sum_{j=l_1'+1}^{l_2'}\xi_j^2, \quad \mu_j' > 0$$

where $l_1' = l_1$ because of (1.3) so that $F_p(\rho)$ has $r-2$ non-zero eigenvalues (with counting multiplicity) with a contradiction. Thus $W(\rho) = \{0\}$ near $\bar{\rho}$ hence no spectral transition occurs. $\square$

In what follows we always work near $\bar{\rho}$ but do not mention this and it should be understood that F_p is defined only on Σ near $\bar{\rho}$.

1.2 A normal form

Denote

$$\Sigma' = \{\phi_j = 0, 1 \leq j \leq d\}$$

and

$$f \overset{\Sigma'}{=} 0 \Longleftrightarrow f = 0 \;\text{ on }\; \Sigma' \;\text{ near }\; \bar{\rho}.$$

Proposition 1.1. *Assume* (1.1), (1.2) *and that the spectral transition occurs at* $\bar{\rho}$ *then one can write*

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \sum_{j=2}^{r} \phi_j^2 + \sum_{j=r+1}^{d} \phi_j^2 \tag{1.4}$$

with a smooth θ (*positively homogeneous of degree* 0) *such that* $1 + \theta > 0$ *where*

$$\{\phi_i, \phi_j\} \overset{\Sigma'}{=} 0, \quad 0 \le i \le d, \;\; j \ge r+1, \quad \{\xi_0 - \phi_1, \phi_j\} \overset{\Sigma'}{=} 0, \quad 0 \le j \le d \tag{1.5}$$

and

$$\{\phi_1, \phi_2\}(\bar{\rho}) \neq 0, \; \{\phi_2, \phi_j\} \overset{\Sigma'}{=} 0, \; 3 \le j \le r, \; \det(\{\phi_i, \phi_j\}(\bar{\rho}))_{3 \le i,j \le r} \neq 0. \tag{1.6}$$

If p *has the form* (1.4) *with* (1.5) *and* (1.6) *then* θ *coincides with the product of nonzero eigenvalues of* $F_p(\rho)$ (*with counting multiplicity*) *on* Σ' *where* $W(\rho) = \{0\}$, *up to a smooth multiplicative factor which is positive at* $\bar{\rho}$. *Moreover* $\theta(\rho) = 0$ *if and only if* $W(\rho) \neq \{0\}$ *and*

$$\begin{aligned} &\theta(\rho) > 0 \Longleftrightarrow p \text{ is noneffectively hyperbolic at } \rho \text{ with } W(\rho) = \{0\}, \\ &\theta(\rho) = 0 \Longleftrightarrow p \text{ is noneffectively hyperbolic at } \rho \text{ with } W(\rho) \neq \{0\}, \\ &\theta(\rho) < 0 \Longleftrightarrow p \text{ is effectively hyperbolic at } \rho. \end{aligned} \tag{1.7}$$

This normal form is not unique. To see this we first note that

Lemma 1.2. *One can rewrite*

$$-(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + \hat{\theta}\tilde{\phi}_1^2$$

with $\tilde{\phi} = (1+\nu)\phi_1$ *and* $\hat{\theta} = (\theta - \nu^2 - 2\nu)/(1+\nu)^2$.

Proof. The proof is clear. □

If $\nu \overset{\Sigma'}{=} 0$, it is clear that (1.5) and (1.6) hold replacing ϕ_1 by $\tilde{\phi}_1$ such that

$$p = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + \hat{\theta}\phi_1^2 + \Sigma_{j=2}^r \phi_j^2 + \Sigma_{j=r+1}^d \phi_j^2$$

is also a normal form. In this case we have $\theta \overset{\Sigma'}{=} \hat{\theta}$. This is no coincidence. Let $\Pi\lambda_j(\rho)$ be the product of nonzero eigenvalues of $F_p(\rho)$ (with counting multiplicity). Note that $\Pi\lambda_j(\rho)$ is real because nonzero eigenvalues of $F_p(\rho)$ are $\pm i\mu$ and possibly $\pm\lambda$ with real μ, λ. From Proposition 1.1 there is a smooth $f(\rho)$ near $\bar{\rho}$ with $f(\bar{\rho}) > 0$ such that

$$\Pi\lambda_j(\rho) = f(\rho)\,\theta(\rho), \quad W(\rho) = \{0\}, \quad \rho \in \Sigma. \tag{1.8}$$

Since $\Pi\lambda_j(\rho)$ is invariant under changes of symplectic coordinates we could say that $\theta|_\Sigma$ is conformally invariant. It is also clear from (1.8) that $\Pi\lambda_j(\rho)$ smoothly extends from the set where $W(\rho) = \{0\}$ to a neighborhood of $\bar{\rho}$ on Σ by setting $\Pi\lambda_j(\rho) = 0$ if $W(\rho) \neq 0$, the extension is given by the right-hand side of (1.8).

Remark 1.1. If $\theta = \tilde{\theta} + \Sigma_{j=1}^{d} c_j\phi_j$ choosing ν such that $2\nu = \Sigma_{j=1}^{d} c_j\phi_j$ in Lemma 1.2 we have

$$\hat{\theta} = \tilde{\theta} + r, \quad r = O^2(\Sigma').$$

Therefore one can write

$$-(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + (\tilde{\theta} + O^2(\Sigma'))\tilde{\phi}_1^2.$$

That is, if $\tilde{\theta}|_{\Sigma'} = \theta|_{\Sigma'}$ one can replace θ by $\tilde{\theta} + O^2(\Sigma')$ in the normal form.

This proposition shows that transitions may occur already in very simple examples. For instance, consider

$$p = -\xi_0^2 + (1 + \theta(x))\xi_1^2 + (x_0 + x_1)^2\xi_n^2, \quad |\theta(x)| < 1, \ \ \theta(0) = 0$$

near $(0, e_n)$, $e_n = (0, \ldots, 0, 1)$ with $n \geq 2$. It is clear that the doubly characteristic set is given by $\Sigma = \{\xi_0 = 0, \xi_1 = 0, x_0 + x_1 = 0\}$. Writing

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta(x)\phi_1^2 + \phi_2^2, \quad \phi_1 = \xi_1, \ \ \phi_2 = (x_0 + x_1)\xi_n \tag{1.9}$$

it is also clear that (1.5) and (1.6) are satisfied. According to the choice of θ, every type of transition occurs.

Proof of Proposition: Denote $A = (\{\phi_i, \phi_j\})_{0 \leq i,j \leq d}$ (note that A is independent of ξ_0) where the rank of $A(\rho)$ is constant r. Then there exists a smooth basis $v_0, v_{r+1}, \ldots, v_d$, defiened on Σ', of Ker$A$. One can assume $v_0 = {}^t(1, v_0')$. To check this it suffices to prove the first entry of some $v_0(\bar{\rho}), v_{r+1}(\bar{\rho}), \ldots, v_d(\bar{\rho})$ is different form 0. Suppose the contrary. Let $X \in C_{\bar{\rho}} \cap T_{\bar{\rho}}\Sigma$ where $C_{\bar{\rho}}$ is the propagation cone. One can write $X = \Sigma_{j=0}^{d} \alpha_j H_{\phi_j}(\bar{\rho})$ since $C_{\bar{\rho}} \subset \mathrm{Im}F_p(\bar{\rho})$. From $X \in T_{\bar{\rho}}\Sigma$ it follows $A\alpha = 0$ hence $\alpha_0 = 0$ by assumption. Writing $X = (\bar{x}, \bar{\xi})$ we have $\bar{x}_0 = 0$. Recall $C_{\bar{\rho}} = \{X; \sigma(X, Y) \leq 0, \ \forall Y \in \Gamma_{\bar{\rho}}\}$. Since for any (y, η') there is η_0 such that $(y, \eta) \in \Gamma_{\bar{\rho}}$ (because $\Gamma_{\bar{\rho}} = \{\eta_0 > (\sum_{j=1}^{r} d\phi_j(y, \eta')^2)^{1/2}\}$) we conclude $X = 0$. This proves that $C_{\bar{\rho}} \cap T_{\bar{\rho}}\Sigma = \{0\}$ hence $\bar{\rho}$ is effectively hyperbolic ([3, Corollary 1.4.7], [5, Lemma 1.1.2]) so that $W(\bar{\rho}) = \{0\}$ contradicting the assumption.

Since one can assume $v_0 = {}^t(1, v_0')$ considering $v_j -$(the first component of v_j)v_0 one may assume

$$v_j = {}^t(0, v_j'(\rho)), \ \ v_j'(\rho) = (v_{j1}', \ldots, v_{jd}'), \ \ \rho \in \Sigma' \quad j = r+1, \ldots, d.$$

We extend $v_{r+1}', \ldots, v_d'$ outside Σ' and orthonormarize them and denote the resulting ones by w_j'. We choose smooth $w_1', \ldots, w_r'$ such that $w_1', \ldots, w_d'$ will be a smooth orthonormal basis of $\mathbb{R}^d$. Denote

$$w_0 = {}^t(1, 0), \ \ w_j = {}^t(0, w_j'), \quad j = 1, \ldots, d$$

then it is clear that $Aw_j \overset{\Sigma'}{=} 0$ for $j = r+1, \ldots, d$ and $w_0, w_1, \ldots, w_d$ is an orthonormal basis of $\mathbb{R}^{d+1}$. Let e_j be the unit vector in $\mathbb{R}^{d+1}$ with $j+1$ th component 1. Denote $(e_0, \ldots, e_d) = (w_0, \ldots, w_d)P$, $P = (p_{ij})$ so that P is orthogonal matrix. It is clear that $P = 1 \oplus P'$. Denote

$$\tilde{\phi}_i = \sum_{k=0}^{d} p_{ik}\phi_k$$

then we have $\tilde{\phi}_0 = \phi_0$ and

$$p = -\xi_0^2 + \sum_{j=1}^{d} \tilde{\phi}_j^2.$$

Noting that A is skew-symmetric we see that $PAP^{-1} \overset{\Sigma'}{=} \tilde{A} \oplus O_{d-r}$ where O_{d-r} is the zero matrix of order $d-r$. Since $(\{\tilde{\phi}_i, \tilde{\phi}_j\}) \overset{\Sigma'}{=} PAP^{-1}$ we have the first assertion of (1.5) replacing $\tilde{\phi}_j$ by ϕ_j. Write $\tilde{\phi}_j \to \phi_j$ and denote

$$\tilde{A} = (\{\phi_i, \phi_j\})_{0\le i,j\le r} = \begin{pmatrix} 0 & {}^t a' \\ -a' & A_1 \end{pmatrix}, \quad {}^t a' = (\{\xi_0, \phi_1\}, \ldots, \{\xi_0, \phi_r\})$$

where $a'(\bar{\rho}) \neq 0$ otherwise $\{\xi_0, \phi_j\}(\bar{\rho}) = 0$ for $j = 1, \ldots, d$ so that $F_p(\bar{\rho})$ is similar to $F_{\xi_0^2} \oplus F_{\Sigma_{j=1}^d \phi_j^2}(\bar{\rho})$ and hence $W(\bar{\rho}) = \{0\}$ contradicting with assumption. We now show

$$A_1 = (\{\phi_i, \phi_j\})_{1\le i,j\le r} \quad \text{is non-singular.} \tag{1.10}$$

Since $P = 1 \oplus P'$ there is $v = {}^t(1, v')$, $v' \neq 0$ such that $\tilde{A}v = 0$, that is $\langle a', v'\rangle = 0$ and $A_1 v' = a'$. Suppose that there is $0 \neq w'$ such that $A_1 w' = 0$. Note that $\langle a', w'\rangle = \langle A_1 v', w'\rangle = -\langle v', A_1 w'\rangle = 0$ for A_1 is skew-symmetric. This shows that $\tilde{A}z = 0$ with $z = {}^t(1, v' + w')$ which contradicts to $\dim \mathrm{Ker}\tilde{A} = 1$ (for the rank of $\tilde{A}$ is r).

Define $\alpha = {}^t(\alpha_1, \ldots, \alpha_r) \neq 0$ and ψ by

$$A_1\alpha = a', \quad \psi = -\sum_{j=1}^{r} \alpha_j \phi_j$$

then since A_1 is skew-symmetric it is clear that

$$(\{\phi_i, \phi_j\})_{0\le i,j\le r} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \tilde{A} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = 0$$

that is, one has

$$\{\xi_0 - \psi, \phi_j\} \overset{\Sigma'}{=} 0, \quad j = 0, \ldots, d. \tag{1.11}$$

Choosing a smooth orthogonal matrix $T = (t_{ij})_{1\le i,j\le r}$ with the first row $-\alpha/|\alpha|$ consider $\tilde{\phi}_j = \sum_{k=1}^{r} t_{jk}\phi_k$ so that $\sum_{j=1}^{r} \tilde{\phi}_j^2 = \sum_{j=1}^{r} \phi_j^2$ where

$$\tilde{\phi}_1 = \psi/|\alpha|$$

hence one can write

$$p = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + \sum_{j=2}^{r} \tilde{\phi}_j^2 + \sum_{j=r+1}^{d} \phi_j^2. \tag{1.12}$$

Here note that

$$(\{\tilde{\phi}_i, \tilde{\phi}_j\})_{1\le i,j\le r} \stackrel{\Sigma'}{=} TA_1T^{-1}, \quad \{\xi_0 - |\alpha|\tilde{\phi}_1, \tilde{\phi}_j\} \stackrel{\Sigma'}{=} 0, \quad 0 \le j \le d.$$

If $\{\tilde{\phi}_1, \tilde{\phi}_j\}(\bar{\rho}) = 0$ for $2 \le j \le r$ then $\{\xi_0, \tilde{\phi}_j\}(\bar{\rho}) = 0$ for $1 \le j \le r$ from which we have $W(\bar{\rho}) = \{0\}$ as before contradicting with the assumption. Thus we may assume $\{\tilde{\phi}_1, \tilde{\phi}_2\}(\bar{\rho}) \neq 0$. Writing $\psi = |\alpha|\tilde{\phi}_1 \to \phi_1$ and $\tilde{\phi}_j \to \phi_j$ $(2 \le j \le r)$ we have

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \sum_{j=2}^{r} \phi_j^2 + \sum_{j=r+1}^{d} \phi_j^2 \tag{1.13}$$

where

$$\theta = (1-|\alpha|^2)/|\alpha|^2, \quad \{\xi_0 - \phi_1, \phi_j\} \stackrel{\Sigma'}{=} 0, \;\; j = 0, \ldots, d, \quad \{\phi_1, \phi_2\}(\bar{\rho}) \neq 0. \tag{1.14}$$

Thus we have the second assertion of (1.5) and the first assertion of (1.6). If we replace $\tilde{\phi}_1$ by $|\alpha|\tilde{\phi}_1$ then $\det(\{\tilde{\phi}_i, \tilde{\phi}_j\})_{1\le i,j\le r}$ is multiplied by $|\alpha|^2$ hence still nonsingular. Therefore we have

$$\det(\{\phi_i, \phi_j\})_{1\le i,j\le r} \neq 0.$$

Consider

$$\tilde{\phi}_j = \sum_{k=2}^{r} t_{jk}\phi_k, \quad 2 \le j \le r$$

and choosing a suitable (smooth) orthogonal matrix $T = (t_{jk})_{2\le j,k\le r}$ with the first row=normalized ${}^t(\{\phi_1, \phi_2\}, \ldots, \{\phi_1, \phi_r\})$ such that

$$\sum_{j=2}^{r} \tilde{\phi}_j^2 = \sum_{j=2}^{r} \phi_j^2, \quad \{\phi_1, \tilde{\phi}_j\} \stackrel{\Sigma'}{=} \sum_{k=2}^{r} t_{jk}\{\phi_1, \phi_k\} \stackrel{\Sigma'}{=} 0, \quad 3 \le j \le r$$

one can assume that

$$\{\phi_1, \tilde{\phi}_2\}(\bar{\rho}) = (\Sigma_{j=2}^{r}\{\phi_1, \phi_j\}^2)^{1/2} \neq 0, \quad \{\phi_1, \tilde{\phi}_j\} \stackrel{\Sigma'}{=} 0, \;\; 3 \le j \le r.$$

Since it is clear that $\{\xi_0 - \phi_1, \tilde{\phi}_j\} \stackrel{\Sigma'}{=} 0$, $0 \le j \le d$ writing $\tilde{\phi}_j \to \phi_j$ we can assume

$$\begin{gathered} p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \Sigma_{j=2}^{r}\phi_j^2 + \Sigma_{j=r+1}^{d}\phi_j^2, \\ \{\xi_0 - \phi_1, \phi_j\} \stackrel{\Sigma'}{=} 0, \quad 0 \le j \le r, \quad \{\phi_1, \phi_j\} \stackrel{\Sigma'}{=} 0, \quad 3 \le j \le r, \\ \{\phi_1, \phi_2\}(\bar{\rho}) \neq 0, \quad \det(\{\phi_i, \phi_j\})_{1\le i,j\le r} \neq 0. \end{gathered} \tag{1.15}$$

Write

$$A_1 = (\{\phi_i, \phi_j\})_{1\le i,j\le r} \overset{\Sigma'}{=} \begin{pmatrix} 0 & {}^t a^{(1)} \\ -a^{(1)} & A_2 \end{pmatrix}, \quad a^{(1)} = {}^t(\{\phi_1, \phi_2\}, 0, \dots, 0)$$

and show that $\dim\mathrm{Ker} A_2 = 1$. Since A_2 is skew symmetric of odd order $r-1$ then $\dim\mathrm{Ker} A_2 \ge 1$ hence $\mathrm{rank}\, A_2 \le r-2$. Suppose $\mathrm{rank}\, A_2 < r-2$ then, expanding $\det A_1$ by cofactors of the first column (or the first row), it is clear that

$$\det A_1 = 0$$

hence contradiction, thus $\mathrm{rank}\, A_2 = r-2$. This proves $\dim\mathrm{Ker} A_2 = 1$ and $\mathrm{Ker} A_2$ is spanned by $\beta = {}^t(\beta_2, \dots, \beta_r) \ne 0$. Note that $\beta_2 \ne 0$. Otherwise $\tilde\beta = {}^t(0, 0, \beta_3, \dots, \beta_r) \ne 0$ verifies $A_1\tilde\beta = 0$ which is a contradiction. Choosing a smooth orthogonal matrix $T = (t_{ij})_{2\le i,j\le r}$ with the first row $\beta/|\beta|$ consider $\tilde\phi_j = \sum_{k=2}^r t_{jk}\phi_k$ so that $\sum_{j=2}^r \tilde\phi_j^2 = \sum_{j=2}^r \phi_j^2$. Then we have

$$\{\tilde\phi_2, \tilde\phi_j\} \overset{\Sigma'}{=} 0, \quad 2 \le j \le r.$$

We also note that

$$\{\phi_1, \tilde\phi_2\}(\bar\rho) = \Sigma_{k=2}^r(\beta_k/|\beta|)\{\phi_1, \phi_k\}(\bar\rho) = (\beta_2/|\beta|)\{\phi_1, \phi_2\}(\bar\rho) \ne 0.$$

Since $(\{\tilde\phi_i, \tilde\phi_j\})_{2\le i,j\le r} = TA_2T^{-1}$ then

$$\dim\mathrm{Ker}(\{\tilde\phi_i, \tilde\phi_j\})_{2\le i,j\le r} = 1.$$

Writing $\tilde\phi_j \to \phi_j$ $(2 \le j \le r)$ one can write

$$\begin{aligned} p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta\phi_1^2 + \Sigma_{j=2}^r \phi_j^2 + \Sigma_{j=r+1}^d \phi_j^2, \\ \{\xi_0 - \phi_1, \phi_j\} \overset{\Sigma'}{=} 0, \ \ 0 \le j \le d, \\ \{\phi_2, \phi_j\} \overset{\Sigma'}{=} 0, \quad 2 \le j \le r, \quad \{\phi_1, \phi_2\}(\bar\rho) \ne 0. \end{aligned} \tag{1.16}$$

Since

$$A_2 = (\{\phi_i, \phi_j\})_{2\le i,j\le r} \overset{\Sigma'}{=} \begin{pmatrix} 0 & 0 \\ 0 & A_3 \end{pmatrix}$$

it is clear that $A_3 = (\{\phi_i, \phi_j\})_{3\le i,j\le r}$ is nonsingular because $\dim\mathrm{Ker} A_2 = 1$. Here note the following

Lemma 1.3. (cf.[1]) *Let p have the form* (1.4) *with* (1.5) *and* (1.6). *Then*

$\theta(\rho) = 0 \Longleftrightarrow W(\rho) \ne \{0\}$,

$\theta(\rho) > 0 \Longleftrightarrow p$ *is non-effectively hyperbolic at* ρ *and* $W(\rho) = \{0\}$,

$(-1 <)$ $\theta(\rho) < 0 \Longleftrightarrow p$ *is effectively hyperbolic at* ρ.

Proof. Write $\delta\phi_1 \to \phi_1$ with $\delta = (1+\theta)^{1/2}$ so that $p = -\xi_0^2 + \Sigma_{j=1}^d \phi_j^2$. If $F_p X = \mu X$ with $\mu \neq 0$ then $X \in \operatorname{Im} F_p$ and hence $X = \Sigma_{j=0}^d \eta_j H_{\phi_j}$. Let $\epsilon_0 = -1$ and $\epsilon_j = 1$ for $1 \le j \le d$ and note that

$$F_p(\sum_{j=0}^d \eta_j H_{\phi_j}) = \sum_{i,j=0}^d \epsilon_i\{\phi_j, \phi_i\}\eta_j H_{\phi_i} = \sum_{i=0}^d \zeta_i H_{\phi_i}, \qquad (1.17)$$
$$\zeta = A\eta, \quad A = (\epsilon_i\{\phi_j, \phi_i\})_{0 \le i,j \le d}.$$

Consider

$$A = A' \oplus O_{d-r}, \quad A' = \begin{pmatrix} 0 & {}^t a \\ a & M \end{pmatrix}.$$

Note that M is nonsingular provided $\delta \neq 0$. From (1.17) a non-zero eigenvalue of $F_p(\rho)$ is also an eigenvalue of A'. Let $a = M\alpha$. Since M is skew-symmetric we see that

$$A' \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} = 0, \quad M\alpha = a = {}^t(\{\xi_0, \phi_1\}, \ldots, \{\xi_0, \phi_r\}). \qquad (1.18)$$

Note that

$$\begin{pmatrix} 1 & {}^t\alpha \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} \mu & {}^t(M\alpha) \\ M\alpha & \mu + M \end{pmatrix} = \begin{pmatrix} \mu & \mu\,{}^t\alpha \\ M\alpha & \mu + M \end{pmatrix}$$

for ${}^tM = -M$ and that

$$\begin{pmatrix} \mu & \mu\,{}^t\alpha \\ M\alpha & \mu + M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1_d \end{pmatrix} = \begin{pmatrix} \mu - \mu|\alpha|^2 & \mu\,{}^t\alpha \\ -\mu\alpha & \mu + M \end{pmatrix}.$$

Hence

$$\det(\mu + A') = \det \begin{pmatrix} \mu - \mu|\alpha|^2 & \mu\,{}^t\alpha \\ -\mu\alpha & \mu + M \end{pmatrix} = \mu \det \begin{pmatrix} 1 - |\alpha|^2 & \mu\,{}^t\alpha \\ -\alpha & \mu + M \end{pmatrix}.$$

Therefore we have

$$\det(\mu + A') = \mu G(\mu), \quad G(0) = (1 - |\alpha|^2)\det M \qquad (1.19)$$

where $\det M > 0$ because M is non-singular skew-symmetric. Assume $W(\rho) = \{0\}$. Thanks to [3, Lemma 1.4.4], non-zero eigenvalues of $F_p(\rho)$ are semisimple (algebraic multiplicity=geometric multiplicity), denoted $\lambda_1, \ldots, \lambda_r$ with (counting multiplicity) linearly independent eigenvectors $v_i = \Sigma_{j=0}^d \eta_{ij} H_{\phi_j} \in \operatorname{Im} F_p$, these $\lambda_1, \ldots, \lambda_r$ are also semisimple eigenvalues of A' with linearly independent eigenvectors $\eta_i = {}^t(\eta_{i0}, \ldots, \eta_{id})$. Since $G(\mu)$ is a monic polynomial of order r we must have $G(\mu) = \Pi_{j=1}^r(\mu - \lambda_j)$. In particluar this proves $|\alpha(\rho)| \neq 1$ and

$$(1 - |\alpha(\rho)|^2)\det M = \Pi_{j=1}^r \lambda_j(\rho), \quad \rho \in \Sigma, \quad W(\rho) = \{0\}. \qquad (1.20)$$

Assume $W(\rho) \neq \{0\}$ then $F_p(\rho)$ has $r-2$ nonzero semi-simple eigenvalues (with counting multiplicity). Suppose $|\alpha(\rho)| \neq 1$ so that A' has r nonzero eigenvalues.

It is clear that not all are semi-simple. So assume $\lambda_j \neq 0$ is not a semi-simple eigenvalue of A' so that there is $0 \neq \eta = {}^t(\eta_0, \ldots, \eta_r)$ such that $(A' - \lambda_j)\eta \neq 0$ and $(A' - \lambda_j)^k\eta = 0$ with some $k \geq 2$. With $v = \Sigma_{j=0}^r \eta_j H_{\phi_j}(\rho)$ we have $(F_p(\rho) - \lambda_j)v \neq 0$ while $(F_p(\rho) - \lambda_j)^k v = 0$ which shows that λ_j is nonzero and non semi-simple eigenvalue of F_p. This contradiction proves that $|\alpha(\rho)| = 1$. Thus

$$|\alpha(\rho)| = 1 \Longleftrightarrow W(\rho) \neq \{0\}. \tag{1.21}$$

Thanks to [3, Theorem] we have either $F_p(\rho) = \{0, \pm i\mu_j(\rho), 1 \leq j \leq r/2\}$ or $F_p(\rho) = \{0, \pm e(\rho), \pm i\mu_j(\rho), 1 \leq j \leq r/2 - 1\}$ if $W(\rho) = \{0\}$ where $e(\rho) > 0, \mu_j(\rho) > 0$. The former case we have

$$\Pi_{j=1}^r \lambda_j = \Pi_{j=1}^{r/2}(i\mu_j)(-i\mu_j) = \Pi_{j=1}^{r/2}\mu_j^2 > 0$$

hence $|\alpha(\rho)| < 1$ and vise versa. In the latter case we have

$$\Pi_{j=1}^r \lambda_j = \Pi_{j=1}^{r/2-1} e(-e)(i\mu_j)(-i\mu_j) = -e^2\Pi_{j=1}^{r/2-1}\mu_j^2 < 0$$

hence $|\alpha(\rho)| > 1$ and vice versa. Therefore we have

$$\begin{gathered} |\alpha(\rho)| < 1 \Longleftrightarrow p \text{ is non-effectively hyperbolic at } \rho \text{ and } W(\rho) = \{0\}, \\ |\alpha(\rho)| > 1 \Longleftrightarrow p \text{ is effectively hyperbolic at } \rho. \end{gathered}$$

In view of (1.5) one has

$$(\{\phi_i, \phi_j\})_{0 \leq i,j \leq r} v \overset{\Sigma'}{=} 0, \quad v = {}^t(1, -1/\delta, 0, \ldots, 0).$$

On the other hand since dimKer $A' = 1$ it follows from (1.18) that $\alpha = {}^t(-1/\delta, 0, \ldots, 0)$ so that $\delta|\alpha| = 1$, that is

$$1 + \theta = 1/|\alpha|^2 > 0.$$

This proves the assertion.

Thus applying this lemma one can prove (1.7). From $\theta = (1 - |\alpha|^2)/|\alpha|^2$ it follows from (1.20) that

$$\Pi_{j=1}^r \lambda_j(\rho) = (|\alpha(\rho)|^2 \det M(\rho))\theta(\rho), \quad \rho \in \Sigma', \;\; W(\rho) = \{0\} \tag{1.22}$$

where $|\alpha(\rho)|^2 \det M(\rho)$ is positive and smooth near $\bar{\rho}$. $\square$

Remark 1.2. Let $q = \Sigma_{j=1}^r \phi_j^2$ where $d\phi_j$ are linearly independent at $\bar{\rho}$. Then

$$\mathrm{Tr}^+ A(\rho) = \mathrm{Tr}^+ F_p(\rho), \quad A = (\{\phi_i, \phi_j\})_{1 \leq i,j \leq r}. \tag{1.23}$$

At the end of this section, we will provide a slightly more general example than (1.9). Consider

$$p = -\xi_0^2 + \xi_1^2 + (x_0 + x_1 - x_0 h(x_2))^2 \xi_n^2 + \xi_2^2, \quad |x_0|, \; |x_1|, \; |h(x_2)| \ll 1$$

near $(0,e_n)$ where $n\geq 3$. It is clear that with $\psi=x_0+x_1-x_0h(x_2)$ the doubly characteristic set is given by $\Sigma=\{\xi_0=\xi_1=\xi_2=0,\psi=0\}$. One can write

$$\begin{aligned}p&=-\xi_0^2+\big(\rho(\xi_1-x_0h'\xi_2)\big)^2+\psi^2\xi_n^2+\big(\rho(x_0h'\xi_1+\xi_2)\big)^2\\&=-\xi_0^2+\tilde\phi_1^2+\tilde\phi_2^2+\tilde\phi_3^2\end{aligned}$$

where $h'=\partial_{x_2}h$ and

$$\rho=1/\sqrt{1+x_0^2(h')^2},\ \ \tilde\phi_1=\rho(\xi_1-x_0h'\xi_2),\ \ \tilde\phi_2=\psi\xi_n,\ \ \tilde\phi_3=\rho(x_0h'\xi_1+\xi_2).$$

Here note that $\{\tilde\phi_3,\tilde\phi_j\}\overset{\Sigma'}{=}0$ for $j=0,1,2$. Next, we rewrite

$$\begin{aligned}&p=-\big(\xi_0+(h-1)\rho^2(\xi_1-x_0h'\xi_2)\big)\big(\xi_0-(h-1)\rho^2(\xi_1-x_0h'\xi_2)\big)\\&+\theta\big((h-1)\rho^2(\xi_1-x_0h'\xi_2)\big)^2+\tilde\phi_2^2+\tilde\phi_3^2,\quad \theta=\frac{2h+x_0^2(h')^2-h^2}{(1-h)^2}.\end{aligned}$$

Then denoting

$$\varphi_1=(1-h)\rho^2(\xi_1-x_0h'\xi_2),\quad \varphi_2=\psi\xi_n,\quad \varphi_3=\rho(x_0h'\xi_1+\xi_2)$$

we arrive at a normal form

$$\begin{gathered}p=-(\xi_0-\varphi_1)(\xi_0+\varphi_1)+\theta\varphi_1^2+\varphi_2^2+\varphi_3^2,\\ \{\xi_0-\varphi_1,\varphi_2\}\equiv 0,\quad \{\xi_0-\varphi_1,\xi_0\}\overset{\Sigma'}{=}0,\quad \{\varphi_3,\varphi_j\}\overset{\Sigma'}{=}0,\ \ j=0,1,2.\end{gathered}$$

2 Extension lemma

As mentioned in Remark 1.1, if $\tilde\theta|_{\Sigma'}=\theta|_{\Sigma'}$ we can write

$$p=-(\xi_0+\tilde\phi_1)(\xi_0-\tilde\phi_1)+(\tilde\theta+O^2(\Sigma'))\tilde\phi_1^2+\Sigma_{j=2}^r\phi_j^2+\Sigma_{j=r+1}^d\phi_j^2$$

which is still a normal form. Therefore the extension of $\theta|_{\Sigma'}$ to a full neighborhood of $\bar\rho$ becomes an important isuue.

2.1 Extension of $C^\infty(\Sigma')$

Lemma 2.1. *Assume that Σ is given by $\phi_j=0$ $(0\leq j\leq d)$ satisfying* (1.5) *and* (1.6). *There are neighborhoods V_i of $\bar\rho$ in $\mathbb{R}^{n+1}\times(\mathbb{R}^n\setminus 0)$ verifying the followings: for any $\theta(x,\xi')\in C^\infty(\Sigma'\cap V_1)$ there exists an extension $\tilde\theta\in C^\infty(V_2)$ such that*

$$\{\phi_1,\tilde\theta\}=c_1\phi_1+c_2\phi_2,\quad \{\phi_2,\tilde\theta\}=c_3\phi_2\quad \text{in } V_2 \tag{2.1}$$

with smooth c_i and

$$\{\phi_j,\tilde\theta\}\overset{\Sigma'}{=}0,\quad 3\leq j\leq r. \tag{2.2}$$

Moreover we can take $\inf_{\Sigma'\cap V_1}\theta\leq\tilde\theta\leq\sup_{\Sigma'\cap V_1}\theta$ *in V_2.*

Proof. Note that (1.5) and (1.6) imply $\{\xi_0, \phi_2\}(\bar\rho) \neq 0$ hence $\partial_{x_0}\phi_2(\bar\rho) \neq 0$ then one can write

$$\phi_2 = e_2(x_0 - \psi(x', \xi')), \quad e_2 \neq 0, \quad x' = (x_1, \ldots, x_n), \;\; \xi' = (\xi_1, \ldots, \xi_n).$$

In view of (1.6) it follows that $d\psi(\bar\rho) \neq 0$ because ϕ_1 is independent of ξ_0. Therefore take $\Xi_0 = \xi_0$, $X_0 = x_0$ and $X_1 = \psi(x', \xi')$ which satisfy the commutation relations and $d\Xi_0$, dX_0, dX_1, $\Sigma_{j=0}^n \xi_j dx_j$ are linearly independent at $\bar\rho$ hence extend to a full homogeneous symplectic coordinates system (X, Ξ) ([2, Theorem 21.1.9]). Changing $(X, \Xi) \to (x, \xi)$ one can assume that

$$\phi_2 = e_2(x_0 - x_1).$$

Since (1.6) implies that $\partial_{\xi_1}\phi_1(\bar\rho) \neq 0$ one can write

$$\phi_1 = e_1(\xi_1 - \psi(x, \xi'')), \quad e_1 \neq 0, \quad \xi'' = (\xi_2, \ldots, \xi_n).$$

Now investigate Σ' which is given by $\Sigma' = \{\phi_j = 0, j = 1, \ldots, d\}$ which is also given by

$$\{\phi_1 = \phi_2 = 0, \phi_j + \beta_j\phi_2 = 0, 3 \leq j \leq r, \phi_{r+1} = \cdots = \phi_d = 0\}$$

where smooth β_j are free. Choosing $\beta_k = \{\phi_1, \phi_k\}/\{\phi_2, \phi_1\}$ we have

$$\{\phi_j, \phi_k + \beta_k\phi_2\} \overset{\Sigma'}{=} 0, \quad 3 \leq k \leq r, \;\; j = 0, 1, 2. \tag{2.3}$$

In fact, if $j = 2$ this is clear from (1.6). If $j = 1$ it is clear from the choice of β_k. The case $j = 0$ is reduced to the case $j = 1$ by (1.5). Writing $\phi_k + \beta_k\phi_2 \to \varphi_k$ $(3 \leq k \leq r)$ and $\phi_j \to \varphi_j$ $(r + 1 \leq j \leq d)$ the manifold Σ' is given by

$$\Sigma' = \{x_0 - x_1 = 0, \xi_1 - \tilde\psi(x', \xi'') = 0, \tilde\varphi_j(x', \xi') = 0, 3 \leq j \leq d\}$$

where

$$\tilde\psi(x', \xi'') = \psi(x_1, x_1, x'', \xi''), \quad \tilde\varphi_j(x', \xi') = \varphi_j(x_1, x_1, x'', \xi').$$

Write $\xi_1 - \tilde\psi(x', \xi'') = e_1^{-1}\phi_1(x, \xi') + (x_0 - x_1)f$ and $\tilde\varphi_j = \varphi_j + (x_0 - x_1)f_j$. From $\{\xi_0, \varphi_j\} \overset{\Sigma'}{=} 0$ $(j \geq 3)$ by (2.3) one has $f_j \overset{\Sigma'}{=} 0$ $(j \geq 3)$. Then we have

$$\{\xi_1 - \tilde\psi, \tilde\varphi_j\} \overset{\Sigma'}{=} 0, \quad 3 \leq j \leq d \tag{2.4}$$

for $\{\phi_1, \varphi_j\} \overset{\Sigma'}{=} 0$ and $\{x_0 - x_1, \varphi_j\} \overset{\Sigma'}{=} 0$. It is also clear that

$$\{\tilde\varphi_i, \tilde\varphi_j\} \overset{\Sigma'}{=} \{\varphi_i, \varphi_j\}, \quad 3 \leq i, j \leq d$$

hence

$$\{\tilde\varphi_i, \tilde\varphi_j\} \overset{\Sigma'}{=} 0, \quad 3 \leq i \leq d, \;\; r + 1 \leq j \leq d. \tag{2.5}$$

Set $\Xi_0 = \xi_0$, $X_0 = x_0$, $X_1 = x_1$ and $\Xi_1 = \xi_1 - \tilde{\psi}(x', \xi'')$ which satisfy the commutation relations and $d\Xi_0$, dX_0, dX_1, $d\Xi_1$, $\Sigma_{j=0}^n \xi_j dx_j$ are linearly independent at $\bar{\rho}$ hence extends to a homogeneous symplectic coordinates system (X, Ξ) ([2, Theorem 21.1.9]). Since $\{\xi_0, \tilde{\varphi}_j\} = 0$ and $\{x_0, \tilde{\varphi}_j\} = 0$ writing $\tilde{\Phi}_j = \tilde{\varphi}_j$ for $3 \le j \le d$ we have $\{\Xi_0, \tilde{\Phi}_j\} = 0$ and $\{X_0, \tilde{\Phi}_j\} = 0$ so that $\tilde{\Phi}_j = \tilde{\Phi}_j(X', \Xi')$. Now Σ' is given by

$$\Sigma' = \{X_0 - X_1 = 0, \Xi_1 = 0, \hat{\Phi}_j(X', \Xi'') = 0, 3 \le j \le d\},$$
$$\hat{\Phi}_j(X', \Xi'') = \tilde{\Phi}_j(X', 0, \Xi'').$$

Denote $\Sigma'' = \{\hat{\Phi}_j(X', \Xi'') = 0, 3 \le j \le d\}$ and show that Σ'' is cylindrical in the X_1 direction. Write

$$\tilde{\Phi}_j(X', \Xi') = \hat{\Phi}_j(X', \Xi'') + \Xi_1 f \tag{2.6}$$

then we have

$$\partial_{X_1} \hat{\Phi}_j(X', \Xi'') = \{\Xi_1, \hat{\Phi}_j\} = \{\Xi_1, \tilde{\Phi}_j - \Xi_1 f\} \overset{\Sigma'}{=} 0$$

because of (2.4) and hence

$$\partial_{X_1} \hat{\Phi}_j(X', \Xi'') \overset{\Sigma''}{=} \Xi_1 h_1 + (X_0 - X_1) h_2.$$

Noting that the left-hand side contains neither Ξ_1 nor X_0 we conclude that

$$\partial_{X_1} \hat{\Phi}_j(X', \Xi'') \overset{\Sigma''}{=} 0, \quad 3 \le j \le d$$

which proves that Σ'' is cylindrical in the X_1 direction and hence

$$\Sigma'' = \{\hat{\Phi}_j(0, X'', \Xi'') = 0, 3 \le j \le d\}.$$

Denote $\tilde{\Sigma} = \{\hat{\Phi}_j(0, X'', \Xi'') = 0, 3 \le j \le r\}$. Since the restriction of the symplectic form to $\tilde{\Sigma}$ has constant rank $r - 2$ in a neighborhood of $\bar{\rho}$. Indeed

$$\det(\{\tilde{\varphi}_i, \tilde{\varphi}_j\})_{3 \le i,j \le r}(\bar{\rho}) = \det(\{\phi_i, \phi_j\})_{3 \le i,j \le r}(\bar{\rho}) \neq 0$$

implies $\mathrm{rank}(\{\tilde{\varphi}_i, \tilde{\varphi}_j\})_{3 \le i,j \le r}(\bar{\rho}) = r - 2$ on $\tilde{\Sigma}$. Thanks to [2, Theorem 21.2.4], there are homogeneous symplectic coordinates X'', Ξ'' such that, denoting $\hat{\Phi}_j(0, X'', \Xi'')$ by $\psi_j(X'', \Xi'')$, $3 \le j \le d$ in these new symplectic cordinates (X'', Ξ''), we have

$$\tilde{\Sigma} = \{\psi_j = 0, 3 \le j \le r\} = \{X_2 = \cdots = X_l = \Xi_2 = \cdots = \Xi_l = 0\}, \quad r = 2l$$

so that Σ' is given by

$$\{X_0 - X_1 = 0, \Xi_1 = 0, X_2 = \cdots = X_l = \Xi_2 = \cdots = \Xi_l = 0, \psi_j(0, \tilde{X}, 0, \tilde{\Xi})$$
$$= 0, r + 1 \le j \le d\}, \quad \tilde{X} = (X_{l+1}, \ldots, X_n),\ \tilde{\Xi} = (\Xi_{l+1}, \ldots, \Xi_n).$$

Here we note that

$$\{X_i,\psi_j\}\stackrel{\Sigma'}{=}0,\quad \{\Xi_i,\psi_j\}\stackrel{\Sigma'}{=}0,\quad 2\le i\le l,\ \ r+1\le j\le d. \tag{2.7}$$

To prove this we first show

$$\{\psi_i,\psi_j\}\stackrel{\Sigma'}{=}0,\quad 3\le i\le d,\ \ r+1\le j\le d. \tag{2.8}$$

Note that $\partial_{X_1}\tilde\Phi_j(X',\Xi')\stackrel{\Sigma'}{=}0$ for $3\le j\le d$ because $\hat\Phi_j(X',\Xi'')=0$ on Σ'' hence are linear combinations of $\hat\Phi_j(0,X'',\Xi'')$, $3\le j\le d$ and (2.6). Therefore we have

$$\{\hat\Phi_i,\hat\Phi_j\}=\{\tilde\Phi_i+\Xi_1 f_i,\tilde\Phi_j+\Xi_1 f_j\}\stackrel{\Sigma'}{=}0,\quad 3\le i\le d,\ \ r+1\le j\le d \tag{2.9}$$

for $\{\tilde\Phi_i,\tilde\Phi_j\}\stackrel{\Sigma'}{=}0$ by (2.5). Since

$$\{\hat\Phi_i(0,X'',\Xi''),\hat\Phi_j(0,X'',\Xi'')\}\stackrel{\Sigma'}{=}\{\hat\Phi_i+X_1g_i,\hat\Phi_j+X_1g_j\}\stackrel{\Sigma'}{=}\{\hat\Phi_i,\hat\Phi_j\}$$

we get (2.8). Since X_i, Ξ_i, $2\le i\le l$ are linear combinations of $\psi_j(X'',\Xi'')$, $3\le j\le r$ we get the assertion.

Denote $\tilde\Sigma'=\{\tilde\psi_j(\tilde X,\tilde\Xi)=\psi_j(0,\tilde X,0,\tilde\Xi)=0, r+1\le j\le d\}$. Write

$$\psi_j=\tilde\psi_j(\tilde X,\tilde\Xi)+\Sigma_{k=2}^l c_{jk}X_k+\Sigma_{k=2}^l c'_{jk}\Xi_k,\quad r+1\le j\le d.$$

It follows from (2.7) that $c_{jk}\stackrel{\Sigma'}{=}0$ and $c'_{jk}\stackrel{\Sigma'}{=}0$ hence we have

$$\tilde\psi_j=\psi_j+O^2(\Sigma'),\quad r+1\le j\le d$$

which proves that $\{\tilde\psi_i,\tilde\psi_j\}\stackrel{\Sigma'}{=}0,\quad r+1\le i,j\le d$. Since $\tilde\psi_j$, $r+1\le j\le d$ contains no $X_0,X_1,\dots,X_l$ and $\Xi_1,\dots,\Xi_l$ we conclude that

$$\{\tilde\psi_i,\tilde\psi_j\}\stackrel{\tilde\Sigma'}{=}0,\quad r+1\le i,j\le d.$$

Since the restriction of the symplectic form to $\tilde\Sigma'$ has rank 0 there are homogeneous symplectic coordinates $\tilde X,\tilde\Xi$ such that $\tilde\Sigma'$ is given by (thanks to [2, Theorem 21.2.4])

$$\Xi_{l+1}=\cdots=\Xi_{d-l}=0.$$

So there exist homogeneous symplectic coordinates (X,Ξ) leaving X_0,Ξ_0 unchanged such that Σ' is given by

$$\{X_0-X_1=0,\Xi_1=0,X_2=\cdots=X_l=\Xi_2=\cdots=\Xi_l=0,\Xi_{l+1}=\cdots=\Xi_{d-l}=0\}.$$

Let $\theta(x,\xi')\in C^\infty(\Sigma')$. Write $\theta(x,\xi')=\Theta(X,\Xi')$. Define the extension $\tilde\Theta(X,\Xi')$ of $\Theta(X,\Xi')$ outside Σ' to a neighborhood of $\bar\varrho$ ($\bar\rho\leftrightarrow\bar\varrho$) by

$$\tilde\Theta(X,\Xi')=\Theta(X_0,X_0,0,\dots,0,X_{l+1},\dots,X_n,0,\dots,0,\Xi_{d-l+1},\dots,\Xi_n).$$

It is clear that $\inf_{\Sigma'} \Theta \le \tilde{\Theta} \le \sup_{\Sigma'} \Theta$. Define the extension $\tilde{\theta}(x,\xi')$ of $\theta(x,\xi')$ outside Σ' by

$$\tilde{\theta}(x,\xi') = \tilde{\Theta}(X,\Xi').$$

Since $\{\phi_2, \tilde{\theta}\} = \{\tilde{e}_2(X_0 - X_1), \tilde{\Theta}\} = c_3(X_0 - X_1)$ for $\{X_0 - X_1, \tilde{\Theta}\} = \partial_{\Xi_1}\tilde{\Theta} = 0$ thus we have

$$\{\phi_2, \tilde{\theta}\} = c\,\phi_2.$$

Note that ϕ_1 is given by $\tilde{e}_1\Xi_1 + \tilde{f}_1(X_0 - X_1)$ then

$$\{\phi_1, \tilde{\theta}\} = \{\tilde{e}_1\Xi_1 + \tilde{f}_1(X_0 - X_1), \tilde{\Theta}\} = \tilde{c}_1\Xi_1 + \tilde{c}_2(X_0 - X_1) = c_1\phi_1 + c_2\phi_2$$

because $\{\Xi_1, \tilde{\Theta}\} = 0$ which proves the assertion.

Since $\{\Xi_j, \tilde{\Theta}\} = 0$ $(1 \le j \le l)$ and $\{X_j, \tilde{\Theta}\} = 0$ $(0 \le j \le d - l)$ and

$$\hat{\Phi}_j(0, X'', \Xi'') = \Sigma_{k=2}^{l} a_{jk}\Xi_k + \Sigma_{k=2}^{l} b_{jk}X_k, \quad 3 \le j \le r$$

and $\hat{\Phi}_j = \hat{\Phi}_j(0, X'', \Xi'') + X_1 g_j$ then noting $d - l \ge l$ we have $\{\hat{\Phi}_j, \tilde{\Theta}\} \overset{\Sigma'}{=} 0$, $3 \le j \le r$. Recalling $\tilde{\Phi}_j(X', \Xi') = \hat{\Phi}_j(X', \Xi'') + \Xi_1 f_j$ we have $\{\tilde{\Phi}_j, \tilde{\Theta}\} \overset{\Sigma'}{=} 0$, $3 \le j \le r$. Therefore we have

$$\{\tilde{\varphi}_j, \tilde{\theta}\} \overset{\Sigma'}{=} 0, \quad 3 \le j \le r.$$

Since $\phi_j = \tilde{\varphi}_j + (x_0 - x_1)g_j$, $3 \le j \le r$ we conclude that $\{\phi_j, \tilde{\theta}\} \overset{\Sigma'}{=} 0$ for $3 \le j \le r$. $\square$

At the end of this section, we reconsider the example (1.9);

$$p = -(\xi_0 + \phi_1)(\xi_0 - \phi_1) + \theta(x)\phi_1^2 + \phi_2^2, \quad \phi_1 = \xi_1, \;\; \phi_2 = (x_0 + x_1)\xi_n.$$

Denoting

$$\tilde{\theta}(x_0, x'') = \theta(x_0, -x_0, x''), \quad x'' = (x_2, \ldots, x_n)$$

one can write $\theta(x) = \tilde{\theta}(x_0, x'') + (x_0 + x_1)\alpha(x)$. Applying Lemma 1.2 we can write with $\tilde{\phi}_1 = (1 + \nu)\phi_1$

$$p = -(\xi_0 + \tilde{\phi}_1)(\xi_0 - \tilde{\phi}_1) + \hat{\theta}\tilde{\phi}_1^2 + \phi_2^2, \quad \hat{\theta} = (\theta - \nu^2 - 2\nu)/(1 + \nu)^2.$$

Choosing $2\nu = (x_0 + x_1)\alpha(x)$ hence $\hat{\theta} = \tilde{\theta} + r$, $r = O((x_0 + x_1)^2)$ it is clear that

$$\{\tilde{\phi}_1, \hat{\theta}\} = O((x_0 + x_1)), \quad \{\phi_2, \hat{\theta}\} = 0.$$

2.2 More about the extension

Denote by $\tilde{\theta}$ the extension of θ given by Lemma 2.1 then we see that

$$\theta_1 = \theta_2 \Longrightarrow \tilde{\theta}_1 = \tilde{\theta}_2, \quad \widetilde{\theta_1 + \theta_2} = \tilde{\theta}_1 + \tilde{\theta}_2, \quad \widetilde{\theta_1\theta_2} = \tilde{\theta}_1\tilde{\theta}_2. \tag{2.10}$$

Let $\theta \in C^\infty(\Sigma')$. Taking that $H_{\xi_0-\phi_1}$ is tangent to Σ' into account assume

$$\theta(\bar\rho)=0, \quad \{\xi_0-\phi_1,\theta\}(\bar\rho)=0, \quad \{\xi_0-\phi_1,\{\xi_0-\phi_1,\theta\}\}\neq 0 \ \text{ at } \ \bar\rho. \tag{2.11}$$

Denote by $\tilde\theta$ the extension of θ. Choosing a symplectic coordinates system such that $\Xi_0=\xi_0-\phi_1$, $X_0=x_0$ and denoting $\Theta(X,\Xi')=\tilde\theta(x,\xi')$ we have $\partial_{X_0}^k\Theta(\bar\varrho)=0$, $k=0,1$ and $\partial_{X_0}^2\Theta(\bar\varrho)\neq 0$ then thanks to Malgrange preparation theorem one can write

$$\Theta(X,\Xi')=E\big((X_0-\Psi(X',\Xi'))^2+G(X',\Xi')\big)=E(F^2+G), \quad E\neq 0, \ G\geq 0$$

where $\partial_{X_0}F=\{\Xi_0,F\}=1$, $\partial_{X_0}G=\{\Xi_0,G\}=0$ and $\partial_{X_0}\Psi=\{\Xi_0,\Psi\}=0$. Turning back to the coordinates (x,ξ) we have

$$\begin{gathered}\tilde\theta=e(f(x,\xi')^2+g(x,\xi')), \quad f(x,\xi')=x_0-\psi(x,\xi'),\\ \{\xi_0-\phi_1,f\}=1, \ \ \{\xi_0-\phi_1,g\}=0.\end{gathered}$$

Denoting by $\tilde e,\tilde f,\tilde g$ the extensions of $e|_{\Sigma'}, f|_{\Sigma'}, g|_{\Sigma'}$ respectively we have by (2.10)

$$\tilde\theta=\tilde e(\tilde f^2+\tilde g)$$

where $\tilde e$, $\tilde f$ and $\tilde g$ verify (2.1) and (2.2) and that

$$\tilde f=x_0-\tilde\psi, \quad \partial_{x_0}\tilde\psi(\bar\rho)=0, \quad \{\xi_0,\tilde g\}\overset{\Sigma'}{=}0, \quad \tilde g\geq 0, \quad \tilde e\neq 0$$

since $\{\xi_0-\phi_1,\tilde f\}=1$ and $\{\tilde f,\phi_1\}\overset{\Sigma'}{=}0$.

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