A direct energy estimates for effectively hyperbolic operators

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Abstract

This paper is devoted to a simpler derivation of energy estimates, compared to previously existing ones, for effectively hyperbolic operators. One of main points is no use of general Fourier integral operators and another point is an efficient use of the Weyl calculus of pseudodifferential operators associated with several different metrics.

1 Introduction

Consider

$$P = -D_t^2 + A_2(t,x,D) + A_0(t,x,D)D_t + A_1(t,x,D) \tag{1.1}$$

where $A_j(t,x,D)$ are classical pseudodifferential operators of order j on $\mathbb{R}^d$ depending smoothly on t. Denote the principal symbol of P by

$$p(t,x,\tau,\xi) = -\tau^2 + a(t,x,\xi)$$

where $a(t,x,\xi)$ is positively homogeneous of degree 2 in ξ which is assumed to be nonnegative for any $(t,x,\xi) \in U \times \mathbb{R}^d$ with some neighborhood U of $(0,0) \in \mathbb{R}^{d+1}$, a necessary condition for the Cauchy problem for P to be C^∞ well-posed near the origin.

In [5], Ivrii and Petkov proved that if the Cauchy problem for P is C^∞ well-posed for any lower order term then the Hamilton map F_p has a pair of non-zero real eigenvalues at every singular point of $p=0$ ([5, Theorem 3]). A singular point of $p=0$ is called *effectively hyperbolic* ([2]) if the Hamilton map has a pair of non-zero real eigenvalues there. In [6], Ivrii has proved that if every singular point is effectively hyperbolic, and p admits a factorization $p = q_1q_2$ nearby with real smooth symbols q_i, then the Cauchy problem is C^∞ well-posed for every lower order term, reducing P to another one with controllable lower order terms, by operator powers of operator.

If a singular point (t,x,τ,ξ) is effectively hyperbolic then τ is a characteristic root of multiplicity at most 3 ([5, Lemma 8.1]) and every multiple characteristic

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root is at most double, the conjecture has been proved in [7, 8], [11]. In [7, 8] the idea of the proof is to reduce P to a perturbation of that treated in [6] by operator powers of operator of which symbol is found applying the Nash-Moser implicit function theorem. On the other hand in [11] (see also [13]) the proof is based on weighted energy estimates with pseudodifferential weights of which symbol stems from a geometric characterization of effectively hyperbolic singular points, after some preliminary transformations by Fourier integral operators. For the Cauchy problem for operators with triple effectively hyperbolic characteristics, where p cannot be smoothly factorized, see [15] and the references given there.

In this paper, though we follow [13] mainly, we derive energy estimates using only changes of local coordinates x and the Weyl calculus of pseudodifferential operators (see [4, Chapter 18]), which makes much simpler the arguments to derive local existence of solution to the Cauchy problem (Theorem 3.2 below) from microlocal energy estimates. On the other hand, in [14] we gave another way to obtain microlocal energy estimates without use of Fourier integral operators where, in spite of C^∞ problem, we need a calculus of Gevrey pseudodifferential operators in the (t, x)-space and a technical verification of support of solutions.

In Section 3 we derive (microlocal) weighted energy estimates and prove local existence result for the Cauchy problem. In Section 2 several lemmas and propositions required in Section 3 are stated without proofs, of which proofs are given in Sections 4. In the last section 5 we give a proof of Proposition 2.1 below for the sake of completeness.

2 Preparations for direct energy estimates

First recall [12, Lemmas 3.1, 3.2] (see also [13, Section 2.1]).

Proposition 2.1. *Assume that $(0, 0, 0, \bar{\xi})$ is effectively hyperbolic. One can choose a local coordinates x with $\bar{\xi} = e_d = (0, \dots, 0, 1)$ and smooth function $\psi(x, \xi)$ such that either $d\psi = d\xi_1$ or $d\psi = \varepsilon dx_1 + c dx_d$ at $(0, e_d)$ where $c \in \mathbb{R}$, $\varepsilon = 0$ or 1, and smooth $\ell(t, x, \xi)$, $q(t, x, \xi) \geq 0$ vanishing at $(0, e_d)$, positively homogeneous in ξ of degree 1, 2 respectively such that*

$$p(t, x, \tau, \xi) = -\tau^2 + \ell^2(t, x, \xi) + q(t, x, \xi), \quad q(t, x, \xi) \geq c(t - \psi)^2|\xi|^2$$

with some $c > 0$ on a conic neighborhood of $(0, e_d)$ where

$$|\{\ell, \psi\}(0, e_d)| < 1, \quad \{\psi, \{\psi, q\}\}(0, e_d) = 0. \tag{2.1}$$

Note that the change of coordinates can be extended to a diffeomorphism on $\mathbb{R}^d$ which is a linear transformation outside a neighborhood of $x = 0$. According to $d\psi = d\xi_1$ or $d\psi = \varepsilon x_1 + c x_d$ at $(0, e_d)$ one can write

$$\psi(x, \xi) = \xi_1/|\xi| + r(x, \xi), \quad \psi(x, \xi) = \varepsilon x_1 + c x_d + r(x, \xi) \tag{2.2}$$

where $dr(0, e_d) = 0$. Since the case $\varepsilon = 0$ is easier than the case $\varepsilon = 1$ and modifications needed are easily seen we assume $\varepsilon = 1$ from now on. Note that $\{\psi, \{\psi, q\}\}(0, e_d) = 0$ implies that

$$\partial_{x_1}^2 q(0, e_d) = 0 \text{ if } d\psi = d\xi_1, \quad \partial_{\xi_1}^2 q(0, e_d) = 0 \text{ if } d\psi = dx_1 + c dx_d. \tag{2.3}$$

We call (a) the coordinates change which leads to $d\psi = d\xi_1$ and call (b) which leads to $d\psi = dx_1 + cdx_d$.

2.1 Localization of symbols

After making a change of coordinates in Proposition 2.1 we localize such obtained symbol (operator) to a neighborhood of $(0, e_d)$. We first localize coordinates functions. Let $\chi(s) \in C^\infty(\mathbb{R})$ be equal to s on $|s| \le 1$, $|\chi(s)|$ is constant for $|s| \ge 2$ and $0 \le d\chi(s)/ds = \chi^{(1)}(s) \le 1$ everywhere. Define $y(x) = (y_1(x), \ldots, y_d(x))$ and $\eta(\xi) = (\eta_1(\xi), \ldots, \eta_d(\xi))$ by

$$y_j(x) = M^{-1}\chi(Mx_j), \;\; \eta_j(\xi) = M^{-1}\chi(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))$$

for $j = 1, 2, \ldots, d$ with $\langle\xi\rangle_\gamma = (\gamma^2 + |\xi|^2)^{1/2}$ where δ_{ij} is the Kronecker's delta and M, γ are large positive parameters constrained

$$\gamma \ge M^4. \tag{2.4}$$

It is easy to see that $(1 - CM^{-1})\langle\xi\rangle_\gamma \le |(\eta + e_d)\langle\xi\rangle_\gamma| \le (1 + CM^{-1})\langle\xi\rangle_\gamma$ and

$$|y(x)| \le CM^{-1}, \quad |\eta(\xi)| \le CM^{-1}, \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \tag{2.5}$$

with some $C > 0$ so that $(y(x), \eta(\xi) + e_d)$ is contained in a neighborhood of $(0, e_d)$, shrinking with M. Note that $(y, (\eta + e_d)\langle\xi\rangle_\gamma) = (x, \xi)$ on the conic neighborhood $W_{M,\gamma}$ of $(0, e_d)$;

$$W_{M,\gamma} = \big\{(x, \xi) \mid |x| \le M^{-1}, \; |\xi/|\xi| - e_d| \le M^{-1}/2, \; |\xi| \ge \gamma M^{1/2}\big\} \tag{2.6}$$

because if $(x, \xi) \in W_{M,\gamma}$ then

$$\begin{gathered}\big|\xi/\langle\xi\rangle_\gamma - e_d\big| \le \big|\xi/\langle\xi\rangle_\gamma - \xi/|\xi|\big| + \big|\xi/|\xi| - e_d\big| \le M^{-1}/2 \\ +\big(\big|\langle\xi\rangle_\gamma - |\xi|\big|\big)/\langle\xi\rangle_\gamma \le M^{-1}/2 + \gamma^2\langle\xi\rangle_\gamma^{-1}(\langle\xi\rangle_\gamma + |\xi|)^{-1} \le M^{-1}.\end{gathered}$$

In what follows we assume that the range of t is constrained such that

$$|t| < T_0 M^{-1} = \delta \tag{2.7}$$

with some fixed $T_0 > 0$.

Let $f(x, \xi) \in S^l_{1,0}(W)$ where W is a conic neighborhood of $(0, e_d)$. We define the localization of f by $f_M(x, \xi) = f(y(x), (\eta(\xi) + e_d)\langle\xi\rangle_\gamma)$ which depends also on γ and coincides with the original f in $W_{M,\gamma}$ if M is large. Denote the coordinates change in Proposition 2.1, extended to $\mathbb{R}^d$, by $x \mapsto \kappa(x)$ and $(Tu)(t,x) = u(t,\kappa(x))$ then the localized symbol of $T^{-1}PT$ is written as

$$-\tau^2 + \ell_M^2(t,x,\xi) + q_M(t,x,\xi) + b_1(t,x,\xi) + b_0(t,x,\xi)\tau$$

which we denote by $\hat{P}$ from now on. All symbols (operators) with which we work in this paper are obtained making two different coordinates changes in Proposition 2.1. To clarify which coordinates change is employed we write

$$\text{assertion}, \quad (a) \quad (\text{respectively } (b))$$

which means that the assertion holds when the coordinates change (a) is chosen (respectively when (b) is chosen). If the assertion contains ϵ we mean that the assertion corresponding to ϵ holds when we choose the coordinates change (ϵ), $\epsilon = a, b$. If the assertion contains neither (a), (b) nor ϵ, it means that the assertion holds for both coordinates changes (a) and (b).

Let

$$G = M^2|dx|^2 + M^2\langle\xi\rangle_\gamma^{-2}|d\xi|^2 = M^2\big(|dx|^2 + \langle\xi\rangle_\gamma^{-2}|d\xi|^2\big).$$

Lemma 2.1. *Let $f(z)$ be a smooth function in a neighborhood of $\bar{z}$ and let $z_j(x,\xi) \in S(M^{-1},G)$ and $f_M(x,\xi) = f(z(x,\xi)+\bar{z})$. Then $f_M(x,\xi) \in S(M^{-r},G)$ if $\partial_z^\alpha f(\bar{z}) = 0$ for $0 \le |\alpha| < r$. In particular $f_M(x,\xi) - f(\bar{z}) \in S(M^{-1},G)$.*

It is clear $y_j \in S(M^{-1},G)$ while we see

$$\begin{aligned} \big|\partial_\xi^\alpha \eta_j(\xi)\big| &\precsim \sum_{|\alpha_i|\ge 1} M^{-1}|\chi^{(s)}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))| \\ &\times|\partial_\xi^{\alpha_1}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))|\cdots|\partial_\xi^{\alpha_s}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))| \\ &\precsim \sum_{s\le|\alpha|} M^{-1}M^s\langle\xi\rangle_\gamma^{-|\alpha|} \precsim M^{-1+|\alpha|}\langle\xi\rangle_\gamma^{-|\alpha|} \end{aligned}$$

so that $\eta_j \in S(M^{-1},G)$ where $A \precsim B$ means that $A \le CB$ with some $C > 0$ independent of M and γ.

Lemma 2.2. *We have $\partial\eta_j/\partial\xi_k - \delta_{jk}\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1} \in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$ for $1 \le j \le d-1$, $1 \le k \le d$.*

By Lemma 2.1 we have $\psi_M(x,\xi) = \psi(y(x),\eta(\xi)+e_d) \in S(M^{-1},G)$ which we denote by $\psi(x,\xi)$ dropping M to simplify notation. Denoting

$$\bar{\ell}(t,x,\xi) = \ell(t,y(x),\eta(\xi)+e_d), \quad \bar{q}(t,x,\xi) = q(t,y(x),\eta(\xi)+e_d) \tag{2.8}$$

we have $\ell_M = \bar{\ell}(t,x,\xi)\langle\xi\rangle_\gamma$ and $q_M = \bar{q}(t,x,\xi)\langle\xi\rangle_\gamma^2$ which we denote by $\ell(t,x,\xi)$ and $q(t,x,\xi)$ dropping M again. Note that $\bar{\ell} \in S(M^{-1},G)$ and $\bar{q} \in S(M^{-2},G)$ in view of Lemma 2.1 and Proposition 2.1 shows that

$$\bar{q}(t,x,\xi) \ge c\,(t-\psi(x,\xi))^2. \tag{2.9}$$

Lemma 2.3. *We have* $q \in S(M^{-2}\langle\xi\rangle_\gamma^2, G)$. *There exists* $C > 0$ *such that*

$$\begin{aligned}&\left|\partial_{x_1} q\right| \le CM^{-1/2}\sqrt{q}\,\langle\xi\rangle_\gamma, \quad \left|\partial_{x_j} q\right| \le C\sqrt{q}\,\langle\xi\rangle_\gamma, \;\; j \ne 1, \quad (a),\\ &\left|\partial_{\xi_j} q\right| \le CM^{-1/2}\sqrt{q}, \;\; j = 1, d, \quad \left|\partial_{\xi_j} q\right| \le C\sqrt{q}, \;\; j \ne 1, d, \quad (b).\end{aligned}$$

Lemma 2.4. *We have* $\psi \in S(M^{-1}, G)$ *and*

$$\left.\begin{aligned}&\psi(x,\xi) - M^{-1}\chi(M\xi_1\langle\xi\rangle_\gamma^{-1}) \in S(M^{-2}, G)\\ &\partial\psi/\partial\xi_k - \delta_{1k}\chi^{(1)}(M\xi_1\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1} \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, G)\end{aligned}\right\} \quad (a),$$

$$\left.\begin{aligned}&\psi(x,\xi) - M^{-1}\chi(Mx_1) - cM^{-1}\chi(Mx_d) \in S(M^{-2}, G),\\ &\partial\psi/\partial x_k - \delta_{1k}\chi^{(1)}(Mx_1) - c\delta_{dk}\chi^{(1)}(Mx_d) \in S(M^{-1}, G)\end{aligned}\right\} \quad (b).$$

Proposition 2.2. *We have* $\left|\{q, \psi\}\right| \le CM^{-1/2}\sqrt{q}\,\langle\xi\rangle_\gamma^{-1}$.

Proof. The proof is clear from Lemmas 2.3 and 2.4. □

Thanks to Lemma 2.4 one sees

Lemma 2.5. *We have* $\{\ell, \psi\} + \kappa\chi^{(1)}(Mx_1)\chi^{(1)}(M\xi_1\langle\xi\rangle_\gamma^{-1}) \in S(M^{-1}, G)$ *where* $\kappa = \partial_{x_1}\ell(0, e_d)$, *(a) or* $\kappa = -\partial_{\xi_1}\ell(0, e_d)$, *(b) and* $|\kappa| < 1$ *by* (2.1).

2.2 Approximate square roots and pseudodifferential weights

Introducing a parameter $\lambda \ge 1$ we denote

$$\bar{b} = \big(\bar{q} + \lambda\langle\xi\rangle_\gamma^{-1}\big)^{1/2}$$

so that $b = \langle\xi\rangle_\gamma\bar{b} = (q + \lambda\langle\xi\rangle_\gamma)^{1/2}$ where λ is constrained

$$\lambda \le \gamma M^{-2}, \qquad \lambda \ge 1 \tag{2.10}$$

such that $\lambda\langle\xi\rangle_\gamma^{-1} \le M^{-2}$. In the end of this section λ will be fixed. Introducing

$$\omega = ((t-\psi)^2 + \langle\xi\rangle_\gamma^{-1})^{1/2} \tag{2.11}$$

where $\langle\xi\rangle_\gamma^{-1/2} \le \omega \le CM^{-1}$ and taking (2.9) into account one has

$$\begin{aligned}b = \big(q + \lambda\langle\xi\rangle_\gamma\big)^{1/2} &\ge \big(c(t-\psi)^2\langle\xi\rangle_\gamma^2 + \lambda\langle\xi\rangle_\gamma\big)^{1/2}\\ &\ge c\,\omega^{-1}\langle\xi\rangle_\gamma\big((t-\psi)^2\omega^2 + \omega^2\langle\xi\rangle_\gamma^{-1}\big)^{1/2}\\ \ge c\,\omega^{-1}\langle\xi\rangle_\gamma\big(|t-\psi|^4 + \langle\xi\rangle_\gamma^{-2}\big)^{1/2} &\ge (c/\sqrt{2})\langle\xi\rangle_\gamma\omega\end{aligned} \tag{2.12}$$

because $\omega^2 \ge \langle\xi\rangle_\gamma^{-1}$. Introduce the metric

$$\bar{g} = \langle\xi\rangle_\gamma|dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2$$

which is one of basic metrics with which we work. Note that $S(m,G)\subset S(m,\bar{g})$ because $M^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}\le (M^2\langle\xi\rangle_\gamma^{-1})^{|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ and $M^2\langle\xi\rangle_\gamma^{-1}\le 1$. The metric $\bar{g}$ is σ temperate *uniformly* in $\gamma\ge M^4\ge 1$ (e.g.[4, Definition18.5.1]) which will be checked later. For a σ temperate metric g, in this paper, we call a positive σ, g temperate function (e.g.[4, Definition 18.5.1]) an admissible weight for g (or admissible for g for short).

Lemma 2.6. *We have* $\bar{b}\in S(\bar{b},\bar{g})$ *and* $\partial_x^\alpha\partial_\xi^\beta\,\bar{b}\in S(\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\,\bar{b},\bar{g})$ *for* $|\alpha+\beta|=1$.

Lemma 2.7. *We have* $\partial_x^\alpha\partial_\xi^\beta\,\bar{b}\in S(\langle\xi\rangle_\gamma^{-|\beta|},\bar{g})$ *for* $|\alpha+\beta|=1$.

It follows easily from Lemma 2.6

Lemma 2.8. *We have* $\partial_x^\alpha\partial_\xi^\beta\,\bar{b}^{-1}\in S(\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\,\bar{b}^{-1},\bar{g})$ *for* $|\alpha+\beta|=1$.

Proposition 2.3. *b is an admissible weight for $\bar{g}$ and $b\in S(b,\bar{g})$.*

Since b and b^{-1} are admissible weights for $\bar{g}$ we have

$$b\#b^{-1}=1-r$$

where $r\in S(\lambda^{-1},\bar{g})$ which follows from Lemmas 2.6 and 2.8. Therefore choosing $\lambda\ge 1$ suitably large we have $\|\mathrm{op}(r)\|_{\mathcal{L}(L^2,L^2)}<1$ so that $(I-\mathrm{op}(r))^{-1}$ exists which is given by $(I-\mathrm{op}(r))^{-1}=\mathrm{op}(\tilde{r})$ with $\tilde{r}\in S(1,\bar{g})$ (see [1], [10]). Thus we have $b\#(b^{-1}\#\tilde{r})=1$ and $(b^{-1}\#\tilde{r})\#b=1$ where $\tilde{b}=b^{-1}\#\tilde{r}\in S(b^{-1},\bar{g})$. We summarize

Proposition 2.4. *One can find $\lambda\ge 1$ independent of M and γ such that there exists $\tilde{b}\in S(b^{-1},\bar{g})$ satisfying $b\#\tilde{b}=\tilde{b}\#b=1$.*

From now on we *fix* such a $\lambda=\bar{\lambda}$ while M and γ remain to be free with the constraints (2.4) and (2.10).

Lemma 2.9. *We have* $\bar{q}\in S(\langle\xi\rangle_\gamma^{-1/2}\,\bar{b},\bar{g})$. *Moreover* $\partial_{x_1}\bar{q}\in S(M^{-1/2}\,\bar{b},\bar{g})$, *(a) and* $\partial_{\xi_j}\bar{q}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{-1}\,\bar{b},\bar{g})$ *for* $j=1,d$, *(b).*

Corollary 2.1. *We have* $\partial_{x_1}\bar{b}\in S(M^{-1/2},\bar{g})$, *(a) and* $\partial_{\xi_j}\bar{b}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{-1},\bar{g})$ *for* $j=1,d$, *(b).*

Corollary 2.2. *We have* $\partial_t\bar{b}\in S(1,\bar{g})$. *Moreover* $\partial_{x_1}\partial_t\bar{b}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{1/2},\bar{g})$, *(a) and* $\partial_{\xi_j}\partial_t\bar{b}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{-1/2},\bar{g})$ *for* $j=1,d$, *(b).*

Define ϕ, the symbol of weight for energy estimates, by

$$\phi=\sqrt{(t-\psi)^2+\langle\xi\rangle_\gamma^{-1}}+t-\psi=\omega+t-\psi$$

and note that

$$M\langle\xi\rangle_\gamma^{-1}/C\le\langle\xi\rangle_\gamma^{-1}/(2\omega)\le\phi\le CM^{-1}. \tag{2.13}$$

Introduce two metrics g_ϵ, $\epsilon = a, b$ associated to the case (a) and (b);

$$g_\epsilon = M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma|dx|^2 + M^{-2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1}|d\xi|^2 \tag{2.14}$$

where $\delta_{\epsilon\epsilon'} = 1$ if $\epsilon = \epsilon'$ and $\delta_{\epsilon\epsilon'} = 0$ otherwise. The metric g_ϵ is σ temperate *uniformly* in $\gamma \geq M^2 \geq 1$ which is checked later. Note that

$$M^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \leq (M^4\langle\xi\rangle_\gamma^{-1})^{|\alpha+\beta|/2}M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}.$$

so that $S(m, G) \subset S(m, g_\epsilon)$ where

$$\epsilon(\alpha, \beta) = |\alpha|\delta_{\epsilon a} + |\beta|\delta_{\epsilon b}.$$

Proposition 2.5. *We have* $\omega^s \in S(\omega^s, g_\epsilon)$ *and* $\phi^s \in S(\phi^s, g_\epsilon)$*. Moreover*

$$\partial_x^\alpha\partial_\xi^\beta\omega^s \in S(M^{-\epsilon(\alpha,\beta)}\omega^{-1}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\omega^s, g_\epsilon),$$
$$\partial_x^\alpha\partial_\xi^\beta\phi^s \in S(M^{-\epsilon(\alpha,\beta)}\omega^{-1}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi^s, g_\epsilon)$$

for $|\alpha + \beta| \geq 1$.

Lemma 2.10. *We have* $\partial_{\xi_j}\phi \in S(M^{-1}\omega^{-1}\langle\xi\rangle_\gamma^{-1}\phi, g_a)$, $\partial_{\xi_j}\omega^s \in S(M^{-1}\omega^{s-1}\langle\xi\rangle_\gamma^{-1}, g_a)$ *for* $j \neq 1$, (a) *and* $\partial_{x_j}\phi \in S(M^{-1}\omega^{-1}\langle\xi\rangle_\gamma^{-1}\phi, g_b)$, $\partial_{x_j}\omega^s \in S(M^{-1}\omega^{s-1}\langle\xi\rangle_\gamma^{-1}, g_b)$ *for* $j \neq 1, d$, (b).

Proposition 2.6. ω, ϕ *are admissible weights for both* g_ϵ *and* $\bar{g}$.

2.3 Some bounds of pseudodifferential operators

In the following lemmas all constants are independent of M and γ.

Lemma 2.11. *Let* m *be admissible for* g_ϵ *and* $p \in S(m, g_\epsilon)$ *satisfy* $p \geq c\,m$ *with some constant* $c > 0$*. Then* $p^{-1} \in S(m^{-1}, g_\epsilon)$ *and there exist* $k, \tilde{k} \in S(M^{-1}, g_\epsilon)$ *such that*

$$p\#p^{-1}\#(1+k) = 1,\ \ (1+k)\#p\#p^{-1} = 1,\ \ p^{-1}\#(1+k)\#p = 1,$$
$$p^{-1}\#p\#(1+\tilde{k}) = 1,\ \ (1+\tilde{k})\#p^{-1}\#p = 1,\ \ p\#(1+\tilde{k})\#p^{-1} = 1.$$

Lemma 2.12. *Let* $q \in S(1, g_\epsilon)$ *satisfy* $q \geq c$ *with some constant* c*. Then there is* $C > 0$ *such that*

$$\big(\mathrm{op}(q)u, u) \geq (c - CM^{-1/2})\|u\|^2.$$

Lemma 2.13. *Let* $q \in S(1, g_\epsilon)$ *then there is* $C > 0$ *such that*

$$\|\mathrm{op}(q)u\| \leq \big(\sup|q| + CM^{-1/2}\big)\|u\|.$$

Lemma 2.14. *Let* $m > 0$ *be admissible for* g_ϵ *and* $m \in S(m, g_\epsilon)$*. Then*

$$(\mathrm{op}(m)u, u) \geq (1 - CM^{-2})\|\mathrm{op}(\sqrt{m})u\|^2.$$

If $q \in S(m, g_\epsilon)$ *then there is* $C > 0$ *such that*

$$\big|(\mathrm{op}(q)u, u)\big| \leq \big(\sup\big(|q|/m\big) + CM^{-1/2}\big)\|\mathrm{op}(\sqrt{m}\,)u\|^2.$$

Lemma 2.15. *Let* $m_i > 0$ *be admissible for* g_ϵ *and assume that* $m_i \in S(m_i, g_\epsilon)$ *and* $m_2 \leq C\,m_1$ *with* $C > 0$*. Then there is* $C' > 0$ *such that*

$$\big\|\mathrm{op}(m_2)u\big\| \leq C'\big\|\mathrm{op}(m_1)u\big\|.$$

3 Direct energy estimates

3.1 Direct energy estimate for localized operators

Let

$$L = \mathrm{op}(\ell), \quad B = \mathrm{op}(b).$$

Since $\ell \in S(M^{-1}\langle\xi\rangle_\gamma, G)$ then $\partial_x^\alpha\partial_\xi^\beta \ell \in S(M^{-1+|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\beta|}, g_\epsilon)$ for $|\alpha+\beta| = 2$ hence $\ell\#\ell - \ell^2 \in S(M^2, g_\epsilon) \subset S(M^{-2}\langle\xi\rangle_\gamma, g_\epsilon)$ because of (2.4), that is

$$\mathrm{op}(\ell^2) = L^2 + \mathrm{op}(r), \quad r \in S(M^{-2}\langle\xi\rangle_\gamma, g_\epsilon). \tag{3.1}$$

On the other hand we have $b\#b = b^2 + \tilde{r} = q + \bar{\lambda}\langle\xi\rangle_\gamma + \tilde{r}$ with $\tilde{r} \in S(\langle\xi\rangle_\gamma, \bar{g})$ thanks to Lemma 2.7. Thus

$$\mathrm{op}(q) = B^2 - \mathrm{op}(r), \quad r = \bar{\lambda}\langle\xi\rangle_\gamma + \tilde{r} \in S(\langle\xi\rangle_\gamma, \bar{g}). \tag{3.2}$$

Taking $(D_t - i\theta)e^{-\theta t} = e^{-\theta t}D_t$ where $\theta > 0$ into account consider

$$\hat{P}_\theta = -A^2 + L^2(t,x,D) + B^2(t,x,D) + B_0(t,x,D)D_t + B_1(t,x,D)$$

with $A = D_t - i\theta$ where $B_i = \mathrm{op}(\tilde{b}_i)$ with some $\tilde{b}_i \in S(\langle\xi\rangle_\gamma^i, \bar{g})$. Since

$$\begin{aligned}\hat{P}(t,x,\tau,\xi) &= -\tau^2 + \ell^2(t,x,\xi) + q(t,x,\xi) + b_1(t,x,\xi)\langle\xi\rangle_\gamma + b_0(t,x,\xi)\tau \\ &= -\tau^2 + \ell^2(t,x,\xi) + b^2(t,x,\xi) + (b_1(t,x,\xi) - \bar{\lambda})\langle\xi\rangle_\gamma + b_0(t,x,\xi)\tau\end{aligned}$$

where $b_j = b_j(t, y(x), \eta(\xi) + e_d)$ hence $\tilde{b}_1$ contains $\bar{\lambda}\langle\xi\rangle_\gamma$ *but* $\bar{\lambda}$ *has been fixed.* Recall that $\hat{P}(t,x,\tau,\xi)$ coincides with the symbol of $T^{-1}PT$ in $W_{M,\gamma}$.

Definition 3.1. We set

$$\Phi = \mathrm{op}(\phi^{-n}), \quad \Phi^\flat = \mathrm{op}(\omega^{1/2}\phi^{-n}), \quad \Phi^\sharp = \mathrm{op}(\omega^{-1/2}\phi^{-n})$$

here and in what follows to simplify notation the power n is not indicated in Φ, $\Phi^\flat$, $\Phi^\sharp$ which depends on n of course.

In this section it is assumed that all constants c, $\hat{c}$, $\bar{c}$, c_i are independent of n, M, γ and θ and every constant C, may change from line to line, is independent of M, γ and θ while may depend on n.

Assume $K^* = K$ (actually we take $K = L$ or $K = B$) then it is easy to see

$$\begin{aligned}2\mathsf{Im}(\Phi K^2u, \Phi Au) &= \partial_t\|\Phi Ku\|^2 + 2\theta\|\Phi Ku\|^2 \\ &+2\mathsf{Im}(\Phi[A,K]u, \Phi Ku) + 2\mathsf{Im}([A,\Phi]Ku, \Phi Ku) \\ &+2\mathsf{Im}([\Phi,K]Au, \Phi Ku) + 2\mathsf{Im}(\Phi Au, [K,\Phi]Ku).\end{aligned} \tag{3.3}$$

Note that $[A,\Phi] = in\,\mathrm{op}(\omega^{-1}\phi^{-n})$ and hence

$$2\mathsf{Im}([A,\Phi]Ku, \Phi Ku) = 2\,n\mathsf{Re}(\mathrm{op}(\omega^{-1}\phi^{-n})Ku, \mathrm{op}(\phi^{-n})Ku).$$

Consider $\mathrm{op}(\phi^{-n})\mathrm{op}(\omega^{-1}\phi^{-n})=\mathrm{op}(\phi^{-n}\#(\omega^{-1}\phi^{-n}))$. Since $\phi^{-n}\#(\omega^{-1}\phi^{-n})=\omega^{-1}\phi^{-2n}+r$ with $r\in S(M^{-1}\omega^{-1}\phi^{-2n},g_\epsilon)$ in view of Proposition 2.5 then thanks to Lemma 2.14 one has $|(\mathrm{op}(r)u,u)|\leq CM^{-1}\|\Phi^\sharp u\|^2$. Thus Lemma 2.14 again gives

$$2\mathsf{Im}([A,\Phi]Ku,\Phi Ku)\geq 2n(1-CM^{-1})\|\Phi^\sharp Ku\|^2. \tag{3.4}$$

Note that $\ell\#\phi^{-n}-\phi^{-n}\#\ell=-i\{\ell,\phi^{-n}\}+r$ with $r\in S(M\phi^{-n},g_\epsilon)$ since $\partial_x^\alpha\partial_\xi^\beta\phi^{-n}\in S(M^{-\epsilon(\alpha,\beta)}\omega^{-1}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi^{-n},g_\epsilon)$ for $|\alpha+\beta|=2$ by Proposition 2.5 and $\partial_x^\beta\partial_\xi^\alpha\ell\in S(M^{-1+|\beta+\alpha|}\langle\xi\rangle_\gamma^{1-|\alpha|},g_\epsilon)$ for $|\alpha+\beta|=2$ and $\omega\geq\langle\xi\rangle_\gamma^{-1/2}$. Note that

$$\{\ell,\phi^{-n}\}=-in\,\omega^{-1}\{\ell,\psi\}\phi^{-n}+in\,\omega^{-1}\{\ell,\langle\xi\rangle_\gamma^{-1}\}\phi^{-n-1}$$

where $\omega^{-1}\{\ell,\langle\xi\rangle_\gamma^{-1}\}\phi^{-n-1}\in S(\phi^{-n},g_\epsilon)$ in view of (2.13). Therefore we have $\ell\#\phi^{-n}-\phi^{-n}\#\ell=in\{\ell,\psi\}\omega^{-1}\phi^{-n}+r$ with $r\in S(M\phi^{-n},g_\epsilon)$. Thanks to Proposition 2.5 one has $\phi^{-n}\#(\{\ell,\psi\}\omega^{-1}\phi^{-n})-\{\ell,\psi\}\omega^{-1}\phi^{-2n}\in S(M^{-1}\omega^{-1}\phi^{-2n},g_\epsilon)$ since $\{\ell,\psi\}\in S(1,g_\epsilon)$. Thus one can write

$$\phi^{-n}\#(\ell\#\phi^{-n}-\phi^{-n}\#\ell)=in\{\ell,\psi\}\,\omega^{-1}\phi^{-2n}+r_1+r_2$$

where $r_1\in S(M^{-1}\omega^{-1}\phi^{-2n},g_\epsilon)$ and $r_2\in S(M\phi^{-2n},g_\epsilon)$. Write

$$(1+k)\#(\omega^{1/2}\phi^n)\#(\{\ell,\psi\}\omega^{-1}\phi^{-2n})\#(\omega^{1/2}\phi^n)\#(1+\tilde{k})=r$$

with $k,\tilde{k}\in S(M^{-1},g_\epsilon)$ such that $(\omega^{-1/2}\phi^{-n})\#r\#(\omega^{-1/2}\phi^{-n})=\{\ell,\psi\}\omega^{-1}\phi^{-2n}$ where $r-\{\ell,\psi\}\in S(M^{-1},g_\epsilon)$ is clear. Recalling Lemma 2.5 and applying Lemma 2.14 we obtain

$$|(\Phi Au,[L,\Phi]Lu)|\leq n(|\kappa|+CM^{-1})\|\Phi^\sharp Au\|\|\Phi^\sharp Lu\|+CM\|\Phi Au\|\|\Phi Lu\|.$$

Since $|([\Phi,L]Au,\Phi Lu)|$ can be estimated in the same way we have

$$\begin{aligned}&2|(\Phi Au,[L,\Phi]Lu)|+2|([\Phi,L]Au,\Phi Lu)|\\&\leq 2n(|\kappa|+CM^{-1})\big(\|\Phi^\sharp Au\|^2+\|\Phi^\sharp Lu\|^2\big)\\&\qquad+CM\big(\|\Phi Au\|^2+\|\Phi Lu\|^2\big).\end{aligned}\tag{3.5}$$

Note that $[A,L]=-i\,\mathrm{op}(\partial_t\ell)$ and $\partial_t\ell\in S(\langle\xi\rangle_\gamma,G)$. Write

$$(1+k_1)\#(\omega^{1/2}\phi^n)\#\phi^{-n}\#\phi^{-n}\#(\partial_t\ell)\#\langle\xi\rangle_\gamma^{-1}\#(\omega^{-1/2}\phi^n)\#(1+k_2)=r \tag{3.6}$$

such that $(\omega^{-1/2}\phi^{-n})\#r\#(\omega^{1/2}\phi^{-n})\#\langle\xi\rangle_\gamma=\phi^{-n}\#\phi^{-n}\#(\partial_t\ell)$ where it is clear that $r-\partial_t\ell\langle\xi\rangle_\gamma^{-1}\in S(M^{-1},g_\epsilon)$. Noting (2.8) and the constraint of the t range (2.7) we have

$$\big|\langle\xi\rangle_\gamma^{-1}\partial_t\ell(t,x,\xi)\big|\leq c_0+CM^{-1},\quad c_0=|\partial_t\ell(0,e_d)|.$$

Then it follows from Lemma 2.13 that

$$|(\Phi[A,L]u,\Phi Lu)|\leq(c_0+CM^{-1})\|\Phi^\flat\langle D\rangle_\gamma u\|\|\Phi^\sharp Lu\|. \tag{3.7}$$

From (3.3), (3.4), (3.5) and (3.7) it follows that

Lemma 3.1. *We have*

$$2\mathsf{Im}(\Phi L^2 u, \Phi A u) \geq \partial_t \|\Phi L u\|^2 + (2\theta - CM)\|\Phi L u\|^2$$
$$+2n(1 - |\kappa| - c_0/2n - CM^{-1})\|\Phi^\sharp L u\|^2 - 2n(|\kappa| + CM^{-1})\|\Phi^\sharp A u\|^2$$
$$-(c_0 + CM^{-1})\|\Phi^\flat \langle D\rangle_\gamma u\|^2 - CM\|\Phi A u\|^2.$$

Note that from Corollary 2.1 and Lemma 2.7 we have

$$\begin{gathered}\partial_x^\alpha \partial_\xi^\beta b \in S(\langle\xi\rangle_\gamma^{1-|\beta|}, \bar{g}), \quad |\alpha + \beta| = 1,\\ \partial_{x_1} b \in S(M^{-1/2}\langle\xi\rangle_\gamma, \bar{g}),\ (a), \quad \partial_{\xi_j} b \in S(M^{-1/2}, \bar{g}),\ \ j = 1, d,\ (b).\end{gathered} \tag{3.8}$$

From Proposition 2.5 and Lemma 2.10 it follows that

$$\begin{gathered}\partial_x^\alpha \partial_\xi^\beta (\omega^{-1/2}\phi^{-n}) \in S(\omega^{-3/2}\langle\xi\rangle_\gamma^{-|\beta|}\phi^{-n}, g_\epsilon),\ \ |\alpha + \beta| = 1,\\ \partial_{\xi_j}(\omega^{-1/2}\phi^{-n}) \in S(M^{-1}\omega^{-3/2}\langle\xi\rangle_\gamma^{-1}\phi^{-n}, g_a), \quad j \neq 1,\ \ (a)\\ \partial_{x_j}(\omega^{-1/2}\phi^{-n}) \in S(M^{-1}\omega^{-3/2}\phi^{-n}, g_b), \quad j \neq 1, d,\ \ (b).\end{gathered} \tag{3.9}$$

Since $g_\epsilon \leq \bar{g}$ and ω and ϕ are $\bar{g}$ admissible weights thanks to Proposition 2.6 one concludes from (3.8) and (3.9) that

$$(\omega^{-1/2}\phi^{-n})\# b - b\#(\omega^{-1/2}\phi^{-n}) \in S(M^{-1/2}\langle\xi\rangle_\gamma \omega^{1/2}\phi^{-n}, \bar{g}) \tag{3.10}$$

where we have used $\omega^2 \geq \langle\xi\rangle_\gamma^{-1}$. Thus an application of Lemma 2.14 shows

$$\|\Phi^\sharp B u\| \geq \|B\Phi^\sharp u\| - CM^{-1/2}\|\Phi^\flat \langle D\rangle_\gamma u\|. \tag{3.11}$$

Let $\tilde{B} = \mathrm{op}(\tilde{b})$ where $\tilde{b}$ is given in Proposition 2.4 such that $B \cdot \tilde{B} = 1$ and $\tilde{B} \cdot B = 1$. In view of (2.12) one sees $\tilde{b}^{-1} \in S(\langle\xi\rangle_\gamma^{-1}\omega^{-1}, \bar{g})$ hence $(\langle\xi\rangle_\gamma\omega)\#\tilde{b} \in S(1, \bar{g})$. Therefore writing $\langle\xi\rangle_\gamma\omega = (\langle\xi\rangle_\gamma\omega)\#\tilde{b}\# b$ there is $\hat{c} > 0$ such that

$$\|\mathrm{op}(\langle\xi\rangle_\gamma\omega)u\| \leq \|Bu\|/\hat{c}. \tag{3.12}$$

Writing $(\langle\xi\rangle_\gamma\omega)\#(\omega^{-1/2}\phi^{-n}) = (1 + k)\#(\omega^{1/2}\phi^{-n})\#\langle\xi\rangle_\gamma$ it results

$$(1 - CM^{-1})\|\Phi^\flat\langle D\rangle_\gamma u\| \leq \|\mathrm{op}(\langle\xi\rangle_\gamma\omega)\Phi^\sharp u\|. \tag{3.13}$$

Replacing u by $\Phi^\sharp u$ in (3.12) we obtain from (3.11) and (3.13) that

Lemma 3.2. *There are $\hat{c} > 0$, $C > 0$ such that*

$$\hat{c}(1 - CM^{-1/2})\|\Phi^\flat\langle D\rangle_\gamma u\| \leq \|\Phi^\sharp B u\|. \tag{3.14}$$

Denoting $\Phi^{\flat\flat} = \mathrm{op}(\omega\phi^{-n})$ the same argument shows that

$$\hat{c}(1 - CM^{-1/2})\|\Phi^{\flat\flat}\langle D\rangle_\gamma u\| \leq \|\Phi B u\|. \tag{3.15}$$

It is clear that $b\#\phi^{-n}-\phi^{-n}\#b\in S(M^{-1/2}\phi^{-n}\omega^{-1},\bar g)$ from the same argument proving (3.10). Write

$$(1+k)\#(\omega^{1/2}\phi^n)\#\phi^{-n}\#(b\#\phi^{-n}-\phi^{-n}\#b)\#(\omega^{1/2}\phi^n)\#(1+\tilde k)=r$$

such that $(\omega^{-1/2}\phi^{-n})\#r\#(\omega^{-1/2}\phi^{-n})=\phi^{-n}\#(b\#\phi^{-n}-\phi^{-n}\#b)$ where $r\in S(M^{-1/2},\bar g)$. Therefore one has

$$\begin{aligned}|(\Phi Au,[B,\Phi]Bu)|&\le \|\Phi^\sharp Au\|\|\mathrm{op}(r)\Phi^\sharp Bu\|\\&\le CM^{-1/2}\big(\|\Phi^\sharp Au\|^2+\|\Phi^\sharp Bu\|^2\big).\end{aligned}$$

Repeating the same arguments again we have

$$\begin{aligned}&|(\Phi Au,[B,\Phi]Bu)|+|([\Phi,B]Au,\Phi Bu)|\\&\qquad\le CM^{-1/2}\big(\|\Phi^\sharp Au\|^2+\|\Phi^\sharp Bu\|^2\big).\end{aligned}\tag{3.16}$$

Write $(1+k)\#(\omega^{1/2}\phi^n)\#\phi^{-n}\#\phi^{-n}\#(\partial_t b)\#\langle\xi\rangle_\gamma^{-1}\#(\omega^{-1/2}\phi^n)\#(1+\tilde k)=r$ such that $(\omega^{-1/2}\phi^{-n})\#r\#(\omega^{1/2}\phi^{-n})\#\langle\xi\rangle_\gamma=\phi^{-n}\#\phi^{-n}\#(\partial_t b)$. Here we note

Lemma 3.3. *Notations being as above we have* $r-\langle\xi\rangle_\gamma^{-1}\partial_t b\in S(M^{-1/2},\bar g)$.

Proof. Write $(1+k)\#(\omega^{1/2}\phi^n)\#\phi^{-n}\#\phi^{-n}=\omega^{1/2}\phi^{-n}+l$ with $l\in S(M^{-1}\omega^{1/2}\phi^{-n},g_\epsilon)$ and $\langle\xi\rangle_\gamma^{-1}\#(\omega^{-1/2}\phi^n)\#(1+\tilde k)=\langle\xi\rangle_\gamma^{-1}\omega^{-1/2}\phi^n+\tilde l$ with $\tilde l\in S(M^{-1}\langle\xi\rangle_\gamma^{-1}\omega^{-1/2}\phi^n,g_\epsilon)$ such that $r=(\omega^{1/2}\phi^{-n}+l)\#(\partial_t b)\#(\langle\xi\rangle_\gamma^{-1}\omega^{-1/2}\phi^n+\tilde l)$. Thanks to Corollary 2.2 and (3.9) it follows that

$$(\omega^{1/2}\phi^{-n})\#(\partial_t b)-(\partial_t b)\#(\omega^{1/2}\phi^{-n})\in S(M^{-1/2}\langle\xi\rangle_\gamma\omega^{1/2}\phi^{-n},\bar g)$$

hence we have

$$r=(\partial_t b)\#(\omega^{1/2}\phi^{-n})\#(\langle\xi\rangle_\gamma^{-1}\omega^{-1/2}\phi^n)+R=(\partial_t b)\#\langle\xi\rangle_\gamma^{-1}+\tilde R\tag{3.17}$$

where $\tilde R\in S(M^{-1/2},\bar g)$. Since $(\partial_t b)\#\langle\xi\rangle_\gamma^{-1}-\langle\xi\rangle_\gamma^{-1}\partial_t b\in S(M^{-1/2},\bar g)$ the proof is completed. □

Since $\langle\xi\rangle_\gamma^{-1}\partial_t b\in S(1,\bar g)$ in view of Corollary 2.2 from the L^2 boundedness theorem there are $c>0$ and $l\in\mathbb N$ such that

$$\|\mathrm{op}(\langle\xi\rangle_\gamma^{-1}\partial_t b)u\|\le c\big|\langle\xi\rangle_\gamma^{-1}\partial_t b\big|^{(l)}_{S(1,\bar g)}\|u\|=c_1\|u\|.\tag{3.18}$$

Then from (3.17) it follows that

$$|(\Phi_n[A,B]u,\Phi Bu)|\le(c_1+CM^{-1/2})\|\Phi^\sharp Bu\|\|\Phi^\flat\langle D\rangle_\gamma u\|.\tag{3.19}$$

From (3.3), (3.4), (3.14), (3.16) and (3.19) we have

Lemma 3.4. *We have*

$$\begin{aligned}2\mathsf{Im}(\Phi B^2u,\Phi Au)&\ge\partial_t\|\Phi Bu\|^2+2\theta\|\Phi B_u\|^2\\&+n(\hat c-c_1/n-CM^{-1/2})\|\Phi^\flat\langle D\rangle_\gamma u\|^2\\&+n(1-c_1/n-CM^{-1/2})\|\Phi^\sharp Bu\|^2-CM^{-1/2}\|\Phi^\sharp Au\|^2.\end{aligned}$$

Since

$$-2\mathsf{Im}(\Phi Au,\Phi u)=\partial_t\|\Phi u\|^2+2\theta\|\Phi u\|^2+2\mathsf{Im}([A,\Phi]u,\Phi u)$$

replacing u by Au it follows from (3.4) that

$$\begin{aligned}-2\mathsf{Im}(\Phi A^2u,\Phi Au)\geq\partial_t\|\Phi Au\|^2+2\theta\|\Phi Au\|^2\\+2n(1-CM^{-1/2})\|\Phi^\sharp Au\|^2.\end{aligned}\tag{3.20}$$

Then from Lemmas 3.1 and 3.4 we conclude

Proposition 3.1. *We have*

$$\begin{aligned}2\mathsf{Im}(\Phi(-A^2+L^2+B^2)u,\Phi Au)\geq\partial_t\big(\|\Phi Lu\|^2+\|\Phi Bu\|^2+\|\Phi Au\|^2\big)\\+(2\theta-CM)\big(\|\Phi Lu\|^2+\|\Phi Bu\|^2+\|\Phi Au\|^2\big)\\+2n(1-|\kappa|-c_0/2n-CM^{-1/2})\|\Phi^\sharp Lu\|^2\\+2n(1-|\kappa|-CM^{-1/2})\|\Phi^\sharp Au\|^2\\+n(\hat{c}-c_0/n-c_1/n-CM^{-1/2})\|\Phi^\flat\langle D\rangle_\gamma u\|^2\\+n(1-c_1/n-CM^{-1/2})\|\Phi^\sharp Bu\|^2.\end{aligned}$$

Since $-2(\Phi Au,\Phi u)\geq\partial_t\|\Phi u\|^2+2\theta\|\Phi u\|^2$ if $CM^{-1/2}\leq 1$ then

$$\|\Phi Au\|^2\geq\theta\partial_t\|\Phi u\|^2+\theta^2\|\Phi u\|^2.\tag{3.21}$$

Consider the lower order term $B_0D_t+B_1=B_0A+B_1+i\theta B_0$. Write

$$(1+k)\#(\omega^{1/2}\phi^n)\#\phi^{-n}\#\phi^{-n}\#\tilde{b}_1\#\langle\xi\rangle_\gamma^{-1}\#(\omega^{-1/2}\phi^n)\#(1+\tilde{k})=r$$

with $r\in S(1,\bar{g})$ such that $(\omega^{-1/2}\phi^{-n})\#r\#(\omega^{1/2}\phi^{-n})\#\langle\xi\rangle_\gamma=\phi^{-n}\#\phi^{-n}\#\tilde{b}_1$. We make a closer look at r.

Lemma 3.5. *Notations being as above we have* $r-\langle\xi\rangle_\gamma^{-1}\tilde{b}_1\in S(M^{-1/2},\bar{g})$.

Proof. First note that $\tilde{b}_1=d_1-\tilde{r}$ with some $d_1\in S(\langle\xi\rangle_\gamma,g_\epsilon)$ and $\tilde{r}=b\#b-b^2$ given in (3.2). Thanks to Corollary 2.1 it follows that

$$\begin{aligned}\partial_x^\alpha\partial_\xi^\beta\partial_{x_1}\bar{b}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},\bar{g}),\quad(a),\\\partial_x^\alpha\partial_\xi^\beta\partial_{\xi_j}\bar{b}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},\bar{g}),\quad j=1,d,\ \ (b)\end{aligned}$$

hence taking Lemma 2.7 into account we see that $\partial_{x_1}\tilde{r}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{3/2},\bar{g})$, (a) and $\partial_{\xi_j}\tilde{r}\in S(M^{-1/2}\langle\xi\rangle_\gamma^{1/2},\bar{g})$ for $j=1,d$, (b). Applying the same arguments proving Lemma 3.3 we conclude the assertion. □

Since $\mathrm{op}(\langle\xi\rangle_\gamma^{-1}\tilde{b}_1)$ is L^2 bounded, denoting the bound by $\bar{c}$, we have

$$\|\mathrm{op}(\langle\xi\rangle_\gamma^{-1}\tilde{b}_1)u\|\leq\bar{c}\,\|u\|\tag{3.22}$$

hence

$$2|(\Phi B_1 u, \Phi A u)| \le (\bar{c} + CM^{-1/2})(\|\Phi^\sharp A u\|^2 + \|\Phi^\flat \langle D\rangle_\gamma u\|^2). \tag{3.23}$$

Writing $\phi^{-n}\#r_0\#\phi^{-n} = \phi^{-n}\#\phi^{-n}\#\tilde{b}_0$ with $r_0 \in S(1, \bar{g})$ it results

$$2|(\Phi B_0 A u, \Phi A u)| \le CM\|\Phi A u\|^2. \tag{3.24}$$

Similarly one has

$$2\theta|(\Phi B_0 u, \Phi A u)| \le CM(\theta^{3/2}\|\Phi u\|^2 + \theta^{1/2}\|\Phi A u\|^2). \tag{3.25}$$

It is also easy to see that

$$\begin{aligned} 2|(\Phi(-A^2 + L + B^2)u, \Phi A u)| &\le M^{1/2}\|\Phi^\flat(-A^2 + L + B^2)u\|^2 \\ &+ M^{-1/2}(1 + CM^{-1/2})\|\Phi^\sharp A u\|^2. \end{aligned}$$

Therefore from Proposition 3.1 and (3.21) we arrive at

$$\begin{aligned} M^{1/2}\|\Phi^\flat \hat{P}_\theta u\|^2 &\ge \partial_t\big(\|\Phi L u\|^2 + \|\Phi B u\|^2 + \|\Phi A u\|^2 + \theta\|\Phi u\|^2\big) \\ &+\theta(1 - CM^2\theta^{-1} - CM\theta^{-1/2})\big(\|\Phi L u\|^2 + \|\Phi B u\|^2 + \|\Phi A u\|^2\big) \\ &+\theta^2(1 - CM\theta^{-1/2})\|\Phi u\|^2 \\ &+2n(1 - |\kappa| - c_0/2n - CM^{-1/2})\|\Phi^\sharp L u\|^2 \\ &+2n(1 - |\kappa| - \bar{c}/2n - CM^{-1/2})\|\Phi^\sharp A u\|^2 \\ &+n(\hat{c} - c_0/n - c_1/n - \bar{c}/n - CM^{-1/2})\|\Phi^\flat\langle D\rangle_\gamma u\|^2 \\ &+n(1 - c_1/n - CM^{-1/2})\|\Phi^\sharp B u\|^2. \end{aligned}$$

Here writing $(\omega^{1/2}\phi^{-n})\#\langle\xi\rangle_\gamma = (1+k)\#(\omega^{1/2}\langle\xi\rangle_\gamma^{1/4})\#\phi^{-n}\#\langle\xi\rangle_\gamma^{3/4}$ and noting $\omega^{1/2}\langle\xi\rangle_\gamma^{1/4} \ge 1$ one has by Lemma 2.13

$$\|\Phi^\flat\langle D\rangle_\gamma u\| \ge (1 - CM^{-1})\|\Phi\langle D\rangle_\gamma^{3/4} u\|.$$

Similarly we see $\|\Phi^{\flat\flat}\langle D\rangle_\gamma u\| \ge (1 - CM^{-1})\|\Phi\langle D\rangle_\gamma^{1/2} u\|$. Thus we first choose n such that

$$\begin{aligned} &1 - |\kappa| - c_0/2n > 0, \quad 1 - |\kappa| - \bar{c}/2n > 0, \\ &\hat{c} - c_0/n - c_1/n - \bar{c}/n > 0, \quad 1 - c_1/n > 0 \end{aligned}$$

and *fix* such a n. Next we choose M such that the above inequalities remain to be positive after subtracting $CM^{-1/2}$ from each inequality and *fix* such a M then choose γ such that $\gamma \ge M^4$ and $\gamma \ge \bar{\lambda}M^2$ and *fix* γ, still θ is assumed to be free. Once M and γ are fixed we have

$$g_0/C \le G \le Cg_0, \quad \langle\xi\rangle^s/C_s \le \langle\xi\rangle_\gamma^s \le C_s\langle\xi\rangle^s$$

where $g_0 = |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$. Now summarize what we have proved

Proposition 3.2. *There exist $C>0$, $c>0$ and $\theta_0>0$ such that*

$$C\|\Phi^\flat \hat{P}_\theta u\|^2 \ge \partial_t\big(\|\Phi Lu\|^2+\|\Phi Bu\|^2+\|\Phi Au\|^2+\theta\|\Phi u\|^2\big)$$
$$+c\,\theta\big(\|\Phi Lu\|^2+\|\Phi Bu\|^2+\|\Phi Au\|^2+\|\Phi\langle D\rangle^{1/2}u\|^2+\theta\|\Phi u\|^2\big)$$
$$+c\left(\|\Phi^\sharp Lu\|^2+\|\Phi^\sharp Au\|^2+\|\Phi^\sharp Bu\|^2+\|\Phi\langle D\rangle^{3/4}u\|^2\right)$$

for $\theta\ge\theta_0$.

Next we estimate $\langle D\rangle^s u$. Since $\langle D\rangle^s\hat{P}_\theta=\hat{P}_\theta\langle D\rangle^s+[\langle D\rangle^s,\hat{P}]$ we study $[\langle D\rangle^s,L^2]$. Since $\ell\#\ell-\ell^2\in S(1,g_0)$ is clear then

$$\langle\xi\rangle^s\#\ell\#\ell-\ell\#\ell\#\langle\xi\rangle^s-(\langle\xi\rangle^2\#\ell^2-\ell^2\#\langle\xi\rangle^s)\in S(\langle\xi\rangle^s,g_0).$$

It is also easy to see that $\langle\xi\rangle^s\#\ell^2-\ell^2\#\langle\xi\rangle^s=a\ell+r$ with $a\in S(\langle\xi\rangle^s,g_0)$ and $r\in S(\langle\xi\rangle^s,g_0)$. Since one can write $a\ell=(a\langle\xi\rangle^{-s})\#\ell\#\langle\xi\rangle^s+\tilde{r}$ with $\tilde{r}\in S(\langle\xi\rangle^s,g_0)$ we conclude that

$$\big|(\Phi[\langle D\rangle^s,L^2]u,\Phi A\langle D\rangle^s u)\big|\le C\big(\|\Phi A\langle D\rangle^s u\|^2+\|\Phi L\langle D\rangle^s u\|^2+\|\Phi\langle D\rangle^s u\|^2\big). \tag{3.26}$$

Lemma 3.6. *We have*

$$\big|(\Phi[\langle D\rangle^s,B^2]u,\Phi A\langle D\rangle^s u)\big|\le C\|\Phi A\langle D\rangle^s u\|\|\Phi B\langle D\rangle^s u\|.$$

Proof. Note that $[\langle D\rangle^s,B^2]=[\langle D\rangle^s,B]B+B[\langle D\rangle^s,B]$. From Lemma 2.7 we see $\langle\xi\rangle^s\#b-b\#\langle\xi\rangle^s\in S(\langle\xi\rangle^s,\bar{g})$. Thanks to Proposition 2.3 one has $r=(\langle\xi\rangle^s\#b-b\#\langle\xi\rangle^s)\#b\in S(b\langle\xi\rangle^s,\bar{g})$. Applying Proposition 2.4 one can write

$$(\langle\xi\rangle^s\#b-b\#\langle\xi\rangle^s)\#b=r\#\langle\xi\rangle^{-s}\#\tilde{b}\#b\#\langle\xi\rangle^s$$

where $r\#\langle\xi\rangle^{-s}\#\tilde{b}\in S(1,\bar{g})$. Then writing $\phi^{-n}\#(r\#\langle\xi\rangle^{-s}\#\tilde{b})=\tilde{r}\#\phi^{-n}$ with $\tilde{r}\in S(1,\bar{g})$ we conclude

$$\big|(\Phi[\langle D\rangle^s,B]Bu,\Phi A\langle D\rangle^s)\big|\le C\|\Phi B\langle D\rangle^s u\|\|\Phi A\langle D\rangle^s u\|.$$

Repeating the same arguments to $B[\langle D\rangle^s,B]$ we end the proof. □

For commutators coming from lower order term it is easy to see

$$\begin{aligned}\big|(\Phi[\langle D\rangle^s,B_0]Au,\Phi A\langle D\rangle^s u)\big|&\le C\|\Phi A\langle D\rangle^s u\|^2,\\ \big|(\Phi[\langle D\rangle^s,B_0]u,\Phi A\langle D\rangle^s u)\big|&\le C\|\Phi\langle D\rangle^s u\|\|\Phi A\langle D\rangle^s_\gamma u\|,\\ \big|(\Phi[\langle D\rangle^s,B_1]u,\Phi A\langle D\rangle^s u)\big|&\le C\|\Phi\langle D\rangle^{s+1/2}u\|\|\Phi A\langle D\rangle^s u\|.\end{aligned} \tag{3.27}$$

It follows from (3.26), (3.27) and Lemma 3.6 that $|(\Phi[\langle D\rangle^s,\hat{P}]u,\Phi A\langle D\rangle^s u)|$ is controlled by the second term on the right-hand side of Proposition 3.2 with $\langle D\rangle^s u$ in place of u, choosing θ suitably large.

Recalling $Ae^{-\theta t}=e^{-\theta t}D_t$ one has from Proposition 3.2 that

$$Ce^{-2\theta t}\|\Phi^\flat\langle D\rangle^s\hat{P}u\|^2\ge\partial_t e^{-2\theta t}\big(\|\Phi L\langle D\rangle^s u\|^2+\|\Phi B\langle D\rangle^s u\|^2+\|\Phi\langle D\rangle^s D_t u\|^2+\theta\|\Phi\langle D\rangle^s u\|^2\big). \tag{3.28}$$

Here we note

Lemma 3.7. *Let $n \ge 1$. There is $C > 0$ such that $C\|\Phi B v\| \ge \|\langle D\rangle v\|$.*

Proof. Since $\phi \le 2\omega$ and $\omega\phi^{-n} \ge 2^{-n}\omega^{-n+1} \ge (2C)^{-n+1}/2$. Thus the proof follows from (3.15) and Lemma 2.15. □

Similarly from (2.13), using Lemma 2.15, we have

$$\|v\|/C \le \|\Phi v\|,\ \|\Phi^\flat v\| \le C\|\langle D\rangle^n v\|,\ \ n \ge 1/2. \tag{3.29}$$

Definition 3.2. We denote $\|u\|_s = \|\langle D\rangle^s u\|$ and by $H^s = H^s(\mathbb{R}^d)$ the L^2 based Sobolev space of order s. Denote by $\mathcal{H}_{-k,s}(\delta_1,\delta_2)$ the set of all f such that

$$(t-\delta_1)^{-k}\langle D\rangle^s f \in L^2((\delta_1,\delta_2)\times\mathbb{R}^d).$$

Assume $D_t^j u \in \mathcal{H}_{-k,s+2-j}(\delta_1,\delta_2)$, $j = 0,1,2$. From this one sees that $\lim_{t\to+\delta_1}\|D_t^j u(t)\|_{s+1-j}$, $j = 0,1$ exists which is 0 for $k > 0$. Using this we see $\lim_{t\to+\delta_1}(t-\delta_1)^{-k}\|D_t^j u(t)\|_{s+1-j} = 0$, $j = 0,1$. Let $\hat\tau$ be any point with $|\hat\tau| < \delta$. Multiply (3.28) by $(t-\hat\tau)^{-2k}$ and integrate in t from $\hat\tau$ to t we obtain

Proposition 3.3. *For any $s \in \mathbb{R}$ there is C such that*

$$\begin{aligned}(t-\hat\tau)^{-2k}\big(\|D_t u(t)\|_s^2 + \|u(t)\|_{s+1}^2\big)\\ + \int_{\hat\tau}^t (\tau-\hat\tau)^{-2k-1}\big(\|D_t u(\tau)\|_s^2 + \|u(\tau)\|_{s+1}^2\big)d\tau\\ \le C\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|\hat P u(\tau)\|_{n+s}^2 d\tau\end{aligned} \tag{3.30}$$

for any u with $D_t^j u \in \mathcal{H}_{-k,n+s+2-j}(\hat\tau,\delta)$, $j = 0,1,2$.

Consider the adjoint $\hat P^*$ of $\hat P$. Denoting $\check\Phi = \mathrm{op}(\phi^n)$, $\check\Phi^\flat = \mathrm{op}(\omega^{1/2}\phi^n)$ and $\check\Phi^\sharp = \mathrm{op}(\omega^{-1/2}\phi^n)$ a repetition of the same argument gives

$$\begin{aligned}Ce^{2\theta t}\|\check\Phi^\flat\langle D\rangle^s\hat P^* u\|^2 \ge -\partial_t e^{2\theta t}\big(\|\check\Phi L\langle D\rangle^s u\|^2 + \|\check\Phi B\langle D\rangle^s u\|^2\\ +\|\check\Phi\langle D\rangle^s D_t u\|^2 + \theta\|\check\Phi\langle D\rangle^s u\|^2\big).\end{aligned} \tag{3.31}$$

Since $2\phi\omega \ge \langle\xi\rangle_\gamma^{-1}$ repeating similar arguments one has

$$\|\langle D\rangle^{-n} v\|/C \le \|\check\Phi v\| \le C\|v\|, \quad \|\langle D\rangle^{-n+1} v\| \le C\|\check\Phi B v\|,\ n \ge 1. \tag{3.32}$$

Multiply (3.31) by $(t-\hat\tau)^{2k+1}$ and integrate in $I = (\hat\tau,\delta)$ we have

Proposition 3.4. *For any $s \in \mathbb{R}$ there is C such that*

$$\begin{aligned}\int_I (\tau-\hat\tau)^{2k}\big(\|D_t u(t)\|_{-n+s}^2 + \|u(t)\|_{-n+s+1}^2\big)dt\\ \le C\int_I (\tau-\hat\tau)^{2k+1}\|\hat P^* u(t)\|_s^2 dt, \quad u \in C_0^\infty(I\times\mathbb{R}^d).\end{aligned}$$

3.2 Local existence theorem

From Proposition 3.4 we have

$$\begin{aligned}|\int_I (f,v)dt| &\le \big(\int_I (t-\hat\tau)^{-2k}\|f\|^2_{n+k+s+1}dt\big)^{1/2}\big(\int_I (t-\hat\tau)^{2k}\|v\|^2_{-n-k-s-1}dt\big)^{1/2}\\ &\le C\big(\int_I (t-\hat\tau)^{-2k}\|f\|^2_{n+k+s+1}dt\big)^{1/2}\big(\int_I (t-\hat\tau)^{2k+1}\|\hat P^* v\|^2_{-2n-k-s-2}dt\big)^{1/2}\end{aligned}$$

for any $v\in C_0^\infty(I\times\mathbb{R}^d)$ and $f\in\mathcal{H}_{-k,n+k+s+1}(I)$. Using the Hahn-Banach theorem to extend the anti-linear form in $\hat P^* v$;

$$\hat P^* v\mapsto \int_I (f,v)dt \tag{3.33}$$

we conclude that there is some $u\in\mathcal{H}_{-k-1/2,2n+k+s+2}(I)$ such that

$$\int_I (f,v)dt=\int_I (u,\hat P^* v)dt,\quad v\in C_0^\infty(I\times\mathbb{R}^d).$$

This implies that $\hat P u=f$. Since $u\in\mathcal{H}_{0,2n+k+s+2}(I)$ and $f\in\mathcal{H}_{0,n+k+s+1}(I)$ it follows from [4, Theorem B.2.9] that $D_t^j u\in\mathcal{H}_{0,n+k+s+3-j}(I)$ for $j=0,1,2,\ldots$. Thus with $w_j=\langle D\rangle^{n+s+2-j}D_t^j u$ one has $D_t^i w_j\in L^2(I\times\mathbb{R}^d)$ for $i=0,\ldots,k+1$ hence $D_t^i w_j(\hat\tau)$ exists in $L^2(\mathbb{R}^d)$ which is 0 for $i=0,\ldots,k$ since $w_j\in\mathcal{H}_{-k-1/2,0}(I)$. Thus one can write $w_j(t)=\int_{\hat\tau}^t (t-\tau)^k\partial_t^{k+1}w_j(\tau)d\tau/k!$. Thus one concludes that $D_t^j u\in\mathcal{H}_{-k-1/2,n+s+2-j}(I)$ for $0\le j\le 2$ then (3.30) holds for this u. Now let $f\in\mathcal{H}_{-k,n+s}(I)$. Take a rapidly decreasing function $\rho(\xi)$ with $\rho(0)=1$ then $f_\epsilon=\rho(\epsilon D)f\in\mathcal{H}_{-k,2n+k+s+2}(I)$ and $f_\epsilon\to f$ in $\mathcal{H}_{-k,n+s}(I)$. As just proved above there is u_ϵ satisfying $\hat P u_\epsilon=f_\epsilon$ and (3.30). Applying (3.30) to $\{u_\epsilon-u_{\epsilon'}\}$ we conclude

Theorem 3.1. *For any $s\in\mathbb{R}$ and any $f\in\mathcal{H}_{-k,n+s}(I)$ there exists a unique u with $D_t^j u\in\mathcal{H}_{-k-1/2,s+1-j}(I)$, $j=0,1,2$, satisfying $\hat P u=f$ and* (3.30).

Thanks to Theorem 3.1 one can define the solution map

$$\hat G(\hat\tau):\mathcal{H}_{-k,n+s}(I)\ni f\mapsto u\in\mathcal{H}_{-k-1/2,s+1}(I),\quad I=(\hat\tau,\delta).$$

We shall keep $\hat\tau$ fixed in the following discussion and therefore we write $\hat G$ dropping $\hat\tau$. This solution operator $\hat G$ verifies

$$\sum_{j=0}^{1}\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k-1}\|D_t^j\hat G f(\tau)\|^2_{s+1-j}\le C\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|f(\tau)\|^2_{n+s} \tag{3.34}$$

and has (microlocal) finite propagation speed. We state this property without proof (for a proof see [13]).

Proposition 3.5. *Notations being as above and let* Γ_i $(i=1,2,3)$ *be open conic sets in* $\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$ *with relatively compact basis such that* $\Gamma_1\Subset\Gamma_2\Subset\Gamma_3$ *and* $h_i(x,\xi)\in S(1,g_0)=S^0$ *with* $\operatorname{supp} h_1\subset\Gamma_1$, $\operatorname{supp} h_2\subset\Gamma_3\setminus\Gamma_2$. *Then there exists* $\delta'=\delta'(\Gamma_i)>0$ *such that for any* r, s *one can find* $C>0$ *such that*

$$\sum_{j=0}^{1}\int_{\hat\tau}^{t}(\tau-\hat\tau)^{-2k-1}\|\mathrm{op}(h_2)D_t^j\hat G\,\mathrm{op}(h_1)f(\tau)\|_{r-j}^2d\tau$$
$$\le C\int_{\hat\tau}^{t}(\tau-\hat\tau)^{-2k}\|f(\tau)\|_s^2d\tau,\quad \hat\tau<t\le\hat\tau+\delta',\ \ f\in\mathcal{H}_{-k,s}(\hat\tau,\hat\tau+\delta').$$

Recall $(Tu)(t,x)=u(t,\kappa(x))$. Let $R_{\bar\xi}=P-T\hat PT^{-1}$ then with $G_{\bar\xi}=T\hat GT^{-1}$ we have

$$PG_{\bar\xi}=I+R_{\bar\xi}G_{\bar\xi}$$

where it is clear that $G_{\bar\xi}$ verifies (3.34). Since $R_{\bar\xi}=T(\sum_{j=1}^2a_j(t,x,D)D_t^{2-j})T^{-1}$ with $a_j\in S^j\cap S^{-\infty}(W_{M,\gamma})$ applying the description of the wave front set of Tu (e.g.[3, Theorem 8.2.4]) one can find a conic neighborhood $W_{\bar\xi}$ of $(0,\bar\xi)$ such that for any $h(x,\xi)\in S^0$ supported in $W_{\bar\xi}$ we have

$$\|R_{\bar\xi}\,\mathrm{op}(h)u\|_p\precsim(\|D_tu\|_{q-1}+\|u\|_q),\quad\forall p,q\in\mathbb{R}.\tag{3.35}$$

It is not difficult to prove that $G_{\bar\xi}$ has (microlocal) finite propagation speed.

Theorem 3.2. *Assume that every singular point of* $p(0,0,\tau,\xi)=0$ *is effectively hyperbolic. Then there exist* $\delta>0$, $n>0$ *and a neighborhood* U *of* $x=0$ *such that for every* $f\in\mathcal{H}_{-k,s}(\hat\tau,\delta)$ *with* $|\hat\tau|<\delta$ *there exists* u *with* $D_t^ju\in\mathcal{H}_{-k,-n+s+1-j}(\hat\tau,\delta)$, $j=0,1$, *satisfying* $Pu=f$ *in* $(\hat\tau,\delta)\times U$.

Proof. Recall that we have proved that for any $|\eta|=1$ one can find a conic neighborhood W_η of $(0,\eta)$, a positive constant $\delta_\eta>0$ and a solution operator $G_\eta(\hat\tau)$ with (microlocal) finite propagation speed satisfying (3.34) such that

$$PG_\eta=I+R_\eta G_\eta,\quad |t|\le\delta_\eta$$

where R_η satisfies (3.35) for $h\in S^0$ with $\operatorname{supp}h\subset W_\eta$. We can choose a finite number of η_i such that $\cup_iW_{\eta_i}\supset U\times(\mathbb{R}^d\setminus\{0\})$, where U is a neighborhood of $x=0$. Now take another open conic covering $\{V_i\}$ of $U\times(\mathbb{R}^d\setminus\{0\})$ with $V_i\Subset W_{\eta_i}$, and a partition of unity $\{\alpha_i(x,\xi)\}$, $\alpha_i\in S^0$ subordinate to $\{V_i\}$ so that $\sum_i\alpha_i(x,\xi)=\alpha(x)$ where $\alpha(x)$ is equal to 1 in a neighborhood of $x=0$. Denoting

$$G=\sum_iG_{\eta_i}\mathrm{op}(\alpha_i)$$

we have $PGf=\sum_iPG_{\eta_i}\mathrm{op}(\alpha_i)f=\alpha(x)f-Rf$ with $R=-\sum_iR_{\eta_i}G_{\eta_i}\mathrm{op}(\alpha_i)$. Now choosing $\chi_i\in S^0$ supported in W_{η_i} such that $V_i\Subset\{\chi_i=1\}$ and writing $R_{\eta_i}G_{\eta_i}\mathrm{op}(\alpha_i)=R_{\eta_i}(\mathrm{op}(\chi_i)+\mathrm{op}(1-\chi_i))G_{\eta_i}\mathrm{op}(\alpha_i)$ it follows from (microlocal) finite propagation speed and (3.35) that there exists $\delta'>0$ such that

$$\int_{\hat\tau}^{t}(\tau-\hat\tau)^{-2k-1}\|Rf(\tau)\|_s^2d\tau\le C\int_{\hat\tau}^{t}(\tau-\hat\tau)^{-2k}\|f(\tau)\|_s^2d\tau$$

for $\hat\tau \le t \le \hat\tau + \delta'$. Choosing $0 < \delta_1 \le \delta'$ such that $\delta_1 C \le 1/2$ one has

$$\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|Rf(\tau)\|_s^2 d\tau \le \frac{1}{2}\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|f(\tau)\|_s^2 d\tau$$

for $f \in \mathcal{H}_{-k,s}(\hat\tau, \hat\tau+\delta_1)$. With $S = \sum_{k=0}^\infty R^k$ we have $Sf \in \mathcal{H}_{-k,s}(\hat\tau, \hat\tau+\delta_1)$ and

$$\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|Sf(\tau)\|_s \le 2\int_{\hat\tau}^t (\tau-\hat\tau)^{-2k}\|f(\tau)\|_s.$$

Let $\gamma(x) \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 near $x = 0$ such that $\operatorname{supp}\gamma \Subset \{\alpha = 1\}$. Since $\gamma(\alpha - R)S = \gamma(I-R)S = \gamma$ we have $\gamma(x)PGSf = \gamma(x)f$, that is $P\big(GSf\big) = f$ on $\{\gamma(x) = 1\}$. With $u = GSf$ one has

$$\sum_{j=0}^1 \int_{\hat\tau}^t (\tau-\hat\tau)^{-2k-1}\|D_t^j u(\tau)\|_{-n+s+1-j}^2 d\tau \le C\int_{\hat\tau}^t \tau^{-2k}\|Sf(\tau)\|_s^2 d\tau$$

which proves the assertion. □

4 Proof of propositions and lemmas

4.1 Proof of lemmas in Section 2.1

Proof of Lemma 2.1: By the Taylor formula one can write

$$f(z(x,\xi)+\bar z) = \sum_{|\alpha|=r}\frac{1}{\alpha!}z(x,\xi)^\alpha\partial_z^\alpha f(\bar z) + (r+1)\sum_{|\alpha|=r+1}\Big[\frac{1}{\alpha!}z(x,\xi)^\alpha \times\int_0^1 (1-\theta)^r\partial_z^\alpha f(\theta z(x,\xi)+\bar z)d\theta\Big]$$

where $z(x,\xi)^\alpha \in S(M^{-r}, G)$ for $|\alpha| = r$. Since $|z(x,\xi)| \le CM^{-1}$ the integral belongs to $S(1,G)$ hence the second term on the right-hand side is in $S(M^{-r-1}, G)$ thus the assertion. □

Proof of Lemma 2.2: Let $j \ne d$. Note that

$$\partial\eta_j/\partial\xi_j = \chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1} - M^{-2}\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})(M\xi_j\langle\xi\rangle_\gamma^{-1})^2\langle\xi\rangle_\gamma^{-1}$$

where $\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})(M\xi_j\langle\xi\rangle_\gamma^{-1})^2 \in S(1,G)$. If $k \ne j$ then

$$\partial\eta_j/\partial\xi_k = -M^{-1}\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})(M\xi_j\langle\xi\rangle_\gamma^{-1})(\xi_k\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1}.$$

Since $\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})(M\xi_j\langle\xi\rangle_\gamma^{-1}) \in S(1,G)$ the assertion is clear. □

Proof of Lemma 2.3: Writing $q(t,y,\eta+e_d) = \tilde q(y,\eta)$ one sees

$$\tilde q(y,\eta) = \sum_{|\alpha+\beta|=2}\frac{1}{\alpha!\beta!}y^\alpha\eta^\beta\partial_y^\alpha\partial_\eta^\beta\tilde q(0,0) + 3\sum_{|\alpha+\beta|=3}\Big[\frac{1}{\alpha!\beta!}y^\alpha\eta^\beta \times\int_0^1(1-\theta)^2\partial_y^\alpha\partial_\eta^\beta\tilde q(\theta y,\theta\eta)d\theta\Big] \tag{4.1}$$

where the sum over $|\alpha+\beta|=2$ contains no η_d because of the Euler's identity. For the case (a) from $\partial_{y_1}^2\tilde{q}(0,0)=0$ the term $\sum_{|\alpha+\beta|=2} y^\alpha\eta^\beta$ contains no y_1 because $\tilde{q}$ is nonnegative. Therefore $\partial_{x_1}^2\bar{q}\in S(M^{-1},G)$ and $\partial_{x_j}^2\bar{q}\in S(1,G)$ by Lemma 2.1. Now the assertion follows from the Glaeser's inequality. For the case (b) from $\partial_{\xi_d}\bar{q}=\partial_{\eta_d}\tilde{q}(y,\eta)r+\sum_{k\neq d}\partial_{\eta_k}\tilde{q}(y,\eta)r_k$ where $r\in S(\langle\xi\rangle_\gamma^{-1},G)$ and $r_k\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$ we have $\big|\partial_{\xi_d}q\big|\le CM^{-1/2}\sqrt{q}$. Since $\partial_{\eta_1}^2\tilde{q}(0,0)=0$ it results $\big|\partial_{\eta_1}\tilde{q}(y,\eta)\big|\le CM^{-1/2}\sqrt{\tilde{q}(y,\eta)}$ which shows $\big|\partial_{\xi_1}q\big|\le CM^{-1/2}\sqrt{q}$ because

$$\partial_{\xi_j}\bar{q}=\partial_{\eta_j}\tilde{q}(y,\eta)\big\{\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1}+r_{1j}\big\}$$
$$+\sum_{k\neq j,d}\partial_{\eta_k}\tilde{q}(y,\eta)r_{2k}+\partial_{\eta_d}\tilde{q}(y,\eta)\partial\eta_d/\partial\xi_j,\quad j\neq d$$

where $r_{ik}\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$ in view of Lemma 2.2. □

Proof of Lemma 2.4: Noting that $|\eta(\xi)+e_d|^2=\sum_{j=1}^{d-1}\eta_j^2+(\eta_d+1)^2=1+k$ with $k\in S(M^{-1},G)$ we see easily $1/|\eta(\xi)+e_d|=1+\tilde{k}$ with $\tilde{k}\in S(M^{-1},G)$ hence $\eta_1(\xi)/|\eta(\xi)+e_d|-\eta_1(\xi)\in S(M^{-1},G)$. Since $\psi(x,\xi)-\eta_1(\xi)/|\eta(\xi)+e_d|\in S(M^{-2},G)$ by Lemma 2.1 this together with Lemma 2.2 proves the case (a). The proof for the case (b) is similar. □

Proof of Lemma 2.5: Applying (4.1) for $\ell(t,y,\eta+e_d)=\tilde{\ell}(y,\eta)$ then the sum over $|\alpha+\beta|=1$ contains no η_d hence $\partial_{\xi_d}\ell\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$. Since $\partial_x^\alpha\psi\in S(M^{-1},G)$, (a) and $\partial_\xi^\alpha\psi\in S(1,G)$, (b) for $|\alpha|=1$ the rest of the proof follows from Lemma 2.4. □

4.2 Proof of Proposition 2.3

Write $z=(x,\xi)$ and $w=(y,\eta)$. Let g be either $\bar{g}$ or g_ϵ in (2.14). Note that if $\langle\eta\rangle_\gamma\le\langle\xi\rangle_\gamma/2\sqrt{2}$ then $|\xi-\eta|\ge(\gamma+|\xi|)/2\ge\langle\xi\rangle_\gamma/2$ hence $|\xi-\eta|^4\langle\eta\rangle_\gamma^{-2}\ge\gamma\langle\xi\rangle_\gamma/2$ and if $\langle\eta\rangle_\gamma\ge2\sqrt{2}\langle\xi\rangle_\gamma$ then $|\xi-\eta|\ge(\gamma+|\eta|)/2\ge\langle\eta\rangle_\gamma/2$ hence $|\xi-\eta|^4\langle\eta\rangle_\gamma^{-2}\ge\gamma\langle\eta\rangle_\gamma/16$. Therefore we have

$$\langle\xi\rangle_\gamma/\langle\eta\rangle_\gamma+\langle\eta\rangle_\gamma/\langle\xi\rangle_\gamma\le C\big(1+\gamma^{-1}\langle\eta\rangle_\gamma^{-2}|\xi-\eta|^4\big),\quad \xi,\eta\in\mathbb{R}^d.$$

Since $g_w(z-w)\ge M^{-2}\langle\eta\rangle_\gamma^{-1}|\xi-\eta|^2\ge\gamma^{-1/2}\langle\eta\rangle_\gamma^{-1}|\xi-\eta|^2$ for $\gamma\ge M^4$ one has

$$\langle\xi\rangle_\gamma/\langle\eta\rangle_\gamma+\langle\eta\rangle_\gamma/\langle\xi\rangle_\gamma\le C\big(1+g_w(z-w)\big)^2 \tag{4.2}$$

hence

$$g_z(X)/g_w(X)+g_w(X)/g_z(X)\le C\big(1+g_w(z-w)\big)^2,\quad 0\neq X\in\mathbb{R}^d\times\mathbb{R}^d \tag{4.3}$$

in particular g is σ temperate uniformly in $\gamma\ge M^4$. Note that (4.3) implies

$$g_{z+w}(w)\le C(1+g_z(w))^3. \tag{4.4}$$

It is clear from (4.2) that $\langle\xi\rangle_\gamma^s$, $s\in\mathbb{R}$ is admissible for g.

In this section $A \precsim B$ means that $A \le CB$ with some C independent of λ, M and γ with constraint (2.10).

Proof of Lemma 2.6: Since $\bar{q} \in S(M^{-2}, G)$ the Glaeser's inequality shows

$$|\partial_x^\alpha \partial_\xi^\beta \bar{q}| \precsim \langle\xi\rangle_\gamma^{-|\beta|}\sqrt{\bar{q}}\,, \quad |\alpha+\beta| = 1. \tag{4.5}$$

Together with $\bar{b} \ge \lambda^{1/2}\langle\xi\rangle_\gamma^{-1/2}$ and $\sqrt{\bar{q}} \le \bar{b}$ this proves that

$$|\partial_x^\alpha \partial_\xi^\beta \bar{b}| \precsim \lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\,\bar{b}\,, \quad |\alpha+\beta| = 1. \tag{4.6}$$

Assume (4.6) holds for $1 \le |\alpha+\beta| \le n$. Since $\bar{b}^2 = \bar{q} + \lambda\langle\xi\rangle_\gamma^{-1}$ then for $|\alpha+\beta| \ge n+1 \ge 2$ we see

$$\bar{b}\,\partial_x^\alpha \partial_\xi^\beta \bar{b} = \sum_{|\alpha'+\beta'|\ge 1} C_{\dots}\partial_x^{\alpha'}\partial_\xi^{\beta'}\bar{b}\cdot\partial_x^{\alpha''}\partial_\xi^{\beta''}\bar{b} + \partial_x^\alpha\partial_\xi^\beta\bar{q} + \lambda\,\partial_x^\alpha\partial_\xi^\beta\langle\xi\rangle_\gamma^{-1}.$$

Here note that

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \bar{q}| &\precsim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \\ &\precsim \bar{b}^2\lambda^{-1}M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \\ \precsim \bar{b}^2\lambda^{-1}(M^2\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} &\precsim \bar{b}^2\lambda^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \end{aligned} \tag{4.7}$$

since $\bar{b}^2\lambda^{-1}\langle\xi\rangle_\gamma \ge 1$ and $M^2\langle\xi\rangle_\gamma^{-1} \le 1$. On the other hand we have

$$\begin{aligned} |\partial_\xi^\beta\lambda\langle\xi\rangle_\gamma^{-1}| &\precsim \lambda\langle\xi\rangle_\gamma^{-1-|\beta|} \precsim \bar{b}^3\lambda^{-1/2}\langle\xi\rangle_\gamma^{1/2-|\beta|} \\ \precsim \bar{b}^3\lambda^{-1/2}\langle\xi\rangle_\gamma^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} &\precsim \bar{b}^2\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \end{aligned}$$

for $\bar{b} \le CM^{-1}$ from which we conclude (4.6) for any $|\alpha+\beta| \ge 1$ by induction. □

Proof of Lemma 2.7: Note that

$$\partial_x^\alpha\partial_\xi^\beta\bar{b} = (\partial_x^\alpha\partial_\xi^\beta\bar{q} + \lambda\partial_x^\alpha\partial_\xi^\beta\langle\xi\rangle_\gamma^{-1})/2\bar{b}.$$

Repeating a similar argument proving (4.7) we obtain

$$\begin{aligned} |\partial_x^{\alpha+\mu}\partial_\xi^{\beta+\nu}\bar{q}| &\precsim M^{-1}M^{|\mu+\nu|}\langle\xi\rangle_\gamma^{-(|\mu+\nu|-1)/2}\langle\xi\rangle_\gamma^{-1/2-|\beta|}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2} \\ &\precsim (M^2\langle\xi\rangle_\gamma^{-1})^{(|\mu+\nu|-1)/2}\langle\xi\rangle_\gamma^{-1/2-|\beta|}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2} \\ &\precsim \lambda^{-1/2}\langle\xi\rangle_\gamma^{-|\beta|}\,\bar{b}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2}, \quad |\alpha+\beta| = 1 \end{aligned}$$

for $|\mu+\nu| \ge 1$. This together with (4.5) shows $\partial_x^\alpha\partial_\xi^\beta\bar{q}/\bar{b} \in S(\langle\xi\rangle_\gamma^{-|\beta|}, g)$ for $|\alpha+\beta| = 1$. On the other hand it is easy to see

$$|\partial_\xi^{\beta+\nu}\lambda\langle\xi\rangle_\gamma^{-1}| \precsim \lambda\langle\xi\rangle_\gamma^{-1-|\beta+\nu|} \precsim \bar{b}^2\langle\xi\rangle_\gamma^{-|\beta|-|\nu|} \precsim \bar{b}M^{-1}\langle\xi\rangle_\gamma^{-|\beta|-|\nu|}$$

from which we conclude the assertion. □

Proof of Proposition 2.3: Note that $|\partial_x^\alpha \partial_\xi^\beta \bar{b}| \precsim \langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta|=1$ in view of (4.5). Assume $|\eta| \le c\,\langle\xi\rangle_\gamma$ hence

$$\langle\xi+s\eta\rangle_\gamma/C \le \langle\xi\rangle_\gamma \le C\langle\xi+s\eta\rangle_\gamma \tag{4.8}$$

where C is independent of $|s|\le 1$. Thus one has

$$|\bar{b}(z+w)-\bar{b}(z)| \le C(|y|+\langle\xi\rangle_\gamma^{-1}|\eta|) \le C\langle\xi\rangle_\gamma^{-1/2}\bar{g}_z^{1/2}(w) \le C\bar{b}(z)\bar{g}_z^{1/2}(w)$$

hence

$$\bar{b}(z+w) \le C\bar{b}(z)(1+\bar{g}_z(w))^{1/2} \tag{4.9}$$

When $|\eta| \ge c\langle\xi\rangle_\gamma$ then $\bar{g}_z(w) \ge c^2\langle\xi\rangle_\gamma$ hence

$$\bar{b}(z+w) \le C \le C\bar{b}(z)\lambda^{-1/2}\langle\xi\rangle_\gamma^{1/2} \le C'\bar{b}(z)(1+\bar{g}_z(w))^{1/2}$$

thus (4.9). Taking (4.4) and $\bar{g}^\sigma=\bar{g}$ into account we see that $\bar{b}$ is admissible for $\bar{g}$ hence so is b. Noting $\langle\xi\rangle_\gamma^s$ is admissible for $\bar{g}$ and $\langle\xi\rangle_\gamma^s \in S(\langle\xi\rangle_\gamma^s,\bar{g})$ and the proof is completed. $\Box$

Proof of Lemma 2.9: It is clear from (4.5) that $|\partial_x^\alpha \partial_\xi^\beta \bar{q}| \precsim \langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\bar{b}$ for $|\alpha+\beta|=1$. For $|\alpha+\beta|\ge 2$ one sees

$$\begin{gathered}|\partial_x^\alpha \partial_\xi^\beta \bar{q}| \precsim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}\\ \precsim \langle\xi\rangle_\gamma^{-1}(M^2\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \precsim \langle\xi\rangle_\gamma^{-1/2}\,\bar{b}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{gathered}$$

which proves the first assertion. In view of Lemma 2.2 it follows from the proof of Lemmas 2.1 and 2.3 that $\partial_{x_1}\bar{q} \in S(M^{-2},G)$, (a) and $\partial_{\xi_j}\bar{q} \in S(M^{-2}\langle\xi\rangle_\gamma^{-1},G)$, $j=1,d$, (b). Repeating the same arguments proving the first assertion we have

$$\begin{gathered}|\partial_x^\alpha \partial_\xi^\beta \partial_{x_1}\bar{q}| \precsim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \precsim M^{-1}\bar{b}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},\;\; (a)\\ |\partial_x^\alpha \partial_\xi^\beta \partial_{\xi_j}\bar{q}| \precsim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-1-|\beta|} \precsim M^{-1}\bar{b}\langle\xi\rangle_\gamma^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},\; j=1,d,\;\; (b)\end{gathered}$$

for $|\alpha+\beta|\ge 1$ which together with Lemma 2.3 completes the proof. $\Box$

Proof of Corollary 2.2: The first assertion is clear from Lemma 2.7. A repetition of the same arguments proving Lemma 2.9 shows $\partial_t\bar{q} \in S(\bar{b},\bar{g})$. Noting

$$\partial_x^\alpha \partial_\xi^\beta \partial_t \bar{b} = \partial_x^\alpha \partial_\xi^\beta \partial_t\bar{q}/(2\bar{b}) - \partial_x^\alpha \partial_\xi^\beta \bar{q}\,\partial_t\bar{q}/(4\bar{b}^3), \quad |\alpha+\beta|=1$$

and $\bar{b} \ge \langle\xi\rangle_\gamma^{-1/2}$ the assertion follows from Lemma 2.9 taking $\partial_{x_1}\partial_t\bar{q} \in S(M^{-1},G)$, (a) and $\partial_{\xi_j}\partial_t\bar{q} \in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$ for $j=1,d$, (b) into account. $\Box$

4.3 Proof of Proposition 2.5

Lemma 4.1. *We have* $\partial_x^\alpha \partial_\xi^\beta \psi \in S(\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g_\epsilon)$ *for* $|\alpha+\beta|\ge 1$. *Hence* $\partial_x^\alpha \partial_\xi^\beta \psi \in S(\omega M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2},g_\epsilon)$ *for* $|\alpha+\beta|=1$.

Proof. Recall that $\psi = \eta_1(\xi) + r$, (a) or $\psi = y_1(x) + cy_d(x) + r$, (b) with $r \in S(M^{-2}, G)$ in view of Lemma 2.4. Let $|\beta| \geq 1$ then

$$|\partial_\xi^\beta \psi| \precsim M^{-1-\delta_{\epsilon b}+|\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \precsim \langle\xi\rangle_\gamma^{-1/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\beta|-1)/2}.$$

Let $|\alpha| \geq 1$ then $|\partial_x^\alpha \psi| \precsim M^{-1-\delta_{\epsilon a}+|\alpha|}$ which is bounded by

$$\langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\alpha|-1)/2}.$$

Let $|\alpha| \geq 1$ and $|\beta| \geq 1$, recalling $\epsilon(\alpha,\beta) = \delta_{\epsilon a}|\alpha| + \delta_{\epsilon b}|\beta| \leq |\alpha+\beta|$, we have

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta\psi| &\precsim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}M^{2|\alpha+\beta|-2}\langle\xi\rangle_\gamma^{-(|\alpha+\beta|-1)/2}\\ &\precsim \langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^4\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-1)/2}.\end{aligned}$$

Since $M^4\langle\xi\rangle_\gamma^{-1} \leq 1$ by (2.4) the assertion follows. The second assertion is clear because $\langle\xi\rangle_\gamma^{-1/2} \leq \omega$. □

Lemma 4.2. *We have* $\partial_x^\alpha\partial_\xi^\beta\omega^s \in S(\omega^{s-1}\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ *for* $|\alpha+\beta| \geq 1$. *In particular* $\omega^s \in S(\omega^s, g_\epsilon)$.

Proof. First show the assertion for $s = 2$. Since $\omega^2 = (t-\psi)^2 + \langle\xi\rangle_\gamma^{-1}$ noting $\omega\langle\xi\rangle_\gamma^{1/2} \geq 1$ one sees for $|\beta| \geq 1$

$$\begin{aligned}\left|\partial_\xi^\beta\omega^2\right| &\precsim \omega M^{-1-\delta_{\epsilon b}+|\beta|}\langle\xi\rangle_\gamma^{-|\beta|} + M^{-2-2\delta_{\epsilon b}+|\beta|}\langle\xi\rangle_\gamma^{-|\beta|} + \langle\xi\rangle_\gamma^{-1-|\beta|}\\ &\precsim \omega\langle\xi\rangle_\gamma^{-1/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\beta|-1)/2}\\ &+\omega\langle\xi\rangle_\gamma^{-1/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\beta|-2)/2} + \omega\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{-|\beta|}\end{aligned}$$

where the second term $M^{-2-2\delta_{\epsilon b}+|\beta|}\langle\xi\rangle_\gamma^{-|\beta|}$ on the right-hand side is absent when $|\beta| = 1$. Let $|\alpha| \geq 1$ then we see

$$\begin{aligned}\left|\partial_x^\alpha\omega^2\right| &\precsim \omega M^{-1-\delta_{\epsilon a}+|\alpha|} + M^{-2-2\delta_{\epsilon a}+|\alpha|}\\ &\precsim \omega\langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\alpha|-1)/2}\\ &+\omega\langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\alpha|-2)/2}\end{aligned}$$

where if $|\alpha| = 1$ then the term $M^{-2-2\delta_{\epsilon a}+|\alpha|}$ on the right-hand side is absent. Let $|\alpha| \geq 1$ and $|\beta| \geq 1$. Then one has

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta\omega^2\right| &\precsim |\omega\partial_x^\alpha\partial_\xi^\beta r| + M^{-4+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|}\\ &\precsim \omega M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} + \omega M^{-4+|\alpha+\beta|}\langle\xi\rangle_\gamma^{1/2-|\beta|}\\ &\precsim \omega\langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^4\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-1)/2}\\ &+\omega\langle\xi\rangle_\gamma^{-1/2}(M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma)^{|\alpha|/2}(M^{2\delta_{\epsilon b}}\langle\xi\rangle_\gamma)^{-|\beta|/2}(M^4\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-2)/2}.\end{aligned}$$

Since $M^4\langle\xi\rangle_\gamma^{-1} \leq 1$ we have the first assertion for $s = 2$. Since $\langle\xi\rangle_\gamma^{-1/2} \leq \omega$ it is clear that $\omega^2 \in S(\omega^2, g_\epsilon)$ from which it follows easily that $\omega^s \in S(\omega^s, g_\epsilon)$ for any $s \in \mathbb{R}$. □

Lemma 4.3. *We have $\phi \in S(\phi, g_\epsilon)$.*

Proof. Let $|\alpha+\beta| = 1$ and write

$$\partial_x^\alpha \partial_\xi^\beta \phi = \frac{-\partial_x^\alpha \partial_\xi^\beta \psi}{\omega}\phi + \frac{\partial_x^\alpha \partial_\xi^\beta \langle\xi\rangle_\gamma^{-1}}{2\omega} = \phi_{\alpha\beta}\phi + \psi_{\alpha\beta}. \tag{4.10}$$

Since $\omega^{-1} \in S(\omega^{-1}, g_\epsilon)$ by Lemma 4.2 then

$$\begin{aligned}\big|\partial_x^\mu \partial_\xi^\nu\big(\psi_{\alpha\beta}\big)\big| &\precsim \omega^{-1}\langle\xi\rangle_\gamma^{-1} M^{-\epsilon(\mu,\nu)}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}\langle\xi\rangle_\gamma^{-|\alpha+\beta|/2}\\ &\precsim \phi M^{-\epsilon(\alpha+\mu,\beta+\nu)}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}\end{aligned}$$

in view of $\langle\xi\rangle_\gamma^{-|\alpha+\beta|/2} \leq M^{-\epsilon(\alpha,\beta)}$ and (2.13). On the other hand thanks to Lemmas 4.1 and 4.2 it follows that

$$|\partial_x^\mu \partial_\xi^\nu \phi_{\alpha\beta}| \precsim M^{-\epsilon(\alpha+\mu,\beta+\nu)}\langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}.$$

Hence using (4.10) the assertion is proved by induction on $|\alpha+\beta|$. □

Lemma 4.4. *We have*

$$\partial_x^\alpha \partial_\xi^\beta \phi \in S(\omega^{-1} M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi, g_\epsilon), \quad |\alpha+\beta| \geq 1.$$

Proof. One has $\phi_{\alpha\beta} \in S(\omega^{-1} M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ for $|\alpha+\beta| \geq 1$ by Lemma 4.1 . From Lemma 4.2 it follows that

$$\big|\partial_x^\mu \partial_\xi^\nu\big(\psi_{\alpha\beta}\big)\big| \precsim \omega^{-1}\langle\xi\rangle_\gamma^{-1-|\beta|} M^{-\epsilon(\mu,\nu)}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2}$$

for $|\alpha+\beta| \geq 1$ because $\partial_x^\alpha \partial_\xi^\beta \langle\xi\rangle_\gamma^{-1} \in S(\langle\xi\rangle_\gamma^{-1-|\beta|}, g_\epsilon)$ is clear. Since $C\phi\langle\xi\rangle_\gamma \geq 1$ and $\langle\xi\rangle_\gamma^{-|\beta|} \leq M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ hence

$$\psi_{\alpha\beta} \in S(\omega^{-1} M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\phi, g_\epsilon), \quad |\alpha+\beta| \geq 1.$$

Since $\phi \in S(\phi, g_\epsilon)$ by Lemma 4.3 we conclude the assertion from (4.10). □

Proof of Lemma 2.10: Assume (a). Since $\partial_{\xi_j}\psi \in S(M^{-1}\langle\xi\rangle_\gamma^{-1}, G)$ for $j \neq 1$ by Lemma 2.4 then the assertion follows from (4.10). The assertion for the case (b) is proved similarly. □

4.4 Proof of Proposition 2.6

We prove

Lemma 4.5. *There is $C > 0$ such that*

$$\omega(z+w) \leq C\omega(z)(1+g_{\epsilon,z}(w)), \;\; \phi(z+w) \leq C\phi(z)(1+g_{\epsilon,z}(w)). \tag{4.11}$$

In particular ω and ϕ are admissible weights for g_ϵ and $\bar{g}$.

Proof. First recall that $\langle\xi\rangle_\gamma^{-1/2} \le \omega \le CM^{-1}$. Assume $|\eta| \ge c\,\langle\xi\rangle_\gamma$ hence $g_{\epsilon,z}(w) \ge c^2M^{-2}\langle\xi\rangle_\gamma \ge c^2M^{-2}\langle\xi\rangle_\gamma^{1/2}\langle\xi\rangle_\gamma^{1/2} \ge \langle\xi\rangle_\gamma^{1/2}$. Therefore

$$\omega(z+w) \le CM^{-1} \le CM^{-1}\langle\xi\rangle_\gamma^{1/2}\omega(z) \le C\omega(z)(1+g_{\epsilon,z}(w)). \tag{4.12}$$

Assume $|\eta| \le c\,\langle\xi\rangle_\gamma$. Denote $f = t-\psi$ and $h = \langle\xi\rangle_\gamma^{-1/2}$ so that $\omega^2 = f^2+h^2$. Note that

$$\begin{aligned} |\omega(z+w)-\omega(z)| &= |\omega^2(z+w)-\omega^2(z)|/|\omega(z+w)+\omega(z)| \\ &\le 2|f(z+w)-f(z)| + 2|h(z+w)-h(z)| \end{aligned} \tag{4.13}$$

because $|f(z+w)+f(z)|/|\omega(z+w)+\omega(z)|$ and $|h(z+w)+h(z)|/|\omega(z+w)+\omega(z)|$ are bounded by 2. It is assumed that constants C may change from line to line but independent of $\gamma \ge M^2 \ge 1$. Noting $|f(z+w)-f(z)| = |\psi(z+w)-\psi(z)|$ it follows from Lemma 4.1 that

$$\begin{aligned} |f(z+w)-f(z)| &\le C(M^{-\delta_{\epsilon a}}|y| + M^{-\delta_{\epsilon b}}\langle\xi+s\eta\rangle_\gamma^{-1}|\eta|) \\ \le C\langle\xi\rangle_\gamma^{-1/2}&(M^{-\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{1/2}|y| + M^{-\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1/2}|\eta|) \le C\omega(z)g_{\epsilon,z}^{1/2}(w). \end{aligned} \tag{4.14}$$

Similarly we see $|h(z+w)-h(z)| \le C\langle\xi\rangle_\gamma^{-1}g_{\epsilon,z}^{1/2}(w) \le C\langle\xi\rangle_\gamma^{-1/2}\omega(z)g_{\epsilon,z}^{1/2}$. Therefore (4.13) gives $|\omega(z+w)-\omega(z)| \le C\omega(z)g_{\epsilon,z}^{1/2}(w)$ hence $\omega(z+w) \le C\omega(z)(1+g_{\epsilon,z}(w))^{1/2}$.

Turn to ϕ. If $|\eta| \ge \langle\xi\rangle_\gamma/2$ then $g_{\epsilon,z}(w) \ge M^{-2}\langle\xi\rangle_\gamma/4$ hence, taking into account (2.13) we have

$$\phi(z+w) \le CM^{-1} \le CM^{-2}\langle\xi\rangle_\gamma\phi(z) \le C\phi(z)(1+g_{\epsilon,z}(w)). \tag{4.15}$$

Assume $|\eta| \le \langle\xi\rangle_\gamma/2$ so that (4.8) holds. Note that $\phi(z+w)-\phi(z)$ is equal to

$$\frac{(f(z+w)-f(z))(\phi(z+w)+\phi(z)) + h^2(z+w)-h^2(z)}{\omega(z+w)+\omega(z)}. \tag{4.16}$$

for $\phi = \omega+f$. From (4.14) it results that $|f(z+w)-f(z)| \le C\langle\xi\rangle_\gamma^{-1/2}g_{\epsilon,z}^{1/2}(w)$. It is easy to see that $|h^2(z+w)-h^2(z)| \le CM\langle\xi\rangle_\gamma^{-3/2}g_{\epsilon,z}^{1/2}(w)$. Taking these into account (4.16) yeilds

$$\begin{aligned} |\phi(z+w)-\phi(z)| \le C\Big(&\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(\phi(z+w)+\phi(z)) \\ &+\frac{M\langle\xi\rangle_\gamma^{-3/2}}{\omega(z+w)+\omega(z)}\Big)(1+g_{\epsilon,z}(w))^{1/2}. \end{aligned} \tag{4.17}$$

Since $\phi(z) \ge M\langle\xi\rangle_\gamma^{-1}/C$ we have

$$\begin{aligned} |\phi(z+w)-\phi(z)| &\le C\Big(\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(\phi(z+w)+\phi(z)) \\ &\quad+\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}\phi(z)\Big)(1+g_{\epsilon,z}(w))^{1/2} \\ &= C(\phi(z+w)+2\phi(z))\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(1+g_{\epsilon,z}(w))^{1/2}. \end{aligned}$$

If $\langle\xi\rangle_\gamma^{-1/2}(1+g_{\epsilon,z}(w))^{1/2}/(\omega(z+w)+\omega(z))<1/3$ then it follows

$$\big|\phi(z+w)/\phi(z)-1\big|\leq(\phi(z+w)/\phi(z)+2)/3$$

from which we have $2\phi(z+w)/5\leq\phi(z)\leq 4\,\phi(z+w)$. If

$$\langle\xi\rangle_\gamma^{-1/2}(1+g_{\epsilon,z}(w))^{1/2}/(\omega(z+w)+\omega(z))\geq 1/3 \tag{4.18}$$

we have, noting $\phi(z)\geq\langle\xi\rangle_\gamma^{-1}/(2\omega(z))$, from (4.18)

$$18(1+g_{\epsilon,z}(w))\geq 4\langle\xi\rangle_\gamma\omega(z+w)\omega(z)\geq\phi(z+w)/\phi(z)$$

in view of an obvious inequality $2\,\omega(z+w)\geq\phi(z+w)$. Thus (4.15). From (4.11) and (4.4) it is clear that ω and ϕ are g_ϵ continuous. Then from (4.11) and (4.4) again ω and ϕ are admissible for g_ϵ since $g_\epsilon\leq g_\epsilon^\sigma$ hence also admissible for $\bar g$ because $g_\epsilon\leq\bar g=\bar g^\sigma$. □

4.5 Proof of lemmas in Section 2.3

Proof of Lemma 2.11: Note that $g_\epsilon\leq\bar g=\bar g^\sigma\leq g_\epsilon^\sigma$. In this proof every constant is independent of $\gamma\geq 1$ and M. It is clear that $p^{-1}\in S(m^{-1},g_\epsilon)$. Write $p\#p^{-1}=1-r$ where $r\in S(M^{-1},g_\epsilon)$. Since

$$|r|^{(l)}_{S(1,\bar g)}=\sup_{|\alpha+\beta|\leq l,(x,\xi)\in\mathbb{R}^{2d}}\big|\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2}\partial_x^\alpha\partial_\xi^\beta r\big|\leq C_lM^{-1}$$

from the L^2-boundedness theorem (see [4, Theorem 18.6.3]) we have $\|\mathrm{op}(r)\|\leq CM^{-1}$. Therefore for large M there exists the inverse $(1-\mathrm{op}(r))^{-1}$ in $\mathcal{L}(L^2,L^2)$ which is given by $1+\sum_{\ell=1}^\infty r^{\#\ell}\in S(1,\bar g)$. (see [1], [10], [9]). Denote $k=\sum_{\ell=1}^\infty r^{\#\ell}\in S(1,\bar g)$ and we will prove $k\in S(M^{-1},g_\epsilon)$. It can be seen from the proof (e.g. [10, Theorem 2.6.27], [9, Theorem I.1]) that for any $l\in\mathbb{N}$ one can find $C_l>0$, independent of γ, such that

$$|k|^{(l)}_{S(1,\bar g)}\leq C_l$$

because $|k|^{(l)}_{S(1,\bar g)}$ depends only on l, $|r|^{(l')}_{S(1,\bar g)}$ with some $l'=l'(l)$ and structure constants of $\bar g$ which is independent of γ. Note that k satisfies $(1-r)\#(1+k)=1$, that is

$$k=r+r\#k. \tag{4.19}$$

Since $r\in S(M^{-1},g_\epsilon)$ and $g_\epsilon\leq\bar g$ it follows from (4.19) that $\big|k\big|^{(l)}_{S(1,\bar g)}\leq C_lM^{-1}$. Assume that

$$\sup\big|\langle\xi\rangle_\gamma^{(|\beta|-|\alpha|)/2}\partial_x^\alpha\partial_\xi^\beta k\big|\leq C_{\alpha,\beta,\nu}M^{-1-l},\quad\epsilon(\alpha,\beta)\geq l \tag{4.20}$$

for $0\leq l\leq\nu$. Let $\epsilon(\alpha,\beta)\geq\nu+1$ and note that

$$\partial_x^\alpha\partial_\xi^\beta k=\partial_x^\alpha\partial_\xi^\beta r+\sum C_{\dots}\big(\partial_x^{\alpha''}\partial_\xi^{\beta''}r\big)\#\big(\partial_x^{\alpha'}\partial_\xi^{\beta'}k\big)$$

where $\alpha' + \alpha'' = \alpha$ and $\beta' + \beta'' = \beta$. From the assumption (4.20) we have $\partial_x^{\alpha'}\partial_\xi^{\beta'} k \in S(M^{-1-\nu}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$ if $\epsilon(\alpha', \beta') \geq \nu + 1$ and if $\epsilon(\alpha', \beta') \leq \nu$ we have $\partial_x^{\alpha'}\partial_\xi^{\beta'} k \in S(M^{-1-\epsilon(\alpha',\beta')}\langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2}, \bar{g})$. Since $r \in S(M^{-1}, g_\epsilon)$ one has

$$\big(\partial_x^{\alpha''}\partial_\xi^{\beta''} r\big)\#\big(\partial_x^{\alpha'}\partial_\xi^{\beta'} k\big) \in S(M^{-1-(\nu+1)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g})$$

which implies that (4.20) holds for $0 \leq l \leq \nu+1$ and hence for all ν by induction on ν. This proves that $k \in S(M^{-1}, g_\epsilon)$. The proof of the assertions for $\tilde{k}$ is similar. □

Proof of Lemma 2.12: One can assume $c = 0$. We see that $q(x,\xi) + M^{-1/2}$ is an admissible weight for $\bar{g}$ and $(q + M^{-1/2})^{1/2} \in S((q + M^{-1/2})^{1/2}, \bar{g})$. Moreover $\partial_x^\alpha\partial_\xi^\beta (q + M^{-1/2})^{1/2} \in S(M^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \bar{g})$ for $|\alpha + \beta| = 1$. Therefore

$$q + M^{-1/2} = (q + M^{-1/2})^{1/2}\#(q + M^{-1/2})^{1/2} + r, \quad r \in S(M^{-1}, \bar{g})$$

which proves the assertion. □

Proof of Lemma 2.14: First note that $m^{\pm 1/2}$ are admissible for g_ϵ and $m^{\pm 1/2} \in S(m^{\pm 1/2}, g_\epsilon)$. Since $m = m^{1/2}\#m^{1/2} - r$ with $r \in S(M^{-2}m, g_\epsilon)$ write

$$\tilde{r} = (1 + k)\#m^{-1/2}\#r\#m^{-1/2}\#(1 + \tilde{k}) \in S(M^{-1}, g_\epsilon)$$

such that $m^{1/2}\#\tilde{r}\#m^{1/2} = r$. Therefore one has $m = m^{1/2}\#(1 + \tilde{r})\#m^{1/2}$ and the first assertion follows from Lemma 2.13. Write

$$\tilde{q} = (1 + k)\#m^{-1/2}\#q\#m^{-1/2}\#(1 + \tilde{k}) \in S(1, g_\epsilon)$$

where $m^{1/2}\#(1 + k)\#m^{-1/2} = 1$ and $m^{-1/2}\#(1 + \tilde{k})\#m^{1/2} = 1$ such that $m^{1/2}\#\tilde{q}\#m^{1/2} = q$. Since k, $\tilde{k} \in S(M^{-1}, g_\epsilon)$ one can write $\tilde{q} = qm^{-1} + r$ with $r \in S(M^{-1}, g_\epsilon)$. Thanks to Lemma 2.13 we have $\|\mathrm{op}(qm^{-1})v\| \leq (\sup\big(|q|/m\big) + CM^{-1/2})\|v\|$ hence

$$\big|(\mathrm{op}(q)u, u)\big| \leq \big|(\mathrm{op}(qm^{-1})\mathrm{op}(m^{1/2})u, \mathrm{op}(m^{1/2})u)\big| + CM^{-1/2}\|\mathrm{op}(m^{1/2})u\|^2$$

proves the second assertion. □

Proof of Lemma 2.15 Write $\tilde{m}_2 = m_2\#m_1^{-1}\#(1 + k)$ such that $m_2 = \tilde{m}_2\#m_1$ with $k \in S(M^{-1}, g_\epsilon)$. Since $\tilde{m}_2 \in S(1, g_\epsilon)$ one has

$$\|\mathrm{op}(m_2)u\| = \|\mathrm{op}(\tilde{m}_2)\mathrm{op}(m_1)u\| \leq C'\|\mathrm{op}(m_1)u\|$$

which proves the assertion. □

5 Proof of Proposition 2.1

5.1 Geometric characterization of effectively hyperbolic singular points

In this subsection, for typographical reason, we write x_0, ξ_0 instead of t, τ respectively and $x = (x_0, x') = (x_0, x_1, \ldots, x_d)$, $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \ldots, \xi_d)$ so that $p(x,\xi) = -\xi_0^2 + a(x,\xi')$. Let $\rho = (0,\bar{\xi})$ be a singular point of $p = 0$ and hence $\bar{\xi}_0 = 0$. We denote $\rho' = (0,\bar{\xi}')$. Consider the Hamilton equation

$$\frac{d}{ds}\begin{bmatrix} x \\ \xi \end{bmatrix} = J\,\nabla p(x,\xi), \quad \nabla p(x,\xi) = \begin{bmatrix} \partial p(x,\xi)/\partial x \\ \partial p(x,\xi)/\partial \xi \end{bmatrix}, \quad J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$$

where I is the identity matrix of order $d+1$. We linearize the Hamilton equation at ρ. It is clear that the linearization is $dX/ds = J\,\nabla^2 p(\rho)X$ with $X = {}^t(x,\xi)$ where $\nabla^2 p(\rho)$ is the Hesse matrix of p at ρ. The coefficient matrix $J\nabla^2 p(\rho)$, denoted by $F_p(\rho)$, is called the Hamilton map of p at ρ. Therefore denoting the quadratic form defined by the Hesse matrix by $Q(X,Y) = \langle X, \nabla^2 p(\rho)Y\rangle$ it is clear that

$$Q(X,Y) = \langle JX, F_p(\rho)Y\rangle = \sigma(X, F_p(\rho)Y)$$

because ${}^tJJ = I_{2d+2}$ where $\sigma(X,Y) = \langle JX, Y\rangle$ is the symplectic two form on $V = \mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$. From the definition we see $p(\rho+\epsilon X) = \epsilon^2 Q(X)/2 + O(\epsilon^3)$ as $\epsilon \to 0$ and Q has the signature $(r,1)$ with some $r \geq 0$ because $a(x,\xi') \geq 0$. Since $a(x,\xi')$ is nonnegative near ρ' from the Morse lemma (e.g.[4, Lemma C.6.2]) one can find $\phi_1,\ldots,\phi_r$ and g vanishing at ρ', homogeneous of degree 1, 2 in ξ' respectively, C^∞ in a conic neighborhood of ρ' such that $\nabla\phi_1,\ldots,\nabla\phi_r$ are linearly independent at ρ', $g \geq 0$, $\nabla^2 g(\rho') = O$ and

$$a(x,\xi') = \sum_{j=1}^{r}\phi_j^2(x,\xi') + g(x,\xi'). \tag{5.1}$$

With $\phi_0 = \xi_0$ it is clear $Q(X,Y) = -\langle\nabla\phi_0, X\rangle\langle\nabla\phi_0, Y\rangle + \sum_{j=1}^r\langle\nabla\phi_j, X\rangle\langle\nabla\phi_j, Y\rangle$. Noting $\langle\nabla\phi_j, X\rangle = \sigma(X, H_{\phi_j})$ where $J\nabla\phi_j = H_{\phi_j}$, it follows that

$$Q(X,Y) = \sigma(X, F_pY) = \sigma\big(X, -\sigma(Y, H_{\phi_0})H_{\phi_0} + \sum_{j=1}^{r}\sigma(Y, H_{\phi_j})H_{\phi_j}\big)$$

and hence $F_pY = -\sigma(Y,H_{\phi_0})H_{\phi_0} + \sum_{j=1}^r \sigma(Y,H_{\phi_j})H_{\phi_j}$. In particular we see

$$\mathrm{Im}F_p = \mathrm{span}\langle H_{\phi_0}, H_{\phi_1},\ldots,H_{\phi_r}\rangle. \tag{5.2}$$

It is clear that $\mathrm{Ker}\,F_p = \{X \in V \mid \sigma(X,H_{\phi_j}) = 0, j = 0,\ldots,r\} = (\mathrm{Im}\,F_p)^\sigma$. Note that if $F_pX_\pm = \pm\lambda X_\pm$ with $\lambda \neq 0$ then $X_\pm \in \mathrm{Im}\,F_p$ so that X in the proof of Lemma 5.1 is a linear combination of H_{ϕ_j}, $j = 0,1,\ldots,r$. Denote by Γ the connected component of $\theta = -H_{x_0} = -J\nabla x_0$ in $\{X \in V \mid Q(X) \neq 0\}$ then

$$\Gamma = \{X = (x,\xi) \mid \xi_0^2 > \sum_{j=1}^{r}\langle\nabla\phi_j(\rho), X\rangle^2,\ \xi_0 > 0\} \tag{5.3}$$

which is an open cone in V. In what follows for $X \in V$ we denote by $\langle X\rangle$ the subspace spanned by X and $C = \{X \in V \mid \sigma(X,Y) \leq 0, Y \in \Gamma\}$ and $\Lambda = \mathrm{Ker}\,F_p$. Here recall [2, Corollary 1.4.7]:

Lemma 5.1. *If $F_p(\rho)$ has a nonzero real eigenvalue then $\Gamma \cap \Lambda^\sigma \neq \{0\}$.*

Proof. Let $\lambda \neq 0$ be a real eigenvalue. Show that $-\lambda$ is also an eigenvalue of F_p. Let $F_pX = \lambda X$, $X \neq 0$. Then from $0 = \sigma((F_p - \lambda)X, Y) = \sigma(X, (-F_p - \lambda)Y)$, $Y \in V$ we see that $F_p + \lambda$ is not surjective proving that $-\lambda$ is also an eigenvalue. Let $F_pX_\pm = \pm\lambda X_\pm$, $X_\pm \neq 0$ then $X_\pm \in \mathrm{Im}F_p = \Lambda^\sigma$. Note that the signature of Q is $(r, 1)$ with $r \geq 1$ otherwise $Q(X)$ would be $-\xi_0^2$ and hence F_p has no nonzero eigenvalues. Write $V = V_0 \oplus \mathrm{Ker}F_p$ (direct sum) and consider Q on V_0. Since Q is nondegenerate on V_0 then Q is of Lorenz signature. We may assume $X_\pm \in V_0$. If $\sigma(X_+, X_-) = 0$ then, since σ is anti-symmetric, Q vanishes on the 2 dimensional subspace in V_0 spanned by X_+ and X_- which is a contradiction. Thus $\sigma(X_+, X_-) \neq 0$. With $X = \alpha X_+ + \beta X_- \in \Lambda^\sigma$ we have

$$Q(X) = \sigma(\alpha X_+ + \beta X_-, \lambda\alpha X_+ - \lambda\beta X_-) = -2\alpha\beta\lambda\sigma(X_+, X_-).$$

Then choosing α, β such that $\alpha\beta\lambda\sigma(X_+, X_-) > 0$ we conclude either X or $-X$ is in Γ . □

Lemma 5.2. *The following three conditions are equivalent;*

(i) $\Gamma \cap \Lambda^\sigma \neq \{0\}$,

(ii) *there is a hyperplane $H \subset V$ such that $H \cap C = \{0\}$ and $\Lambda + \langle\theta\rangle \subset H$,*

(iii) $\Gamma \cap \Lambda^\sigma \cap \langle\theta\rangle^\sigma \neq \{0\}$.

Proof. (i) $\Longrightarrow$ (ii). First assume $\theta \in \Lambda + \Lambda^\sigma$ so that $\theta = X_1 + X_2$ with $X_1 \in \Lambda$ and $X_2 \in \Lambda^\sigma$. Then $0 \neq X_2 \in \Gamma$ since $\Gamma \cap \Lambda = \emptyset$ and $\Gamma + \Lambda \subset \Gamma$. It is clear that $\theta \in \langle X_2\rangle^\sigma$ and $\Lambda \subset \langle X_2\rangle^\sigma$. Suppose $\langle X_2\rangle^\sigma \cap C$ contains some $X \neq 0$. Since Γ is open then $X_2 + Y \in \Gamma$ if $|Y|$ is small hence $\sigma(X_2 + Y, X) = \sigma(Y, X) \leq 0$ for $X \in C$ which is a contradiction. Thus $H = \langle X_2\rangle^\sigma$ is a desired subspace. Next consider the case $\theta \notin \Lambda + \Lambda^\sigma$ and hence $(\Lambda + \Lambda^\sigma) \cap \langle\theta\rangle = \{0\}$. Take $0 \neq Z \in \Gamma \cap \Lambda^\sigma$ then recalling Γ is open we have

$$\Lambda \subset \langle Z\rangle^\sigma, \quad \langle Z\rangle^\sigma \cap C = \{0\}. \tag{5.4}$$

Thus denoting $T = \langle Z\rangle^\sigma \cap (\Lambda + \Lambda^\sigma)$ we see

$$\Lambda \subset T, \quad T \cap C = \{0\}. \tag{5.5}$$

Noting that $C \subset \Lambda^\sigma$ for $\Gamma + \Lambda \subset \Gamma$ it follows from (5.4) that $\Lambda + \Lambda^\sigma \not\subset \langle Z\rangle^\sigma$. This proves that $\dim T = \dim(\Lambda + \Lambda^\sigma) - 1$. Write $V = (\Lambda + \Lambda^\sigma) \oplus W_1$ and $\theta = Y_1 + Y_2$ with $Y_1 \in \Lambda + \Lambda^\sigma$ and $0 \neq Y_2 \in W_1$ and $W_1 = \langle Y_2\rangle \oplus W_2$. Then $H = T + \langle\theta\rangle + W_2$ is of codimension 1. From (5.5) and $C \subset \Lambda^\sigma$ we see $H \cap C = \{0\}$ and hence H is a desired hyperplane.
(ii) $\Longrightarrow$ (iii). Choose $0 \neq Y \in V$ such that $\langle Y\rangle = H^\sigma$ then $\langle Y\rangle \subset \Lambda^\sigma \cap \langle\theta\rangle^\sigma$. Show that Y or $-Y$ belongs to Γ. If not we would have $\langle Y\rangle \cap \Gamma = \emptyset$. Then by the Hahn-Banach theorem there is $0 \neq Z \in V$ such that $\sigma(Z, X) \leq 0, \forall X \in \Gamma$ and $\sigma(Z, X) \geq 0, \forall X \in \langle Y\rangle$. This shows that $Z \in C$ and $Z \in \langle Y\rangle^\sigma = H$ which is a contradictin.
(iii) $\Longrightarrow$ (i) is trivial. □

5.2 Proof of Proposition 2.1

In this subsection we return to the original notation and write t for x_0, $x = (x_1, \dots, x_d)$ and τ for ξ_0, $\xi = (\xi_1, \dots, \xi_d)$. After a suitable linear change of local coordinates x we may assume that $\bar{\xi} = (0, \dots, 0, 1) = e_d$. We write $\rho = (0, 0, e_d) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ and $\rho' = (0, e_d) \in \mathbb{R}^d \times \mathbb{R}^d$. Thanks to Lemma 5.2 one can take $0 \neq X \in \Gamma \cap \Lambda^\sigma \cap \langle\theta\rangle^\sigma$. From $X \in \Lambda^\sigma$, in view of (5.2), X is a linear combination of $H_{\phi_j}(\rho)$ such that $X = \sum_{j=1}^r \alpha_j H_{\phi_j}(\rho) + \alpha_0 H_{\phi_0}(\rho)$. Since $X \in \langle\theta\rangle^\sigma$ we have $\alpha_0 = 0$. We set

$$f(t, x, \xi) = \sum_{j=1}^r \alpha_j \phi_j(t, x, \xi)/|\xi|.$$

Since $H_f(\rho) = X \in \Gamma$, noting (5.3), it is clear that $\partial f/\partial t < 0$ at ρ therefore one can write $f(t, x, \xi) = e(t, x, \xi)(t - \psi(x, \xi))$ where $e(\rho) < 0$. It follows from (5.1)

$$a(t, x, \xi) \geq c\,(t - \psi(x, \xi))^2|\xi|^2 \tag{5.6}$$

with some $c > 0$. Since $-H_{t-\psi}(\rho) \in \Gamma$ we see from (5.3) that

$$1 > \sum_{j=1}^r \langle\nabla\phi_j(\rho), H_{t-\psi}(\rho)\rangle^2 = \sum_{j=1}^r \langle\nabla\phi_j(\rho), J\nabla(t-\psi)(\rho)\rangle^2 = \sum_{j=1}^r \{\phi_j, \psi\}^2(\rho)$$

from which, taking (5.1) and $\nabla^2 g(\rho) = O$ into account, we conclude that

$$\big|\{\psi, \{\psi, a\}\}(\rho)\big| < 2. \tag{5.7}$$

The next lemma is well known.

Lemma 5.3. *Assume $d\psi \neq 0$ and not proportional to dx_d at ρ'. Then one can find a system of local coordinates $x = (x_1, \dots, x_d)$ such that either $d\psi = d\xi_1$ or $d\psi = dx_1 + c\,dx_d$ with some $c \in \mathbb{R}$ at ρ'.*

Proof. Since $\partial_{\xi_d}\psi(\rho') = 0$ by the Euler's identity one can write $\psi(x, \xi) = \langle a', \xi'\rangle + \langle b', x'\rangle + b_d x_d + r(x, \xi)$ where $\xi' = (\xi_1, \dots, x_{d-1})$ and r vanishes at ρ' of order 2. If $a' = 0$ hence $b' \neq 0$ the assertion follows by a linear change of coordinates x'. If $a' \neq 0$ one can assume $\langle a', \xi'\rangle = \xi_1 + \cdots + \xi_k$ renumbering x_j, $1 \leq j \leq d-1$. Replacing the coordinate x_d by $x_d - \sum_{j=1}^k b_j x_j^2/2$ we can assume $\langle b, x\rangle = \sum_{j=k+1}^d b_j x_j$. Replacing again the coordinate x_d by $x_d - x_1\sum_{j=k+1}^d b_j x_j$ we can assume $b = 0$. Then after a linear change of coordinates $(x_1, \dots, x_k)$ the assertion follows. $\square$

In Lemma 5.3 we used coordinates change such that $y = x + q(x)$ where $q(x)$ is a quadratic form in x. If we cut $q(x)$ off outside a neighborhood of $x = 0$ the resulting change of coordinates is a linear transformation outside a neighborhood of $x = 0$.

Proof of Proposition 2.1: If $d\psi = 0$ or proportional to dx_d at ρ' it suffices to take $\ell = 0$ and $q = a$ because $\partial_{\xi_d}^2 a(\rho) = 0$ by the Euler's identity. Assume $d\psi(\rho') \neq 0$

and not proportional to dx_d. Thanks to Lemma 5.3 we may assume $d\psi = d\xi_1$ or $d\psi = dx_1 + cdx_d$. Assume $d\psi = d\xi_1$ at ρ'. If $\partial_{x_1}^2 a(\rho) = 0$ it suffices to take $\ell = 0$ and $b = a$. Otherwise thanks to the Malgrange preparation theorem (e.g.[3, Theorem 7.5.5]) one can write

$$a(t, x, \xi) = e(t, x, \xi)((x_1 - h(t, x', \xi))^2 + g(t, x', \xi)), \quad x' = (x_2, \ldots, x_d)$$

where $e > 0$ and h, g are of homogeneous of degree 0 vanishing at ρ. Choose

$$\ell(t, x, \xi) = e^{1/2}(t, x, \xi)(x_1 - h(t, x', \xi)), \;\; q(t, x, \xi) = e(t, x, \xi)g(t, x', \xi)$$

and set $\psi_1(t, x', \xi) = \psi(h(t, x', \xi), x', \xi)$ then $d\psi_1 = d\psi$ at ρ'. From (5.6) it follows that

$$q(t, x, \xi) \geq c(t - \psi_1(t, x', \xi))^2|\xi|^2, \quad c > 0.$$

Since $\partial\psi_1/\partial t = 0$ at ρ' one can write $t - \psi_1(t, x', \xi) = e'(t, x', \xi)(t - \psi_2(x', \xi))$. Since $d\psi_2 = d\psi_1$ at ρ' then $\{\psi_2, \{\psi_2, q\}\}(\rho) = 0$ hence it follows from (5.7) that $\{\ell, \psi_2\}^2(\rho) < 1$. Thus ψ_2 is a desired one. When $d\psi = dx_1 + cdx_d$ the proof is similar. □

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